# INTUITIONS OF THREE KINDS

# IN GÖDEL'S VIEWS ON THE CONTINUUM

ABSTRACT: Gödel judges certain consequences of the continuum hypothesis to be implausible, and suggests that mathematical intuition may be able to lead us to axioms from which that hypothesis could be refuted. It is argued that Gödel must take the faculty that leads him to his judgments of implausibility to be a different one from the faculty of mathematical intuition that is supposed to lead us to new axioms. It is then argued that the two faculties are very hard to tell apart, and that as a result the very existence of mathematical intuition in Gödel's sense becomes doubtful.

John P. Burgess
Department of Philosophy
Princeton University
Princeton, NJ 08544-1006 USA
jburgess@princeton.edu

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Gödel's views on mathematical intuition, especially as they are expressed in his well-known article on the continuum problem,[1] have been much discussed, and yet some questions have perhaps not received all the attention they deserve. I will address two here.

First, an exegetical question. Late in the paper Gödel mentions several consequences of the continuum hypothesis (CH), most of them asserting the existence of a subset of the straight line with the power of the continuum having some property implying the "extreme rareness" of the set.[2] He judges all these consequences of CH to be implausible. The question I wish to consider is this: *What is the epistemological status of Gödel's judgments of implausibility supposed to be?* In considering this question, several senses of "intuition" will need to be distinguished and examined.

Second, a substantive question. Gödel makes much of the experience of the axioms of set theory "forcing themselves upon one as true," and at least in the continuum problem paper makes this experience the main reason for positing such a faculty as "mathematical intuition." After several senses of "intuition" have been distinguished and examined, however, I wish to address the question: *In order to explain the Gödelian experience, do we really need to posit "mathematical intuition," or will some more familiar and less problematic type of intuition suffice for the explanation?* I will tentatively suggest that Gödel does have available grounds for excluding one more familiar kind of intuition as insufficient, but perhaps not for excluding another.

## 1. Geometric Intuition

In the broadest usage of "intuition" in contemporary philosophy, the term may be applied to any source (or in a transferred sense, to any item) of purported knowledge not obtained by conscious inference from anything more immediate. Sense-perception fits this characterization, but so does much else, so we must distinguish *sensory* from *nonsensory* intuition. Narrower usages may exclude one or the other. Ordinary English tends to exclude sense-perception, whereas Kant scholarship, which traditionally uses "intuition" to render Kant's "Anschauung," makes sense-perception the paradigm case.[3]

If we begin with sensory intuition, we must immediately take note of Kant's distinction between *pure* and *empirical* intuition. On Kant's idealist view, though all objects of outer sense have spatial features and all objects of outer and inner sense alike have temporal features, space and time are features only of things as they appear to us, not of things as they are in themselves. They are forms of sensibility which we impose on the matter of sensation, and it is because they come from us rather than from the things that we can have knowledge of them in advance of interacting with the things. Only empirical, *a posteriori* intuition can provide specific knowledge of specific things in space and time, but pure intuition, spatial and temporal, can provide *a priori* general knowledge of the structure of space and time, which is what knowledge of basic laws of three-dimensional Euclidean geometry and of arithmetic amounts to.

Or so goes Kant's story, simplified to the point of caricature. Kant claimed that his story alone was able to explain how we are able to have the *a priori* knowledge of three-dimensional Euclidean geometry and of arithmetic that we have. But as is well known, not long after Kant's death doubts arose whether we really do have any such *a priori* knowledge in the case of three-dimensional Euclidean geometry, and later doubts also arose as to whether Kant's story is really needed to explain how we are able to have the *a priori* knowledge of arithmetic that we do have. Gödel has a distinctive attitude towards such doubts.

As a result of developments in mathematics and physics from Gauß to Einstein, today one sharply distinguishes mathematical geometry and physical geometry; and while the one may provide *a priori* knowledge and the other knowledge of the world around us, neither provides *a priori* knowledge of the world around us. Mathematical geometry provides knowledge only of mathematical spaces, which are usually taken to be just certain set-theoretic structures. Physical geometry provides only empirical knowledge, and is inextricably intertwined with empirical theories of physical forces such as electromagnetism and gravitation.

And for neither mathematical nor physical geometry does three-dimensional Euclidean space have any longer any special status. For mathematical geometry it is simply one of many mathematical spaces. For physical geometry it is no longer thought to be a good model of the world in which we live and move and have our being. Already with special relativity physical space and time are merged into a four-dimensional physical spacetime, so that it is only relative to a frame of reference that we may speak of three spatial dimensions plus a temporal dimension. With general relativity, insofar as we may speak of space, it is curved and non-Euclidean, not flat and Euclidean; and a personal contribution of Gödel's to twentiethcentury physics was to show that, furthermore, insofar as we may speak of time, it may be circular rather than linear.[4]

The Kantian picture thus seems totally discredited. Nonetheless, while Gödel holds that Kant was wrong on many points, and above all in supposing that physics can supply knowledge only of the world as it appears to us and not as the world really is in itself, still he suggests that Kant may nonetheless have been right about one thing, namely, in suggesting that time is a feature only of appearance and not of reality.[5]

As for intuition, again there is a mix of right and wrong. Gödel writes:

> Geometrical intuition, strictly speaking, is not mathematical, but rather a priori physical, intuition. In its purely mathematical aspect our Euclidean space intuition is perfectly correct, namely it represents correctly a certain structure existing in the realm of mathematical objects. Even physically it is correct 'in the small'.[6]

Elaborating, let us reserve for the pure intuition of space (respectively, of time) "in its physical aspect" the label *spatial* (respectively, *temporal*), intuition, and for the same pure intuition "in its mathematical aspect" let us reserve the label *geometric* (respectively, *chronometric*) intuition. Gödel's view, recast in this terminology, is that spatial intuition is about the physical world, but is only locally and approximately correct, while geometric intuition is globally and exactly correct, but is only about a certain mathematical structure. It would be tempting, but it would also be extrapolating beyond anything Gödel actually says, to attribute to him the parallel view about temporal *versus* chronometric intuition.

If geometric intuition "in its mathematical aspect" is "perfectly correct," can it help us with the continuum problem? The question arises because the continuum hypothesis admits a geometric formulation, thus:

> Given two lines $X$ and $Y$ in Euclidean space, meeting at right angles, say that a region $F$ in the plane they span *correlates* a subregion $A$ of $X$ with a subregion $B$ of $Y$ if for each point $x$ in $A$ there is a unique point $y$ in $B$ such that the point of intersection of the line through $x$ parallel to $Y$ and the line through $y$ parallel to $X$ belongs to $F$, and similarly with the roles of $A$ and $B$ reversed. Say that a subregion $B$ of $Y$ is *discrete* if for every point $y$ of $B$, there is an interval of $Y$ around $y$ containing no other points of $B$. Then for any subregion $A$ of $X$,

there is a region correlating *A* either with the whole of the line *Y* or else with a discrete subregion of *Y*.

