Chapter 1
C^*-algebras

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In this chapter we will collect those basic concepts and facts related to C^*-algebras that will be needed later on. We give complete proofs. In Sects. 1.1, 1.2, 1.3, and 1.6 we follow closely the presentation in [1]. For more information on C^*-algebras, see, e.g. [2–6].

1.1 Basic Definitions

Definition 1. *Let A be an associative $\mathbb{C}$-algebra, let $\|\cdot\|$ be a norm on the $\mathbb{C}$-vector space A, and let $* : A \to A$, $a \mapsto a^*$ be a $\mathbb{C}$-antilinear map. Then $(A, \|\cdot\|, *)$ is called a C^**-algebra, *if $(A, \|\cdot\|)$ is complete and we have for all $a, b \in A$:*

1. $a^{**} = a$ *($*$ is an involution)*
2. $(ab)^* = b^* a^*$
3. $\|ab\| \leq \|a\| \, \|b\|$ *(submultiplicativity)*
4. $\|a^*\| = \|a\|$ *($*$ is an isometry)*
5. $\|a^* a\| = \|a\|^2$ *(C^*-property)*

A (not necessarily complete) norm on A satisfying conditions (1) – (5) is called a C^*-norm.

Remark 1. Note that Axioms 1–5 are not independent. For instance, Axiom 4 can easily be deduced from Axioms 1,3, and 5.

Example 1. Let $(H, (\cdot,\cdot))$ be a complex Hilbert space, let $A = \mathcal{L}(H)$ be the algebra of bounded linear operators on H. Let $\|\cdot\|$ be the *operator norm*, i.e.,

$$\|a\| := \sup_{\substack{x \in H \\ \|x\|=1}} \|ax\|.$$

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Let a^* be the operator adjoint to a, i.e.,

$$(ax, y) = (x, a^*y) \qquad \text{for all } x,\ y \in H.$$

Axioms 1–4 are easily checked. Using Axioms 3 and 4 and the Cauchy–Schwarz inequality we see

$$\begin{aligned} \|a\|^2 &= \sup_{\|x\|=1} \|ax\|^2 = \sup_{\|x\|=1} (ax, ax) = \sup_{\|x\|=1} (x, a^*ax) \\ &\le \sup_{\|x\|=1} \|x\| \cdot \|a^*ax\| = \|a^*a\| \overset{\text{Axiom 3}}{\le} \|a^*\| \cdot \|a\| \overset{\text{Axiom 4}}{=} \|a\|^2. \end{aligned}$$

This shows Axiom 5.

Example 2. Let X be a locally compact Hausdorff space. Put

$$\begin{aligned} A := C_0(X) := \{f : X \to \mathbb{C} \text{ continuous} \mid \forall \varepsilon > 0\, \exists K \subset X \text{ compact, so that} \\ \forall x \in X \setminus K : |f(x)| < \varepsilon\}. \end{aligned}$$

We call $C_0(X)$ the algebra of continuous functions vanishing at infinity. If X is compact, then $A = C_0(X) = C(X)$. All $f \in C_0(X)$ are bounded and we may define

$$\|f\| := \sup_{x \in X} |f(x)|.$$

Moreover let

$$f^*(x) := \overline{f(x)}.$$

Then $(C_0(X), \|\cdot\|, *)$ is a commutative C^*-algebra.

Example 3. Let X be a differentiable manifold. Put

$$A := C_0^\infty(X) := C^\infty(X) \cap C_0(X).$$

We call $C_0^\infty(X)$ the algebra of smooth functions vanishing at infinity. Norm and $*$ are defined as in the previous example. Then $(C_0^\infty(X), \|\cdot\|, *)$ satisfies all axioms of a commutative C^*-algebra except that $(A, \|\cdot\|)$ is not complete. If we complete this normed vector space, then we are back to the previous example of continuous functions.

Definition 2. *A subalgebra A_0 of a C^*-algebra A is called a* C^*-subalgebra *if it is a closed subspace and $a^* \in A_0$ for all $a \in A_0$.*

Any C^*-subalgebra is a C^*-algebra in its own right.

Definition 3. *Let S be a subset of a C^*-algebra A. Then the intersection of all C^*-subalgebras of A containing S is called the* C^*-subalgebra generated by S.

Definition 4. *An element a of a C^*-algebra is called* self-adjoint *if $a = a^*$.*

Remark 2. Like any algebra a C^*-algebra A has at most one unit 1. Now we have for all $a \in A$

$$1^*a = (1^*a)^{**} = (a^*1^{**})^* = (a^*1)^* = a^{**} = a$$

and similarly one sees $a1^* = a$. Thus 1^* is also a unit. By uniqueness $1 = 1^*$, i.e., the unit is self-adjoint. Moreover,

$$\|1\| = \|1^*1\| = \|1\|^2,$$

hence $\|1\| = 1$ or $\|1\| = 0$. In the second case $1 = 0$ and therefore $A = 0$. Hence we may (and will) from now on assume that $\|1\| = 1$.

Example 4. 1. In Example 1 the algebra $A = \mathcal{L}(H)$ has a unit $1 = \mathrm{id}_H$.
2. The algebra $A = C_0(X)$ has a unit $f \equiv 1$ if and only if $C_0(X) = C(X)$, i.e., if and only if X is compact.

Let A be a C^*-algebra with unit 1. We write $A^\times$ for the set of invertible elements in A. If $a \in A^\times$, then also $a^* \in A^\times$ because

$$a^* \cdot (a^{-1})^* = (a^{-1}a)^* = 1^* = 1,$$

and similarly $(a^{-1})^* \cdot a^* = 1$. Hence $(a^*)^{-1} = (a^{-1})^*$.

Lemma 1. *Let A be a C^*-algebra. Then the maps*

$$\begin{aligned} A \times A \to A, &\quad (a, b) \mapsto a + b,\\ \mathbb{C} \times A \to A, &\quad (\alpha, a) \mapsto \alpha a,\\ A \times A \to A, &\quad (a, b) \mapsto a \cdot b,\\ A^\times \to A^\times, &\quad a \mapsto a^{-1},\\ A \to A, &\quad a \mapsto a^* \end{aligned}$$

are continuous.

Proof. (a) The first two maps are continuous for all normed vector spaces. This easily follows from the triangle inequality and from homogeneity of the norm.
(b) *Continuity of multiplication.* Let $a_0, b_0 \in A$. Then we have for all $a, b \in A$ with $\|a - a_0\| < \varepsilon$ and $\|b - b_0\| < \varepsilon$:

$$\begin{aligned} \|ab - a_0b_0\| &= \|ab - a_0b + a_0b - a_0b_0\|\\ &\le \|a - a_0\| \cdot \|b\| + \|a_0\| \cdot \|b - b_0\|\\ &\le \varepsilon\big(\|b - b_0\| + \|b_0\|\big) + \|a_0\| \cdot \varepsilon\\ &\le \varepsilon\big(\varepsilon + \|b_0\|\big) + \|a_0\| \cdot \varepsilon. \end{aligned}$$

(c) *Continuity of inversion.* Let $a_0 \in A^\times$. Then we have for all $a \in A^\times$ with $\|a - a_0\| < \varepsilon < \|a_0^{-1}\|^{-1}$

$$\begin{aligned}\|a^{-1} - a_0^{-1}\| &= \|a^{-1}(a_0 - a)a_0^{-1}\| \\ &\leq \|a^{-1}\| \cdot \|a_0 - a\| \cdot \|a_0^{-1}\| \\ &\leq \left(\|a^{-1} - a_0^{-1}\| + \|a_0^{-1}\|\right) \cdot \varepsilon \cdot \|a_0^{-1}\|.\end{aligned}$$

Thus

$$\underbrace{\left(1 - \varepsilon\|a_0^{-1}\|\right)}_{>0,\ \text{since}\ \varepsilon<\|a_0^{-1}\|^{-1}} \ \|a^{-1} - a_0^{-1}\| \leq \varepsilon \cdot \|a_0^{-1}\|^2$$

and therefore

$$\|a^{-1} - a_0^{-1}\| \leq \frac{\varepsilon}{1 - \varepsilon\|a_0^{-1}\|} \cdot \|a_0^{-1}\|^2.$$

(d) *Continuity of* $*$ is clear because $*$ is an isometry. □

Remark 3. If $(A, \|\cdot\|, *)$ satisfies the axioms of a C^*-algebra except that $(A, \|\cdot\|)$ is not complete, then the above lemma still holds because completeness has not been used in the proof. Let $\bar{A}$ be the completion of A with respect to the norm $\|\cdot\|$. By the above lemma $+$, $\cdot$, and $*$ extend continuously to $\bar{A}$ thus turning $\bar{A}$ into a C^*-algebra.

1.2 The Spectrum

Definition 5. *Let A be a C^*-algebra with unit* 1. *For $a \in A$ we call*

$$r_A(a) := \{\lambda \in \mathbb{C} \mid \lambda \cdot 1 - a \in A^\times\}$$

the resolvent set *of a and*

$$\sigma_A(a) := \mathbb{C} \setminus r_A(a)$$

the spectrum *of a. For $\lambda \in r_A(a)$*

$$(\lambda \cdot 1 - a)^{-1} \in A$$

is called the resolvent *of a at λ. Moreover, the number*

$$\rho_A(a) := \sup\{|\lambda| \mid \lambda \in \sigma_A(a)\}$$

is called the spectral radius *of a.*

Example 5. Let X be a compact Hausdorff space and let $A = C(X)$. Then

$$\begin{aligned} A^\times &= \{f \in C(X) \mid f(x) \neq 0 \text{ for all } x \in X\}, \\ \sigma_{C(X)}(f) &= f(X) \subset \mathbb{C}, \\ r_{C(X)}(f) &= \mathbb{C} \setminus f(X), \\ \rho_{C(X)}(f) &= \|f\|_\infty = \max_{x\in X} |f(x)|.\end{aligned}$$

Proposition 1. *Let A be a C^*-algebra with unit 1 and let $a \in A$. Then $\sigma_A(a) \subset \mathbb{C}$ is a nonempty compact subset and the resolvent*

$$r_A(a) \to A, \qquad \lambda \mapsto (\lambda \cdot 1 - a)^{-1}$$

is continuous. Moreover,

$$\rho_A(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|a^n\|^{\frac{1}{n}} \leq \|a\|.$$

Proof. (a) Let $\lambda_0 \in r_A(a)$. For $\lambda \in \mathbb{C}$ with

$$|\lambda - \lambda_0| < \|(\lambda_0 1 - a)^{-1}\|^{-1} \tag{1.1}$$

the Neumann series

$$\sum_{m=0}^{\infty} (\lambda_0 - \lambda)^m (\lambda_0 1 - a)^{-m-1}$$

converges absolutely because

$$\begin{aligned}\|(\lambda_0 - \lambda)^m (\lambda_0 1 - a)^{-m-1}\| &\leq |\lambda_0 - \lambda|^m \cdot \|(\lambda_0 1 - a)^{-1}\|^{m+1} \\ &= \|(\lambda_0 1 - a)^{-1}\| \cdot \underbrace{\left(\frac{\|(\lambda_0 1 - a)^{-1}\|}{|\lambda_0 - \lambda|^{-1}}\right)^m}_{<1 \text{ by } (1.1)}.\end{aligned}$$

Since A is complete the Neumann series converges in A. It converges to the resolvent $(\lambda 1 - a)^{-1}$ because

$$\begin{aligned}&(\lambda 1 - a) \sum_{m=0}^{\infty} (\lambda_0 - \lambda)^m (\lambda_0 1 - a)^{-m-1} \\ &= [(\lambda - \lambda_0)1 + (\lambda_0 1 - a)] \sum_{m=0}^{\infty} (\lambda_0 - \lambda)^m (\lambda_0 1 - a)^{-m-1} \\ &= -\sum_{m=0}^{\infty} (\lambda_0 - \lambda)^{m+1} (\lambda_0 1 - a)^{-m-1} + \sum_{m=0}^{\infty} (\lambda_0 - \lambda)^m (\lambda_0 1 - a)^{-m} \\ &= 1.\end{aligned}$$

Thus we have shown $\lambda \in r_A(a)$ for all λ satisfying (1.1). Hence $r_A(a)$ is open and $\sigma_A(a)$ is closed.

(b) *Continuity of the resolvent.* We estimate the difference of the resolvent of a at λ_0 and at λ using the Neumann series. If λ satisfies (1.1), then

$$\begin{aligned}
\left\|(\lambda 1-a)^{-1}-(\lambda_0 1-a)^{-1}\right\| &= \left\|\sum_{m=0}^{\infty}(\lambda_0-\lambda)^m(\lambda_0 1-a)^{-m-1}-(\lambda_0 1-a)^{-1}\right\| \\
&\le \sum_{m=1}^{\infty}|\lambda_0-\lambda|^m\,\|(\lambda_0 1-a)^{-1}\|^{m+1} \\
&= \|(\lambda_0 1-a)^{-1}\|\cdot\frac{|\lambda_0-\lambda|\cdot\|(\lambda_0 1-a)^{-1}\|}{1-|\lambda_0-\lambda|\cdot\|(\lambda_0 1-a)^{-1}\|} \\
&= |\lambda_0-\lambda|\cdot\frac{\|(\lambda_0 1-a)^{-1}\|^2}{1-|\lambda_0-\lambda|\cdot\|(\lambda_0 1-a)^{-1}\|} \\
&\to 0 \quad \text{for } \lambda\to\lambda_0.
\end{aligned}$$

Hence the resolvent is continuous.

