Problems on Philosophical Logic

For each of the five logics treated (temporal, modal, conditional, relevantistic, intuitionistic) the list below first collects verifications 'left to the reader' in the corresponding chapter of Philosophical Logic*, then adds other problems, introducing while doing so some supplementary topics not treated in the book. In working problems 'left to the reader' it is appropriate to use any material up to the point in the book where the problem occurs; in working other problems, it is appropriate to use any material in the pertinent chapter. In all cases, it is appropriate to use results from earlier problems in later problems*

Instructors adopting the book as a text for a course may obtain solutions to these problems by e-mailing the author jburgess@princeton.edu

Temporal Logic

Problems 'left to the reader' in chapter 2 of Philosophical Logic

1. From page 22: Show that axiom (24a) is true everywhere in every model.

2. From page 24: Do the conjunction case in the proof of the rule of replacement (Rep).

3. From page 25: The lemma needed for the proof of the rule of duality (Dual) is proved by induction on complexity. If A is an atom, then A^* is just A while A' is $\neg A$, so $\neg A'$ is $\neg\neg A$ and $A^* \leftrightarrow \neg A'$ is $A \leftrightarrow \neg\neg A$, which is a tautology, hence a theorem; thus the lemma holds in the atomic case. Prove that:

(a) show that if A is a negation $\neg B$ and the lemma holds for B it holds for A.

(b) show that if A is a conjunction $B \wedge C$ and the lemma holds for B and for C, then the lemma holds for A. (The case of a disjunction is similar.)

(c) show that if A has the form $\mathsf{H}B$ and the lemma holds for B, then it holds for A. (The cases of $\mathsf{G}B$ and of $\mathsf{P}B$ and of $\mathsf{F}B$ are similar.)

4. From page 26: Prove (40a), at least in the case $n = 2$.

5. From page 32: Prove (58a).

6. From page 37: Explain why the converse Barcan formula holds if nothing ever goes out of existence but may fail otherwise, and that the direct Barcan formula holds if nothing ever comes into existence but may fail otherwise.

Additional problems

7. A frame $(U, \prec)$ that is irreflexive, transitive, and total is called a *total order*. Show that for any positive integer n there is a formula that is valid in a total order with at most n elements, but not in a total order with n+1 elements, by exhibiting such a formula. Briefly explain why your example works.

8. The axiom $A \to \mathsf{GP}A$ together with substitution gives $\mathsf{T}_1 A \to \mathsf{GPT}_1 A$ which together with Becker gives $\mathsf{T}_2\mathsf{T}_1 A \to \mathsf{T}_2\mathsf{GPT}_1 A$ for any two "tenses". In other words, we can *insert* GP at the beginning, at the end, or in the middle of any "tense" and get a formula implied by the original formula. The mirror image tells us we can insert HF, and the dual that we can *delete* FH or PG. Can we also justify in a similar way doing the following? Does it make a difference which tense logic

we are working in?

(a) *expand* G to GG (and similarly for H and F and P)

(b) *contract* GG to G (and similarly for H and F and P)

(c) *weaken* G to F (and similarly H to P)

(d) *flip* FG to GF (and similarly PH to HP)

(e) *flip* GH to HG or vice versa (and similarly FP to PF or vice versa)

(f) *reduce* GH (or HG) to G or H

(g) *enlarge* P (or F) to PF or FP

9. Using transformations as in the preceding problem, show that in the logic of sections 2.6-2.8 each of the following pairs of formulas X and Y are equivalent (in the sense that $X \to Y$ and $Y \to X$ are both theorems):

(a) G F G*A* and F G*A*

(b) G H F*A* and G F*A*

(c) G F H*A* and G H*A*

(d) G P H*A* and P H*A*

(e) G H P*A* and H P*A*

(f) G F P*A* and F P*A*

10. Show that in the logic of section 2.6-2.8:

(a) H*p* does not imply F H*p*

(b) P H*p* does not imply H*p*

(c) H P*p* does not imply P H*p*

(d) P*p* does not imply H P*p*

(e) G P*p* does not imply P*p*

It is enough to produce in each case a model whose frame part is a dense linear order extendible in both direction and a point where the one formula holds and the other doesn't. In fact, in every case the frame can be taken to be the real numbers with their usual order, and the point to be zero, and to specify the model one need only indicate — in words or in a picture — where p is true.