Furthermore, it is not just the continuum hypothesis but many other questions that can be formulated in this style.[7] Among such questions are the problems of descriptive set theory whose status Gödel considers briefly at the end of his monograph on the consistency of the continuum hypothesis.[8] Can geometric intuition help with any of these problems? More specifically, can Gödel's implausibility judgments about the "extreme rareness" results that follow from CH be regarded as geometric intuitions? Some more background will be needed before this question can be answered.

Gödel's student years coincided with the period of struggle — Einstein called it a "frog and mouse battle" — between Brouwer's intuitionism and Hilbert's formalism. It is rather surprising, given the developments in mathematics and physics that tended to discredit Kantianism, that the two rival schools both remained Kantian in outlook. Thus Brouwer describes his intuitionism as "abandoning Kant's apriority of space but adhering the more resolutely to the apriority of time,"[9] while Hilbert proposes to found mathematics on spatial intuition, treating it as concerned with the visible or visualizable properties of visible or visualizable symbols, strings of strokes.[10]

Hans Hahn, Gödel's nominal dissertation supervisor and a member of the Vienna Circle, wrote a popular piece alleging the bankruptcy of intuition in mathematics,[11] and thus by implication separating himself, like a good logical positivist, from both the intuitionist frogs and the formalist mice. Hahn alludes to the developments in mathematics and physics culminating in relativity theory as indications of the untrustworthiness of intuition, but places more weight on such "counterintuitive" discoveries as Weierstraß's curve without tangents and Peano's curve filling space.[12] Do such counterexamples show that geometric intuition is not after all "perfectly correct"?

Gödel in effect insists that there is no real "crisis in intuition" while conceding that there is an apparent one. Thus we writes:

> One may say that many of the results of point-set theory … are highly unexpected and implausible. But, true as that may be, still … in those instances (such as, *e.g.*, Peano's curves) the appearance to the contrary can in general be explained by a lack

> of agreement between our intuitive geometrical concepts and the set-theoretical ones occurring in the theorems.[13]

The appearance of paradox results from a gap between the technical, set-theoretic understanding of certain terms with which Weierstraß, Peano, and other discoverers of pathological counterexamples were working, and the intuitive, geometric understanding of the same terms.

Presumably the key term in the examples under discussion is "curve." The technical, set-theoretic concept of curve is that of a continuous image of the unit interval. The intuitive, geometric concept of curve is of something more than this, though unfortunately Gödel does not offer any explicit characterization for comparison. Unfortunately also, Gödel does not address directly other "counterintuitive" results in the theory of point-sets, where presumably it is some term other than "curve" that is associated with different concepts in technical set-theory and intuitive geometry.[14] Thus he leaves us with little explicit indication of what he takes the intuitive geometric concepts to be like.

But to return to his basic point about the divergence between intuitive geometric notions and technical set-theoretic notions, it is precisely on account of this divergence, and not because of any unreliability of geometric intuition in its proper domain, that Gödel is unwilling to appeal to geometric intuition in connection with the continuum problem. Gödel explicitly declines for just this reason to appeal to geometric intuition in opposition to one of the easier consequences of the continuum hypothesis derived in Sierpinski's monograph on the subject.[15] The consequence in question is that the plane is the union of countably many "generalized curves" or graphs of functions $y = f(x)$ or $x = g(y)$.[16] This may appear "highly unexpected and implausible," but this notion of "generalized curve" is even further removed from the intuitive, geometric notion of curve than is the notion of a curve as any continuous image of the unit interval.[17] Thus no help with the continuum problem is to be expected from geometric intuition. We must conclude that Gödel's implausibility judgments are not intended as reports of geometric intuitions. They must be something else.

## 2. Rational Intuition

It is time to turn to nonsenory as opposed to sensory intuition, which will turn out to be a rather heterogeneous category. Let us proceed straight to the best-known passage in the continuum problem paper, which speaks of "something like a perception" even of objects of great "remoteness from sense experience":

> But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception…[18]

The passage is as puzzling as it is provocative.

Almost the first point Charles Parsons makes in his recent extended discussion of the usage of the term "intuition" in philosophy of mathematics is that it is crucial to distinguish intuition *of* from intuition *that*. One may, for instance, have an intuition *of* a triangle in the Euclidean plane without having an intuition *that* the sum of its interior angles is equal to two right angles.[19] Gödel, by contrast, seems in the quoted passage to leap at once and without explanation from an intuition *of* set-theoretic objects to an intuition *that* set-theoretic axioms are true. What is the connection supposed to be here? It is natural to think that perceiving or grasping set-theoretic *concepts* (set and elementhood) would involve (or even perhaps just consist in) perceiving or grasping *that* certain set-theoretic axioms are supposed to hold; but why should one think the same about perceiving set-theoretic *objects* (sets and classes)? After all, we have not just "something like" a perception but an outright perception of the objects of astronomy, but when we look up at the starry heavens above, no astronomical axioms force themselves upon us as true.[20]

The passage is the more puzzling because one can find, even within the same paper, passages where Gödel seems to distance himself from any claim to have intuition of mathematical objects individually:

> For someone who considers mathematical objects to exist independently of our constructions and of our having an intuition of them individually, and who requires only that the general mathematical concepts must be sufficiently clear for us to be able to recognize their soundness and the truth of the axioms concerning them, there exists, I believe, a satisfactory foundation for Cantor's set theory in its whole original extent and meaning, namely, axiomatics of set theory interpreted in the way sketched below.[21]

There is scarcely room for doubt that Gödel is thinking of himself as such a "someone." If so, then he seems to be insisting only on an understanding of "general mathematical concepts," not a perception of individual mathematical objects.

The general view of commentators, expressed already many years ago by William Tait,[22] is that one should not, on the strength of the puzzling passage, attribute to Gödel the view that we have a kind of ESP by which we can observe the elements of the set-theoretic universe. Tait points to clues to what Gödel may mean by "something like a perception of the objects of set theory" in Gödel's statements in adjoining passages to the effect that (i) even in the case of sense-perception we do not immediately perceive physical objects but form ideas of them on the basis of what we do immediately perceive, and (ii) the problem of the existence of mathematical objects is an exact replica of the problem of the existence of physical objects. Tait does not explicitly say what conclusions about what Gödel meant should be drawn from these passages, except to repudiate the ESP interpretation.

The conclusion one might think suggested would be this: The experience of the axioms forcing themselves upon us is like the experience of receiving sense-impressions, and inferring the set-theoretic objects from the experience of the axioms forcing themselves upon is is like inferring physical objects from sense-impressions. But there is a well-known problem with such a view. From sensations we infer material bodies as their *causes*, but if we are to avoid claims of ESP, we must not suppose that the sets can be inferred as *causes* of our feeling the axioms forced upon us. They are presumably inferrable, once the axioms have forced themselves upon us, only as things behaving as the axioms say sets behave; and the problem is that this will not distinguish the genuine sets from the elements of any isomorphic model, a point familiar from discussions of structuralism in philosophy of mathematics.