(c) We show $\rho_A(a) \le \inf_n \|a^n\|^{\frac{1}{n}} \le \liminf_{n\to\infty}\|a^n\|^{\frac{1}{n}}$. Let $n \in \mathbb{N}$ be fixed and let $|\lambda|^n > \|a^n\|$. Each $m \in \mathbb{N}_0$ can be written uniquely in the form $m = pn + q$, p, $q \in \mathbb{N}_0$, $0 \le q \le n-1$. The series

$$\frac{1}{\lambda}\sum_{m=0}^{\infty}\left(\frac{a}{\lambda}\right)^m = \frac{1}{\lambda}\sum_{q=0}^{n-1}\left(\frac{a}{\lambda}\right)^q\sum_{p=0}^{\infty}\Big(\underbrace{\frac{a^n}{\lambda^n}}_{\|\cdot\|<1}\Big)^p$$

converges absolutely. Its limit is $(\lambda 1-a)^{-1}$ because

$$(\lambda 1-a)\cdot\left(\sum_{m=0}^{\infty}\lambda^{-m-1}a^m\right) = \sum_{m=0}^{\infty}\lambda^{-m}a^m - \sum_{m=0}^{\infty}\lambda^{-m-1}a^{m+1} = 1$$

and similarly

$$\left(\sum_{m=0}^{\infty}\lambda^{-m-1}a^m\right)\cdot(\lambda 1-a) = 1.$$

Hence for $|\lambda|^n > \|a^n\|$ the element $(\lambda 1-a)$ is invertible and thus $\lambda \in r_A(a)$. Therefore

$$\rho_A(a) \le \inf_{n\in\mathbb{N}}\|a^n\|^{\frac{1}{n}} \le \liminf_{n\to\infty}\|a^n\|^{\frac{1}{n}}.$$

(d) We show $\rho_A(a) \ge \limsup_{n\to\infty}\|a^n\|^{\frac{1}{n}}$. We abbreviate $\widetilde{\rho}(a) := \limsup_{n\to\infty}\|a^n\|^{\frac{1}{n}}$.

Case 1: $\widetilde{\rho}(a) = 0$. If a were invertible, then

$$1 = \|1\| = \|a^n a^{-n}\| \le \|a^n\|\cdot\|a^{-n}\|$$

would imply $1 \le \widetilde{\rho}(a)\cdot\widetilde{\rho}(a^{-1}) = 0$, which would yield a contradiction. Therefore $a \notin A^\times$. Thus $0 \in \sigma_A(a)$. In particular, the spectrum of a is nonempty. Hence the spectral radius $\rho_A(a)$ is bounded from below by 0 and thus

$$\widetilde{\rho}(a) = 0 \leq \rho_A(a).$$

Case 2: $\widetilde{\rho}(a) > 0$. If $a_n \in A$ are elements for which $R_n := (1 - a_n)^{-1}$ exist, then

$$a_n \to 0 \quad \Leftrightarrow \quad R_n \to 1.$$

This follows from the fact that the map $A^\times \to A^\times$, $a \mapsto a^{-1}$ is continuous by Lemma 1. Put

$$S := \{\lambda \in \mathbb{C} \mid |\lambda| \geq \widetilde{\rho}(a)\}.$$

We want to show that $S \not\subset r_A(a)$ since then there exists $\lambda \in \sigma_A(a)$ such that $|\lambda| \geq \widetilde{\rho}(a)$ and hence

$$\rho_A(a) \geq |\lambda| \geq \widetilde{\rho}(a).$$

Assume in the contrary that $S \subset r_A(a)$. Let $\omega \in \mathbb{C}$ be an nth root of unity, i.e., $\omega^n = 1$. For $\lambda \in S$ we also have $\lambda / \omega^k \in S \subset r_A(a)$. Hence there exists

$$\left(\frac{\lambda}{\omega^k} 1 - a\right)^{-1} = \frac{\omega^k}{\lambda}\left(1 - \frac{\omega^k a}{\lambda}\right)^{-1}$$

and we may define

$$R_n(a, \lambda) := \frac{1}{n} \sum_{k=1}^{n} \left(1 - \frac{\omega^k a}{\lambda}\right)^{-1}.$$

We compute

$$\begin{aligned}
\left(1 - \frac{a^n}{\lambda^n}\right) R_n(a, \lambda) &= \frac{1}{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \left(\frac{\omega^{k(l-1)} a^{l-1}}{\lambda^{l-1}} - \frac{\omega^{kl} a^l}{\lambda^l}\right)\left(1 - \frac{\omega^k a}{\lambda}\right)^{-1} \\
&= \frac{1}{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\omega^{k(l-1)} a^{l-1}}{\lambda^{l-1}} \\
&= \frac{1}{n} \sum_{l=1}^{n} \frac{a^{l-1}}{\lambda^{l-1}} \underbrace{\sum_{k=1}^{n} (\omega^{l-1})^k}_{= \begin{cases} 0 & \text{if } l \geq 2 \\ n & \text{if } l = 1 \end{cases}} \\
&= 1.
\end{aligned}$$

Similarly one sees $R_n(a,\lambda)\big(1-\frac{a^n}{\lambda^n}\big)=1$. Hence

$$R_n(a,\lambda)=\Big(1-\frac{a^n}{\lambda^n}\Big)^{-1}$$

for any $\lambda\in S\subset r_A(a)$. Moreover for $\lambda\in S$ we have

$$\begin{aligned}
&\Big\|\Big(1-\frac{a^n}{\widetilde{\rho}(a)^n}\Big)^{-1}-\Big(1-\frac{a^n}{\lambda^n}\Big)^{-1}\Big\|\\
&\leq\frac{1}{n}\sum_{k=1}^{n}\Big\|\Big(1-\frac{\omega^k a}{\widetilde{\rho}(a)}\Big)^{-1}-\Big(1-\frac{\omega^k a}{\lambda}\Big)^{-1}\Big\|\\
&=\frac{1}{n}\sum_{k=1}^{n}\Big\|\Big(1-\frac{\omega^k a}{\widetilde{\rho}(a)}\Big)^{-1}\Big(1-\frac{\omega^k a}{\lambda}-1+\frac{\omega^k a}{\widetilde{\rho}(a)}\Big)\Big(1-\frac{\omega^k a}{\lambda}\Big)^{-1}\Big\|\\
&=\frac{1}{n}\sum_{k=1}^{n}\Big\|\Big(\frac{\widetilde{\rho}(a)}{\omega^k}1-a\Big)^{-1}\Big(-\frac{\widetilde{\rho}(a)a}{\omega^k}+\frac{\lambda a}{\omega^k}\Big)\Big(\frac{\lambda}{\omega^k}1-a\Big)^{-1}\Big\|\\
&\leq|\widetilde{\rho}(a)-\lambda|\cdot\|a\|\cdot\sup_{z\in S}\|(z1-a)^{-1}\|^2.
\end{aligned}$$

The supremum is finite since $z\mapsto(z1-a)^{-1}$ is continuous on $r_A(a)\supset S$ by part (b) of the proof and since for $|z|\geq 2\cdot\|a\|$ we have

$$\|(z1-a)^{-1}\|\leq\frac{1}{|z|}\sum_{n=0}^{\infty}\underbrace{\frac{\|a\|^n}{|z|^n}}_{\leq(\frac{1}{2})^n}\leq\frac{2}{|z|}\leq\frac{1}{\|a\|}.$$

Outside the annulus $\overline{B}_{2\|a\|}(0)-B_{\widetilde{\rho}(a)}(0)$ the expression $\|(z1-a)^{-1}\|$ is bounded by $1/\|a\|$ and on the compact annulus it is bounded by continuity. Put

$$C:=\|a\|\cdot\sup_{z\in S}\|(z1-a)^{-1}\|^2.$$

We have shown

$$\|R_n(a,\widetilde{\rho}(a))-R_n(a,\lambda)\|\leq C\cdot|\widetilde{\rho}(a)-\lambda|$$

for all $n\in\mathbb{N}$ and all $\lambda\in S$. Putting $\lambda=\widetilde{\rho}(a)+\frac{1}{j}$ we obtain

$$\Big\|\Big(1-\frac{a^n}{\widetilde{\rho}(a)^n}\Big)^{-1}-\underbrace{\Big(1-\underbrace{\frac{a^n}{(\widetilde{\rho}(a)+\frac{1}{j})^n}}_{\to 0\text{ for }n\to\infty}\Big)^{-1}}_{\to 1\text{ for }n\to\infty}\Big\|\leq\frac{C}{j},$$

thus

$$\limsup_{n\to\infty} \left\| \left(1 - \frac{a^n}{\widetilde{\rho}(a)^n}\right)^{-1} - 1 \right\| \le \frac{C}{j}$$

for all $j \in \mathbb{N}$ and hence

$$\limsup_{n\to\infty} \left\| \left(1 - \frac{a^n}{\widetilde{\rho}(a)^n}\right)^{-1} - 1 \right\| = 0.$$

For $n \to \infty$ we get

$$\left(1 - \frac{a^n}{\widetilde{\rho}(a)^n}\right)^{-1} \to 1$$

and thus

$$\frac{\|a^n\|}{\widetilde{\rho}(a)^n} \to 0. \tag{1.2}$$

On the other hand we have

$$\begin{aligned}
\|a^{n+1}\|^{\frac{1}{n+1}} &\le \|a\|^{\frac{1}{n+1}} \cdot \|a^n\|^{\frac{1}{n+1}} \\
&= \|a\|^{\frac{1}{n+1}} \cdot \|a^n\|^{-\frac{1}{n(n+1)}} \cdot \|a^n\|^{\frac{1}{n}} \\
&\le \|a\|^{\frac{1}{n+1}} \cdot \|a\|^{-\frac{n}{n(n+1)}} \cdot \|a^n\|^{\frac{1}{n}} \\
&= \|a^n\|^{\frac{1}{n}}.
\end{aligned}$$

Hence the sequence $\left(\|a^n\|^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$ is monotonically nonincreasing and therefore

$$\widetilde{\rho}(a) = \limsup_{k\to\infty} \|a^k\|^{\frac{1}{k}} \le \|a^n\|^{\frac{1}{n}} \qquad \text{for all } n \in \mathbb{N}.$$

Thus $1 \le \|a^n\| / \widetilde{\rho}(a)^n$ for all $n \in \mathbb{N}$, in contradiction to (1.2).

(e) *The spectrum is nonempty.* If $\sigma(a) = \emptyset$, then $\rho_A(a) = -\infty$ contradicting $\rho_A(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} \ge 0$. □

Definition 6. *Let A be a C^*-algebra with unit. Then $a \in A$ is called*

- normal, *if $aa^* = a^*a$,*
- *an* isometry, *if $a^*a = 1$, and*
- unitary, *if $a^*a = aa^* = 1$.*

Remark 4. In particular, self-adjoint elements are normal. In a commutative algebra all elements are normal.

Proposition 2. *Let A be a C^*-algebra with unit and let $a, b \in A$. Then the following holds:*

1. $\sigma_A(a^*) = \overline{\sigma_A(a)} = \{\lambda \in \mathbb{C} \mid \overline{\lambda} \in \sigma_A(a)\}$.
2. *If* $a \in A^\times$, *then* $\sigma_A(a^{-1}) = \sigma_A(a)^{-1}$.
3. *If* a *is normal, then* $\rho_A(a) = \|a\|$.
4. *If* a *is an isometry, then* $\rho_A(a) = 1$.
5. *If* a *is unitary, then* $\sigma_A(a) \subset S^1 \subset \mathbb{C}$.
6. *If* a *is self-adjoint, then* $\sigma_A(a) \subset [-\|a\|, \|a\|]$ *and moreover* $\sigma_A(a^2) \subset [0, \|a\|^2]$.
7. *If* $P(z)$ *is a polynomial with complex coefficients and* $a \in A$ *is arbitrary, then*

$$\sigma_A\big(P(a)\big) = P\big(\sigma_A(a)\big) = \{P(\lambda) \mid \lambda \in \sigma_A(a)\}.$$

8. $\sigma_A(ab) - \{0\} = \sigma_A(ba) - \{0\}$.

Proof. We start by showing Assertion 1. A number λ does not lie in the spectrum of a if and only if $(\lambda 1 - a)$ is invertible, i.e., if and only if $(\lambda 1 - a)^* = \overline{\lambda} 1 - a^*$ is invertible, i.e., if and only if $\overline{\lambda}$ does not lie in the spectrum of a^*.

To see Assertion 2 let a be invertible. Then 0 lies neither in the spectrum $\sigma_A(a)$ of a nor in the spectrum $\sigma_A(a^{-1})$ of a^{-1}. Moreover, we have for $\lambda \neq 0$

$$\lambda 1 - a = \lambda a(a^{-1} - \lambda^{-1} 1)$$

and

$$\lambda^{-1} 1 - a^{-1} = \lambda^{-1} a^{-1}(a - \lambda 1).$$

Hence $\lambda 1 - a$ is invertible if and only if $\lambda^{-1} 1 - a^{-1}$ is invertible.

To show Assertion 3 let a be normal. Then $a^* a$ is self-adjoint, in particular normal. Using the C^*-property we obtain inductively

$$\begin{aligned}
\|a^{2^n}\|^2 &= \|(a^{2^n})^* a^{2^n}\| = \|(a^*)^{2^n} a^{2^n}\| = \|(a^* a)^{2^n}\| \\
&= \|(a^* a)^{2^{n-1}} (a^* a)^{2^{n-1}}\| = \|(a^* a)^{2^{n-1}}\|^2 \\
&= \cdots = \|a^* a\|^{2^n} = \|a\|^{2^{n+1}}.
\end{aligned}$$

Thus

$$\rho_A(a) = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \lim_{n\to\infty} \|a\| = \|a\|.$$

To prove Assertion 4 let a be an isometry. Then

$$\|a^n\|^2 = \|(a^n)^* a^n\| = \|(a^*)^n a^n\| = \|1\| = 1.$$

Hence

$$\rho_A(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 1.$$

For Assertion 5 let a be unitary. On the one hand we have by Assertion 4

$$\sigma_A(a) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}.$$

On the other hand we have

$$\sigma_A(a) \overset{(1)}{=} \overline{\sigma_A(a^*)} = \overline{\sigma_A(a^{-1})} \overset{(2)}{=} \overline{\sigma_A(a)}^{-1}.$$

Both combined yield $\sigma_A(a) \subset S^1$.

