sliding-mode control strategy in the DSS controller allows us to obtain a robust closed-loop system and therefore to eliminate the problems of a disturbance or a nondeterministic behavior. We consider two examples, a robot and a double integrator, and show that sliding-mode control similar to continuous state case can provide a closed-loop system insensitive to disturbances.

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An Alternative Proof for Convergence of Stochastic Approximation Algorithms

S. R. Kulkarni and C. S. Horn

***Abstract*—An alternative proof for convergence of stochastic approximation algorithms is provided. The proof is completely deterministic, very elementary (involving only basic notions of convergence), and direct in that it remains in a discrete setting. An alternative form of the Kushner–Clark condition is introduced and utilized and the results are the first to prove necessity for general gain sequences in a Hilbert space setting.**

I. INTRODUCTION

Stochastic approximation is a widely studied technique that is used in a variety of areas, such as on-line optimization, system identification, and adaptive control. The original and most widely studied stochastic approximation algorithms are the Robbins–Monro [12] and Kiefer–Wolfowitz [6] algorithms. The basic problems are to sequentially find the zero (Robbins–Monro) or extremum point (Kiefer–Wolfowitz) of a function under observations of the function that are corrupted by noise. Much work has been done since the original papers on extending and analyzing stochastic approximation procedures (e.g., see [1], [8], [9], and further references contained therein). In particular, in [9], Walk provides a recent and extensive bibliography on the subject.

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In this paper, we focus on the Robbins–Monro algorithm for finding the zero of a function on a Hilbert space, $\mathbb{H}$, based on

$$x_{n+1} = x_n - a_n(f(x_n) + e_n) \tag{1}$$

where $x_n \in \mathbb{H}$ is the estimate for the location of the zero, x^*, of the function $f\colon \mathbb{H} \to \mathbb{H}$, a_n is a sequence of positive constants tending to zero, and $e_n \in \mathbb{H}$ represents measurement noise. Let $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathbb{H}$, and $|\cdot|$ denote the corresponding induced norm. The one-dimensional (1-D) version of this algorithm was introduced by Robbins and Monro [12] and a number of variants and extensions have been extensively studied. The standard results make statements about the convergence of the sequence x_n to the zero of f under certain regularity assumptions on f, rate conditions on the gain sequence a_n, and assumptions on the noise sequence e_n.

One common analysis technique for stochastic approximation is to start with stochastic assumptions on the noise sequence from the outset. Another standard analysis technique is the ordinary differential equation (ODE) method (e.g., see [8] and [10]). This method employs an embedding of the function f into a differential equation of the form $\dot{x} = -f(x)$, and proves that if the sequence of estimates, x_n, is bounded, then it converges to an asymptotically stable equilibrium point of the differential equation. The approach of Kushner and Clark, whose book [8] is a standard reference on the subject, is based on a deterministic result which involves interpolating the x_n sequence into a continuous parameter process and applying the Arzela–Ascoli theorem to extract convergent subsequences of a sequence of left shifts of the interpolated process. The focus of much of the subsequent work has been on relaxing conditions on the function and the noise sequence. Under suitable regularity conditions on f and the usual assumptions on the gain sequence that $a_n \to 0$ and $\Sigma_n a_n = \infty$, it is known that the following condition on the noise is sufficient for the convergence of $x_n \to x^*$ (see, for example, [8] and [11]).

Definition 1 (Kushner–Clark Condition): The noise sequence e_n is said to satisfy the Kushner–Clark (KC) condition if for every $T > 0$

$$\lim_{n\to\infty} \left(\sup_{n \le p \le m(n,T)} \left| \sum_{i=n}^{p} a_i e_i \right| \right) = 0 \tag{2}$$