D. A. Martin has looked closely at the puzzling passage about "something like a perception of the objects of set theory" with a structuralist point of view in mind, denying like other commentators that Gödel is committed to the perceptibility of individual sets, and if I read him aright suggesting that Gödel may be speaking of the perception of the *structure* of the set-theoretic universe, rather than its elements.[23] The interpretation of Gödel as a structuralist may, however, seem anachronistic to some. A slightly different interpretation is available. For in the course of his study Martin collects textual evidence from a variety of Gödelian sources to show that Gödel does not, as Frege does, think of "objects" and "concepts" as non-overlapping categories, but rather thinks of concepts as a species within the genus of objects. This makes it at least conceivable that when Gödel speaks of a perception of the objects of set theory, he has in mind perception of the concepts of set theory, and that it does not seem as odd to him as it would to some of us to call these concepts "objects."

Parsons, too, seems to take Gödel to be including concepts among the "objects of set theory" in the passage under discussion.[24] In what follows I will take it that for Gödel we have something like a perception of the concept of set, bringing with it (or even perhaps just consisting in) axioms forcing themselves upon us. Such a reading makes Gödel an adherent of the view that there is a faculty resembling sense-perception but directed towards abstract ideas rather than concrete bodies. Commentators call such a faculty *rational* intuition.[25]

Rational intuition as applied specifically to mathematical concepts may be called *mathematical* intuition. Mathematical intuition as applied specifically to set-theoretic concepts may be called *set-theoretic* intuition. The geometric and chronometric intuitions encountered in the preceding section really should be reclassified as forms of mathematical intuition. Gödel does not tell us much about forms of mathematical intuition other than set-theoretic and geometric, let alone about forms of rational intuition other than mathematical; nor does he consider forms of nonsensory intuition other than rational (of which more below).

Belief in such a faculty as rational intuition is hardly original with or unique to Gödel. Thus Diogenes Laertius relates the following tale of an exchange between his namesake Diogenes the Cynic and Plato:

> As Plato was conversing about Ideas and using the nouns "tablehood" and "cuphood," he [the Cynic] said, "Table and cup I see; but your tablehood and cuphood, Plato, I can nowise see." "That's readily accounted for," said Plato, "for you have eyes to see the visible table and cup; but not the understanding by which ideal tablehood and cuphood are discerned."[26]

Nowadays the label "Platonist" is bandied about rather loosely in philosophy of mathematics, but Gödel the label really fits.

Ever a Platonist in this sense, Gödel became first something of a Leibnizian and then more of a Husserlian as he sought a home in a systematic philosophy for his basic belief in rational intuition. Gödel reportedly took up Husserl between the appearance of the first version of the second versions of the continuum problem paper. Commentators more familiar with Husserl and phenomenology than I am have seen evidence of Husserlian influence in some of the new material added to the second version.[27] The suggestion seems to be that the study of phenomenology may have led Gödel to put less emphasis on the supposed independent existence of mathematical objects, and more on other respects in which what I am calling rational intuition of concepts is supposed to resemble sensory intuition of objects.

Some respects come easily to mind, and can be found mentioned more or less explicitly in Gödel. Like sense-perceptions, rational intuitions are not the product of conscious inference, being observations rather than conclusions. Like sense-perceptions, rational intuitions constrain what we can think about the items they are perceptions or intuitions *of*, since we must think of those items as having the properties we observe them to have. Like sense-perceptions, rational intuitions seem open-ended, seem to promise a series of possible further observations. Like sense-perception, rational intuition can be cultivated, since through experience one can develop abilities for closer and more accurate observation.

One important point of resemblance needs to be added to the list: Like sense-perceptions, rational intuitions are fallible, and errors of observation sometimes lead us astray. Gödel emphasizes this feature more in his paper on Russell, where he naturally has to say something about the paradoxes, than in the one on Cantor. He describes Russell as

> …bringing to light the amazing fact that our logical intuitions (*i.e.*, intuitions concerning such notions as: truth, concept, being, class, etc.) are self-contradictory.[28]

Apparently, then, the mere fact that Russell found a contradiction in Frege's system would not in itself necessarily count for Gödel as conclusive evidence that Frege did not have a genuine rational intuition in favor of his Law V. A similar remark would presumably apply to the well-known minor fiasco in Gödel's declining years, when he proposed an axiom intended to lead to the conclusion that the power of the continuum is $\aleph_2$ but actually implying that it is $\aleph_1$.[29]

It may be mentioned that if rational intuition is really to be analogous to sensory intuition, then there must not only be cases where rational intuition is incorrect, but also cases where it is indistinct, like vision in dim light through misty air. And there is something like dim, misty perception of a concept in Gödel. For instance, Gödel seems to see, looming as in a twilight fog beyond the rather small large cardinal axioms he is prepared to endorse (inaccessible and Mahlo cardinals), further principles or maybe one big principle that would imply the existence of much larger cardinals, but that he is not yet in a position to articulate.[30]

The crucial philosophical question about rational intuition, however, is not how bright or dim it is; nor even how reliable or treacherous it is; nor yet how long or short the list of analogies with sense-perception is; and least of all whether "rational intuition" is right or wrong as a label for it. The crucial philosophical question is simply whether there is any real need to posit a special intellectual faculty in order to account for the experiences of the kind Gödel describes, where axioms "force themselves upon us," or whether on the contrary such experiences can be explained in terms of faculties already familiar and less problematic. For there are other, more mundane, varieties of nonsensory intuition, and a skeptic might suspect that one or another of them is what is really behind Gödelian experiences.

There is, for instance, *linguistic* intuition. Linguistic intuitions are simply the more or less immediate judgments of competent speakers to the effect that such-and-such a sentence is or isn't syntactically or semantically in order. In both scientific linguistics and philosophical analysis such intuitions provide the data against which syntactic or semantic rules and theories are evaluated. Even theorists who suppose that competent speakers arrive at their linguistic intuitions by unconsciously applying syntactic or semantic rules don't suppose that there is any psychoanalytic procedure to bring these unconscious rules to consciousness. The only way to divine what the rules must be is to formulate hypotheses, test them against the data that *are* conscious, namely, linguistic intuitions, then revise, retest, and so on until the dialectic reaches stable equilibrium.

Is familiar linguistic intuition enough to explain Gödelian experiences when axioms "force themselves upon us," or do we need to posit a more problematic rational intuition? Perhaps we should ask first just what the difference between appeal to one and appeal to the other amounts to. The two appeals seem to go with two different pictures, both starting from something like Gödel's exposition of the cumulative hierarchy or iterative conception of sets.[31]

On the linguistic picture, from that exposition and the meanings of the words in it we deduce by logic set-theoretic axioms, and then from these by more logic we deduce mathematical theorems. Since as competent speakers we know the meanings of the words in the exposition, and since we are finite beings, the meanings must themselves be in some sense finite. The mathematical theorems we can deduce are thus deducible by logic from a fixed finite basis.

