Pset 5 solutions

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1. We know that f is injective, i.e. $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. We want to show that there is a function $g : Y \to X$ such that $gf = 1_X$. First, fix some $c \in X$ (since X is non-empty we know such an element exists). Next, note that if $y \in f(X)$, then there is an $x_y \in X$ such that $f(x_y) = y$. Since f is injective, x_y is the unique element of X that satisfies this criterion. So we define
$$g(y) = \begin{cases} x_y \text{ if } y \in f(X) \\ c \text{ otherwise} \end{cases}$$
Note that $(gf)(x) = g(f(x)) = x$ for all $x \in X$, so $gf = 1_X$ as desired.

2. By problem 1, there is an $f : X \to Y$ such that $gf = 1_X$. Thus g is split epi. So g is epi by the lemma proved in class. So g is surjective, since epi implies surjective (by proposition 2.7 in the notes).

3. (a) For the models of T we have: the valuation v^* which assigns 0 to all the propositional constants, plus the valuations that assign 1 to exactly one propositional constant. (Note that if a valuation assigns 1 to two or more propositional constants, it cannot assign 1 to all the axioms.) We can specify a function $f : \mathrm{Mod}(T) \to \mathbb{N}$:
$$f(v) = \begin{cases} 0 \text{ if } v = v^*, \\ i + 1 \text{ if } v(p_i) = 1 \end{cases}$$
This is a bijection. (Surjection is obvious. For injection, note that if $f(v_1) = f(v_2) = i + 1$ then $v_1(p_i) = v_2(p_i) = 1$, but we have already shown there is a single valuation that assigns p_i to 1, namely the valuation that assigns all the other proposition constants 0.) So $\mathrm{Mod}(T)$ is countably infinite.

   For T' we have the unique model v^* which assigns p_0 the value 1 (recall the hint for problem 2 in pset 1), and any valuation which assigns p_0 the value 0. Using the notation of pset 4, $\mathrm{Mod}(T') = \{v^*\} \cup C_{\neg p_0}$. Recall, using problem 3(b) of pset 4, that $C_{\neg p_0}$ is uncountably infinite. So, since $\mathrm{Mod}(T') \supseteq C_{\neg p_0}$, $\mathrm{Mod}(T')$ is uncountably infinite.

(b) Suppose toward contradiction that f is essentially surjective. By problem 1(a) on pset 3, $f^* : \mathrm{Mod}(T') \to \mathrm{Mod}(T)$ is injective. This entails that $|\mathrm{Mod}(T')| \leq |\mathrm{Mod}(T)|$. But, by part (a) of this problem, $\mathrm{Mod}(T)$ is countable and $\mathrm{Mod}(T')$ is uncountable, so $|\mathrm{Mod}(T')| > |\mathrm{Mod}(T)|$, a contradiction.