To show Assertion 6 let a be self-adjoint. We need to show $\sigma_A(a) \subset \mathbb{R}$. Let $\lambda \in \mathbb{R}$ with $\lambda^{-1} > \|a\|$. Then $|-i\lambda^{-1}| = \lambda^{-1} > \rho(a)$ and hence $1 + i\lambda a = i\lambda(-i\lambda^{-1} + a)$ is invertible. Put

$$U := (1 - i\lambda a)(1 + i\lambda a)^{-1}.$$

Then $U^* = ((1 + i\lambda a)^{-1})^*(1 - i\lambda a)^* = (1 - i\lambda a^*)^{-1} \cdot (1 + i\lambda a^*) = (1 - i\lambda a)^{-1} \cdot (1 + i\lambda a)$ and therefore

$$\begin{aligned} U^*U &= (1 - i\lambda a)^{-1} \cdot (1 + i\lambda a)(1 - i\lambda a)(1 + i\lambda a)^{-1} \\ &= (1 - i\lambda a)^{-1}(1 - i\lambda a)(1 + i\lambda a)(1 + i\lambda a)^{-1} \\ &= 1. \end{aligned}$$

Similarly $UU^* = 1$, i.e., U is unitary. By Assertion 5 $\sigma_A(U) \subset S^1$. A simple computation with complex numbers shows that

$$|(1 - i\lambda\mu)(1 + i\lambda\mu)^{-1}| = 1 \quad \Leftrightarrow \quad \mu \in \mathbb{R}.$$

Thus $(1 - i\lambda\mu)(1 + i\lambda\mu)^{-1} \cdot 1 - U$ is invertible if $\mu \in \mathbb{C} \setminus \mathbb{R}$. From

$$\begin{aligned} &(1 - i\lambda\mu)(1 + i\lambda\mu)^{-1} \cdot 1 - U \\ &\quad = (1 + i\lambda\mu)^{-1}\big((1 - i\lambda\mu)(1 + i\lambda a)1 - (1 + i\lambda\mu)(1 - i\lambda a)\big)(1 + i\lambda a)^{-1} \\ &\quad = 2i\lambda(1 + i\lambda\mu)^{-1}(a - \mu 1)(1 + i\lambda a)^{-1} \end{aligned}$$

we see that $a - \mu 1$ is invertible for all $\mu \in \mathbb{C} \setminus \mathbb{R}$. Thus $\mu \in r_A(a)$ for all $\mu \in \mathbb{C} \setminus \mathbb{R}$ and hence $\sigma_A(a) \subset \mathbb{R}$. The statement about $\sigma_A(a^2)$ now follows from part 7.

To prove Assertion 7 decompose the polynomial $P(z) - \lambda$ into linear factors

$$P(z) - \lambda = \alpha \cdot \prod_{j=1}^{n} (\alpha_j - z), \qquad \alpha, \alpha_j \in \mathbb{C}.$$

We insert an algebra element $a \in A$:

$$P(a) - \lambda 1 = \alpha \cdot \prod_{j=1}^{n} (\alpha_j 1 - a).$$

Since the factors in this product commute the product is invertible if and only if all factors are invertible.[1] In our case this means

$$\begin{aligned}\lambda \in \sigma_A\big(P(a)\big) &\Leftrightarrow \text{at least one factor is noninvertible}\\ &\Leftrightarrow \alpha_j \in \sigma_A(a) \text{ for some } j\\ &\Leftrightarrow \lambda = P(\alpha_j) \in P\big(\sigma_A(a)\big).\end{aligned}$$

If c is inverse to $1-ab$, then $(1+bca)\cdot(1-ba) = 1-ba+bc(1-ab)a = 1$ and $(1-ba)\cdot(1+bca) = 1-ba+b(1-ab)ca = 1$. Hence $1+bca$ is inverse to $1-ba$, which finally yields Assertion 8. □

Corollary 1. *Let $(A, \|\cdot\|, *)$ be a C^*-algebra with unit. Then the norm $\|\cdot\|$ is uniquely determined by A and $*$.*

Proof. For $a \in A$ the element a^*a is self-adjoint and hence

$$\|a\|^2 = \|a^*a\| \overset{2(3)}{=} \rho_A(a^*a)$$

depends only on A and $*$. □

1.3 Morphisms

Definition 7. *Let A and B be C^*-algebras. An algebra homomorphism*

$$\pi : A \to B$$

is called $*$-morphism *if for all $a \in A$ we have*

$$\pi(a^*) = \pi(a)^*.$$

A map $\pi : A \to A$ is called $*$-automorphism *if it is an invertible $*$-morphism.*

Corollary 2. *Let A and B be C^*-algebras with unit. Each unit-preserving $*$-morphism $\pi : A \to B$ satisfies*

$$\|\pi(a)\| \le \|a\|$$

for all $a \in A$. In particular, π is continuous.

Proof. For $a \in A^\times$

$$\pi(a)\pi(a^{-1}) = \pi(aa^{-1}) = \pi(1) = 1$$

[1] This is generally true in algebras with unit. Let $b = a_1 \cdots a_n$ with commuting factors. Then b is invertible if all factors are invertible: $b^{-1} = a_n^{-1} \cdots a_1^{-1}$. Conversely, if b is invertible, then $a_i^{-1} = b^{-1} \cdot \prod_{j \neq i} a_j$ where we have used that the factors commute.

holds and similarly $\pi(a^{-1})\pi(a) = 1$. Hence $\pi(a) \in B^\times$ with $\pi(a)^{-1} = \pi(a^{-1})$. Now if $\lambda \in r_A(a)$, then

$$\lambda 1 - \pi(a) = \pi(\lambda 1 - a) \in \pi(A^\times) \subset B^\times,$$

i.e., $\lambda \in r_B(\pi(a))$. Hence $r_A(a) \subset r_B(\pi(a))$ and $\sigma_B(\pi(a)) \subset \sigma_A(a)$. This implies the inequality

$$\rho_B(\pi(a)) \leq \rho_A(a).$$

Since π is a $*$-morphism and a^*a and $\pi(a)^*\pi(a)$ are self-adjoint we can estimate the norm as follows:

$$\begin{aligned}\|\pi(a)\|^2 &= \|\pi(a)^*\pi(a)\| = \rho_B\big(\pi(a)^*\pi(a)\big) = \rho_B\big(\pi(a^*a)\big)\\ &\leq \rho_A(a^*a) = \|a\|^2.\end{aligned}$$

□

Corollary 3. *Let A be a C^*-algebra with unit. Then each unit-preserving $*$-automorphism $\pi : A \to A$ satisfies for all $a \in A$:*

$$\|\pi(a)\| = \|a\|.$$

Proof.

$$\|\pi(a)\| \leq \|a\| = \|\pi^{-1}\big(\pi(a)\big)\| \leq \|\pi(a)\|.$$

□

If $P(z) = \sum_{j=0}^n c_j z^j$ is a polynomial of one complex variable and a an element of an algebra A, then $P(a) = \sum_{j=0}^n c_j a^j$ is defined in an obvious manner. We now show how to define $f(a)$ if f is a continuous function and a is a normal element of a C^*-algebra A. This is known as *continuous functional calculus.*

Proposition 3. *Let A be a C^*-algebra with unit. Let $a \in A$ be normal.*

Then there is a unique $*$-morphism $C(\sigma_A(a)) \to A$ denoted by $f \mapsto f(a)$ such that $f(a)$ has the standard meaning in case f is the restriction of a polynomial. Moreover, the following holds:

1. $\|f(a)\| = \|f\|_{C(\sigma_A(a))}$ *for all* $f \in C(\sigma_A(a))$.
2. *If B is another C^*-algebra with unit and $\pi : A \to B$ a unit-preserving $*$-morphism, then $\pi(f(a)) = f(\pi(a))$ for all $f \in C(\sigma_A(a))$.*
3. $\sigma_A(f(a)) = f(\sigma_A(a))$ *for all* $f \in C(\sigma_A(a))$.[2]

[2] Recall from the proof of Corollary 2 that $\sigma_B(\pi(a)) \subset \sigma_A(a)$. Strictly speaking, the statement is $\pi(f(a)) = (f|_{\sigma_B(\pi(a))})(\pi(a))$.

Proof. For any polynomial P we have that $P(a)$ is also normal and hence by Proposition 2

$$\begin{aligned}\|P(a)\| = \rho_A(P(a)) &= \sup\{|\mu| \mid \mu \in \sigma_A(P(a))\} \\ &= \sup\{|P(\lambda)| \mid \lambda \in \sigma_A(a)\} = \|P\|_{C(\sigma_A(a))}.\end{aligned} \tag{1.3}$$

Thus the map $P \mapsto P(a)$ extends uniquely to a linear map from the closure of the polynomials in $C(\sigma_A(a))$ to A. Since the polynomials form an algebra containing the unit, containing complex conjugates, and separating points, this closure is all of $C(\sigma_A(a))$ by the Stone–Weierstrass theorem. By continuity this extension is a $*$-morphism and Assertion 1 follows from (1.3).

Assertion 2 clearly holds if f is a polynomial. It then follows for continuous f because π is continuous by Corollary 2.

As to Assertion 3 let $\lambda \in \sigma_A(a)$. Choose polynomials P_n such that $P_n \to f$ in $C(\sigma_A(a))$. By Proposition 2 we have $P_n(\lambda) \in \sigma_A(P_n(a))$, i.e., $P_n(a) - P_n(\lambda) \cdot 1 \notin A^\times$. Since the complement of $A^\times$ is closed we can pass to the limit and we obtain $f(a) - f(\lambda) \cdot 1 \notin A^\times$. Hence $f(\lambda) \in \sigma_A(f(a))$. This shows $f(\sigma_A(a)) \subset \sigma_A(f(a))$. Conversely, let $\mu \notin f(\sigma_A(a))$. Then $g := (f - \mu)^{-1} \in C(\sigma(a))$. From $g(a)(f(a) - \mu \cdot 1) = (f(a) - \mu \cdot 1)g(a) = 1$ one sees $f(a) - \mu \cdot 1 \in A^\times$, thus $\mu \notin \sigma(f(a))$. □

We extend Corollary 3 to the case where π is injective but not necessarily onto. This is not a direct consequence of Corollary 3 because it is not a priori clear that the image of a $*$-morphism is closed and hence a C^*-algebra in its own right.

Proposition 4. *Let A and B be C^*-algebras with unit. Each injective unit-preserving $*$-morphism $\pi : A \to B$ satisfies*

$$\|\pi(a)\| = \|a\|$$

for all $a \in A$.

Proof. By Corollary 2 we only have to show $\|\pi(a)\| \geq \|a\|$. Once we know this inequality for self-adjoint elements it follows for all $a \in A$ because

$$\|\pi(a)\|^2 = \|\pi(a)^*\pi(a)\| = \|\pi(a^*a)\| \geq \|a^*a\| = \|a\|^2.$$

Assume there exists a self-adjoint element $a \in A$ such that $\|\pi(a)\| < \|a\|$. By Proposition 2, we have $\sigma_A(a) \subset [-\|a\|, \|a\|]$ and $\rho_A(a) = \|a\|$, hence $\|a\| \in \sigma_A(a)$ or $-\|a\| \in \sigma_A(a)$. Similarly, $\sigma_B(\pi(a)) \subset [-\|\pi(a)\|, \|\pi(a)\|]$.

Choose a continuous function $f : [-\|a\|, \|a\|] \to \mathbb{R}$ such that f vanishes on $[-\|\pi(a)\|, \|\pi(a)\|]$ and $f(-\|a\|) = f(\|a\|) = 1$. From Proposition 3 we conclude $\pi(f(a)) = f(\pi(a)) = 0$ because $f|_{\sigma_B(\pi(a))} = 0$ and $\|f(a)\| = \|f\|_{C(\sigma_A(a))} \geq 1$. Thus $f(a) \neq 0$. This contradicts the injectivity of π. □

Remark 5. Any element a in a C^*-algebra A can be represented as a linear combination $a = a_1 + ia_2$ of self-adjoint elements by setting $a_1 := \frac{1}{2} \cdot (a + a^*)$ and $a_2 := \frac{1}{2i} \cdot (a - a^*)$.

Lemma 2. *Let $a \in A$ be a self-adjoint element in a unital C^*-algebra A. Then the following three statements are equivalent:*

1. $a = b^2$ *for a self-adjoint element* $b \in A$.
2. $a = c^*c$ *for an arbitrary element* $c \in A$.
3. $\sigma_A(a) \subset [0, \infty)$.

Proof. If $a = b^2$ for a self-adjoint element, we have by Proposition 3

$$\sigma_A(a) = \sigma_A(b^2) = \{\lambda^2 \mid \lambda \in \sigma_A(b)\} \subset [0, \infty),$$

which proves the implication "1 $\Rightarrow$ 3."

If $\sigma_A(a) \subset [0, \infty)$, we can define the element $b := \sqrt{a}$ using the continuous functional calculus from Proposition 3. We then have $b^* = b$ and $b^2 = a$, which proves the implication "3 $\Rightarrow$ 1."

The implication "1 $\Rightarrow$ 2" is trivial.

