PHI 312 : pset 8

1. Let B be a Boolean algebra, and let a be a nonzero element of B that has no atoms below it. i.e. there is no atom $b \in B$ such that $b \leq a$. Show that there are infinitely many distinct ultrafilters on B containing a.

   [Hint: Show that the algebra B is infinite; in particular, there is an infinite sequence $a > a_1 > a_2 > \cdots$. Then consider the filters $\uparrow (a_{i-1} \wedge \neg a_i)$, where $\uparrow (x)$ denotes the set $\{y \in B \mid x \leq y\}$.]

2. Let N be a countably infinite set. We say that $E \subseteq N$ is **cofinite** just in case $N \backslash E$ is finite. Let $\mathscr{F}$ be the family of all cofinite subsets of N. We call $\mathscr{F}$ the **cofinite filter** on N.

   (a) Show that $\mathscr{F}$ is a filter on the Boolean algebra $\mathscr{P}N$ of all subsets of N.

   (b) Show that if $E \subseteq N$ is infinite, then E is compatible with $\mathscr{F}$ in the sense that $E \cap X$ is nonempty for each $X \in \mathscr{F}$.

   (c) Show that there are infinitely many distinct ultrafilters containing $\mathscr{F}$.