On the rationalist picture, the only function of the original exposition is to get us to turn our rational intuition in the direction of the concept of set. Once we perceive it, we can go back to it again and again and perceive more and more about it. Hence, though at any stage we will have perceived only finitely much, still we have access to a potentially infinite set of set-theoretic axioms, from which to deduce by logic mathematical theorems. The mathematical theorems we can deduce are thus not restricted to those deducible by logic from a fixed finite basis.[32]

Now it is a consequence of Gödel's first incompleteness theorem that deduction by first-order logical rules from a fixed finite basis of first-order non-logical axioms will leave some mathematical questions unanswered, whatever the fixed finite basis may be. One cannot speak of strict entailments in connection with the kind of broad-brush picture-painting we have been engaged in, but one can say that, in view of Gödel's result, the linguistic picture *tends to suggest* that there must be absolutely undecidable mathematical questions, while the rationalist picture *tends to suggest* that there need not be.[33]

Or perhaps that overstates the matter. On the one hand, since semantic rules are not directly available to consciousness, and definitions doing full justice to the conventional linguistic meaning of a word are not always easy to find — witnesses decades of attempts by analytic philosophers to define "*S* knows that *p*" — when accepted axioms fail to imply an answer to some question, it is conceivable that they simply fail to incorporate everything that is part of the conventional linguistic meaning of some key term, and that appropriate use of linguistic intuition may lead to new axioms. On the other hand, even if it is assumed we have a rational intuition going beyond linguistic intuition, training this intellectual vision once again on the key concept may not be enough to give an answer to a question not decided by accepted axioms, since presumably there are limits to the acuity of metaphorical as much as to literal vision. It remains, however, that in any specific case of a question left undecided by current axioms, the one picture tends to inspire pessimism and the other optimism about the prospects for finding an answer.

That may be a reason to *hope* that the rationalist rather than the linguistic picture is the correct one, but have we any reason to *believe* it is? Gödel does not really address this question, but it seems clear to what evidence he would point, and what kind of claim he would have to make about it, namely, the claim that the standard axioms of set theory plus some large cardinals "force themselves upon us," *even though they are not strictly* *rigorously logically implied by the literal conventional linguistic meaning of his exposition* of the cumulative hierarchy or iterative conception of sets.

Readers may exercise their own intuitions to evaluate such a claim. For what it is worth, I myself do feel that, say, inaccessible cardinals are implied by the *spirit* but not the *letter* of Gödel's exposition. To me, deciding for or against inaccessibles seems a bit like a judge deciding one way or the other in a kind of case that was never anticipated by the legislature and which the literal meaning of the words of the applicable law does not settle unambiguously one way or the other. A decision in one direction may be in the spirit of the law and in the other contrary to it, even though it cannot be said that the letter of the law strictly implies the one or contradicts the other.

If all this is so, then the alleged instances of rational intuition that Gödel cites cannot be explained as instances of linguistic intuition. But explaining apparent rational intuitions as really linguistic intuitions is not the only alternative to recognizing a special faculty of rational intuition. For there may remain yet other kinds of intuition to be considered. After all, *something* has led Gödel to his implausibility judgments about "extreme rareness" results. It is certainly not linguistic intuition, and unless it can be claimed to be rational intuition, it must be something *else* that we have not yet considered.

Nothing Gödel says suggests that he takes his implausibility judgments to be clear rational intuitions. If they were, then he would presumably advocate the denials of the consequences of CH judged implausible as new axioms, comparable to the new axioms of inaccessible and Mahlo cardinals; and this he does not do. Nothing Gödel says even suggests that he has a dim, misty perception of any *potential* for new axioms out in the *direction* of these implausibility judgments.[34] The only directions from which Gödel even hints that a solution to the continuum problem might be sought is from large cardinals or something of the sort,[35] and that remains the most important direction being pursued today.[35]

It may also be pointed out that, while we have seen Gödel speak of "mathematical intuition" in the passage quoted at the beginning of this section, he never applies the term "intuition" to his implausibility judgments.[37] We must conclude that Gödel's implausibility judgments are not intended as reports of rational intuitions. They must be something else.

## 3. Heuristic Intuition

Gödel is interested in rational intuition as the source of axioms from which mathematical deductions can proceed, but he shows very little interest in the source of the deductions themselves. How are they discovered? As a creative mathematician of the highest power, Gödel will have known from personal experience, and far better than any commentator, that a deduction is not discovered by first discovering the first step, second discovering the second step, and so on. But he does not display anything like Hadamard's or Polya's interest in the psychology of discovery in the mathematical field, insofar as this pertains to discovery of proofs from axioms rather than of the axioms themselves.

Hadamard emphasizes the role of the unconscious, while Polya emphasizes the role of induction and analogy in mathematical proof-discovery and problem-solving. Both tend to make the thought-processes of the working mathematician rather resemble those of the empirical scientist.[38] Gödel in his reflections on mathematical epistemology takes little note of such matters. Nor does he take much note of the fact that the body of theorems of mathematics comes accompanied by a body of conjectures for which no rigorous proof has yet been found.[39] When mathematicians speak of "intuition," however, it is perhaps most often in connection with just these matters that Gödel neglects.

Thus it is said to be by intuition that one comes to suspect what the answer to a question must be before one finds the proof, or that a proof is to be sought in *this* rather than *that* direction. This use of "intuition" for the faculty of arriving at hunches, and "intuitions" for the hunches arrived at, is by no means confined to mathematics, but is probably closer to the ordinary sense of the word than any of the special senses of "intuition" considered by philosophers. In contrast to other kinds let me call this *heuristic* intuition.

There is, perhaps, less appearance of immediacy with heuristic than with sensory or linguistic intuitions. To be sure, no kind of intuition is immediate in an absolute sense. A great deal of processing goes on at a subconscious or "subpersonal" level during the very short interval between the exposure of the retina to light and the resulting visual experience, or between our exposure to a sentence and our judgment that it is or isn't good English. But none of this processing, even if one wants to say that it is in some sense a process of "inference," is the kind of inference that could be brought to consciousness.

By contrast, when one has a hunch, in mathematics or elsewhere, when "something makes one think that" such-and-such is the case, often it turns out to be possible after some effort to articulate, at least partially, what the *something* is, and then the heuristic intuition becomes no more an intuition but a heuristic *argument*. Typically the premises in a heuristic argument are a mix, with some that have been rigorously proved and others that are merely plausible conjectures (that is to say, that are themselves heuristic intuitions), while the inferential steps are a mix of logically valid and merely plausible transitions (such as reasoning by induction or analogy). It is reasonable to suspect that in all cases some kind of unconscious reasoning by induction or analogy or some other form of merely plausible inference from mathematic facts with which one is familiar underlies heuristic intuitions.

