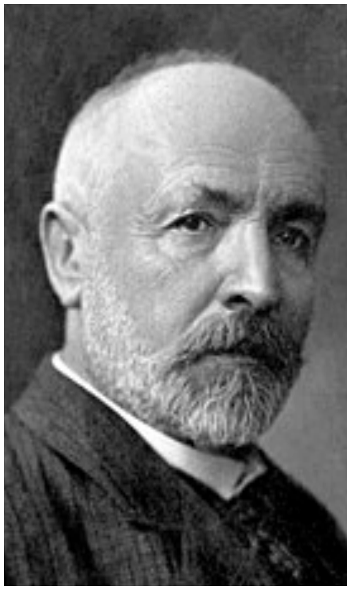
LECTURE 1

Georg Cantor (1845-1918)   Ernst Zermelo (1871-1953)

Set theory has two aspects

•. as a branch of mathematics in its own right, more related to some branches rather than others
with its distinctive subject matter being infinity (a traditional subject of philosophical speculation)
cardinal numerals one, two, three, *answer* "how many of them are there?"
ordinal numerals first, second, third, *answer* "at which place in the series of them does it occur?"
Cantor in papers of the 1870's and 1880's show the natural numbers extend into the infinite two ways
transfinite cardinals and *transfinite ordinals* each with their own arithmetic
(probably true but won't assume students know something already about transfinite cardinals)
bio at en.wikipedia.org/wiki/Georg_Cantor is reasonably accurate
(wanted job in Berlin but stuck in Halle owing to opposition of Kronecker to his theories,
organizer of first International Congress of Mathematicians,
repeatedly hospitalized for bipolar disorder)

• as a framework/foundation for all branches of mathematics realizing the ideal of *rigor*:
all new results must be logically deduced from earlier ones, ultimately from axioms
notions and notations ubiquitous in mathematics (and branches of philosophy that apply mathematics)
set-theoretic paradoxes showed the need for a higher standard of rigor than Cantor's
the axioms of *ZFC*, due mainly to Zermelo turned out adequate not only for set theory but for all math
bio at en.wikipedia.org/wiki/Ernst_Zermelo is reasonably accurate but incomplete
(besides work in set theory on axiom of choice and structure of the set-theoretic universe
in early career offered a "disproof" of the atomic hypothesis, which his boss Planck still doubted,
based on false but common assumption of the universal validity of the 2nd law of thermodynamics,
had alcohol problem, but fired under rules of Heidegger for failing to give Hitler salute)
it is essentially what is found in volume 1 of Bourbaki's *Elements of Mathematic*
an encyclopedia rigorously developing the core of modern mathematics
the main result of PHI 312 Intermediate Logic (*NOT* presupposed by this course) is
Gödel's incompleteness theorem: *for any axiom system for mathematics*
there are mathematical propositions that cannot be proved or refuted (disproved) from those axioms
this applies to ZFC: questions undecidable in ZFC are undecidable by all accepted mathematical means
Gödel's examples of undecidable questions are contrived, set theory provides a natural one,
Cantor's "continuum hypothesis" (CH) about the relation between cardinals and ordinals
(a.k.a. Hilbert's first problem); Gödel himself proved its irrefutability, Cohen its unprovability

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raises philosophical question: do we need more axioms? where would we get them?
Logical order would be to begin with axioms and show how mathematics is developed within them
before turning to theory of cardinals and ordinals and other specifically set-theoretic topics
because first stages of axiomatic development are rather dry, will postpone beginning it until second lecture
and here give a simplified historical sketch, showing how set theory fits into the history of mathematics

The oldest mathematical documents are a few Egyptian papyri and many Babylonian clay tablets
mostly discovered only in 19th century; before then one would have though Egyptian math was geometric,
Babylonian was algebraic; in fact, both were both, but can be seen independent in treatment of fractions
Contemporary cosmopolitan mathematics (with an International Mathematic Union containing societies
from every continent but Antarctica) goes back to Italian Renaissance mathematics, which inherited
mathematics from a half-dozen earlier civilizations south and east of Italy
Euclid's *Elements* (at first available only in Arabic translation, then in Greek original)
transmitted ideal of rigor and the theoretical science of geometry ("earth measurement")
that Greeks before Euclid had created from the practical craft of Egyptian surveyors
Muhammed al-Khwarizmi's [cf "algorithm"] *De Numero Indorum* (Latin translation, Arabic original lost)
introduced decimal "Hindu-Arabic" numerals from India (influenced by Babylonia, probably also China)
more importantly *al-Jabr w'al Muqabala* [cf. "algebra"] expounds systematic methods for solving
simultaneous linear equations and single quadratic equations
"geometry has proofs, algebra has techniques" (still the case in my junior high school in the 1960s)

