Pset 9 solutions

Exercise 1. By Proposition 6.5, $X = E \cup E'$ where E' denotes the set of limit points of E. We know that f and g agree on E so it suffices to show that f and g agree on all limit points of E. Suppose $f(x) \neq g(x)$ for some $x \in E'$. Because Y is Hausdorff, there are disjoint open neighborhoods U of $f(x)$ and V of $g(x)$ in Y. Because f and g are continuous, $f^{-1}(U)$ and $g^{-1}(V)$ are open in X so their intersection $f^{-1}(U) \cap g^{-1}(V)$ is an open neighborhood of $x \in E'$. Since x is a limit point, the set $f^{-1}(U) \cap g^{-1}(V)$ contains a point $p \in E$ not equal to x. But then $f(p) = g(p)$, but $f(p)$ lies in U and $g(p)$ lies in V, which is impossible because U and V are disjoint. Thus f and g agree on all limit points of E so $f = g$.

(2) Let $f : X \to Y$ be a continuous map where X and Y are compact Hausdorff spaces. Further suppose that $f(X)$ is dense in Y. Then we will show that $f(X) = Y$. First we'll need the lemma that if $E \subset Y$ is closed then $\overline{E} = E$. This is true because $\overline{E}$ is the intersection of all closed sets containing E. On one hand, this means that $E \subset \overline{E}$ because E is contained in each set in the intersection. On the other hand, since E is a closed set containing E we have that $\overline{E} \subset E$. Hence, $\overline{E} = E$, which is what we wanted.

We now turn to the problem. Since X is compact and f is continuous we have that $f(X) \subset Y$ is compact by proposition 6.12. Since Y is Hausdorff, this gives that $f(X)$ is closed by proposition 6.10. Moreover, $\overline{f(X)} = Y$ because $f(X)$ is dense. But by the lemma $\overline{f(X)} = f(X)$. This gives that $f(X) = \overline{f(X)} = Y$, which is what we wanted.

(3)

Proof. Since X is a Stone space, then X is totally separated. Thus by definition there exists a clopen subset of X, say Cl_x, containing x but not y. Then define the function $f : X \to \{0, 1\}$ such that

$$f(a) = \begin{cases} 1 & \text{if } a \in Cl_x \\ 0 & \text{otherwise} \end{cases}$$

Clearly $f(x) = 1 \neq 0 = f(y)$, and we claim that this is a continuous function. Indeed, $f^{-1}(\{1\}) = Cl_x$ is open, $f^{-1}(\{0\}) = X - Cl_x$ is open as the complement of a closed set, $f^{-1}(\{0, 1\}) = X$ is open and so is $f^{-1}(\emptyset) = \emptyset$. Since a map is continuous iff the pre-image of open sets is open, we are done. □