Maddy has usefully surveyed just about all the common heuristic arguments for and against CH.[40] Her survey, needless to say, includes the arguments against CH from Gödel's paper. The role of the implausibility judgments in that argument looks just like the role of heuristic intuitions in other arguments, and I am prepared to classify the implausibility judgments as heuristic intuitions, thus answering the exegetical question with which I began: What is the epistemological status of Gödel's judgments of implausibility supposed to be?

Of course, Gödel himself does not use this terminology, but the very title of the section of the paper where these implausibility judgments are advanced — "In What Sense and in Which Direction May a Solution of the Continuum Problem be Expected?" — seems to indicate that Gödel understands his implausibility judgments to be just the sort of plausibility or implausibility judgments that mathematicians typically come up with when discussing a famous conjecture pro and con. The "only" difference is that mathematicians generally anticipate that the famous conjecture they are discussing will eventually be proved or disproved, whereas (so far as proof from currently accepted axioms is concerned) Gödel knows that CH cannot be disproved, and suspects that CH cannot be proved.[41]

Now if we are told that there is supposed to be a difference in status between rational intuitions in favor of large cardinals and merely heuristic intuitions against extremely rare sets of the power of the continuum, we may wonder how one is supposed to be able to *tell*, when one has a pro- or anti-feeling about a given proposition, whether one is experiencing the one kind of intuition or the other. For it does not seem easy to do so.

Looking through Maddy's collection of "rules of thumb," one may guess that Gödel might consider some of them mere heuristic principles and others rational intuitions,[42] though ones too dim and misty to issue in rigorously-formulated axioms rather than roughly-formulated principles. But at least in my own case, I am able to guess this only because of my knowledge of attitudes Gödel has expressed in his writings, not because I myself have any sense when contemplating "rules of thumb" that *this* one is rational, *that* one heuristic.

Since Gödel does not discuss questions of heuristics explicitly, he never confronts the question of how to distinguish genuine, if fallible, perceptions of concepts from mere hunches perhaps based on subconscious inductive or analogical reasoning. But it seems there will be a serious weakness in his position unless a satisfactory answer can be given. At the very least, there will be a serious disanalogy between rational and sensory intuition. For when it comes to the visible properties of visible objects, there is no mistaking *seeing* what they are from having a *hunch* about what they must be.[43]

In the case of intuitions of concepts, unless there is a comparably unmistakable contrast between rational and heuristic intuition, the analogy between the former and vision will not be good. And while the length of the list of good analogies that can be discerned may not in itself matter much, in this particular instance a breakdown in the analogy would have a direct bearing on the question that does matter, the substantive question with which I began: In order to explain the Gödelian experience, do we really need to posit "rational intuition," or will some more familiar and less problematic type of intuition suffice for the explanation?

For if there is no criterion to distinguish rational from heuristic intuition, skeptics are likely to doubt that there is any such thing as rational intuition, over and above heuristic intuition, and to suggest that cases where axioms "force themselves upon us" are simply cases of very forceful heuristic intuition, perhaps based subconsciously on some very forceful analogical thinking. The search for new axioms for set theory, which Gödel urged on us, will then appear to skeptics rather as the law appeared to the greatest skeptic of all:

> Sometimes, the interests of society may require a rule of justice in a particular case; but may not determine any particular rule…In that case, the slightest *analogies* are laid hold of, in order to prevent that indifference and ambiguity, which would be the source of perpetual dissention…Many of the reasonings of lawyers are of this analogical nature, and depend on very slight connexions of the imagination.[44]

The Humean picture is very different from the Platonic, on which judges, in order to decide cases where the letter of the law does not unambiguously imply a decision, should direct the inner gaze of the understanding to the contemplation of the Form of Justice. The Gödelian picture on which set theorists, in order to decide questions where currently accept axioms do not imply a answer, should direct the inner gaze of their mathematical intuition to contemplation of the Concept of Set, is threatened by a skeptical suggestion similar to Hume's suggestion about the law, that supposed mathematical intuition is no more than laying hold of slight analogies and connections of the imagination. The absence of much explicit discussion of heuristics in Gödel leaves me not knowing from which direction to expect a Gödelian defense against such threats.[45]

SUMMARY: TABLE 1 lists the types of intuition distinguished in this paper, numbering the unsubdivided types. The exegetical question considered was whether Gödel's implausibility judgments are supposed to be cases of (iv) or (vi) or (viii). My answer was (viii). The substantive question considered was whether the phenomenon Gödel would explain by appeal to (vi) could be explained by appeal to (vii) or (viii). My answer was a tentative "no" for (vii) and a tentative "yes" for (viii).

TABLE 1. *Taxonomy of Intuition*

| | |
|---|---|
| I. *sensory* | |
| A. *empirical* | (i) |
| B. *pure* | |
| 1. *spatial* | (ii) |
| 2. *temporal* | (iii) |
| II. *nonsensory* | |
| A. *rational* | |
| 1. *mathematical* | |
| a. *geometric* | (iv) |
| b. *chronometric* | (v) |
| c. *set-theoretic* | (vi) |
| d. *other mathematical* | |
| 2. *other rational* | |
| B. *linguistic* | (vii) |
| C. *heuristic* | (viii) |
| D. *other nonsensory (if any)* | |

## NOTES

[1] "What is Cantor's continuum problem?" first version, *Mathematical Monthly*, vol. 9 (1947), pp. 515-525; second version, in P. Benacerraf & H. Putnam, eds., *Philosophy of Mathematics: Selected Readings*, Englewood Cliffs: Prentice-Hall, 1964, pp. 258-273. The second version is reprinted in the second edition of Benacerraf & Putnam, (Cambridge: Cambridge University Press), 1983, pp. 470-485. The massive work S. Feferman, J. W. Dawson, et al., eds., *Kurt Gödel: Collected Works* (Oxford: Oxford University Press), reprints all works published by Gödel during his lifetime in vols. I (1986) & II (1990), a substantial selection from his *Nachlaß* in vol. III (1993), and from his correspondence in vols. IV & V (both 2003). Both versions of the continuum problem paper are reprinted in vol. II, pp. 176-187 & 254-270. Quotations here will be from the second version, and the pagination in citations will be that of the first printing thereof.

[2] Gödel in two paragraphs in the first version (pp. 523-524) mentions seven consequences of CH in all, but the last one mentioned in the first paragraph follows almost immediately from the first one mentioned in the second paragraph, and hardly needs to be counted separately. The last two are rather special, and I'll set them aside. The remaining four are all "extreme rarity" results. For the *cognoscenti*, two pertain to Baire category and being topologically small, while two pertain to Lebesgue measure and being metrically small. The "extreme rarity" properties involved are these:

a universalized version of the property of being first category

a universalized version of the property of having measure zero having countable intersection with any first category set

having countable intersection with any measure zero set

The second version of the paper (p. 267) drops the last of these.

[3] Translators of ordinary, non-philosophical German, would perhaps most often render "Anschauung" as "view." The use of "intuition" for sense-perception conforms less well to the ordinary English meaning of the word than to its Latin etymology (from *intueri*, "to look").

