Pset 7 Solutions, compiled by Alex Meehan

Problem 1

We wish to show that if $E_\phi = E_{\phi'}$, then $E_{\neg\phi} = E_{\neg\phi'}$. We assume $E_\phi = E_{\phi'}$. By definition, $E_\phi := \{\psi \mid \phi \equiv \psi\}$. By reflexivity of equivalence relations, $\phi \in E_\phi$ and $\phi' \in E_{\phi'}$. Because $E_\phi = E_{\phi'}$, we know that $\phi' \in E_\phi$, as well. Thus, by definition of E_ϕ, we know that $\phi \equiv \phi'$.

By definition on page 12 of notes, $\phi \equiv \psi$ iff $T \vDash \phi \leftrightarrow \psi$. Thus, $T \vDash \phi \leftrightarrow \phi'$. By definition of $\vDash$, we see that in all models m of T, $m(\phi) = m(\phi')$. Thus, $m(\neg\phi) = \neg m(\phi) = \neg m(\phi') = m(\neg\phi')$ (by interpretation definition given on page 3 of note set 1). Thus, in all models m of T, we have $m(\neg\phi) = m(\neg\phi')$. Thus, by definition of $\vDash$, we have $T \vDash \neg\phi \leftrightarrow \neg\phi'$.

By definition on page 12 of notes, $\neg\phi \equiv \neg\phi'$. Assume for contradiction that $E_{\neg\phi} \neq E_{\neg\phi'}$. Then there is either some $\psi^* \in E_{\neg\phi}$ not in $E_{\neg\phi'}$, or there is some $\psi^* \in E_{\neg\phi'}$ not in $E_{\neg\phi}$. Consider the first case: $\psi^* \in E_{\neg\phi}$. By definition, $\neg\phi \equiv \psi^*$. By symmetry and transitivity of equivalence relations, we have $\neg\phi' \equiv \psi^*$, and so $\psi^* \in E_{\neg\phi'}$, by definition. Consider the second case: $\psi^* \in E_{\neg\phi'}$. Then $\psi^* \equiv \neg\phi'$, and by symmetry and transitivity of equivalence relations, $\psi^* \equiv \neg\phi$, and so by symmetry and definition, $\psi^* \in E_{\neg\phi}$.

Thus, we see that $\psi^* \in E_{\neg\phi} \implies \psi^* \in E_{\neg\phi'}$, and $\psi^* \in E_{\neg\phi'} \implies \psi^* \in E_{\neg\phi}$ Our assumption (for contradiction) is thus contradicted. Thus, $E_{\neg\phi} = E_{\neg\phi'}$. $\square$

Exercise 2. Let T be a theory with signature Σ, and let $L(T)$ be its Lindenbaum algebra. Suppose $E_\phi \le E_\psi$ where $\phi, \psi \in \mathrm{Sent}(\Sigma)$. Then $E_\phi \wedge E_\psi = E_\phi$ by definition. By Fact 4.7, it follows that $E_{\phi\wedge\psi} = E_\phi$ so $T \vdash (\phi \wedge \psi) \leftrightarrow \phi$. In particular $T \vdash \phi \to (\phi \wedge \psi)$ so by the rule of assumptions, $\to$ elimination, and $\wedge$ elimination $T, \phi \vdash \psi$.

Conversely, suppose $T, \phi \vdash \psi$ where $\phi, \psi \in \mathrm{Sent}(\Sigma)$. Then by the rule of assumptions, $\wedge$ introduction and $\to$ introduction $T \vdash \phi \to (\phi \wedge \psi)$. From the rule of assumptions $T, \phi \wedge \psi \vdash \phi \wedge \psi$ so by $\wedge$ elimination and $\to$ introduction we have $T \vdash (\phi\wedge\psi) \to \phi$. Thus $T \vdash (\phi\wedge\psi) \leftrightarrow \phi$ so $E_\phi = E_{\phi\wedge\psi} = E_\phi \wedge E_\psi$. It follows that $E_\phi \le E_\psi$.

3. We first verify that $\forall \phi, \psi$, $\bar{f}(-E_\phi) = -\bar{f}(E_\phi)$ and that $\bar{f}(E_\phi \cup E_\psi) = \bar{f}(E_\phi) \cup \bar{f}(E_\psi)$.

- We have that $\forall \phi$, $\bar{f}(-E_\phi) = \bar{f}(E_{\neg\phi}) = f(\neg\phi)$, and that $-\bar{f}(E_\phi) = -f(\phi)$. But since f is an interpretation, we have that $f(\neg\phi) = -f(\phi)$. Therefore $\bar{f}(-E_\phi) = -\bar{f}(E_\phi)$, as desired.
- We have that $\forall \phi, \psi$, $\bar{f}(E_\phi \cup E_\psi) = \bar{f}(E_{\phi\vee\psi}) = f(\phi \vee \psi)$, and that $\bar{f}(E_\phi) \cup \bar{f}(E_\psi) = f(\phi) \cup f(\psi)$. But since f is an interpretation, we have that $f(\phi \vee \psi) = f(\phi) \cup f(\psi)$. Therefore $\bar{f}(E_\phi \cup E_\psi) = \bar{f}(E_\phi) \cup \bar{f}(E_\psi)$, as desired.

But since $\forall \phi, \psi$, $\bar{f}(-E_\phi) = -\bar{f}(E_\phi)$ and $\bar{f}(E_\phi \cup E_\psi) = \bar{f}(E_\phi) \cup \bar{f}(E_\psi)$, we have that, by problem 1 of homework 6, $\bar{f}$ must be a Boolean homomorphism.

Exercise 4. Let B be a Boolean algebra with T_B the theory described in Proposition 4.10. The signature Σ_B of T_B is B viewed as a set. We have a conservative interpretation $e_B \colon T_B \to B$ sending every propositional constant $p \in \Sigma_B$ to $p \in B$. Suppose $\phi \in \mathrm{Sent}(\Sigma_B)$. Then $e_B(\phi)$ is some element p in the Boolean algebra B. But then p is an element of Σ_B because $B = \Sigma_B$ as sets. Certainly $e_B(p) = p$ by the definition of e_B so $e_B(\phi) = e_B(p)$. Because e_B is a conservative interpretation, we know by Lemma 4.1 that $T_B \vdash p \leftrightarrow \phi$