PHI 312 : pset 3

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① a) Suppose $f^* v = f^* w$ for some $v, w \in Mod(T')$.
We want to show $v = w$.

Since f is essentially surjective, $\forall \psi \in Sent(\Sigma')$,
$\exists \phi \in sent(\Sigma)$ such that $T' \vdash \psi \leftrightarrow f(\phi)$.

(*) Since $v \in Mod(T')$, $v(\psi \leftrightarrow f(\phi)) = 1$ (by soundness),
and similarly $w(\psi \leftrightarrow f(\phi)) = 1$.

Since $f^* v = f^* w$, we have, $\forall \phi \in Sent(\Sigma)$:

$$(f^* v)(\phi) \underset{\substack{\uparrow \\ \text{defn of } f^*}}{=} v(f(\phi)) = (f^* w)(\phi) \underset{\substack{\uparrow \\ \text{defn of } f^*}}{=} w(f(\phi))$$

Thus $v(f(\phi)) = w(f(\phi))$.

But by (*), $v(f(\phi)) = v(\psi)$ and $w(f(\phi)) = w(\psi)$
using the definition of $\leftrightarrow$.

So, $v(\psi) = w(\psi)$ $\forall \psi \in sent(\Sigma')$, i.e. $v = w$.

b) Suppose $\exists$ translation $g: T \to T'$ s.t.
$gf \simeq 1_T$ and $fg \simeq 1_{T'}$. We want to
Show f is essentially surjective.

Let $\psi \in Sent(\Sigma')$.

$T' \vdash 1_{T'}(\psi) \leftrightarrow fg(\psi)$ by defn of $\simeq$ extended to sentences.

So $T' \vdash \psi \leftrightarrow f(g(\psi))$ by defn of $1_{T'}$.

So $\forall \psi \in Sent(\Sigma')$, $\exists \phi \in Sent(\Sigma)$, namely $g(\psi)$, such that $T' \vdash \psi \leftrightarrow f(\phi)$.

(2) a) Reflexivity: $\vdash \phi \to \phi$ ROA

$\vdash (\phi \to \phi) \wedge (\phi \to \phi)$ $\wedge$-intro

$\vdash \phi \leftrightarrow \phi$ defn of $\leftrightarrow$

Symmetry: Suppose $\vdash (\phi \to \psi) \wedge (\psi \to \phi)$

$\vdash (\phi \to \psi)$ $\wedge$-elim

$\vdash (\psi \to \phi)$ $\wedge$-elim

$\vdash (\psi \to \phi) \wedge (\phi \to \psi)$ $\wedge$-intro.

Transitivity: Suppose $\vdash \phi \leftrightarrow \psi$, $\vdash \psi \leftrightarrow \varphi$.

$\vdash \phi \to \psi$ $\wedge$-elim, $\vdash \psi \to \varphi$ $\wedge$-elim

$\cancel{\vdash \psi \to \varphi \quad \wedge\text{-elim}}$

$\phi \vdash \psi$ $\to$-elim, $\vdash \psi \to \varphi$

$\phi \vdash \varphi$ $\to$-elim

$\vdash \phi \to \varphi$ $\to$-intro

Using similar reasoning, we also get $\vdash \varphi \to \phi$.

So $\vdash \varphi \leftrightarrow \phi$ $\wedge$-intro.

b) There are exactly two valuations on Σ, v and w, with $v(p) = 1$ and $w(p) = 0$.

By soundness and completeness, $\vdash \phi \leftrightarrow \psi$ iff $\models \phi \leftrightarrow \psi$.

Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \text{Sent}(\Sigma)$ such that:

$$\boxed{\begin{array}{ll} v(\varphi_1) = 1, & w(\varphi_1) = 1 \\ v(\varphi_2) = 1, & w(\varphi_2) = 0 \\ v(\varphi_3) = 0, & w(\varphi_3) = 1 \\ v(\varphi_4) = 0, & w(\varphi_4) = 0 \end{array}}$$

<u>We know such sentences exist</u>:

Let $\varphi_1 = p \vee \neg p$

$\varphi_2 = p$

$\varphi_3 = \neg p$

$\varphi_4 = p \wedge \neg p$

We claim $[\varphi_1]_R$, $[\varphi_2]_R$, $[\varphi_3]_R$, $[\varphi_4]_R$ are disjoint, and that $\bigcup_{i=1}^{4} [\varphi_i]_R = \text{Sent}(\Sigma)$.