Let $a = c^*c$ and suppose $\sigma_A(-a) \subset [0, \infty)$. By Assertion 8 from Proposition 2, we have $\sigma_A(-cc^*) = \sigma_A(-c^*c) - \{0\} \subset [0, \infty)$. Writing $c = c_1 + ic_2$ with self-adjoint elements c_1, c_2, we find $c^*c + cc^* = 2c_1^2 + 2c_2^2$, hence $c^*c = 2c_1^2 + 2c_2^2 - cc^*$, which implies $\sigma_A(c^*c) \subset [0, \infty)$. Hence $\sigma_A(c^*c) = \{0\}$, which implies $c^*c = a = 0$.

Now suppose $a = c^*c$ for an arbitrary element $c \in A$. Since $a = c^*c$ is self-adjoint and $\sigma_A(a^2) \subset [0, \infty)$, by the continuous functional calculus from Proposition 3, there exists a unique element $|a| := \sqrt{a^2}$ with

$$\sigma_A(d) = \{\sqrt{\lambda} \mid \lambda \in \sigma_A(a^2)\} \subset [0, \infty).$$

By the same argument, the elements $a_+ := \frac{1}{2} \cdot (|a| + a)$ and $a_- := \frac{1}{2} \cdot (|a| - a)$ are self-adjoint and satisfy $\sigma_A(a_i) \subset [0, \infty)$. We then have $a = a_+ - a_-$. Further, for the element $d := ca_-$, we compute

$$-d^*d = -a_-c^*ca_- = -a_-(a_+ - a_-)a_- = -a_-a_+a_- + (a_-)^3 = (a_-)^3,$$

since $a_+a_- = \frac{1}{4}(|a| + a) \cdot (|a| - a) = \frac{1}{4}(|a|^2 - a^2) = 0$. We thus have $\sigma_A(-d^*d) = \sigma_A((a_-)^3) \subset [0, \infty)$, which yields $d = 0$. Hence $c = 0$ or $a_- = 0$, thus $a = a_+$ and $\sigma_A(a) = \sigma_A(a_+) \subset [0, \infty)$. This proves the implication "1 $\Rightarrow$ 3." □

Definition 8. *A self-adjoint element $a \in A$ is called* positive, *if one and hence all of the properties in Lemma 2 hold.*

Remark 6. By the reasoning of the preceding proof, any self-adjoint element $a \in A$ can be represented as a linear combination $a = a_+ - a_-$ with positive elements $a_+ := \frac{1}{2} \cdot (|a| + a)$ and $a_- := \frac{1}{2} \cdot (|a| - a)$ satisfying $a_+a_- = 0$. Combining this observation with Remark 5, we conclude that any $*$-subalgebra of A is spanned by its positive elements (of norm ≤ 1).

1.4 States and Representations

Let $(A, \|\cdot\|, *)$ be a C^*-algebra and H a Hilbert space.

Definition 9. *A* representation *of A on H is a $*$-morphism $\pi : A \to \mathcal{L}(H)$. A representation is called* faithful, *if π is injective. A subset $U \subset H$ is called* invariant *under A, if*

$$\pi(A)U := \{\pi(a) \cdot u \mid a \in a, u \in U\} \subset U .$$

A representation is called irreducible, *if the only closed vector subspaces of H invariant under A are $\{0\}$ and H.*

Remark 7. Let $\pi_\lambda : A \to \mathcal{L}(H_\lambda)$, $\lambda \in \Lambda$ be representations of A. Then

$$\pi = \bigoplus_{\lambda\in\Lambda} \pi_\lambda : A \to \mathcal{L}(\bigoplus_{\lambda\in\Lambda} H_\lambda),$$
$$\pi(a)\big((x_\lambda)_{\lambda\in\Lambda}\big) = \big(\pi_\lambda(a) \cdot x_\lambda\big)_{\lambda\in\Lambda},$$

is called the *direct sum representation.*

Definition 10. *Two representations $\pi_1 : A \to \mathcal{L}(H_1)$, $\pi_2 : A \to \mathcal{L}(H_2)$ are called* unitarily equivalent, *if there exists a unitary operator $U : H_1 \to H_2$, such that for every $a \in A$:*

$$U \circ \pi_1(a) = \pi_2(a) \circ U .$$

Definition 11. *A vector $\Omega \in H$ is called* cyclic *for a representation π, if*

$$\{\pi(a) \cdot \Omega \mid a \in A\} \subset H$$

is a dense subset.

Example 6. The commutative C^*-algebra $A = C(X)$ of continuous functions on a compact Hausdorff space has a natural representation on the Hilbert space $H = L^2(X)$ by multiplication. The constant function $\Omega = 1$ is a cyclic vector since the continuous functions are dense in $L^2(X)$.

Lemma 3. *If (H, π) is an irreducible representation, then either π is the zero map or every non-zero vector $\Omega \in H$ is cyclic for π.*

Proof. For every vector $\Omega \in H$, the space $\pi(A)\Omega$ is invariant under A, hence its closure is either $\{0\}$ or H. If Ω is non-zero then either $\pi(A)\Omega = \{0\}$, so that the one-dimensional subspace $\mathbb{C}\cdot\Omega$ is invariant under A, whence $H = \mathbb{C}\cdot\Omega$ and $\pi = 0$, or there exists an element $a \in A$ such that $\pi(a)\Omega \neq 0$, so that $\pi(A) \cdot \Omega$ is dense in H and hence Ω is cyclic. □

Definition 12. *A* state *on a C^*-algebra A is a linear functional $\tau : A \to \mathbb{C}$ with*

1. $\|\tau\| := \sup\{|\tau(a)| \mid a \in A, \|a\| = 1\} = 1$ *(τ has norm 1).*
2. $\tau(a^*a) \geq 0 \, \forall a \in A$ *(τ is positive).*

The set of all states on A is denoted by $S(A)$.

Example 7. Let X be a compact Hausdorff space, $A = C(X)$. Let μ be a Borel probability measure on X, i.e., a measure on the Borel sigma algebra of X with $\int_X d\mu = 1$. Then

$$\begin{aligned} \tau_\mu : A &\to \mathbb{C} \\ f &\mapsto \int_X f \, d\mu \end{aligned}$$

is a state. For instance, the state $\mu_{\delta_{x_0}}$ corresponding to the Dirac measure at x_0 is the evaluation at x_0:

$$\mu_{\delta_{x_0}}(f) = f(x_0)\,.$$

Example 8. On the C^*-algebra $A = \mathrm{Mat}(n \times n; \mathbb{C})$ of complex matrices, we have the state

$$\tau(A) := \frac{1}{n} \cdot \mathrm{tr}(A)\,.$$

Example 9. On $A = \mathcal{L}(H)$, a vector $\Omega \in H$ with $\|\Omega\| = 1$ yields a so-called *vector state*

$$\tau(A) := \langle A \cdot \Omega, \Omega \rangle\,.$$

Proposition 5. *Let $\tau : A \to \mathbb{C}$ be a state on a C^*-algebra A with unit. Then we have the following:*

1. *$A \times A \to \mathbb{C}$, $(a, b) \mapsto \tau(b^*a)$ is a positive semi-definite, Hermitian sesquilinear form.*
2. $|\tau(b^*a)|^2 \leq \tau(a^*a) \cdot \tau(b^*b) \quad \forall a, b \in A$ (Cauchy–Schwarz inequality).
3. $\tau(a^*) = \overline{\tau(a)} \quad \forall a \in A$.
4. $|\tau(a)|^2 \leq \tau(a^*a) \quad \forall a \in A$.
5. $\tau(1) = \|\tau\| = 1$.

Proof. It follows immediately from the definitions that the form $(a, b) \mapsto \tau(b^*a)$ is sesquilinear and positive semi-definite. To show that it is Hermitian, we set $c = a \cdot z + b$ for some $z \in \mathbb{C}$ and compute

$$\begin{aligned} 0 &\leq \tau(c^*c) \\ &= \bar{z} \cdot z \cdot \tau(a^*a) + \bar{z} \cdot \tau(a^*b) + z \cdot \tau(b^*a) + \tau(b^*b)\,. \end{aligned} \tag{1.4}$$

It follows that $\operatorname{Im}\big(\bar{z} \cdot \tau(a^*b) + z \cdot \tau(b^*a)\big) = 0$. Setting $z = 1$, we obtain $\operatorname{Im}\tau(a^*b) = -\operatorname{Im}\tau(b^*a)$, setting $z = i$, we obtain $\operatorname{Re}\tau(a^*b) = \operatorname{Re}\tau(b^*a)$. Thus $\tau(a^*b) = \overline{\tau(b^*a)}$.

Setting $z = -\frac{\tau(a^*b)}{\tau(a^*a)}$, (1.4) implies the Cauchy–Schwarz inequality:

$$0 \leq \frac{|\tau(a^*b)|^2}{\tau(a^*a)} - \frac{|\tau(a^*b)|^2}{\tau(a^*a)} - \frac{|\tau(a^*b)|^2}{\tau(a^*a)} + \tau(b^*b).$$

Since A has a unit, we have

$$\tau(a^*) = \tau(a^*1) = \overline{\tau(1^*a)} = \overline{\tau(a)}.$$

To show Assertion 4, we compute

$$\begin{aligned} |\tau(a)|^2 = |\tau(1^*a)|^2 &\leq \tau(1^*1) \cdot \tau(a^*a) = \tau(1) \cdot \tau(a^*a) \\ &\leq \|\tau\| \cdot \|1\| \cdot \tau(a^*a) \leq \tau(a^*a). \end{aligned}$$

Using $\tau(1) = \tau(1^*1) \geq 0$ and $\tau(1) \leq 1$, we compute

$$|\tau(a)|^2 \leq \tau(1^*1) \cdot \tau(a^*a) \leq \tau(1) \cdot \|\tau\| \cdot \|a^*a\| = \tau(1) \cdot \|a\|^2.$$

We thus have

$$1 = \|\tau\|^2 \leq \sup_{\substack{a \in A \\ a \neq 0}} \frac{|\tau(a)|^2}{\|a\|^2} \leq \tau(1),$$

hence $\tau(1) = 1$. □

Remark 8. The proof of Assertion 5 shows that $\varphi(1) = \|\varphi\|$ holds for every positive linear functional φ.

Corollary 4. *Let $\tau_1, \ldots, \tau_n$ be states and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{j=1}^n \lambda_j = 1$. Then the* convex combination *$\tau = \sum_{j=1}^n \lambda_n \cdot \tau_n$ is also a state.*

Let τ_i, $i \in \mathbb{N}$ be states and define $\tau(a) := \lim_{i\to\infty} \tau_i(a)$ provided the limit exists. Then τ is a state.

Proof. For the convex combinations, we have

$$\tau(a^*a) = \sum_{j=1}^n \underbrace{\lambda_j}_{\geq 0} \cdot \underbrace{\tau_j(a^*a)}_{\geq 0} \geq 0$$

and

$$\|\tau\| = \tau(1) = \sum_{j=1}^n \lambda_j \cdot \tau_j(1) = \sum_{j=1}^n \lambda_j = 1.$$

Similarly, for the pointwise convergence, we have

$$\tau(a^*a) = \lim_{i\to\infty} \tau_i(a^*a) \geq 0$$

and

$$\tau(1) = \lim_{i\to\infty} \tau_i(1) = \lim_{i\to\infty} 1 = 1.$$

□

Example 10. Let $\tau_1, \ldots, \tau_n$ be vector states for vectors $\Omega_1, \ldots, \Omega_n \in H$. Then for the state $\tau = \sum_{j=1}^n \lambda_j \tau_j$ with $\lambda_j \geq 0$, $\sum_{j=1}^n \lambda_j = 1$, we find

$$\tau(a) = \sum_{j=1}^n \lambda_j \cdot \tau_j(a) = \sum_{j=1}^n \lambda_j \cdot (a \cdot \Omega_j, \Omega_j) = \mathrm{tr}(\varrho \cdot a)\,.$$

Here $\varrho \in \mathcal{L}(H)$ is an operator of finite-dimensional range with eigenvectors Ω_j and eigenvalues λ_j.

More generally, a positive trace class operator $\varrho \in \mathcal{L}(H)$ defines a state τ on $A = \mathcal{L}(H)$ by $\tau(a) := \mathrm{tr}(\varrho \cdot a)$. States of this form are called *normal.*

Lemma 4. *Let τ be a state on a C^*-algebra A. Then the following holds:*

1. $\tau(a^*a) = 0 \Leftrightarrow \tau(ba) = 0$ *for any* $b \in A$.
2. $\tau(b^*a^*ab) \leq \|a^*a\| \cdot \tau(b^*b)$.

Proof. 1. Suppose $\tau(a^*a) = 0$. Then the Cauchy–Schwarz inequality

$$|\tau(ba)|^2 \leq \underbrace{\tau(a^*a)}_{=0} \cdot \tau(bb^*) = 0$$

implies $\tau(ba) = 0$. The other direction is obvious.

2. If $\tau(b^*b) = 0$, then $\tau(cb) = 0$ for any $c \in A$, especially for $c = b^*a^*a$. We thus assume $\tau(b^*b) > 0$ and set $\varrho(c) := \frac{\tau(b^*cb)}{\tau(b^*b)}$. Then ϱ is a positive linear functional with $\|\varrho\| = \varrho(1) = 1$. Hence ϱ is a state, and from Proposition 5 we have $\varrho(a^*a) \leq \|a^*a\|$.