11. Let $(U, \prec)$ be a total order, extendible in both directions. It is *discrete* iff every element u has an *immediate successor* (an element v for which we have $u \prec v$ but do not have $u \prec w$ and $w \prec v$ for any w) and an *immediate predecessor* (an element u for which we have $u \prec v$ but do not have $u \prec w$ and $w \prec v$ for any w). Show that the formula

$$((\mathsf{H}p \wedge p) \rightarrow \mathsf{FH}p) \wedge ((\mathsf{G}p \wedge p) \rightarrow \mathsf{PG}p)$$

characterizes discreteness in the sense that it is true at all times in all models based on $(U, \prec)$ just in case $(U, \prec)$ is discrete.

12. (a) Give an example of a total order, extendible in both directions, that is neither dense nor discrete.

(b) Is there a formula that characterizes those total orders, extendible in both directions, that are either-dense-or-discrete, in the way the formula of the preceding problem characterizes those that are discrete? If so, give an example and explain briefly why it works; if not, explain briefly why not.

13. Let $(U, \prec)$ be a total order that is right-extendible. It is a *well-ordering* iff every nonempty subset V of U has a first element (an element u such that $u \prec v$ for every other element v of V). Show that the formula

$$\mathsf{H}(\mathsf{H}p \rightarrow p) \rightarrow \mathsf{H}p$$

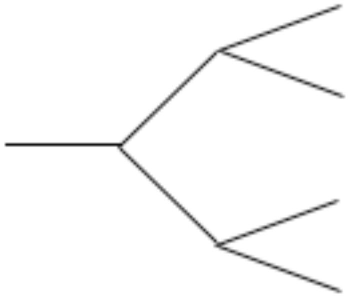
characterizes well-orders.

14. Let $(U, \prec)$ be a total order that is extendible in both directions. An *upper bound* for a nonempty subset V of U is an element u of U such that $v \prec u$ or $v = u$ for every v in V. (Here u itself may or may not be in V.) A *least upper bound* for a nonempty subset V of U is an upper bound u for V such that no w with $w \prec u$ is also an upper bound for V. The order has the *least upper bound property* iff every nonempty subset of U that has an upper bound has a least upper bound. (The real numbers have the least upper bound property, while the rational numbers do not, a counterexample being the set of rational numbers less than $\sqrt{2}$.) Show that the formula

$$(\mathsf{PF}p \wedge \mathsf{FG}\neg p) \rightarrow \mathsf{PF}(\mathsf{G}\neg p \wedge \neg\mathsf{PG}\neg p)$$

characterizes the least upper bound property.

15. A frame $(U, \prec)$ that is irreflexive, transitive, extendible in both directions and L-total is called a *tree*. Such an order looks something like this:



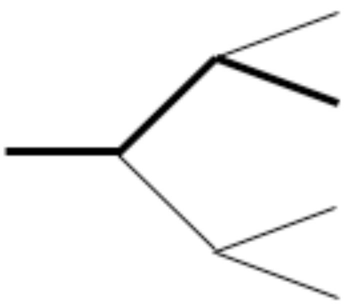
Which of the following are true everywhere in every tree:

(a) $\mathsf{GH}A \rightarrow \mathsf{HG}A$ (b) $\mathsf{HG}A \rightarrow \mathsf{GH}A$

(c) $\mathsf{PF}A \rightarrow \mathsf{FP}A$ (d) $\mathsf{FP}A \rightarrow \mathsf{PF}A$

Briefly explain your answers.

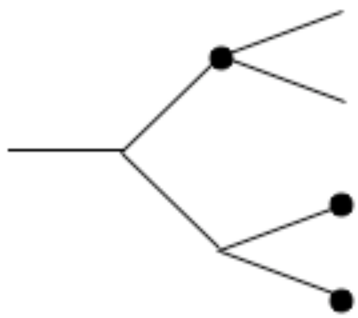
16. A *branch* through a tree $(U, \prec)$ is a non-empty subset V of U such that R-totality holds for V (meaning that for all v and w in V we have $v \prec w$ or $v = w$ or $w \prec v$) but for any W with $V \subseteq W$ and $V \neq W$, R-totality does not hold for W. Such a branch looks something like this:



For trees we may add to the usual operators of temporal logic a *strong future* operator $\mathbf{F}$ such that

$(U, \prec) \vDash \mathbf{F}A[u]$ iff for every branch V with u in V $(U, \prec) \vDash A[v]$ for some v in V with $u \prec v$

For instance, if u is the leftmost or next-to-leftmost point in this picture and p is true at the three points marked here:



then the strong $\mathbf{F}p$ is true at u, but if we made p to be true only at one or two of the marked points, then only the weak $\mathsf{F}p$ would be true. (Some philosophers have thought that while there is one past behind us, there are many possible futures in front of us, making time "garden of forking paths" or tree; and a branch through the tree represents a possible course of history. Intuitively, the weak future "it may happen that p" means there is some course of history going through the actual present with an occurrence of p in the future, while the strong future "it must happen that p" means that for any course of history going through the actual present there is an occurrence of p in the future.) Which of the following are true everywhere in every tree with extendibility in both directions?

(a) $\mathsf{P}\mathbf{F}A \to (\mathbf{F}A \vee A \vee \mathsf{P}A)$ (b) $(\mathbf{F}A \vee A \vee \mathsf{P}A) \to \mathbf{F}\mathsf{P}A$

(c) $(\mathbf{F}A \vee A \vee \mathsf{P}A) \to \mathsf{P}\mathbf{F}A$ (d) $\mathbf{F}\mathsf{P}A \to (\mathbf{F}A \vee A \vee \mathsf{P}A)$

Briefly explain your answers.

17. Continuing the preceding problem, answer the same question for:

(e) $\mathsf{F}\mathbf{F}A \to \mathbf{F}\mathsf{F}A$ (f) $\mathbf{F}\mathsf{F}A \to \mathsf{F}\mathbf{F}A$

(g) $\neg\mathbf{F}\neg\mathsf{P}A \to \mathbf{F}\mathsf{P}A$ (h) $\mathbf{F}\mathsf{P}A \to \neg\mathbf{F}\neg\mathsf{P}A$

Briefly explain your answers.

18. Write $\Box A$ for $\mathsf{G}A \wedge A$. Which of the following a theorems of the minimal tense logic? of the tense logic of classical physics?

(a) $\Box(A \to B) \to (\Box A \to \Box B)$ (b) $\Box A \to A$

(c) $\Box A \to \Box\Box A$ (d) $\neg\Box\neg A \to \Box\neg\Box\neg A$

Briefly explain your answers.

Modal Logic

Problems 'left to the reader' in chapter 3 of Philosophical Logic

1. From page 50: Show that (37) is a theorem of **S5**.

2. From page 52: Show that (45) is a theorem of **S5**.

3. From page 57: Prove (49a).

4. From page 57: Prove (50).

5. From page 58: Prove the conjunction case of (52).

6. From page 62: For k sentence letters, how many models are there with universe $\{1, 2, \ldots, n\}$?

Additional problems

7. Show that $\Box A \rightarrow A$ is not deducible from the other axioms of **S5** (or in other words, in the system **K** + (35) + (38)).

8. Modal logic with the axiom $\Box A \rightarrow A$ weakened to $\Box A \rightarrow \Diamond A$ is often called *deontic* logic, and then $\Box$ is read "obligatory". There is a deontic system corresponding to each modal system **T**, **S4**, **S4.2**, **S4.3**. But there is no deontic system corresponding to the modal system **S5**. Show why by showing that $\Box A \rightarrow A$ is provable given the other **S5** axioms plus $\Box A \rightarrow \Diamond A$.

9. The system **S4.4** of modal logic adds to the system **S4** the axiom $(A \wedge \Diamond\Box A) \rightarrow \Box A$. Show that the axiom (37) of **S4.3** is a theorem of **S4.4**.

10. Show that the theorem $\Diamond\Box A \rightarrow \Box A$ of **S5** is *not* a theorem of **S4.4**.

11. A formula A is *fully modalized* if every occurrence of an atom in A occurs as part of subformula of A beginning with a $\Box$. (Thus every formula beginning with a box trivially counts as fully modalized.) Show by induction on complexity that any fully modalized formula is provably equivalent in **S5** to a formula beginning with a $\Box$.