where

$$m(n,T) \triangleq \max\{k\colon a_n + \cdots + a_k \le T\}.$$

In this paper, we introduce a new completely deterministic proof for convergence of stochastic approximation algorithms. Our convergence proof is direct and very elementary, involving only basic notions of convergence. We do not assume that the x_n sequence is bounded from the outset, and do not require the notion of asymptotic stability or the application of the Arzela–Ascoli theorem. Hence, our proof technique provides an additional approach for analyzing stochastic approximation algorithms that complements other existing analysis methods. Our approach and subsequent direct approaches such as [2] show the strength of elementary and completely deterministic analyses. In our proof, we also introduce an alternative form of the Kushner–Clark condition, which may be of interest in its own right as it leads to the use of different tools in verifying the condition for stochastic noise. The equivalence of our condition and the standard Kushner–Clark condition (as well as other forms) was shown in [13] and [14]. Another contribution is that a result is provided which is both necessary and sufficient for convergence. To our knowledge, our results are the first to provide necessary and sufficient conditions for general gain sequences and in a general Hilbert space setting. Other works which deal with necessary and sufficient conditions are Clark [4] and Chen *et al.* [3], who have obtained results in $\mathbb{R}^d$ for the special case $a_n = 1/n$. We provide results for all positive gain sequences that converge to zero and sum to infinity, and our results hold in both finite- and infinite-dimensional Hilbert spaces.

II. Alternative Form of the Kushner–Clark Condition and Main Result

The intuition behind our alternative form of the Kushner–Clark condition and our proof technique is as follows. Suppose f has a unique zero at x^* and satisfies suitable regularity conditions. Given that we are at x_n at time n, there is a natural restoring motion toward x^* due to the $a_n f(x_n)$ term. Assume that the regularity assumptions on f and the gain sequence a_n are chosen so that without any disturbance, x_n would converge to x^* as desired. Now, suppose an adversary has limited resources but wishes to impose disturbances (noise) on the measurements in order to force x_n to fail to converge to x^*. One strategy the adversary could use is to impose no noise for long periods to conserve resources (rest intervals), but periodically over some intervals add large amounts of noise to disturb convergence of the algorithm (active intervals). During the rest intervals, x_n may get arbitrarily close to x^*. However, to ensure that $x_n \nrightarrow x^*$, the adversary need only force x_n outside some fixed ball around x^* infinitely often. Hence, in the active intervals the noise sequence must be sufficiently strong to overcome the natural restoring force. If I denotes an active interval, the magnitude of the motion due to the adversary's disturbances is $|\Sigma_{n\in I} a_n e_n|$, while the motion due to the natural restoring force is $\Sigma_{n\in I}\, a_n f(x_n)$. One of the regularity assumptions we require on f is that outside any neighborhood of x^*, $f(x)$ has a component toward x^* which is bounded away from zero. Hence, since x_n is being forced some fixed distance away from x^*, it can be argued (as will be done in a precise manner in Theorem 1 below) that with the active intervals appropriately defined, the magnitude of the motion due to the natural restoring force is at least $\alpha\ \Sigma_{n\in I}\, a_n$ for some $\alpha > 0$. To make x_n move outside some fixed ball around x^* in the active interval I, the net motion must be greater than some positive constant, so that for some $\beta > 0$ the adversary would like to have $|\Sigma_{n\in I}\, a_n e_n| \geq \alpha\ \Sigma_{n\in I}\, a_n + \beta$. Finally, recall that to force x_n to fail to converge, the adversary needs an infinite number of active intervals. It turns out that the condition arrived at through this argument is equivalent to the Kushner and Clark condition and is both a necessary and sufficient condition for the algorithm to fail to converge. These ideas are made precise in Definition 2 and Theorem 1 below.

Definition 2 (Alternative Form of Kushner–Clark Condition): The noise sequence e_n is said to satisfy the alternative form of the Kushner–Clark (KC′) condition if $\forall \alpha, \beta > 0$ and infinite sets of nonoverlapping intervals $\{I_k\}$

$$\left|\sum_{n\in I_k} a_n e_n\right| < \alpha \sum_{n\in I_k} a_n + \beta \tag{3}$$

for all but a finite number of k's.

It is also useful to consider that negative of the above statement, namely e_n, does not satisfy (KC′) if $\exists$ constants $\alpha, \beta > 0$ and an infinite set of nonoverlapping intervals $\{I_k\}$ such that for all k, $|\Sigma_{n\in I_k} a_n e_n| \geq \alpha\ \Sigma_{n\in I_k}\, a_n + \beta$. A noise sequence that does not satisfy (KC′) is also referred to as a persistently disturbing noise sequence, in the sense that such a noise sequence persistently disturbs the natural convergence properties of the algorithm as described in the intuition above.