□

From every state τ on a C^*-algebra A we can construct a representation of A by making the product $(b, a) \mapsto \tau(b^*a)$ nondegenerate. By Assertion 1 in Lemma 4, the null space

$$N_\tau := \{a \in A \mid \tau(a^*a) = 0\}$$

is a closed linear subspace of A. By Assertion 2 in Lemma 4, N_τ is a left ideal in A. Therefore, the pairing

$$A/N_\tau \times A/N_\tau \to \mathbb{C}\,,$$
$$([a],[b]) \mapsto \tau(b^*a)$$

is a well-defined Hermitian scalar product. Let H_τ be the completion of the pre-Hilbert space A/N_τ. Then the map

$$\pi_\tau : A \to \mathcal{L}(A/N_\tau)\,,$$
$$\pi_\tau(a)\cdot[b] := [ab]$$

satisfies

$$\|\pi_\tau(a)\cdot[b]\|^2 = \tau(b^*a^*ab) \le \|a^*a\|\cdot\tau(b^*b) = \|a\|^2\cdot\|[b]\|^2\,,$$

so $\|\pi_\tau(a)\| \le \|a\|$ and $\|\pi_\tau\| \le 1$. The map π_τ thus extends to a representation

$$\pi_\tau : A \to \mathcal{L}(H_\tau)\,.$$

The scalar product induced by $([a],[b]) \mapsto \tau(b^*a)$ on H_τ will be denoted by $\langle\cdot,\cdot\rangle_\tau$.

Definition 13. *Let τ be a state on a C^*-algebra A. The representation $(H_\tau, \langle\cdot,\cdot\rangle_\tau, \pi_\tau)$ constructed above is called the* Gelfand–Naimark–Segal representation *or* GNS representation *in short.*

Example 11. For $A = C(X)$ with a state τ_μ given by a probability measure μ as $\tau_\mu(f) = \int_X f\,d\mu$, the representation space of the GNS representation is $H_{\tau_\mu} = L^2(X,\mu)$.

Remark 9. Let τ be a state on a C^*-algebra A with unit. Then we have the following:

1. The vector $\Omega_\tau := [1] \in H_\tau$ is cyclic for π_τ, since

$$\pi_\tau(A)\cdot\Omega_\tau = A/N_\tau \subset H_\tau$$

is dense.
2. τ can be represented as a vector state on the GNS representation because

$$\tau(a) = \tau(1^*a1) = \langle[a1],[1]\rangle_\tau = \langle\pi_\tau(a)\cdot\Omega_\tau, \Omega_\tau\rangle_\tau.$$

Definition 14. *Let A be a C^*-algebra A. The direct sum representation*

$$\bigoplus_{\tau\in S(A)} \pi_\tau : A \to \mathcal{L}\Big(\bigoplus_{\tau\in S(A)} H_\tau\Big)$$

is called the universal representation *of A.*

Remark 10. The universal representation is faithful. Hence every C^*-algebra A is isomorphic to a subalgebra of the algebra $\mathcal{L}(H)$ of bounded linear operators on a Hilbert space H.

Definition 15. *A state τ on a C^*-algebra A is called* pure, *if for every positive linear functional $\varrho : A \to \mathbb{C}$ with $\varrho(a^*a) \leq \tau(a^*a)$ $\forall a \in A$, there exists $\lambda \in [0, 1]$ with $\varrho = \lambda \cdot \tau$.*

Remark 11. A pure state τ cannot be written as a convex combination of different states $\tau_1 \neq \tau_2$. If $\tau = \lambda \cdot \tau_1 + (1 - \lambda) \cdot \tau_2$ with $\lambda \in [0, 1]$, then $\tau \geq \lambda \cdot \tau_1$ implies $\lambda = 0$ and $\tau = \tau_2$ or $\lambda = 1$ and $\tau = \tau_1$.

Example 12. The trace as a state of the algebra $A = \mathrm{Mat}(n \times n; \mathbb{C})$ (see Example 8) is not pure unless $n = 1$, namely it can be written as $\frac{1}{n}\mathrm{tr} = \sum_{i=1}^{n} \frac{1}{n}\tau_i$ where τ_i is the vector state for the ith standard unit vector of $\mathbb{C}^n$.

Definition 16. *Let $S \subset \mathcal{A}$ be a subset of a C^*-algebra A. The space $S' := \{a \in A \mid [a, s] = 0 \ \forall a \in A, s \in S\}$ is called the* commutant *of S. Here $[a, s] := as - sa$ is the* commutator *of a and s.*

Remark 12. If $S \subset A$ is a $*$-invariant subset, i.e., $S^* := \{s^* \mid s \in S\} \subset S$, then the commutant S' is also $*$-invariant. S' is closed, since for every $s \in S$, the map $A \to A$, $a \mapsto [a, s]$ is continuous. Hence S' is a C^*-subalgebra of A.

Theorem 1. *Let (H, π) be a representation of a unital C^*-algebra A. Then the following two statements are equivalent:*

1. *π is irreducible.*
2. $(\pi(A))' = \mathbb{C} \cdot \mathrm{id}_H$.

Proof. Suppose π is irreducible and $b \in \mathcal{L}(H)$ commutes with all elements of $\pi(A)$. By Remark 5, we may write $b = b_1 + ib_2$ with self-adjoint elements $b_1, b_2 \in \mathcal{L}(H)$. We need to show that $\sigma_A(b_1)$ and $\sigma_A(b_2)$ each consist of a single point. Suppose, to the contrary, that $\sigma_A(b_1)$ contains two different numbers $\lambda \neq \mu$. Then we choose functions $f, g \in C(\sigma_A(b_1))$ such that $f(\lambda) = g(\mu) = 1$ and $f \cdot g = 0$. By the continuous functional calculus from Proposition 3 in the C^*-algebra $(\pi(A))'$, we have $f(b_1) \cdot g(b_1) = (f \cdot g)(b_1) = 0$ and $f(b_1), g(b_1) \neq 0$. Since $g(b_1)$ commutes with every element of $\pi(A)$ and π is irreducible, $g(b_1) \cdot H$ is an A-invariant, dense subspace of H. The vanishing of $f(b_1)$ on this subspace implies $f(b_1) = 0$, which contradicts the fact that the continuous functional calculus from Proposition 3 is an isometry. Thus $\sigma_A(b_1)$ consists of a single point, hence $C(\sigma_A(b_1))$ is one dimensional. Since the continuous functional calculus $C(\sigma_A(b_1)) \to A$ is an isometric embedding with b_1, id_H in its image, we conclude that $b_1 = \lambda \mathrm{id}_H$ for a $\lambda \in \mathbb{C}$. By the same argument, b_2 and hence b is a multiple of the identity.

Now suppose $(\pi(A))' = \mathbb{C} \cdot \mathrm{id}_H$. Let $K \subset H$ be a closed subspace invariant under A, and let p be the orthogonal projection from H onto K. The invariance property $\pi(A)K \subset K$ yields that p commutes with every operator in $\pi(A)$. Hence p is of the form $p = \lambda \cdot \mathrm{id}_H$, $\lambda \in \mathbb{C}$. Since p is a projection, $p^2 = p$; thus $\lambda^2 = \lambda$. Hence $K = \{0\}$ or $K = H$. □

Theorem 2. *Let τ be a state on a C^*-algebra A. Then the following two statements are equivalent:*

1. *τ is a pure state.*
2. *The GNS representation (H_τ, π_τ) is* irreducible, *i.e., H_τ has no nontrivial closed A-invariant subspace.*

Example 13. The GNS representation of the algebra $A = \mathrm{Mat}(n \times n; \mathbb{C})$ for a vector state for $\Omega \in \mathbb{C}^n$ is the standard representation of A on $\mathbb{C}^n$, hence irreducible. Therefore, such vector states are pure.

Proof (of Theorem 2). Suppose τ is a pure state and $v \in \mathcal{L}(H)$ is a positive element of norm ≤ 1 that commutes with every element in $\pi_\tau(A)$. Then the function

$$\varrho : A \to \mathbb{C}, a \mapsto \langle \pi_\tau(a) \cdot v\Omega_\tau, \Omega_\tau \rangle$$

is a positive linear functional on A, satisfying $\varrho(a^*a) \leq \tau(a^*a)$ for all $a \in A$. Hence $\varrho = \lambda \cdot \tau$ for a $\lambda \in [0, 1]$. Thus for arbitrary $a, b \in A$, we obtain in the pre-Hilbert space A/N_τ:

$$\begin{aligned}
\langle v \cdot (a + N_\tau), (b + N_\tau) \rangle_\tau &= \langle v \cdot \pi_\tau(a)\Omega_\tau, \pi_\tau(b)\Omega_\tau \rangle_\tau \\
&= \langle v \cdot \pi_\tau(b^*a)\Omega_\tau, \Omega_\tau \rangle_\tau \\
&= \varrho(b^*a) \\
&= \lambda \cdot \tau(b^*a) \\
&= \langle \lambda \mathrm{id}_{N_\tau} \cdot (a + N_\tau), (b + N_\tau) \rangle_\tau \,.
\end{aligned}$$

This implies $v = \lambda \mathrm{id}_{H_\tau}$, since A/N_τ is dense in H_τ. By Proposition 1, we conclude that π_τ is irreducible.

Now suppose that π_τ is irreducible. Let ϱ be a positive linear functional on A such that $\varrho(a^*a) \leq \tau(a^*a)$ for all $a \in A$. Then the pairing

$$(a + N_\tau, b + N_\tau) \mapsto \varrho(b^*a)$$

is a positive semi-definite, Hermitian sesquilinear form on A/N_τ. Being majorized by $\langle \cdot, \cdot \rangle_\tau$, it extends to an inner product $\langle \cdot, \cdot \rangle_\varrho$ on the Hilbert space H_τ. Hence there exists a bounded positive operator $m \in \mathcal{L}(H)$ such that

$$\langle x, y \rangle_\varrho = \langle x, my \rangle_\tau \qquad \forall x, y \in H_\tau \,.$$

Now the estimate

$$0 \leq \varrho(a^*a) = \langle \pi_\tau(a)\Omega, m\pi_\tau(a)\Omega \rangle_\tau \leq \tau(a^*a) = \langle \pi_\tau(a)\Omega, \pi_\tau(a)\Omega \rangle_\tau$$

yields $\|m\| \leq 1$. For every $a, b, c \in A$, we have

$$
\begin{aligned}
\langle \pi_\tau(a)\Omega, m\pi_\tau(b)\pi_\tau(c)\Omega\rangle_\tau &= \varrho(a^*bc) \\
&= \varrho((b^*a)^*c) \\
&= \langle \pi_\tau(b)^*\pi_\tau(a)\Omega, m\pi_\tau(c)\Omega\rangle_\tau \\
&= \langle \pi_\tau(a)\Omega, \pi_\tau(b)m\pi_\tau(c)\Omega\rangle_\tau\,.
\end{aligned}
$$

Hence m commutes with every $\pi(b)$, $b \in A$.

By Theorem 1, m is a multiple of the identity and thus ϱ is a multiple of the state τ. This shows that τ is pure. □

Lemma 5. *In a unital C^*-algebra A, every state is a pointwise limit of convex combinations of pure states.*

Proof. By Corollary 4, convex combinations of pointwise limits of states are states. Hence $S(A)$ is a bounded closed convex set in the topology of pointwise convergence. By the Banach–Alaoglu theorem from functional analysis, $S(A)$ is thus a compact subset of the closed unit ball in the dual space of A (in the topology of pointwise convergence). The Krein–Milman theorem then implies that $S(A)$ is the closed convex hull of its extreme points, which by Remark 11 contain all pure states.

It remains to show that all extreme points in $S(A)$ are pure. Let $\tau \in S(A)$ be an extreme point of $S(A)$, ϱ a positive linear functional on A satisfying $\varrho(a^*a) \leq \tau(a^*a)$ for all $a \in A$, and suppose $\tau \neq \varrho \neq 0$. Then setting $t = \|\varrho\| \in (0, 1)$, we find

$$
\tau = t \cdot \frac{\varrho}{\|\varrho\|} + (1-t) \cdot \frac{(\tau - \varrho)}{\|\tau - \varrho\|}\, s,
$$

since $\|\tau - \varrho\| = \tau(1) - \varrho(1) = \|\tau\| - \|\varrho\| = 1 - t$ by Remark 8. Hence $\varrho/\|\varrho\| = (\tau - \varrho)/\|\tau - \varrho\| = \tau$, since by assumption τ is an extreme point of $S(A)$. Thus $\varrho = t\tau$ and hence τ is a pure state. □

Remark 13. The restriction of a pure state to a subalgebra need not be pure. For example, let $A = \mathrm{Mat}(4 \times 4; \mathbb{C})$. Then the vector state τ for the unit vector $\Omega = 2^{-1/2}(1, 0, 0, 1)$ is pure. Now embed $B = \mathrm{Mat}(2 \times 2; \mathbb{C})$ as a subalgebra into A via $b \mapsto \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$. The restriction of τ to B yields the trace state of B which is not pure.

The converse can also happen. If τ_1 is the vector state of A for $(1, 0, 0, 0)$ and τ_2 for $(0, 0, 1, 0)$, then $\tau = \frac{1}{2}\tau_1 + \frac{1}{2}\tau_2$ is not pure as a state of A, but it restricts to a pure state for B (the vector state for $(1, 0)$).

1.5 Product States

In this section, we consider states on the tensor product of C^*-algebras. The norms making the algebraic tensor product into a C^*-algebra are highly nonunique. However, the norm making the algebraic tensor product of Hilbert spaces into a pre-Hilbert space is unique. So it seems natural to study norms on the algebras by means of norms on representation spaces. We will work here with the finest norm topology making the algebraic tensor product of C^*-algebras into a C^*-algebra. Throughout this section, we assume the C^*-algebras in question to have a unit.

Remark 14. Let $(H, \langle\cdot,\cdot\rangle_H)$ and $(K, \langle\cdot,\cdot\rangle_K)$ be Hilbert spaces. Then there is a unique inner product $\langle\cdot,\cdot\rangle$ on the algebraic tensor product of H and K such that

$$\langle x \otimes y, x' \otimes y'\rangle = \langle x, x'\rangle_H \cdot \langle y, y'\rangle_K \qquad \forall x, x' \in H, y, y' \in K\,.$$

The completion of the algebraic tensor product with respect to this inner product is called the tensor product of the Hilbert spaces and is denoted by $H \otimes K$. Moreover, for bounded operators $a \in \mathcal{L}(H)$ and $b \in \mathcal{L}(K)$, there exists a unique operator $a \otimes b \in \mathcal{L}(H \otimes K)$ such that

$$(a \otimes b)(x \otimes y) = a(x) \otimes b(y) \qquad \forall x \in H, y \in K\,.$$

This operator satisfies $\|a \otimes b\| = \|a\|_H \cdot \|b\|_K$.