Early modern contributions were all on the non-rigorous side:
√ Cardano et al. (cubic and quartic equations) involved recognizing negatives, introducing imaginaries
characteristic of modern mathematics to solve problems about one system by introducing auxiliary system
("irreducible case" of cubic, with three distinct real roots, got at through complex numbers)
√ Viète ["symbolic" as opposed to "rhetorical" or "syncopated" using words or abbreviations,
essentially the notation we still use]
√ Descartes/Fermat coordinate methods make algebra applicable to geometry
(Omar Khayyam had done the reverse, applying Apollonius on conic sections to cubic equations)
in the process compromising the rigor of geometry
characteristic of modern mathematics to connect different mathematical structures
(e.g. plane or space with real numbers)
√ Considering a finite object as a sum of infinitely many infinitely small ones (known to Archimedes
but not mentioned in his rigorous proofs and found in MS discovered only in 20th century)
used to solve some classical geometric tangency and area problems
√ Newton, Leibniz differentiation and integration, used in tangency and area problems
are inverse operations and relate position, speed, acceleration — basis for Newtonian physics

Probably the mathematics of 18th century physics (differential equations) would not have been discovered
if mathematicians had insisted on rigor as strictly as they were doing by ca 1900
Probably the mathematics of 20th century physics (differential geometry, functional analysis)
would not have been discovered if mathematicians had been as loose about rigor as they were ca 1800
One main motive for rigor: generality. Introduction of unfamiliar structures where intuition unreliable:
e.g. non-Euclidean *hyperbolic space* in geometry, non-commutative *quaternions* in algebra
In analysis "vibrating string controversy" (involving Euler, d'Alembert) show lack of agreement
on what is a "function"; eventually most general notion adopted, with requirement that any special properties
assumed in a given theorem be explicitly stated in hypotheses.
Cantor's work was part of the rigorization process.

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Related to two of the most ancient topics in pure and applied mathematics.

I. Cantor on trigonometric series

History goes back to the Pythagorean discovery that musical octave *do/do'*, fifth *do/so*, fourth *do/fa* are related to the ratios 1:2, 2:3, 3:4; pitch and volume correspond to frequency and amplitude
fundamental + overtones, timbre of an instrument determined by the pattern
mathematical representation

$$\sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx) + C$$

"vibrating string problem"concerned which functions could be represented in this form
cf. Fourier on heat
many 19th century mathematicians were involved in rigorous development of existence theory
Cantor's colleague Heine suggested he should work on related uniqueness question

basic theorem: if the sequence converges at every point to zero, the coefficients must all be zero
"if the function is well-behaved at every point, the desired theorem holds"
there can be one exceptional point where the function is not well-behaved
there can be any finite number of exceptional points
there can be infinitely many exceptional points provided they are all isolated from each other:
for each exceptional x there is a little interval $(x - \varepsilon, x + \varepsilon)$ containing no other exceptional points
there can be one doubly exceptional point not isolated from the other exceptional points
there can be any finite number of doubly exceptional points
there can be infinitely many doubly exceptional points provided they are all isolated from each other
… transition from plural "exceptional points" to singular "set of exceptional points"
the set is a "one" formed from a "many"
what does it mean to say it is treated as an object? that one can apply operations to it
set derivative X' remove isolated points
if E is the set of exceptional points then theorem holds