[4] That is, there are solutions to the field equations of general relativity in which there are closed time-like paths. "An example of a new type of cosmological solutions of Einstein's field equations of gravitation," *Reviews of Modern Physics*, vol. 21 (1949), pp. 447-450. (The existence of such paths, a feature of some but not all of Gödel's models, is generally considered less significant for physics than the "rotating universe" feature of all his models.)

[5] "A remark about the relationship between relativity theory and idealistic philosophy," in P. Schilpp, ed., *Albert Einstein: Philosopher Scientist* (Evanston: Library of Living Philosophers), 1949, pp. 555-562. Gödel argues that while the conclusion that time is subjective and not objective is suggested already by special relativity, the case is not conclusive without his own results in general relativity.

[6] Letter to Marvin Jay Greenberg, October 2, 1973, in *Collected Works*, vol. IV, pp. 453-454. It is followed by a reply from Greenberg and a draft of a reply to that by Gödel, expressing doubt about the notion of any *non*-Euclidean geometric intuition. It is more than possible that Greenberg does not understand "intuition" in the same sense as Gödel, but in the sense discussed in §3 of the present paper.

[7] As explained, with figures, in §II.A.5.b "Non-Empirical Physics," in J. Burgess & G. Rosen, *A Subject with No Object* (Oxford: Oxford University Press), 1997, pp. 118-123, in connection with "geometric nominalism."

[8] *The Consistency of the Continuum Hypothesis* (Princeton: Princeton University Press), 1940, p. 67, Note 1. Actually, this note mentions explicitly just the existence of a projective well-ordering of the real numbers in order type $\omega_1$, only alluding to and not discussing further implications. For a filling in of Gödel's sketch and explicit treatment of further implications see John W. Addison, "Some consequences of the axiom of constructibility," *Fundamenta Mathematicæ*, vol. 46 (1959), pp. 337-357.

[9] L. E. J. Brouwer, "Intuitionism and Formalism, English translation by A. Dresden, originally in the *Bulletin of the American Mathematical Society*, vol. 20 (1913), pp. 81-96; reprinted in Benacerraf & Putnam, pp. 66-77, with the quoted passage on p. 69. The paper dates from between the discovery of special relativity and that of general relativity. Kant's views on time as regards *outer* sense seem discredited by special relativity and the discovery that the temporal order of distant events is in general not absolute but relative

to a frame of reference. But it is clear from the continuation of the passage that Brouwer is speaking of adhering to Kant's views on time only as regards *inner* sense. If those views, too, are threatened by developments in physics, it is by Gödel's results in general relativity.

[10] More precisely, Hilbert proposes to found *finitist* mathematics in this way; but finitist mathematics is for him the only "real" or *inhaltlich* mathematics. Charles Parsons has objected that though Hilbert regarded exponentiation as a legitimate operation of finitist arithmetic on a par with addition, there is a crucial difference. See his *Mathematical Thought and Its Objects* (Cambridge: Cambridge University Press), 2008, especially chapter 7 "Intuitive Arithmetic and Its Limits." The objection of Parsons is that while addition as an operation of strings of strokes can be visualized as juxtaposition, exponentiation seems to be visualizable only as a *process* rather than an *object*. But it remains that Hilbert's professed orientation, despite his deep interest in general relativity, is still quasi- or neo-Kantian to the same degree as Brouwer's. Of course, Hilbert does not make mathematics depend on geometric intuition in the way that Frege was driven to do after the collapse of his logicist program in contradiction: He does not revert to Newton's conception of real numbers as abstracted ratios of geometric properties, whose basic laws are to be derived from theorems of Euclidean geometry.

[11] "The Crisis in Intuition." Originally a lecture in German, it is very well known in the English speaking world from its appearance in print in English — no translator is named

— in James R. Newman's anthology, *The World of Mathematics* (New York: Simon & Schuster), 1956, vol. III, 1956-1976.

[12] The "counterintuitiveness" of these examples has been disputed by Benoît Mandelbrot in *The Fractal Geometry of Nature* (New York: W. H. Freeman), 1977, *passim*. His appears, however, to be a minority view.

[13] "What is Cantor's Continuum Problem?" p. 267. The importance of this passage has been noted by both of the commentators whose work has most influenced the present paper, Penelope Maddy and D. A. Martin, in their papers cited below. (Maddy in particular explicitly reaches the conclusion stated in the last two sentences of the present section.)

[14] For instance, despite his ringing endorsement of the axiom of choice as in all respects equal in status to the other axioms of set theory ("What is Cantor's continuum problem?" p. 259, footnote 2), he does not discuss one of its most notorious geometrical consequences, the Banach-Tarski paradox, and this even though he cites the paper in which the word "paradox" was first applied to the Banach-Tarski result. (L. M. Blumenthal, "A Paradox, a Paradox, a Most Ingenious Paradox," *American Mathematical Monthly*, vol. 47 (1940), pp. 346-353.) The absence of an explicit Gödelian treatment of this example is especially regrettable because one suspects that what Gödel would have said about this case, where "intuitions" contrary to set-theoretic results seem to be based on the assumption that any region of space must have a well-defined volume, might well

extend to the "intuitions" appealed to in Chris Freiling's infamous argument against the continuum hypothesis ("Axioms of symmetry: throwing darts at the real number line," *Journal of Symbolic Logic*, vol. 51 (1986), pp. 190-200), which commentators have seen as assuming that any event must have a well-defined probability.

[15] See “What is Cantor's continuum problem?” p. 273, the second of four numbered remarks at the beginning of the supplement added to the second version, for Gödel's remarks on Waclaw Sierpinski, *L'Hypothèse du Continu*." The particular consequence alluded to is among the *equivalents* of CH listed in the book, where it is named $\mathbf{P}_2$. Gödel cites both the first edition, (Warsaw: Garasinski), 1934, and the second, (New York: Chelsea), 1956.

[16] The continuum hypothesis implies that there is an ordering of the real numbers in which for each $x$ there are only countably many $y$ less than $x$. The axiom of choice allows us to pick for each $x$ a function $h_x$ from the natural numbers onto the set of such $y$. Then we may define functions $f_n(x) = h_x(n)$, and the graphs of these functions, plus their reflections in the diagonal $y = x$, plus the diagonal itself, give countably many "generalized curves" filling the plane.

[17] Even Mandelbrot's more expansive conception of what is intuitive seems to take in only $F_\sigma$ or $G_\delta$ or anyhow low-level Borel sets (to which classifications his "fractals" all belong), not arbitrary "generalized curves."

[18] "What is Cantor's continuum problem" p. 271. This passage comes from the supplement added to the second version of the paper.

[19] See *Mathematical Thought and Its Objects*, p. 8. This book has had a greater influence on the present paper than will be evident from my sporadic citations of it. Inversely, Parsons holds, as a consequence of his structuralism, that we can have an intuition *that* every natural number has a successor, though we have no intuition *of* natural numbers. See *Mathematical Thought and Its Objects*, §37 "Intuition of numbers denied," pp. 222-224.