12. Show that if B is fully modalized, then $\Box(A \rightarrow \Box B)$ is provably equivalent in **S5** to $\Diamond A \rightarrow B$.

13. Recall that $A \Rightarrow B$ abbreviates $\Box(A \rightarrow B)$. Much of the early literature of modal logic was concerned with the status of formulas having $\Rightarrow$ as their only logical operator.

(a) Show that $A \Rightarrow (B \Rightarrow A)$ is not a theorem of **S5**.

(b) Show that $(((A \Rightarrow D) \Rightarrow B) \Rightarrow C) \Rightarrow (((A \Rightarrow D) \Rightarrow C) \Rightarrow C)$ is not a theorem of **S4**.

14. (a) Show that the following variant of the formula in problem 10(a):

$(A \Rightarrow C) \Rightarrow (B \Rightarrow (A \Rightarrow C))$

is a theorem of **S4**.

(b) Show that the formula in problem 13(b) is a theorem of **S5**.

15. In section 3.6 it is shown that any formula A that is not a theorem of **K** is untrue somewhere in some Kripke model, by considering the set of all maximal consistent sets u of formulas and the relation $u \lhd v$ that holds iff whenever a formula $\Box B$ is in u, B is in v. It was further shown that if we add the axiom (34), then the relation $\lhd$ is reflexive, and if we add the axiom (35) then the relation $\lhd$ is transitive, thus proving that any formula A that is not a theorem of **S4** is untrue somewhere in some reflexive, transitive Kripke model. In section 3.7 it is shown that any formula A that is not a theorem of **K** is untrue somewhere in some *finite* Kripke model by considering the set of all maximal consistent sets u of A-formulas and the relation $u \lhd v$ that holds iff whenever an A-formula $\Box B$ is in u, B is in v. The argument that if we add (34), then $\lhd$ is reflexive still goes through in this new context. The argument that if we add (35), then $\lhd$ is transitive does not.

(a) Why does the argument not go through?

(b) How can we fix the argument?

16. Consider the minimal tense logic of chapter 2, and define maximal consistent sets analogously to the definition for the minimal modal logic in chapter 3. For maximal consistent sets u and v, define $u \angle v$ iff whenever $\mathsf{G}A$ is in u, A is in v. Then equivalently $u \angle v$ iff whenever A is in v, $\mathsf{F}A$ is in u. (This equivalence was proved in the text for modal logic with box in place of G and diamond in place of F and the proof goes through unchanged.) Show that also $u \angle v$ iff whenever $\mathsf{H}A$ is in v, A is in u.

17. Starting now from the tense logic for classical physics instead of the minimal tense logic, show that if $u \angle w$ then there exists v such that $u \angle v$ and $v \angle w$.

18. Show that in **S4** or any stronger system, $\sim\Box(A \leftrightarrow \sim\Box A)$ is a theorem.

Conditional Logic

Problems 'left to the reader' in chapter 4 of Philosophical Logic

1. From page 86: Prove (41c).

2. From page 88: Prove that (37a) holds in all models.

3. From page 88: Prove that (38a) holds in all models.

4. From page 88: Prove (46) in the two-premise case. (Hint: Let E be $(A_2 \wedge \neg B_2) \vee (C \wedge D)$ and let F be $(A_1 \wedge \neg B_1) \vee E$. The assumption that the Adams test is aced means that $(A_1 \vee A_2 \vee C) \supset F$ is a tautology, so we can begin our deduction with $A_1 \vee A_2 \vee C \rightarrow F$. As an intermediate goal, try to get to $A_2 \vee C \rightarrow E$.)

5. From page 97: Show that $(p \vee q) \Rightarrow \neg p$ and $(p \vee r) \Diamond\!\!\Rightarrow p$ *do* imply $(q \vee r) \Rightarrow \neg r$ for total models (even including infinite ones).

6. From page 97: Show that $(p \vee q) \Rightarrow \neg p$ and $(p \vee r) \Diamond\!\!\Rightarrow p$ do *not* imply $(q \vee r) \Rightarrow \neg r$ for *all* models (or even for all finite ones).

