When can non-commutative statistical inference be Bayesian?

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Abstract

Based on recalling two characteristic features of Bayesian statistical inference in commutative probability theory, a stability property of the inference is pointed out, and it is argued that that stability of the Bayesian statistical inference is an essential property which must be preserved under generalization of Bayesian inference to the non-commutative case. Mathematical no-go theorems are recalled then which show that, in general, the stability can not be preserved in non-commutative context. Two possible interpretations of the impossibility of generalization of Bayesian statistical inference to the non-commutative case are offered, none of which seems to be completely satisfying.

1 Two features of Bayesian statistical inference

Bayesian statistical (or stochastic) inference in commutative context is characterized by two features:

1. The probability measures p, which are supposed to be defined on a Boole algebra S of events, are interpreted as the measures of rational belief of an abstract rational person (usually and hereafter called "agent") who is supposed to be capable of ideally logical thinking;

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2. The changes in the agent's degree of belief, $p \to p'$, on his learning an event E in S to be true (i.e having probability one) are given by conditionalization on E via the Bayes' rule:

$$p'(x) = p(xE)/p(E) \equiv p(x/E) \quad \text{for all } x \text{ in } S$$

or, more generally, if the agent happens to learn the probabilities of events of a whole subfield S_0 of S rather than the probability of a single event E, then

$$p'(x) = p(Tx) \qquad x \in S$$

where T is the m-conditional expectation from the commutative von Neumann algebra of essentially bounded m-measurable functions $L^\infty(S, m)$ onto its von Neumann subalgebra $L^\infty(S_0, m_0)$ with respect to an "a priori" probability measure m on S (and its restriction m_0 to S_0). The measure m is to be interpreted as the representative of the agent's background information and it is preserved under T (for the theory of classical i.e. commutative conditional expectation we refer to [Loeve 1960]).

2 Stability of Bayesian inference

There are a number of arguments in favour of reasonability of both (1) and (2) (for instance "Dutch book" arguments, see e.g. [Teller 1976]), which I do not intend to assess here. What I whish to point out is the following stability property of Bayesian statistical inference: If the agent reviews his *new* degrees of belief (p') in the light of the *same* evidence E *again*, he must conclude that

$$p''(x) = p'(xE)/p'(E) = [p(xEE)/p(E)]/[p(xE)/p(E)] = p'(x)$$

that is, the agent's newly revised degrees of belief (p'') do not differ from what he had already inferred from p on the basis of learning E. In the general case this stability is expressed and ensured by the fact that T is an m-preserving *projection* from $L^\infty(S, m)$ onto $L^\infty(S_0, m_0)$: $TTf = Tf$ (for all f in $L^\infty(S, m)$). (The events, i.e. the elements of S are identified with the characteristic functions in $L^\infty(S, m)$) .

This stability seems to be essential in Bayesian statistical inference, for let us assume that it does not hold. In this case the agent would find himself in a very frustrating situation every time after having learned the probabilities of the events in S_0, since he would have his new degrees of belief being p' after the first inference, but, looking at p' at a second time, and without having gained any new evidence, he would have to conclude that his new degrees of rational beliefs

should be rather $p'' \neq p'$ than p'. Given m, p and S_0 as the exclusive basis for the inference, the agent could not decide rationally between p' and p'', i.e. either he should choose one of them irrationally (by tossing a coin for instance), in which case the chosen new degrees of belief could no longer be considered as degrees of *rational* belief, or the agent's degrees of belief become indefinite. In either case, obviously, the inference would no longer be Bayesian in the sense of the above definition. In short: if the probability measures are interpreted as measures of degree of rational belief then the stochastic inference $p \rightarrow p'$ must be stable.

