Final Exam **PHI 340 Philosophical Logic**

Instructions: Please complete **two** problems. This exam is open book and open notes; but you should not consult with other persons. If you cite a result from homework, please include a copy of your proof. **The exam is due by 5pm on Monday, January 21**. When you are finished, put your exam in Professor Halvorson's mailbox in Room 212, 1879 Hall.

Problems:

1. The language **B4** has the same syntax as classical propositional logic (**C**). (We take "$\wedge$" and "$\neg$" as primitive connectives, and define the others by the standard equivalences.) The valuations of **B4** are functions from sentences to the set $\{0, 1, b, n\}$ satisfying:

$$\odot_{-}(1) = 0; \qquad \odot_{-}(0) = 1; \qquad \odot_{-}(b) = n; \qquad \odot_{-}(n) = b;$$

$$\odot_{\wedge}(b, n) = \odot_{\wedge}(n, b) = 0;$$

and for all x,

$$\odot_{\wedge}(1, x) = \odot_{\wedge}(x, 1) = x \qquad \odot_{\wedge}(0, x) = \odot_{\wedge}(x, 0) = 0.$$

(Please see pages 27–28 of *Possibilities and Paradox* for an explanation of this notation.) The designated values are 1 and b.

   (a) Show that each valuation of **C** is also a valuation of **B4**.

   (b) Show that if v is a valuation of **B4** that assigns an undesignated value to a sentence A, then there is a corresponding valuation v' of **C** that assigns the value 0 to A.

   (c) Define a single turnstile $\vdash$ for **B4** (i.e. construct either a natural deduction system or a tableaux system), and prove that it is sound and complete.

2. Show that there is no finite functional semantics relative to which the natural deduction system **RM** is both sound and complete.

3. For two natural deduction systems X and Y (formulated with the same syntax), let $\vdash_X$ and $\vdash_Y$ be the respective turnstiles. We say that Y is an

extension of X (written $X \leq Y$) just in case: if $\Gamma \vdash_X A$ then $\Gamma \vdash_Y A$. We say that Y is a proper extension of X (written $X < Y$) just in case Y is an extension of X, and there are Δ and B such that $\Delta \vdash_Y B$, but $\Delta \not\vdash_X B$. Show that **RW** $<$ **R** $<$ **RM** $<$ **C**. (Warning: It may not be so easy to show that **R** is a proper extension of **RW**.) You may say that "it is obvious" to justify the claim that adding new structural rules to a logic produces an extension of that logic.

4. Let **I** be intuitionistic propositional logic, and let **RM** be the relevance propositional logic that assumes contraction and mingle. True or False:

   (a) $\mathbf{I} \leq \mathbf{RM}$.

   (b) $\mathbf{RM} \leq \mathbf{I}$.

   If you say True, then please prove that the second logic is an extension of the first. If you say False, please provide an example of an argument that is provable in the first logic, but not in the second logic.

5. A sentence A of modal propositional logic is said to be *fully modalized* just in case each atomic sentence in A is within the scope of a modal connective (either $\Box$ or $\Diamond$). Let $\langle W, R, v\rangle$ be an **S5** model. For each $w \in W$, define the set $S(w)$ of worlds accessible to w by

$$S(w) = \{w' \in W : \langle w, w'\rangle \in R\}.$$

   Show that the truth value of any fully modalized sentence A is constant in $S(w)$; i.e. if $w_1, w_2 \in R(w)$ then $v(w_1, A) = v(w_2, A)$. (Hint: Use induction on sentence structure.)

6. Consider the following two natural deduction rules for modal logic:

$$\frac{\Gamma \vdash A \qquad \Gamma \text{ fully modalized}}{\Gamma \vdash \Box A}$$

$$\frac{\begin{array}{ll}\Gamma \vdash \Diamond A & \\ \Delta \cup \{A\} \vdash B & \qquad \Delta \text{ and } B \text{ fully modalized}\end{array}}{\Gamma \cup (\Delta - \{A\}) \vdash B}$$

   Show that these rules are both sound relative to the **S5** semantics; but one of the two rules is *not* sound relative to the **S4** semantics.

7. Show that a valuation of the modal logic K does not typically have a base (where "base" is defined on page 27 of *Possibilities and Paradox*).

8. We say that "$\circ$" is a *fusion connective* just in case the following is valid for all sentences A, B, C:

$$\vdash [A \to (B \to C)] \leftrightarrow [(A \circ B) \to C]$$

Show that for **RW**, **R**, and **RM**, a fusion connective can be defined in terms of the connectives $\to$ and $\neg$.

9. Let F be the standard embedding of intuitionistic propositional logic (**I**) into classical propositional logic (**C**), and let G be the Gödel translation of **C** into **I**.

   (a) True or false (justify your answer): for all sentences A of **C**, $FG(A)$ implies A, and A implies $FG(A)$.

   (b) True or false (justify your answer): for all sentences A of **I**, $GF(A)$ implies A, and A implies $GF(A)$.

   (c) Show that G is not "essentially surjective" (i.e. show that there is a sentence B of **I** such that B is not equivalent to any sentence of the form $G(A)$, with A a sentence of **C**).

10. Let **I** be intuitionistic logic, and let **S4** be the normal modal logic whose accessibility relation is reflexive and transitive. We will use $\neg$ to denote the negation connective of **I**, and $\sim$ to denote the negation connective of **S4**. Similarly, we will use $\to$ to denote the conditional of **I**, and $\supset$ to denote the material conditional of **S4**.

    Define a translation $G : \mathbf{I} \to \mathbf{S4}$ by setting $G(A) = A$ when A is atomic, $G(\neg A) = \Box \sim A$, $G(A \to B) = \Box(A \supset B)$, and $G(A * B) = A * B$ for any other binary connective $*$. Determine if each of the following claims is true:

    (a) If $\Delta \vdash_{\mathbf{I}} A$ then $G(\Delta) \vdash_{\mathbf{S4}} G(A)$.

    (b) If $G(\Delta) \vdash_{\mathbf{S4}} G(A)$ then $\Delta \vdash_{\mathbf{I}} A$.

    If the claim is true, prove it. If it is not true, provide a counterexample.