Additional problems

7. Show how to translate any conditional $A \rightarrow B$, with A and B truth-functional into a modal formula $(A \rightarrow B)^*$ in such a way that the following

(i) $A_1 \rightarrow B_1, \ldots, A_n \rightarrow B_n \vdash C \rightarrow D$

will be deducible in conditional logic iff the following

(ii) $((A_1 \rightarrow B_1)^* \wedge \ldots \wedge (A_n \rightarrow B_n)^*) \rightarrow (C \rightarrow D)^*$

is deducible in **S4**. (Note that in the modal system you are supposed to translate *into*, unlike the conditional system you are supposed to translate *out of*, the arrow simply represents the material conditional.) Briefly explain why the translation works.

8. Show that in S4.3, $\Box(A \rightarrow \Diamond(A \wedge \Box(A \rightarrow B)))$ is equivalent to $\Diamond A \rightarrow \Diamond(A \wedge \Box(A \rightarrow B))$. Briefly, what does this have to do with conditional logic?

9. Give a deduction for the following in conditional logic:

$$(A \vee C) \rightarrow C,\ A \rightarrow B \quad \vdash \quad C \rightarrow (A \supset B)$$

10. Give a deduction for the following in conditional logic:

$$(A \vee B) \rightarrow A,\ (B \vee C) \rightarrow B \quad \vdash \quad (A \vee C) \rightarrow A$$

11. Show that the item in problem 9 is deducible in conditional logic by showing that it passes the Adams test.

12. Show that the item in problem 9 is deducible in conditional logic by showing that it is model-theoretically valid.

13. Show that the item in problem 9 is deducible in conditional logic by showing that it is probabilistically valid.

14. Show that the item in problem 10 is deducible in conditional logic by one of the three methods used in problems 11, 12, 13.

15. In considering whether $(p \vee q) \Rightarrow q$ and $(p \vee r) \Diamond\!\!\Rightarrow p$ imply $(q \vee r) \Diamond\!\!\Rightarrow q$, does it make a difference whether we consider only total models, as in problem 5, or consider also non-total models, as in problem 6? Briefly explain.

16. In conditional logic we considered only arguments with conditional premises and conditional conclusions. We can consider arguments where some or all of the premises or the the conclusion may be simply truth-functional compounds of atoms by treating such a premise or conclusion A as equivalent to the conditional $\top \rightarrow A$ where $\top$ abbreviates $p \vee \neg p$. With this understanding show that $A \rightarrow B$ is deducible from $A \wedge B$.

17. Show that $A \supset B$ is deducible from $A \rightarrow B$.

18. Show that $A \rightarrow B$ is not deducible from $A \supset B$.

Relevantistic Logic

Problems 'left to the reader' in chapter 5 of Philosophical Logic

1. From page 101: Show that if A and B are classical formulas, then $A \supseteq B$ is valid iff A classically entails B and the set of sentence letters occurring in A overlaps with the set of sentence letters occurring in B.

2. From page 105: If $A_1, \dots, A_m$ perfectly entail B and B perfectly entails $C_1, \dots, C_m$, must $A_1, \dots, A_m$ perfectly entail $C_1, \dots, C_m$?

3. From page 108: Verify that $(p \wedge q)$ r-entails p, but that $\neg(p \wedge q) \wedge p$ does not r-entail $\neg q$.

4. From page 111: How should the assignment of *pairs* of values 0 or 1 to sentence letters be extended to arbitrary formulas if we wish to ensure that A r-entails B iff every valuation of that makes A true makes B true?

5. From page 117: Derive (10) using just MP and Ded.

6. From page 118: Complete the proof of the starred deduction theorem for relevance/relevant logic in section 5.8.

Additional problems

7. Check the status of each of the following

i	$A \wedge B$	entails	A
ii	$\neg(A \wedge B) \wedge A$	entails	$\neg B$
iii	$(A \vee B) \wedge (\neg A \wedge \neg B)$	entails	$A \wedge B$
iv	$(A \vee B) \wedge (\neg A \vee \neg B)$	entails	$(A \wedge \neg B) \vee (\neg A \wedge B)$
v	$(A \wedge \neg B) \vee (\neg A \wedge B)$	entails	$(A \vee B) \wedge (\neg A \vee \neg B)$
vi	A	entails	$A \vee (B \wedge \neg B)$
vii	$A \vee (B \wedge \neg B)$	entails	A

for each of the following logics:

(a) the logic requiring overlap of topic between premise and conclusion

(b) the logic requiring containment of topic of conclusion in subject matter of premise

(c) the logic requiring containment of topic of premise in topic of conclusion

8. Continuing the preceding problem, check the status of each of i-vii for each of the following logics:

(d) perfectionist logic

(e) relevance/relevant logic

9. (a) Classically, if B and C entail each other, then for any A and D the two premises A, B entail D iff the two premises A, C entail D. Does this principle hold for perfectionist logic? (If so, explain why; if not, give a counterexample.)

