

② a) Reflexivity: $\vdash \phi \rightarrow \phi$ ROA
 $\vdash (\phi \rightarrow \phi) \wedge (\phi \rightarrow \phi)$ \wedge -intro
 $\vdash \phi \leftrightarrow \phi$ defn of \leftrightarrow

Symmetry: Suppose $\vdash (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$
 $\vdash (\phi \rightarrow \psi)$ \wedge -elim
 $\vdash (\psi \rightarrow \phi)$ \wedge -elim
 $\vdash (\psi \rightarrow \phi) \wedge (\phi \rightarrow \psi)$ \wedge -intro

Transitivity: Suppose $\vdash \phi \leftrightarrow \psi$, $\vdash \psi \leftrightarrow \chi$
 $\vdash \phi \rightarrow \psi$ \wedge -elim, $\vdash \psi \rightarrow \chi$ \wedge -elim
 ~~$\vdash \psi \rightarrow \psi$ \wedge -intro~~
 $\phi \vdash \psi$ \rightarrow -elim, $\vdash \psi \rightarrow \chi$
 $\phi \vdash \chi$ \rightarrow -elim

$\vdash \phi \rightarrow \chi$ \rightarrow -intro
 Using similar reasoning, we also
 get $\vdash \chi \rightarrow \phi$.
 So $\vdash \chi \leftrightarrow \phi$ \wedge -intro.

b) There are exactly two valuations on Σ ,
 v and w , with $v(p)=1$ and $w(p)=0$.
 By soundness and completeness, $\vdash \phi \leftrightarrow \psi$ iff $\vDash \phi \leftrightarrow \psi$.
 Let $\psi_1, \psi_2, \psi_3, \psi_4 \in \text{Sent}(\Sigma)$ such that:

$v(\psi_1) = 1$,	$w(\psi_1) = 1$
$v(\psi_2) = 1$,	$w(\psi_2) = 0$
$v(\psi_3) = 0$,	$w(\psi_3) = 1$
$v(\psi_4) = 0$,	$w(\psi_4) = 0$

We know such sentences exist:
 Let $\psi_1 = p \vee \neg p$
 $\psi_2 = p$
 $\psi_3 = \neg p$
 $\psi_4 = p \wedge \neg p$

We claim $[\varphi_1]_R, [\varphi_2]_R, [\varphi_3]_R, [\varphi_4]_R$ are disjoint, and that $\bigcup_{i=1}^4 [\varphi_i]_R = \text{Sent}(\Sigma)$.

Suppose $\phi \in [\varphi_i]_R \cap [\varphi_j]_R$ with $i \neq j$.
~~(Without loss of generality,~~ consider e.g. $i=1, j=2$;
 the other cases are similar.

$$\phi \in \{ \psi \in \text{Sent}(\Sigma) : \models \psi \Leftrightarrow \varphi_1 \} \Rightarrow v(\phi) = 1, w(\phi) = 1$$

$$\phi \in \{ \psi \in \text{Sent}(\Sigma) : \models \psi \Leftrightarrow \varphi_2 \} \Rightarrow v(\phi) = 1, w(\phi) = 0$$

So $w(\phi) = 1$ and $w(\phi) = 0$, a contradiction.

Since ϕ was arbitrary, we conclude $[\varphi_1]_R \cap [\varphi_2]_R = \emptyset$.

Suppose $\phi \in \text{Sent}(\Sigma)$. Then v and w must both make an assignment to ϕ .

If e.g. $v(\phi) = 1$ and $w(\phi) = 1$, then $\phi \in [\varphi_1]_R$, and similarly for the other 3 possible cases. Thus,

$$\phi \in \bigcup_{i=1}^4 [\varphi_i]_R \text{ as desired.}$$

c) We want to show there is a unique function $\bar{v} : X \rightarrow \{0,1\}$ such that $v = \bar{v} \circ \varrho$ (by defn of what it means for the diagram to commute), and that there is a unique function $\bar{w} : X \rightarrow \{0,1\}$ such that $w = \bar{w} \circ \varrho$. (Here we refer to the two models named in part b.)