if E has no elements: is the empty set $\emptyset$

if $E' = \emptyset$, if $E'' = \emptyset$, and so on

let $E^{(0)} = E$, $E^{(1)} = E'$, $E^{(n+1)} = E^{(n)\prime}$
then theorem holds if $E^{(n)} = \emptyset$ for some n
actually the $E^{(n)}$ can all have points in them so long as there is no point in all of them
$E^{(\omega)} = \bigcap_n E^{(n)} = \emptyset$
can go on beyond iterations 0, 1, 2, … , ω, to
$\omega + 1$, $\omega + 2$, $\omega + 3$, … , $\omega + \omega = \omega \cdot 2$
$\omega \cdot 3$, $\omega \cdot 4$, $\omega \cdot 5$, … , $\omega \cdot \omega = \omega^2$
ω^3, ω^4, ω^5, … , ω^ω , and so on
"transfinite ordinal numbers"
(in grammar, *one, two, three* are "cardinal", *first*, *second*, *third*, are "ordinal")
as notation indicates, he developed an arithmetic of these numbers, generalizing ordinary arithmetic

N.B. natural numbers have the *well-ordering* property: any nonempty subset has a least element
all the ordinals mentioned so far can be represented by rearrangements of the natural numbers still having this well-ordering property, e.g.

1	2	3	4	5	…	$\omega + 1$	$\omega + 2$	$\omega + 3$	$\omega + 4$	$\omega + 5$	…
1	3	5	7	9	…	2	4	6	8	10	…

they are "countable" ordinals

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II Canor on transcendental numbers

the ideal of rigor conveyed to modern mathematics mainly by Euclid's *Elements*
geometry, which means "earth-measurement" had begun, according to all Greek sources
in Egypt with the practical work of surveyors, called "rope-stretchers"
("line" still means "rope" in nautical English)
a very long historical development gave the theoretical science of Euclid, Archimedes, Apollonios
which also bequeathed "classical problems" of straightedge-and-compass construction
duplication of the cube, *trisection of the angle*, *squaring of the circle*
the other "great book" at the beginning of modern mathematics was
al-Khwarizmi's *al-jabr w'al muqabala* (whence "algorithm" and "algebra")
which did not have a deductive organization but presented methods of solving
linear and quadratic equations (going back by way of India to Babylonia)
Descartes and Fermat develop coordinate methods making algebra applicable to geometry
turning the geometric constructions into algebraic problems
circa 1800 duplication and trisection proved impossible by showing that
$\sqrt[3]{2}$ and sin 10° are not obtainable by solving linear and quadratic equations with rational coefficients
the impossibility of squaring requires showing the same for π
π in fact is a *transcendental* not an *algebraic* number:
not a root of any polynomial with rational coefficients but this was for a long time an open problem
Liouville proved the existence of transcendental numbers
a rational number $r = m/n$ in lowest terms is a good approximation to an irrational α
if $|\alpha - m/n|$ is small compared to $1/n$
Liouville showed there are bounds on how well an algebraic number can be approximated
but there are numbers that can be approximated better than that
.110001000000000000000001… (with 1's in places 1!, 2!, 3!, 4!, …)
Hermite then showed e is transcendental, and finally Lindemann showed π is transcendental
between Hermite and Lindemann, Cantor showed *most* real numbers are transcendental
when do two infinite sets have the same number of elements?
when they can be matched up one-to-one, like whole numbers and even numbers
(matching each number with its double)
Cantor in effect showed that finite sequences from a finite alphabet of symbols
can be matched up with the positive integers
(we would say today they can be represented by a sequence of zeros and ones,
which is the binary numeral for a code number for the sequence; details later)
but an algebraic equation of degree n has at most n roots, so finite expressions like
"the 2nd solution of $x^3 + x^2 + x + 1 = 0$" give all the algebraic numbers
but the real numbers., even those in the unit interval (or: infinite zero-one sequences)
cannot be matched up with positive integers or listed
by the "diagonal" argument: any list leaves something out

$\underline{0}$	0	0	0	0	0	0	…
1	$\underline{1}$	1	1	1	1	1	…
0	1	$\underline{0}$	1	0	1	0	…
1	0	1	$\underline{0}$	1	0	1	…

invert diagonal (underlined) terms $\underline{0100}$… to get 1011… not on the list
so we have at least two sizes of infinite sets the positive integers $\aleph_0$ and the real numbers c
Cantor showed there are ones bigger than c, asked whether there are ones between $\aleph_0$ and c