The interest in alternative equivalent conditions arises from the fact that intuition and work involved in checking when certain properties hold may be easier to see with different forms of the condition, and different tools may be more easily applied. For example, checking that certain stochastic noise satisfies our form of the Kushner–Clark condition can be accomplished by a simple application of Markov's inequality and the Borel–Cantelli lemma [7]. Also, it can be seen that the following decomposition condition leads to a slightly simpler proof of necessity in Theorem 1 than using (KC′) directly (which was pointed out to us by an anonymous referee).

Definition 3 (Decomposition Condition): The noise sequence e_n is said to satisfy the decomposition (DC) condition if there exist sequences f_n and g_n such that $e_n = f_n + g_n$ and $\Sigma_{n=1}^{N}\, a_n f_n$ converges and $g_n \to 0$.

The equivalence between these and other conditions on the noise sequence can be found in [13] and [14].

We now state our main result, which is composed of three parts. Part a) is a positive statement which gives a necessary and sufficient condition for convergence in the limit of the Robbins–Monro algorithm to the desired value for one class of functions, while Part b) is a negative statement which gives a necessary and sufficient condition for lack of convergence of the algorithm to the desired value for a second class of functions. Finally, Part c) is a combination of Parts a) and b). Our assumptions are composed of boundedness and smoothness assumptions on the function f and rate assumptions on the a_n sequence.

Theorem 1: Consider the set of conditions:

A1) $|f(x)| \leq K_f \forall x$;

A2) $a_n > 0, a_n \to 0$ and $\Sigma_{n=1}^{\infty}\, a_n = \infty$;

B1) $\exists x^* \in \mathbb{H}$ s.t. $\forall \delta > 0, \exists h_\delta > 0$ s.t. $|x - x^*| \geq \delta \Rightarrow \langle f(x), x - x^*\rangle \geq h_\delta |x - x^*|$;

C1) $\exists x^* \in \mathbb{H}$ s.t. $f(x^*) = 0$ and $f(x)$ is continuous at $x = x^*$;

and consider the families of functions

$$\mathcal{F}_1 = \{f\colon f \text{ satisfies A1) and B1)}\}$$
$$\mathcal{F}_2 = \{f\colon f \text{ satisfies A1) and C1)}\}.$$

Let a_n satisfy A2) and suppose x_n is generated according to the Robbins–Monro algorithm (1). Then

a) $x_n \to x^*$ for every $f \in \mathcal{F}_1$ and every $x_1 \in \mathbb{H}$ iff the noise sequence e_n satisfies (KC′).

b) $x_n \nrightarrow x^*$ for every $f \in \mathcal{F}_2$ and every $x_1 \in \mathbb{H}$ iff the noise sequence e_n does not satisfy (KC′).

c) For any $f \in \mathcal{F}_1 \cap \mathcal{F}_2$ and any $x_1 \in \mathbb{H}$, $x_n \to x^*$ iff the noise sequence e_n satisfies (KC′).

Before proving this theorem in the following section, a few remarks on the assumptions and statement of the theorem are in order. Assumption A1) requires that each function be bounded in magnitude by some constant, but the constant may depend on the function, so that the functions are not required to be uniformly bounded. Typically, in place of A1), only growth rate conditions are imposed on the function. At present, our simple proof uses A1) for sufficiency, but as can be seen from the proof given in the next section, A1) is not required for necessity. Assumption A2) is a usual requirement for Robbins–Monro procedures. Note that, as Theorem 1 is completely deterministic, we do not require the common assumption that $\Sigma_{n=1}^{\infty}\, a_n^2 < \infty$.

Assumption B1) [for Part a)] is a form of a standard Lyapunov-type assumption. It is a weak assumption, but it ensures that there is a sufficient natural restoring force and makes sure that f does not come arbitrarily close to zero away from x^*. If the restoring force of f is allowed to come arbitrarily close to zero away from x^*, then x_n may fail to converge to x^* even though the noise sequence satisfies (KC$'$). On the other hand, Assumption C1) [for Part b)] is somewhat strict. However, without continuity, we can have $x_n \to x^*$ even though the noise sequence does not satisfy (KC$'$).