Suppose $\phi \in [\varphi_i]_R \cap [\varphi_j]_R$ with $i \neq j$. ~~Without loss of generality,~~ (consider e.g. $i=1$, $j=2$; the other cases are similar.)

$\phi \in \{\psi \in \text{Sent}(\Sigma) : \models \psi \leftrightarrow \varphi_1\} \Rightarrow v(\phi)=1, w(\phi)=1$

$\phi \in \{\psi \in \text{Sent}(\Sigma) : \models \psi \leftrightarrow \varphi_2\} \Rightarrow v(\phi)=1, w(\phi)=0$

So $w(\phi)=1$ and $w(\phi)=0$, a contradiction.

Since ϕ was arbitrary, we conclude $[\varphi_1]_R \cap [\varphi_2]_R = \emptyset$.

Suppose $\phi \in \text{Sent}(\Sigma)$. Then v and w must both make an assignment to ϕ. If e.g. $v(\phi)=1$ and $w(\phi)=1$, then $\phi \in [\varphi_1]_R$, and similarly for the other 3 possible cases. Thus, $\phi \in \bigcup_{i=1}^{4} [\varphi_i]_R$ as desired.

c) We want to show there is a unique function $\bar{v} : X \to \{0,1\}$ such that $v = \bar{v} \circ q$ (by defn of what it means for the diagram to commute), and that there is a unique function $\bar{w} : X \to \{0,1\}$ such that $w = \bar{w} \circ q$. (Here we refer to the two models named in part b).)

For v, let $\bar{v}([\varphi_1]_R) = 1$

$\bar{v}([\varphi_2]_R) = 1$

$\bar{v}([\varphi_3]_R) = 0$

$\bar{v}([\varphi_4]_R) = 0$

This clearly specifies a unique function $\bar{v}: X \to \{0,1\}$.

Let $\phi \in \mathrm{Sent}(\Sigma)$. Since q is the quotient function, $q(\phi) = [\varphi_i]_R$ iff $\phi \in [\varphi_i]_R$. And $\phi \in [\varphi_1]_R$ only if $v(\phi) = 1$, $\phi \in [\varphi_2]_R$ only if $v(\phi) = 1$, $\phi \in [\varphi_3]_R$ only if $v(\phi) = 0$, and $\phi \in [\varphi_4]_R$ only if $v(\phi) = 0$. Thus $v = \bar{v} \circ q$.

A similar argument can be made for w.

We have the pullback square.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "k"'] & B \arrow[d, tail, "m"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

By definition, this square is commutative, i.e., $\boxed{fk = mg}$ (1)

Now, consider two $a, b : Z \to A$ s.t. $ka = kb$. We want to show that $a = b$.

The diagram now is:

$$\begin{tikzcd} Z \arrow[dr, shift left, "b"] \arrow[dr, shift right, "a"'] \arrow[drr, "gb"] \arrow[ddr, "ka"'] & & \\ & A \arrow[r, "g"] \arrow[d, "k"] & B \arrow[d, tail, "m"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

By def. of the pullback, $fka = mgb$.

By eq (1), (composed w/ a on the right), $fka = mga$.

So, $mgb = mga$. But m is a monomorphism.

$\Rightarrow gb = ga$. So we get the following diagram:

$$\begin{tikzcd} Z \arrow[dr, dashed] \arrow[drr, "gb=ga"] \arrow[ddr, "ka=kb"'] & & \\ & A \arrow[r, "g"] \arrow[d, "k"] & B \arrow[d, tail, "m"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

By the universal property of the pullback, function in red s.t. diagram commutes is unique. But both a & b satisfy this description.

$\Longrightarrow \boxed{a = b.}$ QED