Given two C^*-algebras $(A, \|\cdot\|_A, *)$ and $(B, \|\cdot\|_B, *)$, we want to construct C^*-norms on the algebraic tensor product $A \otimes B$. The simplest way to do so is by using the universal representations. The $\mathbb{C}$-antilinear map $* : A \otimes B \to A \otimes B$ defined by $(a \otimes b)^* := a^* \otimes b^*$ on homogeneous elements and extended bilinearly to $A \otimes B$ makes the algebraic tensor product into an involutive algebra.

Lemma 6. *Let (H, φ) and (K, ψ) be representations of A and B, respectively. Then there is a unique $*$-homomorphism $\pi : A \otimes B \to \mathcal{L}(H \otimes K)$ such that*

$$\pi(a \otimes b) = \varphi(a) \otimes \psi(b) \qquad \forall a \in A, b \in B\,.$$

Moreover, if the representations φ and ψ are faithful, then so is π.

Proof. The map $A \times B \to \mathcal{L}(H \otimes K)$, $(a, b) \mapsto \varphi(a) \otimes \psi(b)$ is bilinear and thus yields a unique linear map $\pi : A \otimes B \to \mathcal{L}(H \otimes K)$ as claimed, which is indeed a $*$-morphism. If both φ and ψ are injective and $z \in A \otimes B$ satisfies $\pi(z) = 0$, then by writing $z = \sum_{j=1}^n a_j \otimes b_j$ with linearly independent b_j, we conclude $\varphi(a_j) = 0$ for $j = 0, \dots, n$. Hence $a_j = 0$ for $j = 1, \dots, n$ and thus $z = 0$. □

By this lemma, it is natural to make use of the universal representation from Definition 14 to obtain a C^*-norm on the algebraic tensor product.

Definition 17. *Let $(A, \|\cdot\|_A, *)$ and $(B, \|\cdot\|_B, *)$ be C^*-algebras with the universal representations $\pi^A : A \to \mathcal{L}(H)$ and $\pi^B : B \to \mathcal{L}(K)$. The* injective C^*-norm *$\|\cdot\|_\iota$ on the algebraic tensor product is defined by*

$$\|c\|_\iota := \|\pi(c)\|\,,$$

where $\pi : A \otimes B \to \mathcal{L}(H \otimes K)$ is the unique $*$-morphism induced by $\pi^A \times \pi^B$ as in Lemma 6. The completion of the algebraic tensor product with respect to the C^*-norm $\|\cdot\|_\iota$ is called the injective C^*-tensor product *and is denoted by $A \otimes_\iota B$.*

Since the $*$-morphism π constructed via the universal representations is injective, $\|\cdot\|_\iota$ is indeed a C^*-norm on $A \otimes B$. Another natural C^*-norm on $A \otimes B$ is constructed by taking the supremum over all C^*-norms. By Remark 2, a unit-preserving $*$-morphism π from the algebraic tensor product $A \otimes B$ to a C^*-algebra C satisfies $\|\pi(x)\| \leq \|x\|_\gamma$ with respect to any C^*-norm $\|\cdot\|_\gamma$ on $A \otimes B$. This yields the following characterization of the maximal C^*-norm on $A \otimes B$:

Definition 18. *Let $(A, \|\cdot\|_A, *)$ and $(B, \|\cdot\|_B, *)$ be C^*-algebras. The* projective C^*-norm $\|\cdot\|_\pi$ *on the algebraic tensor product $A \otimes B$ is defined by*

$$\|c\|_\pi := \inf \left\{ \sum_{j=1}^{n} \|a_j\|_A \cdot \|b_j\|_B \;\middle|\; c = \sum_{j=1}^{n} a_j \otimes b_j \right\}.$$

The completion of $A \otimes B$ with respect to the C^*-norm $\|\cdot\|_\pi$ is called the projective C^*-tensor product *and is denoted by $A \otimes_\pi B$.*

Remark 15. The projective C^*-norm $\|\cdot\|_\pi$ satisfies $\|a \otimes b\|_\pi = \|a\|_A \cdot \|b\|_B$ for all $a \in A$, $b \in B$. Clearly, any other C^*-norm $\|\cdot\|_\gamma$ on $A \otimes B$ satisfies $\|c\|_\gamma \leq \|c\|_\pi$ for all $c \in A \otimes B$, hence the projective C^*-norm is maximal among all C^*-norms on $A \otimes B$. One can show that the injective C^*-norm $\|\cdot\|_\iota$ on the other hand is minimal among all C^*-norms on $A \otimes B$.

The projective C^*-tensor product has the following universal property.

Lemma 7. *Let A, B, and C be C^*-algebras and let $\varphi : A \to C$ and $\psi : B \to C$ be $*$-morphisms such that $\varphi(a)$ and $\psi(b)$ commute for all $a \in A$, $b \in B$. Then there exists a unique $*$-morphism $\pi : A \otimes_\pi B \to C$ such that*

$$\pi(a \otimes b) = \varphi(a) \cdot \psi(b) \qquad \forall a \in A, b \in B. \tag{1.5}$$

Proof. The bilinear map $A \times B \to C$, $(a, b) \mapsto \varphi(a) \cdot \psi(b)$ induces a unique linear map $\pi : A \otimes B \to C$ satisfying (1.5). This map is a $*$-morphism. The map $\|\cdot\|_\gamma : A \otimes B \to \mathbb{R}$, $c \mapsto \|\pi(c)\|_\gamma$ is a C^*-norm; hence it satisfies $\|c\|_\gamma \leq \|c\|_\pi$ for all $c \in A \otimes B$. Hence π is continuous with respect to the projective C^*-norm and thus uniquely extends from the dense subset $A \otimes B$ to the projective C^*-tensor product $A \otimes_\pi B$. □

Now we study states on the projective C^*-tensor product $A \otimes_\pi B$. Taking linear functionals $\mu : A \to \mathbb{C}$ and $\nu : B \to \mathbb{C}$, setting

$$(\mu \otimes \nu)(a \otimes b) := \mu(a) \cdot \nu(b)$$

on homogeneous elements and extending bilinearly, we obtain a linear functional on $A \otimes B$. In the projective C^*-norm, we have $\|\mu \otimes \nu\|_\pi = \|\mu\|_A \cdot \|\nu\|_B$. Furthermore, for the homogeneous elements $a \otimes b$, we have

$$(\mu \otimes \nu)\big((a \otimes b)^*(a \otimes b)\big) = (\mu \otimes \nu)(a^*a \otimes b^*b) = \mu(a^*a) \cdot \nu(b^*b).$$

Hence the functional $\mu \otimes \nu : A \otimes B \to \mathbb{C}$ is positive, if μ and ν are.

Definition 19. *Let A and B be C^*-algebras and let $\mu \in S(A)$ and $\nu \in S(B)$ be states. The unique extension of $\mu \otimes \nu$ to a state on the projective C^*-tensor product $A \otimes_\pi B$ is called a* product state.

Since A and B have a unit, we can restrict a state $\tau \in S(A \otimes B)$ to one of the factors by setting

$$\begin{aligned} \tau^A(a) &:= \tau(a \otimes 1) \qquad \forall a \in A \\ \tau^B(b) &:= \tau(1 \otimes b) \qquad \forall b \in B \end{aligned}$$

Obviously, for any two states $\mu \in S(A)$ and $\nu \in S(B)$, there is a state $\tau \in S(A \otimes_\pi B)$ such that $\tau^A = \mu$ and $\tau^B = \nu$, namely the product state $\tau = \mu \otimes \nu$. Hence in this case, $\tau = \tau^A \otimes \tau^B$, i.e., the measurement in the state τ of an observable in $A \otimes_\pi B$ simply results in the product of measurements in the states τ^A and τ^B, respectively. In general this is not the case, so we set the following.

Definition 20. *A state $\tau \in S(A \otimes_\pi B)$ is called* correlated, *if there exists $a \in A$ and $b \in B$ such that $\tau(a \otimes b) \neq \tau^A(a) \cdot \tau^B(b)$.*

Definition 21. *A state $\tau \in S(A \otimes_\pi B)$ is called* decomposable, *if it is the pointwise limit of convex combinations of product states. A state $\tau \in S(A \otimes_\pi B)$ is called* entangled, *if it is not decomposable.*

Remark 16. In the literature, the pointwise limit of linear functionals is referred to as the weak-* limit. Stated this way, the set of decomposable states is the weak-* closure of the convex hull of the product states.

Example 14. A pure state on $A \otimes_\pi B$ cannot be written as convex combination of different states. Nor can it be written as a pointwise limit of such convex combinations. Hence a pure state is decomposable if and only if it is a product state.

The set of decomposable states is a convex subset of the set of all (positive) linear functionals on the projective C^*-tensor product $A \otimes_\pi B$. One aims at a characterization of this convex set by inequalities. While a complete characterization is unknown, a simple such inequality has been deduced from the work of Bell in the late 1950s on the Einstein–Podolsky–Rosen paradox. Therefore, inequalities of this type are often referred to as (generalized) *Bell's inequalities.* See also [9].

Lemma 8. *Let A and B be C^*-algebras and let τ be a decomposable state on the projective C^*-tensor product $A \otimes_\pi B$. Then*

$$|\tau(a \otimes (b - b'))| + |\tau(a' \otimes (b + b'))| \leq 2 \tag{1.6}$$

holds for all self-adjoint elements $a, a' \in A$, $b, b' \in B$ of norm ≤ 1.

Proof. For a product state $\tau = \mu \otimes \nu$, we have

$$\begin{aligned}\tau(a \otimes (b - b')) &= \mu(a) \cdot \nu(b) - \mu(a) \cdot \nu(b') \\ &= \mu(a) \cdot \nu(b) \cdot \big(1 \pm \mu(a') \cdot \nu(b')\big) - \mu(a) \cdot \nu(b') \cdot \big(1 \pm \mu(a') \cdot \nu(b)\big).\end{aligned}$$

By assumption, $|\mu(a)|, |\mu(a')|, |\nu(b)|, |\nu(b')| \leq 1$, so we have

$$\begin{aligned}|\tau(a \otimes (b - b'))| &\leq |1 \pm \mu(a') \cdot \nu(b')| + |1 \pm \mu(a') \cdot \nu(b)| \\ &= 1 \pm \mu(a') \cdot \nu(b') + 1 \pm \mu(a') \cdot \nu(b) \\ &= 2 \pm \tau(a' \otimes (b + b')).\end{aligned}$$

Hence, (1.6) holds for all product states. If τ is a convex combination of product states, $\tau = \sum_{j=1}^{n} \lambda_j \mu_j \otimes \nu_j$, we obtain

$$\begin{aligned}&|\tau(a \otimes (b - b'))| + |\tau(a' \otimes (b + b'))| \\ &\leq \sum_{j=1}^{n} \lambda_j \cdot \Big\{(\mu_j \otimes \nu_j)(a \otimes (b - b')) + (\mu_j \otimes \nu_j)(a' \otimes (b + b'))\Big\} \\ &\leq 2.\end{aligned}$$

Taking pointwise limits of convex combinations, the inequality holds by continuity. □

Example 15. Let $A = B = \mathrm{Mat}(2 \times 2; \mathbb{C})$ be matrix algebras. Let e_1, e_2 be the standard basis of $\mathbb{C}^2$. On $A \otimes B$, we have the Bell state τ, which is the vector state with the vector

$$\Omega := \frac{1}{\sqrt{2}}\big(e_1 \otimes e_1 + e_2 \otimes e_2\big).$$

It is easy to see that the Bell state is entangled. For instance, the observables

$$a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; a' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; b = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \; b' = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

in the state τ yield

$$\begin{aligned}\tau(a \otimes (b - b')) = \sqrt{2}\,\tau(a \otimes a) &= \sqrt{2}\,\langle (a \otimes a)(\Omega), \Omega \rangle \\ &= \sqrt{2}\,\langle \Omega, \Omega \rangle = \sqrt{2}\end{aligned}$$

and similarly

$$\tau(a' \otimes (b + b')) = \sqrt{2},$$

hence

$$|\tau(a \otimes (b - b'))| + |\tau(a' \otimes (b + b'))| = 2\sqrt{2} > 2.$$

Thus the state τ violates Bell's inequality and is therefore entangled by Lemma 8.

Bell's inequalities are often referred to as inequalities which a priori hold for all states in a classical system. The existence of entangled states may thus be considered as a characterizing phenomenon of quantum systems. In fact, if one of the observable algebras is abelian – e.g., if it corresponds to a classical system – then there are no entangled states in $A \otimes_\pi B$.

Proposition 6. *Let A and B be C^*-algebras with unit. If A or B is abelian, then all states on the projective C^*-tensor product $A \otimes_\pi B$ are decomposable.*

Proof. By Theorem 2, it suffices to show that every pure state τ on $A \otimes_\pi B$ is a product state. We first claim that $\tau(xy) = \tau(x) \cdot \tau(y)$ holds for all $x \in A \otimes_\pi B$ and all $y \in Z(A \otimes_\pi B)$, where $Z(A \otimes_\pi B)$ denotes the center of $A \otimes_\pi B$. Since $Z(A \otimes_\pi B)$ is spanned by its positive elements of norm ≤ 1, it suffices to prove the claim for y positive, i.e., $y = z^2$ for a self-adjoint $z \in Z(A \otimes_\pi B)$, with $\|y\| \leq 1$. If $\tau(y) = 0$, the Cauchy–Schwarz inequality

$$\begin{aligned} |\tau(xy)|^2 &= |\tau((zx^*)^* z)|^2 \leq \tau(xz^*zx^*) \cdot \tau(z^*z) \\ &= \tau(xyx^*) \cdot \tau(y) \end{aligned}$$

implies $\tau(xy) = 0$. If $\tau(y) = 1$, then $\tau(1 - y) = 0$; thus $0 = \tau(x(1 - y)) = \tau(x) \cdot \tau(y) - \tau(xy)$.