[20] The most plausible account to date of how and in what sense we might be said to perceive sets is that of Penelope Maddy in *Realism in Mathematics* (Oxford: Oxford University Press), 1990, especially chapter 2 , "Perception and Intuition." But on this account set-theoretic perception is mainly of small sets of medium-sized physical objects, just as sense-perception is mainly of medium-sized physical objects themselves. The theoretical extrapolation to infinite sets then seems to have the same status as the theoretical extrapolation to subvisible physical particles, and this would seem to leave the axiom of infinity with the same status as the atomic hypothesis: historically a daring conjecture, which by now has led to so much successful theorizing that we can hardly imagine doing without it, but still not something that "forces itself upon us."

[21] The passage comes from §3 of the paper (p. 262) and leads into Gödel's exposition of the cumulative hierarchy or iterative conception of set (which is what the phrase "the way sketched below" in the quotation refer to).

[22] "Truth and Proof: The Platonism of Mathematics," *Synthese*, vol. 69 (1986), pp. 341-370. See note 3, pp. 364-365.

[23] "Gödel's conceptual realism," *Bulletin of Symbolic Logic*, vol. 2 (2005), pp. 207-224. I will not be doing justice to this study, which would require extended discussion of structuralism. In particular I will not be discussing what real difference, if any, there would be between perceiving the *structure* of the universe of sets as Martin understand it and perceiving the *concept* of set as Gödel understands concepts. (Both are clearly different from perceiving the individual sets that occupy positions in the structure and exemplify the concepts.)

[24] See "Platonism and mathematical intuition in Kurt Gödel's thought," *Bulletin of Symbolic Logic*, vol. 1 (1995), pp. 44-74, where he discusses the passage at issue on p. 65. In helpful comments on a preliminary version of the present study, Parsons remarks, "One piece of evidence … is that Gödel frequently talks [elsewhere] of perception of concepts but hardly at all about perception or intuition of sets. It may be that any perception of sets that he would admit is derivative from perception of concepts," here alluding to the suggestion made in footnote 43 of the cited paper that those sets, such as the ordinal ω that individually definable may be "perceived" by perceiving the concepts

that identify them uniquely — though, of course, what it identifies uniquely is really only the position of the ordinal in the set-theoretic universe.

[25] See Parsons, *Mathematical Thought and Its Objects*, §52 "Reason and 'rational intuition'" for some healthy skepticism about the appropriateness of this traditional term.

[26] Diogenes Laertius, with English translation by R. D. Hicks, *Lives of Eminent Philosophers*, Loeb Classical Library (Cambridge: Harvard University Press), 1925, Book VI, Diogenes, p. 55.

[27] In particular, Kai Hauser in a talk at the 2009 NYU conference in philosophy of mathematics cited as evidence of Husserlian influence the following somewhat concessive passage (which has also drawn the attention of earlier commentators):

> However, the question of the objective existence of the objects of mathematical intuition … is not decisive for the problem under consideration here. The mere psychological fact of the existence of an intuition which is sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis. (penultimate paragraph of the supplement, p. 272)

[28] "Russell's mathematical logic," in Benacerraf & Putnam, pp. 221-232, with the quoted passage on pp. 215-216. Gödel's "Platonism" or "realism" is nearly as evident in this

work as in the continuum problem paper. Parsons, in correspondence, while agreeing that Gödel acknowledged the fallibility of rational intuition, and emphasizing that in so acknoweldging Gödel was departing from the earlier rationalist tradition, nonetheless warns against reading too much into the quoted passage, on the grounds that Gödel's usage of "intuition" may have been looser than at the time of the Russell paper than it later became.

[29] The documents (two notes and an unsent letter by Gödel), and an informative discussion of the unedifying episode by Robert Solovay, can be found in *Collected Works*, vol. III, pp. 405-425. Another example of the fallibility of intuition may perhaps be provided by the fact mention by Solovay, that the pioneering descriptive set theorist Nikolai Luzin, who disbelieved CH, connected his disbelief with "certainty" that every subset of the reals of size $\aleph_1$ is coanalytic. We now know, however, that assuming a measurable cardinal, if CH fails then *no* set is of size $\aleph_1$ is coanalytic (since assuming a measurable cardinal, every coanalytic set is either countable or of the power of the continuum).

[30] His formulations, however, in "What is Cantor's continuum problem?" p. 264, footnote 20 and the text to which it is attached, are rather cautious, and he mentions on the next page that "there may exist … other (hitherto unknown) axioms."

[31] In §3 "Restatement of the problem…" or in other expositions of the same kind, several of which can be found in §IV "The concept of set" of the second edition of Benacerraf &

Putnam. Note, however, that two of the contributors there, George Boolos ("The iterative conception of set," pp. 486-502) and Charles Parsons ("What is the iterative conception of set?" pp. 503-529) in effect deny the reality of Gödelian experiences, deny that the axioms do "force themselves upon us." They do so also in other works (Boolos in "Must we believe in set theory?" in *Logic, Logic, and Logic* (Cambridge: Harvard University Press), 1998, pp. 120-132. Parsons in *Mathematical Thought and Its Objects*, §55 "Set theory," pp.338-342). In this paper I will not debate this point, but will simply grant for the sake of argument that Gödel is right and in fact there occurs such a phenomenon as the axioms "forcing themselves upon one." The issue I wish to discuss is, granting that in fact such experiences occur, whether we need to posit rational intuition to explain their occurrence.

[32] The kind of view I am attributing to Gödel resembles the kind of view Tyler Burge attributes to Frege. See "Frege on sense and linguistic meaning," in *Truth, Thought, Reason* (Oxford: Clarendon Press), 2005, pp. 242-269. Frege sometimes says that everyone has a grasp of the concept of number and sometimes says that even very eminent mathematicians before him lacked a sharp grasp of the concept of number. Burge proposes to explain Frege's speaking now one way, now the other, by suggesting that Frege distinguishes the kind of minimal grasp of the associated concept possessed by anyone who knows the fixed, conventional linguistic meaning of an expression, with the ever sharper and sharper grasp to which not every competent speaker of the language, by any means, can hope to achieve.

[33] Something like the contrast I have been trying to describe was, I suspect, ultimately the issue between Gödel and Carnap, but examination of that relationship in any detail is out of the question here. A complication is that Gödel sometimes uses "meaning" related terms in idiosyncratic senses, so that he ends up saying that mathematics is "analytic" and thus *sounding* like Carnap, though he doesn't at all *mean* by "analytic" what Carnap would. Martin and Parsons both discuss examples of this usage.

