MAT 313 Category Theory

Take home final exercise

Instructions: You may use all passive sources of information (e.g. books, internet). You may not discuss your solutions with other people. The exam is due by 5pm on Monday, January 17.

Definition 1. Let $G : A \longrightarrow X$ be a functor, and let $f, g : a \longrightarrow b$ be arrows in A. We say that the pair (f, g) is a *G-split-coequalizer pair* just in case the pair (Gf, Gg) is a split fork in X.

Definition 2. Consider an adjunction $\begin{tikzcd} A \arrow[r, bend left, "G"] \arrow[r, phantom, "\top"] & X \arrow[l, bend left, "F"] \end{tikzcd}$ and let $K : A \longrightarrow X^T$ be the comparison functor. We say that G is *of descent type* just in case K is full and faithful.

Definition 3. An arrow $f : a \longrightarrow b$ is called a *regular epimorphism* just in case it is the coequalizer of a parallel pair $g, h : x \longrightarrow a$.

Definition 4. An epimorphism $f : a \longrightarrow b$ is called *strong* just in case it is orthogonal to all monomorphisms, in the following sense: whenever there are arrows $g : a \longrightarrow c$ and $h : b \longrightarrow d$ and a monomorphism $m : c \longrightarrow d$ such that $m \circ g = h \circ f$, then there is a unique arrow $x : b \longrightarrow c$ such that $m \circ x = h$ and $x \circ f = g$.

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[d, "h"] \arrow[dl, dashed, "x"] \\
c \arrow[r, tail, "m"'] & d
\end{tikzcd}$$

Definition 5. An epimorphism $f : a \longrightarrow b$ is called *extremal* just in case whenever $f = m \circ g$ with m an monomorphism, then m is in fact an isomorphism.

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[dr, "g"'] & b \\
 & c \arrow[u, tail, "m"']
\end{tikzcd}$$

Problems

1. Suppose that $G : A \longrightarrow X$ has a left adjoint F. Show that G is of descent type if and only if G reflects coequalizers of all G-split-coequalizer pairs.

2. Show that strong epimorphisms are extremal.

3. Show that regular epimorphisms are strong.

4. Show that if f is both a monomorphism and an extremal epimorphism then it is an isomorphism.

5. Consider the following category REL: The objects of REL are sets, but an arrow $A \longrightarrow B$ is a relation on A, B, i.e. a subset of $A \times B$. If R is a relation on A, B and S is a relation on B, C, then the composite $S \circ R$ is defined by: $\langle x, y \rangle \in S \circ R$ if and only if there is a $z \in B$ such that $\langle x, z \rangle \in R$ and $\langle z, y \rangle \in S$. The identity arrow on an object A is just the diagonal relation $\{\langle x, x \rangle : x \in A\}$. Prove or refute: the category REL has finite limits.