For $0 < \tau(y) < 1$, we have

$$\tau(x) = \tau(y) \cdot \underbrace{\frac{1}{\tau(y)} \cdot \tau(xy)}_{=:\tau_1(x)} + (1 - \tau(y)) \cdot \underbrace{\frac{1}{1 - \tau(y)} \cdot \tau(x(1 - y))}_{=:\tau_2(x)} \quad \forall x \in A \otimes_\pi B\,.$$

Since $y \in Z(A \otimes_\pi B)$, we have $\tau_1(x^*x) = \frac{1}{\tau(y)} \cdot \tau(x^*xy) = \frac{1}{\tau(y)} \cdot \tau((zx)^*zx) \geq 0$. Similarly, $\tau_2(x^*x) = \frac{1}{1-\tau(y)} \cdot \tau(x^*x(1 - y)) \geq 0$, since

$$\tau(x^*xy) = \tau(x^*z^*zx) \leq \|z^*z\| \cdot \tau(x^*x) \leq \tau(x^*x)\,.$$

Clearly, $\tau_1(1) = \tau_2(1) = 1$; hence τ_1 and τ_2 are states on $A \otimes_\pi B$. Since τ is a pure state by assumption, we conclude $\tau = \tau_1 = \tau_2$. Hence $\tau_1(x) = \tau_2(x)$ for all $x \in A \otimes_\pi B$, which yields $\tau(xy) = \tau(y) \cdot \tau(x)$.

Now if A is abelian, then $A \otimes_\pi \{1\} \subset Z(A \otimes_\pi B)$. As we have seen, every pure state τ on $A \otimes_\pi B$ satisfies

$$\tau(a \otimes b) = \tau((a \otimes 1) \cdot (1 \otimes b)) = \tau^A(a) \cdot \tau^B(b) \quad \forall a \in A, b \in B\,.$$

Hence τ is a product state. □

1.6 Weyl Systems

In this section we introduce Weyl systems and CCR representations. They formalize the "canonical commutator relations" from quantum field theory in an "exponentiated form." The main result of this section is Theorem 3 which says that for each symplectic vector space there is an essentially unique CCR representation. Our approach follows ideas in [7]. A different proof of this result may be found in [8, Sect. 5.2.2.2].

Let (V, ω) be a *symplectic vector space*, i.e., V is a real vector space of finite or infinite dimension and $\omega : V \times V \to \mathbb{R}$ is an antisymmetric bilinear map such that $\omega(\phi, \psi) = 0$ for all $\psi \in V$ implies $\phi = 0$.

Definition 22. *A* Weyl system *of* (V, ω) *consists of a* C^**-algebra* A *with unit and a map* $W : V \to A$ *such that for all* $\phi, \psi \in V$ *we have*

(i) $W(0) = 1$,
(ii) $W(-\phi) = W(\phi)^*$,
(iii) $W(\phi) \cdot W(\psi) = e^{-i\omega(\phi,\psi)/2}\, W(\phi + \psi)$.

Condition (iii) says that W is a representation of the additive group V in A up to the "twisting factor" $e^{-i\omega(\phi,\psi)/2}$. Note that since V is not given a topology there is no requirement on W to be continuous. In fact, we will see that even in the case when V is finite dimensional and so V carries a canonical topology W will in general not be continuous.

Example 16. We construct a Weyl system for an arbitrary symplectic vector space (V, ω). Let $H = L^2(V, \mathbb{C})$ be the Hilbert space of square-integrable complex-valued functions on V with respect to the counting measure, i.e., H consists of those functions $F : V \to \mathbb{C}$ that vanish everywhere except for countably many points and satisfy

$$\|F\|_{L^2}^2 := \sum_{\phi \in V} |F(\phi)|^2 < \infty.$$

The Hermitian product on H is given by

$$(F, G)_{L^2} = \sum_{\phi \in V} \overline{F(\phi)} \cdot G(\phi).$$

Let $A := \mathcal{L}(H)$ be the C^*-algebra of bounded linear operators on H as in Example 1. We define the map $W : V \to A$ by

$$(W(\phi)F)(\psi) := e^{i\omega(\phi,\psi)/2}\, F(\phi + \psi).$$

Obviously, $W(\phi)$ is a bounded linear operator on H for any $\phi \in V$ and $W(0) = \mathrm{id}_H = 1$. We check (ii) by making the substitution $\chi = \phi + \psi$:

$$
\begin{aligned}
(W(\phi)F, G)_{L^2} &= \sum_{\psi\in V} \overline{(W(\phi)F)(\psi)}\, G(\psi) \\
&= \sum_{\psi\in V} \overline{e^{i\omega(\phi,\psi)/2}\, F(\phi+\psi)}\, G(\psi) \\
&= \sum_{\chi\in V} \overline{e^{i\omega(\phi,\chi-\phi)/2}\, F(\chi)}\, G(\chi-\phi) \\
&= \sum_{\chi\in V} \overline{e^{i\omega(\phi,\chi)/2}} \cdot \overline{F(\chi)} \cdot G(\chi-\phi) \\
&= \sum_{\chi\in V} \overline{F(\chi)} \cdot e^{i\omega(-\phi,\chi)/2} \cdot G(\chi-\phi) \\
&= (F, W(-\phi)G)_{L^2}.
\end{aligned}
$$

Hence $W(\phi)^* = W(-\phi)$. To check (iii) we compute

$$
\begin{aligned}
(W(\phi)(W(\psi)F))(\chi) &= e^{i\omega(\phi,\chi)/2}\,(W(\psi)F)(\phi+\chi) \\
&= e^{i\omega(\phi,\chi)/2}\, e^{i\omega(\psi,\phi+\chi)/2}\, F(\phi+\chi+\psi) \\
&= e^{i\omega(\psi,\phi)/2}\, e^{i\omega(\phi+\psi,\chi)/2}\, F(\phi+\chi+\psi) \\
&= e^{-i\omega(\phi,\psi)/2}\,(W(\phi+\psi)F)(\chi).
\end{aligned}
$$

Thus $W(\phi)W(\psi) = e^{-i\omega(\phi,\psi)/2}\, W(\phi+\psi)$. Let $\mathrm{CCR}(V,\omega)$ be the C^*-subalgebra of $\mathcal{L}(H)$ generated by the elements $W(\phi)$, $\phi \in V$. Then $\mathrm{CCR}(V,\omega)$ together with the map W forms a Weyl system for (V,ω).

Proposition 7. *Let (A, W) be a Weyl system of a symplectic vector space (V,ω). Then*

1. *$W(\phi)$ is unitary for each $\phi \in V$,*
2. *$\|W(\phi) - W(\psi)\| = 2$ for all $\phi, \psi \in V$, $\phi \neq \psi$,*
3. *the algebra A is not separable unless $V = \{0\}$,*
4. *the family $\{W(\phi)\}_{\phi\in V}$ is linearly independent.*

Proof. From $W(\phi)^*W(\phi) = W(-\phi)\,W(\phi) = e^{i\omega(-\phi,\phi)}W(0) = 1$ and similarly $W(\phi)\,W(\phi)^* = 1$ we see that $W(\phi)$ is unitary.

To show Assertion 2 let $\phi, \psi \in V$ with $\phi \neq \psi$. For arbitrary $\chi \in V$ we have

$$
\begin{aligned}
W(\chi)\,W(\phi-\psi)\,W(\chi)^{-1} &= W(\chi)\,W(\phi-\psi)\,W(\chi)^* \\
&= e^{-i\omega(\chi,\phi-\psi)/2}\, W(\chi+\phi-\psi)\,W(-\chi) \\
&= e^{-i\omega(\chi,\phi-\psi)/2}\, e^{-i\omega(\chi+\phi-\psi,-\chi)/2}\, W(\chi+\phi-\psi-\chi) \\
&= e^{-i\omega(\chi,\phi-\psi)}\, W(\phi-\psi).
\end{aligned}
$$

Hence the spectrum satisfies

$$
\sigma_A(W(\phi-\psi)) = \sigma_A(W(\chi)\,W(\phi-\psi)\,W(\chi)^{-1}) = e^{-i\omega(\chi,\phi-\psi)}\,\sigma_A(W(\phi-\psi)).
$$

Since $\phi - \psi \neq 0$ the real number $\omega(\chi, \phi - \psi)$ runs through all of $\mathbb{R}$ as χ runs through V. Therefore the spectrum of $W(\phi - \psi)$ is U(1)-invariant. By Assertion 5 of Proposition 2 the spectrum is contained in S^1 and by Proposition 1 it is nonempty. Hence $\sigma_A(W(\phi - \psi)) = S^1$ and therefore

$$\sigma_A(e^{i\omega(\psi,\phi)/2}\, W(\phi - \psi)) = S^1.$$

Thus $\sigma_A(e^{i\omega(\psi,\phi)/2}\, W(\phi - \psi) - 1)$ is the circle of radius 1 centered at -1. Now Assertion 3 of Proposition 2 says

$$\|e^{i\omega(\psi,\phi)/2}\, W(\phi - \psi) - 1\| = \rho_A\left(e^{i\omega(\psi,\phi)/2}\, W(\phi - \psi) - 1\right) = 2.$$

From $W(\phi) - W(\psi) = W(\psi)(W(\psi)^*\, W(\phi) - 1) = W(\psi)(e^{i\omega(\psi,\phi)/2}\, W(\phi - \psi) - 1)$ we conclude

$$\begin{aligned}
&\|W(\phi) - W(\psi)\|^2 \\
&= \|(W(\phi) - W(\psi))^*(W(\phi) - W(\psi))\| \\
&= \|(e^{i\omega(\psi,\phi)/2}\, W(\phi - \psi) - 1)^*\, W(\psi)^*\, W(\psi)\,(e^{i\omega(\psi,\phi)/2}\, W(\phi - \psi) - 1)\| \\
&= \|(e^{i\omega(\psi,\phi)/2}\, W(\phi - \psi) - 1)^*\,(e^{i\omega(\psi,\phi)/2}\, W(\phi - \psi) - 1)\| \\
&= \|e^{i\omega(\psi,\phi)/2}\, W(\phi - \psi) - 1\|^2 \\
&= 4.
\end{aligned}$$

This shows part 2. Assertion 3 now follows directly since the balls of radius 1 centered at $W(\phi)$, $\phi \in V$ form an uncountable collection of mutually disjoint open subsets.

We show Assertion 4. Let $\phi_j \in V$, $j = 1, \ldots, n$ be pairwise different and let $\sum_{j=1}^{n} \alpha_j W(\phi_j) = 0$. We show $\alpha_1 = \cdots = \alpha_n = 0$ by induction on n. The case $n = 1$ is trivial by Assertion 1. Without loss of generality assume $\alpha_n \neq 0$. Hence

$$W(\phi_n) = \sum_{j=1}^{n-1} \frac{-\alpha_j}{\alpha_n} W(\phi_j)$$

and therefore

$$\begin{aligned}
1 &= W(\phi_n)^* W(\phi_n) \\
&= \sum_{j=1}^{n-1} \frac{-\alpha_j}{\alpha_n} W(-\phi_n) W(\phi_j) \\
&= \sum_{j=1}^{n-1} \frac{-\alpha_j}{\alpha_n} e^{-i\omega(-\phi_n,\phi_j)/2} W(\phi_j - \phi_n) \\
&= \sum_{j=1}^{n-1} \beta_j W(\phi_j - \phi_n),
\end{aligned}$$

where we have put $\beta_j := \frac{-\alpha_j}{\alpha_n} e^{i\omega(\phi_n,\phi_j)/2}$. For an arbitrary $\psi \in V$ we obtain

$$\begin{aligned}1 &= W(\psi) \cdot 1 \cdot W(-\psi)\\ &= \sum_{j=1}^{n-1} \beta_j W(\psi) W(\phi_j - \phi_n) W(-\psi)\\ &= \sum_{j=1}^{n-1} \beta_j e^{-i\omega(\psi,\phi_j-\phi_n)} W(\phi_j - \phi_n).\end{aligned}$$

From

$$\sum_{j=1}^{n-1} \beta_j W(\phi_j - \phi_n) = \sum_{j=1}^{n-1} \beta_j e^{-i\omega(\psi,\phi_j-\phi_n)} W(\phi_j - \phi_n)$$

we conclude by the induction hypothesis

$$\beta_j = \beta_j e^{-i\omega(\psi,\phi_j-\phi_n)}$$

for all $j = 1, \dots, n-1$. If some $\beta_j \neq 0$, then $e^{-i\omega(\psi,\phi_j-\phi_n)} = 1$, hence

$$\omega(\psi, \phi_j - \phi_n) = 0$$

for all $\psi \in V$. Since ω is nondegenerate $\phi_j - \phi_n = 0$, a contradiction. Therefore all β_j and thus all α_j are zero, a contradiction. □

Remark 17. Let (A, W) be a Weyl system of the symplectic vector space (V, ω). Then the linear span of the $W(\phi)$, $\phi \in V$, is closed under multiplication and under $*$. This follows directly from the properties of a Weyl system. We denote this linear span by $\langle W(V)\rangle \subset A$. Now if (A', W') is another Weyl system of the same symplectic vector space (V, ω), then there is a unique linear map $\pi : \langle W(V)\rangle \to \langle W'(V)\rangle$ determined by $\pi(W(\phi)) = W'(\phi)$. Since π is given by a bijection on the bases $\{W(\phi)\}_{\phi\in V}$ and $\{W'(\phi)\}_{\phi\in V}$ it is a linear isomorphism. By the properties of a Weyl system π is a $*$-isomorphism. In other words, there is a unique $*$-isomorphism such that the following diagram commutes:

$$\begin{tikzcd} & \langle W'(V)\rangle \\ V \arrow[ur, "W_2"] \arrow[r, "W_1"'] & \langle W(V)\rangle \arrow[u, "\pi"'] \end{tikzcd}$$

Remark 18. On $\langle W(V)\rangle$ we can define the norm

$$\Big\| \sum_\phi a_\phi W(\phi) \Big\|_1 := \sum_\phi |a_\phi|.$$

This norm is not a C^*-norm but for every C^*-norm $\|\cdot\|_0$ on $\langle W(V)\rangle$ we have by the triangle inequality and by Assertion 1 of Proposition 7

$$\|a\|_0 \leq \|a\|_1 \tag{1.7}$$

for all $a \in \langle W(V)\rangle$.