3 Difficulty with stability of Bayesian statistical inference in the non-commutative case

Now the question I wish to raise is whether the definition (1)-(2) can be generalized to the non-commutative case, where S is replaced by a non- commutative event structure, by the lattice $L(N)$ of projections of a non- commutative von Neumann algebra N, the latter one taking the role of $L^\infty(S, m)$, and where m is replaced by a state ϕ on N. As it turns out, the generalization of (1)-(2) to this non-commutative case can not be done, if one insists on the stability of the inference: if N_0 is an arbitrary von Neumann subalgebra of N, a ϕ-preserving conditional expectation, i.e. a ϕ -preserving *projection* from N onto N_0, does not exist in general (see. eg. [Takesaki 1972], [Accardi and Cecchini 1982]).

   For instance if N is assumed to be the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on the Hilbert space $\mathcal{H}$, and $L(\mathcal{B}(\mathcal{H}))$ is the lattice of projections, then there does not exist a conditional expectation from $\mathcal{B}(\mathcal{H})$ onto a subalgebra B_0, if B_0 is generated by the spectral projectors of a selfadjoint operator A having non-discrete spectrum ([Davies 1976] Chapter 4.3). Translated into the language of quantum physics, where $L(\mathcal{B}(\mathcal{H}))$ describes the simplest case of a quantum event structure and A represents an observable physical quantity with non discrete spectra, the non-existence of conditional expectation means that it is far from clear, what probabilities of physical events belonging to the lattice $L(\mathcal{B}(\mathcal{H}))$ of $\mathcal{B}(\mathcal{H})$ (but not lying in B_0) should an observer (agent) infer on the basis of knowing the probabilities of *all* events of the form "the physical quantity A has its value in the set d of real numbers".

4 Possible interpretations of violation of stability of non-commutative statistical inference

How can one interpret this mathematical fact? A possible conclusion could be that probability measures defined on a non-commutative event structure can not be interpreted as degrees of rational belief.

In particular, this conclusion would entail that the probabilities occurring in quantum mechanics can not be considered as degrees of rational belief, or, identifying the numbers $R(E) = 1 - p(E)$ with the measure of lack of knowledge concerning the quantum event E (where $p(E)$ is the degree of rational belief), one could equally say that quantum probabilities can not be reasonably viewed as measures of ignorance. This would be a further argument against the reasonability of what is called the "ignorance interpretation" of quantum state, and which has already been criticized on quite different grounds [Van Frassen 1980], [Ochs 1981], [Rédei 1985].

Another position could be that non-commutative probabilities can be interpreted as measures of rational belief, but the agent is able perform unfrustrated statistical inference only under special circumstances, namely only if a projection T from N onto N_0 does exist.

A ϕ-preserving conditional expectation from N onto N_0 is known to exist (and to be unique), if and only if the subalgebra N_0 is invariant for the $D(\phi)$ dynamic defined in a unique way by the a priori measure i.e. by the state ϕ ("Takesaki Theorem"). This dynamic is given as a one parameter group of automorphisms on N, and it also defines a dynamic in the state space of N i.e in the set of all probability measures on $L(N)$. (For the technically involved details of the Takesaki Theorem we refer to [Takesaki 1972]).

One can try to interpret the Takesaki Theorem for instance in the following way: While the agent's background information (represented by the state ϕ) remains the same, his degrees of rational belief are changing in time spontaneously (according to the dynamic $D(\phi)$) even without performing any inference based on gaining new knowledge. If the set N_0 of events whose probabilities are presented to the agent is not invariant for the natural change in time of the agent's degrees of belief, then he is not even able to associate the new probabilities to a well defined set of events, and so he is unable to make unambigious inference. In other words, the agent can perform statistical inference only on the basis of knowing the probabilities of events whose collection forms a constant, recognizable unit with respect to the natural dynamic in his measures of degrees of rational belief.

I regard this latter position very reasonable but also highly spec-

ulative, since, after all, one does not expect mathematics, especially particular mathematical theorems, to give insight into the psychic processes of human mind. Also, however, the categorical rejecting the possibility to interpret non-commutative probabilities as degrees of rational belief seems to be unsatisfactory because it is too strong a claim; furthemore, it is at least as speculative as the other position and for the same reason.

Thus, I think, one ought to look for further answers to the question:

When can a non-commutative statistical inference be Bayesian?

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