Problem Set 8: Solutions

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Problem 1

Assume there is no such element $a_1 \neq 0$ such that $a_1 < a$. Then if $b \neq 0$ and $b \leq a$, it must be that $b = a$. So $b = a$ is an atom such that $b \leq a$, which is a contradiction. So there must exist an element $a_1 \in B$ where $a_1 < a$.

We then use induction. Assume there is no element $a_{n+1} \neq 0$ such that $a_{n+1} < a_n$. Then if $b \neq 0$ and $b \leq a_n$, we get $b = a_n$, so a_n is an atom. But $a_n \leq a$, which is a contradiction. So there exists a_{n+1} such that $a_{n+1} < a_n$. By induction there is an infinite sequence $1 \geq a > a_1 > a_2 > \cdots$, so B is infinite.

Consider the filter $F_i = \uparrow(a_{i-1} \wedge \neg a_i)$, for $i \geq 2$. Because $a_{i-1} \leq a$, it is also true that $a_{i-1} \wedge \neg a_i \leq a \wedge \neg a_i$, that $a_{i-1} \wedge \neg a_i \leq a$, and that $a \in F_i$.

We have previously shown the inequality that $x \leq y \Leftrightarrow x \wedge \neg y = 0$ (PS6). We know that $a_i < a_{i-1}$. Thus $a_{i-1} \not\leq a_i$, so $a_{i-1} \wedge \neg a_i \neq 0$. Thus, $0 \notin F$, since $a_{i-1} \wedge \neg a_i \leq 0$ would imply $a_{i-1} \wedge \neg a_i = 0$, which is a contradiction. Then F at least contains $a_{i-1} \wedge \neg a_i$ and is non-empty. So F is a proper filter.

Let us show each such filter is distinct. Consider $a_k \wedge (a_{i-1} \wedge \neg a_i)$ $(\star)$. If $k \leq i-1$, then since $a_{i-1} \leq a_k$, we get $a_k \wedge a_{i-1} = a_{i-1}$ and this simplifies $(\star)$ to $a_{i-1} \wedge \neg a_i$, which is nonzero. If $k = i$, then this simplifies to $a_i \wedge a_{i-1} \wedge \neg a_i = 0$. This behavior is different for each filter because each one has a different i, so no two filters can be the same.

Because each filter is different, then each one expands to a distinct ultrafilter (expansion possible by UF axiom), as either $\neg a$ or a is in each ultrafilter, due to the bijection with homomorphisms $B \to 2$. So since the number of F_i is infinite, the number of ultrafilters containing a is also infinite. $\square$

Problem 2a & 2b

In this problem, we will use $\subseteq$ and $\leq$ interchangeably since they mean the same thing in the Boolean Algebra of all subsets of the natural numbers.

(a) Suppose that $A, B \in \mathcal{F}$. This means that $N \setminus A$ and $N \setminus B$ are finite. Then $N \setminus (A \cap B) = (N \setminus A) \cup (N \setminus B)$ is finite, so $A \cap B \in \mathcal{F}$. Next, suppose $A \in \mathcal{F}$ and $A \leq B$, which means that $A \subseteq B$. Then $N \setminus A$ is finite. Now $N \setminus B \subseteq N \setminus A$ since $A \subseteq B$. Therefore $N \setminus B$ is finite, so $B \in \mathcal{F}$. Therefore $\mathcal{F}$ is a filter.

(b) Suppose E is infinite and $E \cap X = \varnothing$ for some $X \in \mathcal{F}$. Then $E \subseteq N \setminus X$, which is impossible since E is infinite and $N \setminus X$ is finite. Therefore for all $X \in \mathcal{F}$, $E \cap X \neq \varnothing$.

Problem 2c

Proof of (c). Let $\uparrow(X, \mathscr{F}) = \{C \in \mathscr{P}N : A \cap B \subseteq C, A \in \uparrow(X), B \in \mathscr{F}\}$. I claim that $\uparrow(X, \mathscr{F})$ is a filter. If $C_0, C_1 \in \uparrow(X, \mathscr{F})$, then $A_i \cap B_i \subseteq C_i$, where $A_i \in \uparrow(X), B_i \in \mathscr{F}$. It follows that $(A_0 \cap A_1) \cap (B_0 \cap B_1) \subseteq C_0 \cap C_1$, so $C_0 \cap C_1 \in \uparrow(X, \mathscr{F})$. Now let $C_0 \in \uparrow(X, \mathscr{F})$ and $A_0 \cap B_0 \subseteq C_0$. If $C_1 \supseteq C_0$, then clearly $A_0 \cap B_0 \subseteq C_1$ by the transitivity of $\subseteq$. Thus, $\uparrow(X, \mathscr{F})$ is a filter.

To prove the claim, we construct ultrafilters from a sequence of filters $\{\uparrow(X_n, \mathscr{F})\}$. (Since N is countably infinite, assume without loss of generality that $N = \mathbb{N}$.) Let $X'_n = \{x \in N : x = p2^n, p \in \mathbb{N}\}$, and let $X_n = X'_{n-1} \setminus X'_n$ for $n \geq 1$. Since each X_n is infinite, $X_n \cap X \neq 0$ for all $X \in \mathscr{F}$ by Problem 8.2(b). Thus, $\uparrow(X_n, \mathscr{F})$ is a proper filter. Using **UF**, we extend each $\uparrow(X_n, \mathscr{F})$ to an ultrafilter $\mathscr{U}_n$ using the same arguments as in Problem 8.1.

I claim that $\mathscr{U}_n \neq \mathscr{U}_m$ if $n \neq m$. Assume without loss of generality that $n < m$. Then $n \leq m - 1$, so if $x \in X'_{m-1}$, then $x = p2^{m-1}$ for some p. It follows that $x = p2^n 2^{m-1-n} = p'2^n$ for some p'. Thus, $x \in X'_n$, and $X'_n \supseteq X'_{m-1}$. Now, $X'_{m-1} = X'_{m-1} \cap N \in \uparrow(X_{m-1}, \mathscr{F})$ and $\neg X'_n = \neg X'_n \cap N \in \uparrow(X_n, \mathscr{F})$ by Lemma 8.1. $X'_n \in \uparrow(X_m, \mathscr{F})$ by the definition of a filter. By the same reasoning as in Problem 8.1, $\mathscr{U}_n \neq \mathscr{U}_m$.

I also claim that $\mathscr{F} \subseteq \mathscr{U}_n$ for all n. This follows immediately from the fact that, for all $X \in \mathscr{F}$, $N \cap X = X \leq X$. Thus, $X \in \uparrow(X_n, \mathscr{F})$ and $\mathscr{F} \subseteq \uparrow(X_n, \mathscr{F}) \subseteq \mathscr{U}_n$.

We have shown there are infinitely many distinct $\mathscr{U}_n$. Each $\mathscr{U}_n$ contains $\mathscr{F}$. This concludes the proof. □