[34] It would be very difficult to formulate any such new axiom about extreme rarity, since nothing is more common in point-set theory than to find that sets small in one sense are large in another. Right at the beginning of the subject comes the discovery of the Cantor set, which is small topologically (first category) and metrically (measure zero), but large in cardinality (having the power of the continuum). Another classic result is that the unit interval can be written as the union of a first category set and a measure zero set. See John C. Oxtoby, *Measure and Category* (Berlin: Springer), 1971, for more information (The particular result just cited appears as Corollary 1.7, p. 5.) The difficulty of finding a rigorous formulation, however, is only to be expected with dim and misty rational intuitions.

[35] Here "something of the sort" may be taken to cover the suggestion of looking for some sort of maximal principle, made in footnote 23, p. 266. Gödel also mentions (p. 265) the possibility of justifying a new axiom not by rational intuitions in its favor, but by verification of striking consequences. Gödel cites no candidate example and even today it is not easy to think of one, if one insists that the striking consequences be not just

æsthetically pleasing, like the pattern of structural and regularity properties for projective sets that follow from the assumption of projective determinacy, but *verified*. The one case I can think of is Martin's proof of Borel determinacy (as a corollary of analytic determinacy) assuming a measurable cardinal before he found a more difficult proof without that assumption. And in this example the candidate new axiom supported is still a large cardinal axiom.

[36] To be sure, in the wake of Cohen's work, Azriel Levy and Solovay showed that no solution to the continuum problem is to be expected from large cardinal axioms of a straightforward kind. (See their "Measurable cardinals and the continuum hypothesis," *Israel Journal of Mathematics*, vol. 5 (1967), pp. 233-248.) But the present-day Woodin program can nonetheless be considered as in a sense still pursuing the direction to which Gödel pointed. According to Woodin's talk at the 2009 NYU conference in philosophy of mathematics, one of the possible outcomes of that program would be the adoption of a new axiom implying (1) that power of the continuum is $\aleph_2$ and (2) that Martin's Axiom (MA) holds. (1) is something Gödel came, at least for a time, to believe (in connection with the unedifying square axioms incident alluded to earlier). (2) is shown by Martin and Solovay, in the paper in which MA was first introduced ("Internal Cohen extensions," *Annals of Mathematical Logic*, vol. 2 (1970), pp. 143-178; see especially §5.3 "Is A true?" pp. 176-177), to imply many of the same consequences as CH. In particular, MA implies several of the consequences about extreme rarity that Gödel judges implausible, plus a modified version of another that Gödel might well have judged nearly equally implausible.

[37] The implausibility judgments are at least indirectly classified as "intuitions" by commentators. Martin and Solovay contrast Gödel's opinion with their own "intuitions," thus:

> If one agrees with Gödel that [the extreme rareness results] are implausible, then one must consider [MA] an unlikely proposition. The authors, however, have virtually no intuitions at all about [the extreme rareness results]… (p. 176)

Martin ("Hilbert's First Problem: The Continuum Hypothesis," in F. Browder, ed., *Mathematical Developments Arising from Hilbert Problems*, Proceedings of Symposia in Pure Mathematics, vol. 28 (Providence: American Mathematical Society), pp. 81-92) refers to Gödel's judgments as "intuitions" as he expresses dissent from them, thus:

> While Gödel's intuitions should never be taken lightly, it is very hard to see that the situation is different from that of Peano curves, and it is even hard for some of us to see why the examples Gödel cites are implausible at all.

The usage of the commentators here is in conformity with the kind of usage of "intuition" in mathematics to be discussed in the next section; but it seems Gödel's usage is more restricted than that.

[38] Jacques Hadamard, *The Psychology of Invention in the Mathematical Field* (New York: Dover), 1945. George Polya, *Mathematics and Plausible Reasoning* (Princeton: Princeton University Press), 1954, vol. I *Induction and Analogy in Mathematics*, vol. II

*Patterns of Plausible Inference*. The resemblance between mathematical and scientific methodology is most conspicuous in Polya's second volume, where the patterns of plausible inference Polya detects in mathematical thought closely resemble the rules of Bayesian probabilistic inference often cited in work on the epistemology of science. It is, however, difficult to view them as literal instances, since the Bayesians often require that all logicomathematical truths be assigned probability one.

[39] There are as well principles for which we do not even have a rigorous statement, let alone a rigorous proof. Such is the case with the Lefschetz principle, or Littlewood's three principles, for instance. Rigorous formulations of *parts* of such principles are possible, but always fall short of their full content. The "rules of thumb" in set theory identified by Maddy ("Believing the axioms," *Journal of Symbolic Logic*, vol. 53 (1988), part I pp. 481-511, part II pp. 736-764) may also be considered to be of this type.

[40] "Believing the Axioms, " §II.3 "Informed opinion," pp. 494-500. To give an example not in Maddy's collection, one might argue heuristically against the continuum hypothesis as follows. CH implies not only that all uncountable subsets of the line have the same number of elements, but also that all partitions of the line into uncountably many pieces have the same number of pieces. But even looking at very simple partitions (those for which the associated equivalence relation, considered as a subset of the plane, is analytic) with uncountably many pieces, we find what seem two quite different kinds. For it can be proved that the number of pieces is exactly $\aleph_1$ and that there is no perfect set of pairwise inequivalent elements, while for others it can proved that there *is* such a perfect set and

(hence) that the number of pieces is the power of the continuum. (Compare Sashi Mohan Srivastava, *A Course on Borel Sets* (Berlin: Springer), 1988, chapter 5.)

[41] The suspicion was confirmed by Cohen just a little too late for any more discussion than a very short note at the end to be incorporated into the paper.

[42] Especially the one Maddy calls "Maximize." This looks closely related to Gödel's thinking in footnote 23, p.266, already cited.

[43] This formulation may need a slight qualification. Suppose you are walking through a city you have never visited before, and are approaching a large public building, but are still a considerable distance away, and that the air is full of dust. Despite distance and dust, you are able to form some visual impression of the building. You are equally able to make conjectures about the appearance of the building by induction and analogy, taking into account the features of the lesser buildings you are passing, which you can see much better, and of large public buildings in other cities in the same country that you have recently visited under more favorable viewing conditions. Owing to the influence of expectation on perception, it is just barely possible, if the building is distant enough and the air dusty enough, to mistake such a conjecture for a visible impression, and think one is seeing what one is in fact only imagining must be there. But these are marginal cases.

[44] David Hume, *Enquiry Concerning the Principles of Morals*, §III, part II, ¶ 10. (In version edited by J. Schneewind (Indianapolis: Hackett), 1983, the passage appears on p.

29.) There is, of course, this difference from the situation described by Hume, that it isn't so clear that the interests of society or even of mathematics demand a ruling on the status of the continuum hypothesis.

[45] Parsons, in correspondence, suggests that Gödel might emphasize that potential new axioms force themselves upon us *as flowing from the very concept of set*, something that is rather obviously not the case with his implausibility judgments, though it is equally obviously not the case with the "square axioms" Gödel was later to propose. The danger I see with emphasizing this feature, in order to distinguish rational from heuristic intuition, is that it may make it more difficult to distinguish rational from linguistic intuition.