Problem Set 6: Solutions

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Problem 1

1. We assume that $f : A \to B$ is a function such that (1) $f(a \vee b) = f(a) \vee f(b)$ and (2) $f(\neg a) = \neg f(a)$. To show that f is a homomorphism, we also need to show that $f(0) = 0$, $f(1) = 1$, and $f(a \wedge b) = f(a) \wedge f(b)$.

   - We show $f(1) = 1$:

$$\begin{aligned} f(1) &= f(a \vee \neg a) && \text{(Excluded Middle)} \\ &= f(a) \vee f(\neg a) && \text{(Assumption 1)} \\ &= f(a) \vee \neg f(a) && \text{(Assumption 2)} \\ &= 1 && \text{(Excluded Middle)} \end{aligned}$$

   - We show $f(0) = 0$:

$$\begin{aligned} f(0) &= f(\neg 1) && \text{(Proposition 2.8)} \\ &= \neg f(1) && \text{(Assumption 2)} \\ &= \neg 1 && \text{(Previous Result)} \\ &= 0 && \text{(Proposition 2.8)} \end{aligned}$$

   - We show $f(a \wedge b) = f(a) \wedge f(b)$:

$$\begin{aligned} f(a \wedge b) &= f(\neg\neg(a \wedge b)) && \text{(Proposition 2.10)} \\ &= f(\neg(\neg a \vee \neg b)) && \text{(De Morgan's Law)} \\ &= \neg f(\neg a \vee \neg b) && \text{(Assumption 2)} \\ &= \neg(f(\neg a) \vee f(\neg b)) && \text{(Assumption 1)} \\ &= \neg f(\neg a) \wedge \neg f(\neg b) && \text{(De Morgan's Law)} \\ &= \neg\neg f(a) \wedge \neg\neg f(b) && \text{(Assumption 2 twice)} \\ &= f(a) \wedge f(b) && \text{(Proposition 2.10 twice)} \end{aligned}$$

Problem 2

Let us show that for all $x. u \in B$. that $x < u \Leftrightarrow x \wedge \neg u = 0$.

$(\Rightarrow)$

$$
\begin{aligned}
x \wedge \neg y &= 0 \vee (x \wedge \neg y) && \text{(bottom)} \\
&= (x \wedge \neg x) \vee (x \wedge \neg y) && \text{(excluded middle)} \\
&= x \wedge (\neg x \vee \neg y) && \text{(distributive)} \\
&= x \wedge \neg(x \wedge y) && \text{(De Morgan)} \\
&= x \wedge \neg x && (x \leq y) \\
&= 0 && \text{(excluded middle)}
\end{aligned}
$$

$(\Leftarrow)$

$$
\begin{aligned}
x &= x \wedge (y \vee \neg y) && \text{(top, excluded middle)} \\
&= (x \wedge y) \vee (x \wedge \neg y) && \text{(distributive)} \\
&= (x \wedge y) \vee 0 && (x \wedge \neg y = 0) \\
&= x \wedge y && \text{(bottom)} \\
x &\leq y && \text{(def.)}
\end{aligned}
$$

If $f(x) = 1$, then $a \leq x$, so $a \wedge \neg x = 0 \neq a$, so $a \not\leq \neg x$, so $f(\neg x) = 0$.

If $f(x) = 0$, then $a \not\leq x$, so $a \wedge \neg x \neq 0$. Let $a \wedge \neg x = b$, where $0 \neq b$. Then $b \wedge a = (\neg x \wedge a) \wedge a = \neg x \wedge a = b$, so $b \leq a$. So then because a is an atom and $b \neq 0$, then $b = a$. Thus, $a \wedge \neg x = a$, and $a \leq \neg x$. This means $f(\neg x) = 1$.

Combining the above two cases, we have $f(\neg x) = \neg f(x)$.

Let us now show that $f(x \vee y) = f(x) \vee f(y)$.

If $f(x) = 0$ and $f(y) = 0$, then by the negation property of f just proven, $f(\neg x) = 1$ and $f(\neg y) = 1$. So $a \leq \neg x$ and $a \leq \neg y$. By Proposition 2.2 in the notes, $a \leq \neg x \wedge \neg y$, so $f(\neg x \wedge \neg y) = 1$. By the negation property of f, $f(x \vee y) = f(\neg(\neg x \wedge \neg y)) = 0$. So $f(x \vee y) = 0 = 0 \vee 0 = f(x) \vee f(y)$.

The other case is where at least one of $f(x)$ and $f(y)$ is 1. Without loss of generality, take $f(x) = 1$. So $a \leq x$. But also, by absorption, $x \wedge (x \vee y) = x$, so $x \leq x \vee y$. So by transitivity, $a \leq x \vee y$. So $f(x \vee y) = 1 = 1 \vee 0 = 1 \vee 1 = f(x) \vee f(y)$.

So in all cases, $f(x \vee y) = f(x) \vee f(y)$ and $f(\neg x) = \neg f(x)$. So by the result of problem 1, $f(0) = 0$, $f(1) = 1$, and $f(x \wedge y) = f(x) \wedge f(y)$. So f is a homomorphism.

Problem 3a

Show that there is no Boolean algebra with exactly three elements.

Suppose there is a Boolean algebra $B = \{0, 1, a\}$ such that $a \neq 0$ and $a \neq 1$. By the definition of a Boolean algebra, it must be the case that $\neg a \in B$. There are three cases: (1) $\neg a = a$, (2) $\neg a = 0$, and (3) $\neg a = 1$. We need to show that each case results in a contradiction, thereby showing that B does not exist. We will assume $\neg 0 = 1$.

1.
   - $\neg a = a$
   - $\neg a \vee a = a \vee a$ — $\vee a$ on both sides
   - $1 = a \vee a$ — Excluded Middle
   - $1 = a$ — Idempotence
2.
   - $\neg a = 0$
   - $\neg\neg a = \neg 0$ — $\neg$ on both sides
   - $a = 1$ — $\neg\neg a = a, \neg 0 = 1$
3.
   - $\neg a = 1$
   - $\neg\neg a = \neg 1$ — $\neg$ on both sides
   - $a = 0$ — $\neg\neg a = a, \neg 1 = 0$

These all contradict the requirements that $a \neq 0$ and $a \neq 1$. Hence, there is no Boolean algebra with three elements.

Problem 3b

Suppose A and B are two Boolean algebras with four elements. So let us write $A = \{0, a, \neg a, 1\}$ and $B = \{0, b, \neg b, 1\}$. Now, let $f : A \to B$ be an isomorphism from A to B. In particular, since f is a homomorphism, $f(0) = 0$ and $f(1) = 1$. Note too that since f is a homomorphism then $f(\neg a) = \neg f(a)$. So to fully determine f one needs only to specify $f(a)$. Note that neither $f(a) = 0$ nor $f(a) = 1$: for either case would mean that f is not surjective (and so not an isomorphism). So either $f(a) = b$ or $f(a) = \neg b$. Both choices lead to isomorphisms. So we have two isomorphisms between A and B, as we wanted.