(b) Classically the two premises A, B entail C iff the conjunction $A \wedge B$ entails C. Does this principle hold for perfectionist logic? (If so, explain why; if not, give a counterexample.)

(c) Show that if A classically entails C, then there is a B such that A perfectibly entails B and B perfectibly entails C.

10. Consider the positive logic of section 5.8. Show that the following a theorems of positive logic by deriving them using just MP and the deduction theorem:

(a) $A \rightarrow ((A \rightarrow B) \rightarrow B)$

(b) $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

(c) $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$

(d) $(A \rightarrow B) \rightarrow (A \rightarrow (A \rightarrow B))$

11. Show that if either of the following is added as a further axiom to positive logic, the other becomes a theorem:

i $((A \rightarrow B) \rightarrow C) \rightarrow ((A \rightarrow C) \rightarrow C)$

ii $((A \rightarrow B) \rightarrow A) \rightarrow A$

12. Consider deductions of $\rightarrow$ formulas in relevance/relevant logic. In a *relevant* deduction of C from $A_1, \ldots, A_n$ and B, B must be used at least once (as a premise in an application of MP) in getting to C. In a *super-relevant* deduction, B must be used *exactly* once (as a premise in an application of MP) in getting to C. State and prove an analogue of the starred deduction theorem for relevance/relevant logic of section 5.8 for super-relevance/super-relevant logic, with the following axioms:

i $A \rightarrow A$

ii $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

iii $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$

13. Show that the class of theorems is not changed if in the preceding problem we replace axiom ii by the following:

ii' $(A \to B) \to ((B \to C) \to (A \to C))$

14. Which of the derivations in problem 10 is acceptable in

(a) relevance/relevant logic?

(b) *super*-relevance/*super*-relevant logic?

Briefly explain.

15. The Moh-Church logic has as axioms (i)-(iii) of problem 12, plus the axiom

iv $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

and as rule MP. Show that axiom (iv) can be replaced by

iv' $(A \to (A \to B)) \to (A \to B)$

without altering the class of theorems.

16. An *Urquhart model* is a pair (S, V) where S is a family of subsets of some set I with the properties that

i the empty set Ø is in S

ii if x and y are in S, then their union $x \cup y$ is in S

and where V assigns to each x in S a set $V(x)$ of atoms. For formulas A built up from atoms using only $\to$ we define what it is for A to be true in the model (S, V) at an element x of S by induction on complexity: An atom A is true at x iff it is in $V(x)$, while $A \to B$ is true at x iff for every y in S, if A is true at y, B is true at $x \cup y$. Show that the axioms (ii)-(iv) of problems 12 and 15 are true at Ø in any Urquhart model.

17. Show that if A is a theorem of Moh-Church logic, then A is true at Ø in any Urquhart model.

18. Show that the following are not theorems of Moh-Church logic:

(a) $p \to (q \to p)$

(b) $(p \to q) \to (p \to (p \to q))$

Intuitionistic Logic

Problems 'left to the reader' in chapter 6 of Philosophical Logic

1. From page 126: Verify the Heyting axioms (8) and (9), given the explanations (3) and (5) of the intended meaning of the intuitionistic $\to$ and $\neg$.

2. From page 127: Show that (22) and (23) are theorems of **I**.

3. From page 132: Prove the case of (8) in the proof of the soundness of **I** for Kripke models.

4. From page 136:

(a) Show that $(A \to \neg A) \to \neg A$ is a theorem of **I**

(b) Show that $(\neg A \to A) \to \neg\neg A$ is a theorem of **I**

5. From page 136:

(a) Show that (51) is not a theorem of **KC** by finding a reflexive, transitive, R-convergent model where $(p \to q) \vee (q \to p)$ fails.

(b) Show that (50) is a theorem of **LC**.

