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Craig's Theorem

1. Introduction

In mathematical logic, Craig's Theorem (not to be confused with Craig's Interpolation Theorem) states that any recursively enumerable theory is recursively axiomatizable. Its epistemological interest concerns its possible use as a method of eliminating "theoretical content" from scientific theories.

Proof of Craig's Theorem: Assume that S is a deductively closed set of sentences, whose elements may be recursively enumerated thus $F(0)$, $F(1)$, …, $F(n)$, …, where F is a recursive function from natural numbers to sentences (we assume that expressions, sentences, etc., have been Gödel-coded in some manner). The set of theorems of an axiomatic theory is automatically recursively enumerable. But, in general, a recursively enumerable set is not automatically recursive. An example of a recursively enumerable set which is non-recursive is the set of logical truths in a first-order language with a single dyadic predicate (this result is known as Church's Theorem). However, by a trick devised by Craig, we can define a recursive set Craig(S) whose deductive closure is S. Let A be a sentence and n a natural number. Let A^n be the (n+1)-fold conjunction $A \wedge \ldots. \wedge A$. The sentence A^n is logically interdeducible with A. Next, consider sentences of the form $F(n)^n$. Define Craig(S) to be $\{F(n)^n: n \in N\}$. The deductive closure of Craig(S) must be S, since each element of Craig(S) is equivalent to an element of S. Next, we give an informal decision procedure for membership in Craig(S). Given a sentence A, to decide whether $A \in$ Craig(S), first check if A has the form B^n, for some sentence B and number n. By unique readability, this is checkable, and if A is not of this form, then $A \notin$ Craig(S). So, suppose that A is of the form B^n. We calculate $F(n)$, and if B is indeed $F(n)$, then $A \in$ Craig(S). And otherwise, $A \notin$ Craig(S). The existence of a decision procedure for membership in Craig(S) implies that Craig(S) is recursive. The set Craig(S) is therefore a recursive axiomatization of the theory S.

2. Craigian Elimination

The logical positivists held that, under a logical reconstruction, a scientific theory is an axiom system formulated in a language $L(O, T)$, where extra-logical predicates and function symbols are classified as either O-terms, for observational properties, or T-terms, for theoretical properties. Statements in $L(O, T)$ can be classified as observational, theoretical, or mixed, depending upon the presence or absence of O-terms or T-terms. Deleting theoretical terms yields a sublanguage $L(O)$, whose sentences express observational or empirical claims about the world. Assume that the property of being an $L(O)$-sentence is recursive. Consider a recursively enumerable theory S in $L(O, T)$. The empirical content of S is the set of $L(O)$-theorems of S. This is a subtheory of S obtained by a restriction on a recursive property. So, it is recursively enumerable too. By Craig's Theorem, there is a recursive set of $L(O)$-sentences whose deductive closure is the empirical content of S. On these assumptions, we can therefore recursively axiomatize the empirical content of any given scientific theory S, obtaining a recursive axiom system Craig(S), known as the Craigian reaxiomatization of S's empirical content.

3. Philosophical Significance of Craigian Elimination

Instrumentalism or positivism about science involves a scepticism towards the non-observational content of a scientific theory. Lacking such content, the Craigian reaxiomatization Craig(S) thus provides an object of rational belief compatible with instrumentalist or positivist scruples. Note that this elimination method need not be based on an observation/theory distinction. By obvious modification, it can be used as a way of eliminating, say, mathematical content from a scientific theory formulated using mathematical predicates and quantification over sets, functions, etc., or as a way of eliminating theoretical content from a psychological theory that refers to mental states, and so on. So, Craigian reaxiomatization offers a possible elimination strategy for a variety of instrumentalist positions.

4. Criticisms of Craigian Elimination

There are two methodological criticisms. First, even if the original theory *S* is presented in a simple manner, the reaxiomatization Craig(S) will be quite complex, and thus will violate the canon of *simplicity* which we might impose on admissible theories. Second, Craig(S) is *parasitic* upon the original theory *S*, and so does not really stand alone from the original theory. Indeed, Craig(S) is a bizarre theory, having infinitely many axioms of the form A^n, where A is an empirical consequence of *S*. Hartry Field refers to Craigian reaxiomatization as "bizarre trickery" and complains that Craig(S) is "obviously uninteresting, since [it] does nothing towards explaining the phenomenon in question in terms of a small number of basic principles" (Field 1980, p. 8). A third criticism is that Craigian elimination rests on a mistaken conception of scientific theories, namely a *syntactic view* of theories. This criticism has been urged by Bas van Fraassen, who writes "empirical import cannot be isolated syntactically … the reduced theory Craig(S) is not a description of the observable part of the world of *S*; rather it is a hobbled and hamstrung version of *S*'s description of everything" (van Fraassen 1976, pp. 87-88). A final criticism attacks the tenability of the observation/theory distinction required. For a simple example, although 'red' seems a paradigmatic observational term, we can nonetheless talk of red blood cells, which are too small to be visible to the naked eye (see Putnam 1962).

Modulo certain assumptions discussed above concerning the notion of "empirical content", Craig's Theorem tells us that we can reaxiomatize the empirical content of a scientific theory, thereby eliminating apparent reference to unobservable objects and properties. However, this elimination procedure has not found many adherents, and it seems safe to say that the significance of Craigian elimination is primarily pedagogical.

Bibliography

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