Completeness

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In what follows, we work with a fragment of first-order logic where the only connectives are $\wedge$ and $\neg$ and the only quantifier is $\exists$. This restriction is made only for convenience: every theorem we prove for this fragment of first-order logic also holds for full first-order logic. The reason that this claim is true is because full-first order logic is reducible — in a precise sense — to this fragment.

For convenience, we introduce a new 0-place predicate symbol (i.e. a sentence) $\perp$. For this new sentence $\perp$, we add new derivation rules and new clauses to our definition of an L structure.

	introduction	elimination
$\perp$	$\dfrac{\Gamma \Rightarrow \phi,\ \Delta \Rightarrow \neg\phi}{\Gamma \cup \Delta \Rightarrow \perp}$	$\dfrac{\Gamma \cup \{\phi\} \Rightarrow \perp}{\Gamma \Rightarrow \neg\phi}$

For any L structure M, we require that $\perp^M = \emptyset$. In other words, $\perp$ is false in every structure.

Definition. Let L be a first-order language and let T be a set of sentences in L. We say that (L, T) *has the C properties* just in case:

C1. For every closed term t of L, $t = t \in T$;

C2. For any closed terms t and t' of L, if $\phi(t) \in T$ and $t = t' \in T$ then $\phi(t') \in T$;

C3. If $\phi \wedge \psi \in T$ then $\phi \in T$ and $\psi \in T$;

C4. If $\phi \in T$ and $\psi \in T$ then $\phi \wedge \psi \in T$;

C5. If $\phi(t) \in T$, where t is a closed term, then $\exists x\phi(x) \in T$;

C6. If $\exists x \phi(x) \in T$ then $\phi(t) \in T$ for some closed term t of L;

C7. If $\phi \in T$ then $\neg\phi \notin T$;

C8. If $\phi \notin T$ then $\neg\phi \in T$.

Definition. Let T be a set of sentences. We say that T is *consistent* just in case $T \not\Rightarrow \perp$. Recall that, by the definition of $\Rightarrow$, $T \Rightarrow \phi$ if and only if $F \Rightarrow \phi$ for some finite subset F of ϕ. Hence $T \not\Rightarrow \perp$ if and only if there is no finite subset $F \subseteq T$ such that $F \Rightarrow \perp$.

Lemma 1. *If T is consistent and satisfies C8, then T satisfies all of the C properties except possibly C6.*

Exercise 1. Prove this lemma.

Lemma 2. *If T is a consistent set of sentences, then T is contained in a set T^+ that is consistent and such that either $\phi \in T^+$ or $\neg\phi \in T^+$ for every sentence ϕ.*

Exercise 2. Prove this lemma.

Lemma 3. *Suppose that T is a consistent set of sentences in the language L. Then there is a language $L^w \supseteq L$ and a consistent set $T^w \supseteq T$ of sentences in L^w such that if $\exists x \phi(x) \in T$ then $\phi(c) \in T^w$ for some constant symbol c of L^w.*

Exercise 3. Prove this lemma.

Exercise 4. Show that $T \not\Rightarrow \phi$ if and only if $T \cup \{\neg\phi\}$ is consistent.

Exercise 5. Show that $T \not\models \phi$ if and only if $T \cup \{\neg\phi\}$ is satisfiable.

Theorem 4 (Completeness). *If $T \models \phi$ then $T \Rightarrow \phi$.*

Exercise 6. Explain how completeness can be proved from the Lemmas in this handout, combined with the results from the Compactness handout. Hint: It is enough to show that if T is consistent then T is satisfiable.