Lemma 9. *Let (A, W) be a Weyl system of a symplectic vector space (V, ω). Then*

$$\|a\|_{\max} := \sup\{\|a\|_0 \mid \|\cdot\|_0 \text{ is a } C^*\text{-norm on } \langle W(V)\rangle\}$$

defines a C^*-norm on $\langle W(V)\rangle$.

Proof. The given C^*-norm on A restricts to one on $\langle W(V)\rangle$, so the supremum is not taken on the empty set. Estimate (1.7) shows that the supremum is finite. The properties of a C^*-norm are easily checked, e.g., the triangle inequality follows from

$$\begin{aligned}
\|a+b\|_{\max} &= \sup\{\|a+b\|_0 \mid \|\cdot\|_0 \text{ is a } C^*\text{–norm on } \langle W(V)\rangle\} \\
&\leq \sup\{\|a\|_0 + \|b\|_0 \mid \|\cdot\|_0 \text{ is a } C^*\text{–norm on } \langle W(V)\rangle\} \\
&\leq \sup\{\|a\|_0 \mid \|\cdot\|_0 \text{ is a } C^*\text{–norm on } \langle W(V)\rangle\} \\
&\quad + \sup\{\|b\|_0 \mid \|\cdot\|_0 \text{ is a } C^*\text{–norm on } \langle W(V)\rangle\} \\
&= \|a\|_{\max} + \|b\|_{\max}.
\end{aligned}$$

The other properties are shown similarly. □

Lemma 10. *Let (A, W) be a Weyl system of a symplectic vector space (V, ω). Then the completion $\overline{\langle W(V)\rangle}^{\max}$ of $\langle W(V)\rangle$ with respect to $\|\cdot\|_{\max}$ is* simple, *i.e., it has no nontrivial closed two-sided $*$-ideals.*

Proof. By Remark 17 we may assume that (A, W) is the Weyl system constructed in Example 16. In particular, $\langle W(V)\rangle$ carries the C^*-norm $\|\cdot\|_{\mathrm{Op}}$, the operator norm given by $\langle W(V)\rangle \subset \mathcal{L}(H)$ where $H = L^2(V, \mathbb{C})$.

Let $I \subset \overline{\langle W(V)\rangle}^{\max}$ be a closed two-sided $*$-ideal. Then $I_0 := I \cap \mathbb{C}\cdot W(0)$ is a (complex) vector subspace in $\mathbb{C}\cdot W(0) = \mathbb{C}\cdot 1 \cong \mathbb{C}$ and thus $I_0 = \{0\}$ or $I_0 = \mathbb{C}\cdot W(0)$. If $I_0 = \mathbb{C}\cdot W(0)$, then I contains 1 and therefore $I = \overline{\langle W(V)\rangle}^{\max}$. Hence we may assume $I_0 = \{0\}$.

Now we look at the projection map

$$P : \langle W(V)\rangle \to \mathbb{C}\cdot W(0), \quad P(\sum_\phi a_\phi W(\phi)) = a_0 W(0).$$

We check that P extends to a bounded operator on $\overline{\langle W(V)\rangle}^{\max}$. Let $\delta_0 \in L^2(V, \mathbb{C})$ denote the function given by $\delta_0(0) = 1$ and $\delta_0(\phi) = 0$ otherwise. For $a = \sum_\phi a_\phi W(\phi)$ and $\psi \in V$ we have

$$
\begin{aligned}
(a \cdot \delta_0)(\psi) &= (\sum_{\phi} a_\phi W(\phi)\delta_0)(\psi) \\
&= \sum_{\phi} a_\phi \, e^{i\omega(\phi,\psi)/2} \delta_0(\phi + \psi) \\
&= a_{-\psi} \, e^{i\omega(-\psi,\psi)/2} \quad = \quad a_{-\psi},
\end{aligned}
$$

and therefore

$$
(\delta_0, a \cdot \delta_0)_{L^2} = \sum_{\psi \in V} \overline{\delta_0(\psi)}(a \cdot \delta_0)(\psi) = (a \cdot \delta_0)(0) = a_0.
$$

Moreover, $\|\delta_0\| = 1$. Thus

$$
\|P(a)\|_{\max} = \|a_0 W(0)\|_{\max} = |a_0| = |(\delta_0, a \cdot \delta_0)_{L^2}| \leq \|a\|_{\mathrm{Op}} \leq \|a\|_{\max},
$$

which shows that P extends to a bounded operator on $\overline{\langle W(V) \rangle}^{\max}$.

Now let $a \in I \subset \overline{\langle W(V) \rangle}^{\max}$. Fix $\epsilon > 0$. We write

$$
a = a_0 W(0) + \sum_{j=1}^{n} a_j \, W(\phi_j) + r,
$$

where the $\phi_j \neq 0$ are pairwise different and the remainder term r satisfies $\|r\|_{\max} < \epsilon$. For any $\psi \in V$ we have

$$
I \ni W(\psi) \, a \, W(-\psi) = a_0 W(0) + \sum_{j=1}^{n} a_j \, e^{-i\omega(\psi,\phi_j)} \, W(\phi_j) + r(\psi),
$$

where $\|r(\psi)\|_{\max} = \|W(\psi) r W(-\psi)\|_{\max} \leq \|r\|_{\max} < \epsilon$. If we choose ψ_1 and ψ_2 such that $e^{-i\omega(\psi_1,\phi_n)} = -e^{-i\omega(\psi_2,\phi_n)}$, then adding the two elements

$$
\begin{aligned}
a_0 W(0) + \sum_{j=1}^{n} a_j \, e^{-i\omega(\psi_1,\phi_j)} \, W(\phi_j) + r(\psi_1) \in I \\
a_0 W(0) + \sum_{j=1}^{n} a_j \, e^{-i\omega(\psi_2,\phi_j)} \, W(\phi_j) + r(\psi_2) \in I
\end{aligned}
$$

yields

$$
a_0 W(0) + \sum_{j=1}^{n-1} a'_j \, W(\phi_j) + r_1 \in I,
$$

where $\|r_1\|_{\max} = \|\frac{r(\psi_1)+r(\psi_2)}{2}\|_{\max} < \frac{\epsilon+\epsilon}{2} = \epsilon$. Repeating this procedure we eventually get

$$a_0\, W(0) + r_n \in I,$$

where $\|r_n\|_{\max} < \epsilon$. Since ϵ is arbitrary and I is closed we conclude

$$P(a) = a_0\, W(0) \in I_0,$$

thus $a_0 = 0$.

For $a = \sum_\phi a_\phi\, W(\phi) \in I$ and arbitrary $\psi \in V$ we have $W(\psi)a \in I$ as well, hence $P(W(\psi)a) = 0$. This means $a_{-\psi} = 0$ for all ψ, thus $a = 0$. This shows $I = \{0\}$. □

Definition 23. *A Weyl system (A, W) of a symplectic vector space (V, ω) is called a* CCR representation *of (V, ω) if A is generated as a C^*-algebra by the elements $W(\phi)$, $\phi \in V$. In this case we call A a* CCR-algebra *of (V, ω).*

Of course, for any Weyl system (A, W) we can simply replace A by the C^*-subalgebra generated by the elements $W(\phi)$, $\phi \in V$, and we obtain a CCR representation.

Existence of Weyl systems, and hence CCR representations, has been established in Example 16. Uniqueness also holds in the appropriate sense.

Theorem 3. *Let (V, ω) be a symplectic vector space and let (A_1, W_1) and (A_2, W_2) be two CCR representations of (V, ω).*

Then there exists a unique $*$-isomorphism $\pi : A_1 \to A_2$ such that the diagram

$$\begin{tikzcd} & A_2 \\ V \arrow[ur, "W_2"] \arrow[r, "W_1"'] & A_1 \arrow[u, dotted, "\pi"'] \end{tikzcd}$$

commutes.

Proof. We have to show that the $*$-isomorphism $\pi : \langle W_1(V)\rangle \to \langle W_2(V)\rangle$ as constructed in Remark 17 extends to an isometry $(A_1, \|\cdot\|_1) \to (A_2, \|\cdot\|_2)$. Since the pullback of the norm $\|\cdot\|_2$ on A_2 to $\langle W_1(V)\rangle$ via π is a C^*-norm we have $\|\pi(a)\|_2 \le \|a\|_{\max}$ for all $a \in \langle W_1(V)\rangle$. Hence π extends to a $*$–morphism $\overline{\langle W_1(V)\rangle}^{\max} \to A_2$. By Lemma 10 the kernel of π is trivial, hence π is injective. Proposition 4 implies that $\pi : (\overline{\langle W_1(V)\rangle}^{\max}, \|\cdot\|_{\max}) \to (A_2, \|\cdot\|_2)$ is an isometry.

In the special case $(A_1, \|\cdot\|_1) = (A_2, \|\cdot\|_2)$ where π is the identity this yields $\|\cdot\|_{\max} = \|\cdot\|_1$. Thus for arbitrary A_2 the map π extends to an isometry $(A_1, \|\cdot\|_1) \to (A_2, \|\cdot\|_2)$. □

From now on we will call CCR(V, ω) as defined in Example 16 *the* CCR-algebra of (V, ω).

Corollary 5. *CCR-algebras of symplectic vector spaces are simple, i.e., all unit-preserving $*$-morphisms to other C^*-algebras are injective.*

Proof. Direct consequence of Corollary 2 and Lemma 10. □

Corollary 6. *Let (V_1, ω_1) and (V_2, ω_2) be two symplectic vector spaces and let $S : V_1 \to V_2$ be a* symplectic linear map, *i.e., $\omega_2(S\phi, S\psi) = \omega_1(\phi, \psi)$ for all $\phi, \psi \in V_1$.*

Then there exists a unique injective $*$-morphism $\mathrm{CCR}(S) : \mathrm{CCR}(V_1, \omega_1) \to \mathrm{CCR}(V_2, \omega_2)$ *such that the diagram*

$$\begin{tikzcd} V_1 \arrow[r, "S"] \arrow[d, "W_1"'] & V_2 \arrow[d, "W_2"] \\ \mathrm{CCR}(V_1, \omega_1) \arrow[r, "\mathrm{CCR}(S)"] & \mathrm{CCR}(V_2, \omega_2) \end{tikzcd}$$

commutes.

Proof. One immediately sees that $(\mathrm{CCR}(V_2, \omega_2), W_2 \circ S)$ is a Weyl system of (V_1, ω_1). Theorem 3 yields the result. □

From uniqueness of the map $\mathrm{CCR}(S)$ we conclude that $\mathrm{CCR}(\mathrm{id}_V) = \mathrm{id}_{\mathrm{CCR}(V,\omega)}$ and $\mathrm{CCR}(S_2 \circ S_1) = \mathrm{CCR}(S_2) \circ \mathrm{CCR}(S_1)$. In other words, we have constructed a functor

$$\mathsf{CCR} : \mathsf{SymplVec} \to \mathsf{C^*Alg},$$

where SymplVec denotes the category whose objects are symplectic vector spaces and whose morphisms are symplectic linear maps, i.e., linear maps $A : (V_1, \omega_1) \to (V_2, \omega_2)$ with $A^*\omega_2 = \omega_1$. By C*Alg we denote the category whose objects are C^*-algebras and whose morphisms are *injective* unit-preserving $*$–morphisms. Observe that symplectic linear maps are automatically injective.

In the case $V_1 = V_2$ the induced $*$-automorphisms $\mathrm{CCR}(S)$ are called *Bogoliubov transformation* in the physics literature.

References

1. Bär, C., Ginoux, N., Pfäffle, F.: Wave equations on Lorentzian manifolds and quantization. EMS Publishing House, Zürich (2007)
2. Bratteli, O., Robinson, D.W.: Operator Algebras and Quantum Statistical Mechanics I. Springer, Berlin Heidelberg (2002)
3. Davidson, K.: C_-algebras by example. AMS, Providence (1997)
4. Dixmier, J.: Les C*-algèbres et leurs représentations, 2nd edition, Gauthier-Villars Éditeur, Paris (1969)
5. Murphy, G.: C*-algebras and operator theory. Academic Press, Boston (1990)

6. Takesaki, M.: Theory of Operator Algebra I, Springer, Berlin, Heidelberg, New York (2002)
7. Manuceau, J.: C*-algèbre de relations de commutation. Ann. Inst. H. Poincaré Sect. A (N.S.) **8**, 139 (1968)
8. Bratteli, O., Robinson, D.W.: Operator Algebras and Quantum Statistical Mechanics II. Springer, Berlin Heidelberg 2002
9. Baez, F.: Bell's inequality for C^*-Algebras. Lett. Math. Phys. **13**(2), 135–136 (1987)