For v , let $\bar{v}([\varphi_1]_R) = 1$

$$\bar{v}([\varphi_2]_R) = 1$$

$$\bar{v}([\varphi_3]_R) = 0$$

$$\bar{v}([\varphi_4]_R) = 0$$

This clearly specifies a unique function $\bar{v}: X \rightarrow \{0,1\}$.

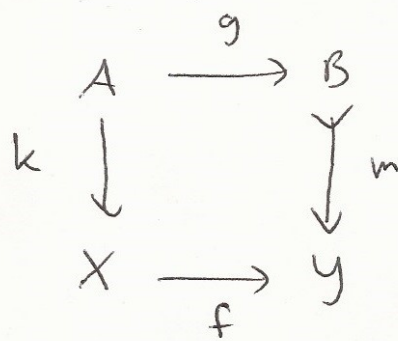
Let $\phi \in \text{Sent}(\Sigma)$. Since q is the quotient function, $q(\phi) = [\varphi_i]_R$ iff $\phi \in [\varphi_i]_R$. And

$\phi \in [\varphi_1]_R$ only if $v(\phi) = 1$, $\phi \in [\varphi_2]_R$ only if $v(\phi) = 1$, $\phi \in [\varphi_3]_R$ only if $v(\phi) = 0$, and

$\phi \in [\varphi_4]_R$ only if $v(\phi) = 0$. Thus $v = \bar{v} \circ q$.

A similar argument can be made for w .

We have the pullback square.

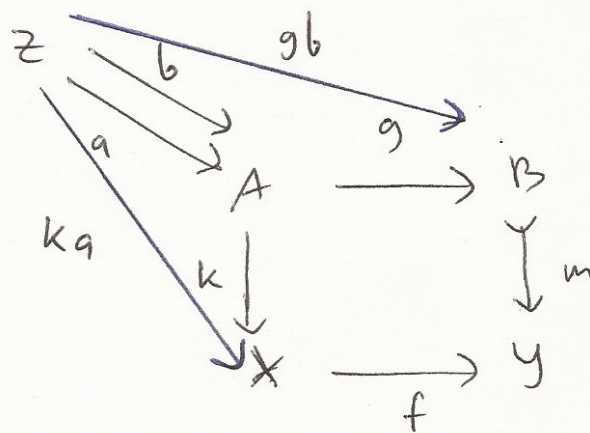


By definition, this square is commutative, i.e., $\boxed{fk = mg}$ (1)

Now, consider two $a, b: Z \rightarrow A$ s.t.

$ka = kb$. We want to show that $a = b$.

The diagram now is:

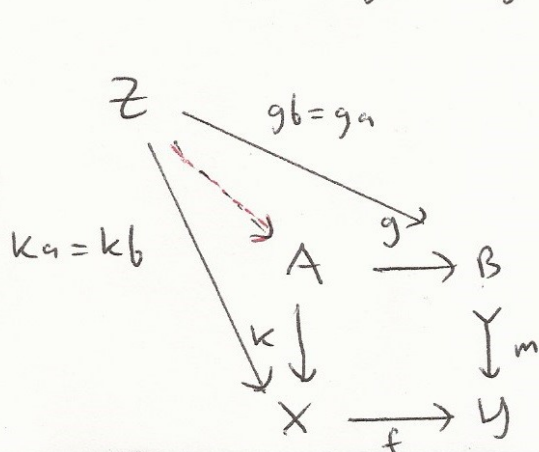


By def. of the pullback, $fka = mgb$.

By eq (1), (composed w/ a on the right), $fka = mga$.

So, $mgb = mga$. But m is a monomorphism.

$\Rightarrow gb = ga$. So we get the following diagram:



By the universal property of the pullback, function a is the unique s.t. diagram commutes. But both a & b satisfy the descriptor.

$\Rightarrow \boxed{a = b}$

QED