6. From page 136: Show that for **LC**, if u, v, w are virtuous and $u \subseteq v$ and $u \subseteq w$, then either $v \subseteq w$ or $w \subseteq v$.

Additional problems

7. Show that $(A \vee B) \to ((A \to B) \to B)$ is a theorem of intuitionistic logic

(a) directly, by indicating a proof of it from the Heyting axioms

(b) indirectly, by showing that it holds everywhere in every Kripke model

8. Show that the following are *not* theorems of intuitionistic logic by showing that each fails somewhere in some Kripke model:

(a) $(\neg p \to p) \to p$

(b) $(((p \to q) \to p) \to p)$

(c) $(\neg p \to \neg q) \to (q \to p)$

(d) $(\neg\neg p \to q) \to ((\neg p \to q) \to q)$

(e) $((p \to q) \to q) \to (((q \to p) \to p) \to (p \vee q))$

9. (a) Which of (a)-(e) in problem 8 is a theorem of **KC**?

(b) Which of (a)-(e) in problem 9 is a theorem of **LC**?

10. Examination of the completeness proof for intuitionist logic shows that if a formula A is not an intuitionist thesis, then A fails somewhere in some Kripke model $(U, \prec)$ that is (i) *finite*, and (ii) *anti-symmetric*, meaning that we can have $u \prec v$ and $v \prec u$ only when $u = v$. An element u of a frame $(U, \prec)$ is called *maximal* iff we can have $u \prec v$ only when $u = v$.

(a) Show that in any finite, antisymmetric model, $\neg\neg A$ holds at u iff A holds at every maximal v with $u \prec v$.

(b) Show that in any model, if u is maximal and A is a classical tautology, then A holds at u.

These facts can be used to give an alternate, model-theoretic proof that $\neg\neg A$ is a theorem of intuitionistic logic whenever A is a classical tautology.

11. (a) Show that if in place of the axiom (50) of **KC** one took the following:

$(\neg\neg A \to B) \to ((\neg A \to B) \to B)$

one would obtain an equivalent system.

(b) Show that if in place of the axiom (51) of **LC** one took the following:

$((A \to B) \to B) \to (((B \to A) \to A) \to (A \vee B))$

one would obtain an equivalent system.

12. Drop $\neg$ from intuitionistic logic, add a symbol $\perp$ that can appear in the same places as sentence letters, and reintroduce $\neg A$ as an abbreviation for $A \to \perp$.

(a) Show that the Heyting axiom (9) is a theorem, even without any axioms for $\perp$.

(b) Show that the Heyting axiom (10) is a theorem, given as a new axiom $\perp \to A$.

(c) Show that $A \vee \neg A$ becomes a theorem given as a further new axiom $(((A \to B) \to A) \to A)$.

13. Show that in S4, $\Box(\Box A \mathbin{\S} \Box B)$ is equivalent to $\Box A \mathbin{\S} \Box B$ if § is $\wedge$ or § is $\vee$ but not if § is $\to$. Briefly, what does this have to do with intuitionistic logic?

14. Add to the minimal modal logic an operator $\bigcirc$ for "it is known that". Add to minimal modal logic the axioms $\bigcirc A \to A$ ("everything known is true") $\bigcirc(A \wedge B) \to \bigcirc A \wedge \bigcirc$ B ("if a conjunction is known, so are both conjuncts"), and $A \to \Diamond\bigcirc A$ ("everything true is knowable"). Show that $A \to \bigcirc A$ ("everything true is known") follows. Briefly, what does this have to do with intuitionistic logic?

15. For each of the following, indicate what logic we get if we add it as an axiom to intuitionistic logic:

(a) $((A \rightarrow B) \rightarrow C) \rightarrow ((A \rightarrow C) \rightarrow C)$

(b) $((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)$

(c) $((A \rightarrow B) \rightarrow B) \rightarrow (A \vee B)$

Briefly explain.

16. Is $((A \rightarrow B) \rightarrow (B \rightarrow A)) \rightarrow (B \rightarrow A)$ an intuitionistic theorem? Explain why or why not.

17. Use Kripke models to show that in intuitionistic logic $\neg\neg\exists x\neg Fx$ does not imply $\exists x\neg Fx$.

18. Use Kripke models to show that if a disjunction is theorem of intuitionistic sentential logic, then one of its disjuncts is.