Note that the positive statement [Part a)] is the interesting statement from the perspective of an agent trying to design a convergent algorithm, while the negative statement [Part b)] is of interest from the perspective of an adversary wishing to force failure of the algorithm. Also, including both Assumptions B1) and C1), as done in Part c), allows a very strong statement to be made for the smaller family of functions $\mathcal{F}_1 \cap \mathcal{F}_2$. Namely, for each fixed noise sequence either we have convergence for every f and every initial point x_1, or else we fail to converge for every f and every x_1. By relaxing B1) [respectively, C1)], the weaker statements of Parts a) and b) can be made which guarantee convergence (respectively, failure to converge) for every f in some family and every x_1 if the noise sequence e_n satisfies a certain condition, but if e_n does not satisfy the condition, then we can say only that x_n fails to converge to x^* (respectively, converges to x^*) for some f and some x_1. However, the families of functions for which these weaker statements can be made are larger.

III. Alternative Proof

Now let us turn to the proof of Theorem 1. We will argue that it is enough to consider two basic categories, namely when there is a gross imbalance between the force due to the noise and the force due to the function, and when a detailed balance between the forces must be studied in a situation where the movement of the algorithm is directed in magnitude away from x^*. The following three lemmas will help us in this regard. The proof of Theorem 1 follows the three lemmas.

The first lemma says that if there is an interval on which the movement in the interval is too big to be accounted for by the force due to the function, then e_n does not satisfy (KC$'$) on that interval.

Lemma 1 (Gross Imbalance): Assume A1) and suppose $\exists c > 0$ and an interval $I_k = [L_k, M_k - 1]$ such that $K_f \sum_{n\in I_k} a_n \le c/4$ and $|x_{M_k} - x_{L_k}| \ge c$. Then $|\sum_{n\in I_k} a_n e_n| \ge 2K_f \sum_{n\in I_k} a_n + c/4$.

Proof: By iteration of (1), we have $x_{M_k} = x_{L_k} - \sum_{n\in I_k} a_n(f(x_n) + e_n)$. Therefore

$$\begin{aligned}\left|\sum_{n\in I_k} a_n e_n\right| &= \left|x_{L_k} - x_{M_k} - \sum_{n\in I_k} a_n f(x_n)\right| \ge c - \frac{c}{4}\\ &= \frac{c}{2} + \frac{c}{4} \ge 2K_f \sum_{n\in I_k} a_n + \frac{c}{4}.\end{aligned}$$

□

Fig. 1 describes the idea that the force due to the function would only allow x_{M_k} to lie somewhere in a $c/4$ ball around x_{L_k}. The fact that x_{M_k} in fact lies a distance away means that the force due to the noise has a sufficient strength that can be quantified as above.

The second lemma is a statement of two facts which hold in inner product spaces and which we will find useful. These facts basically quantify two relations comparing inner products involving a change in one of the arguments from one vector to a nearby vector.

Lemma 2:

a) Let $x, y, z \in \mathbb{H}$ with $|z| < M$ and let $h > 0$. Then $\forall \eta \in (0,1)$ and $c \in (0, \eta h/M]$, we have that if $\langle z, y\rangle \ge h$ and $|y - x| \le c$, then $\langle z, x\rangle \ge (1-\eta)h$.

b) Let $\delta > 0, \epsilon \ge 0$, and $c > 0$. Let $x, y \in \mathbb{H}$ with $|x| \ge \delta, |y| \ge \delta, |y| \ge |x| - 2\epsilon$, and $|y - x| \le c$. Then $\langle y - x, x/|x|\rangle \ge -c^2/(2\delta) - 2\epsilon$.

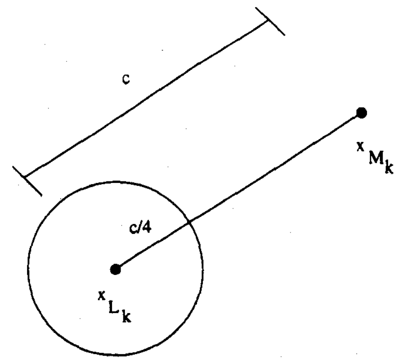

Fig. 1. Figure for Lemma 1.

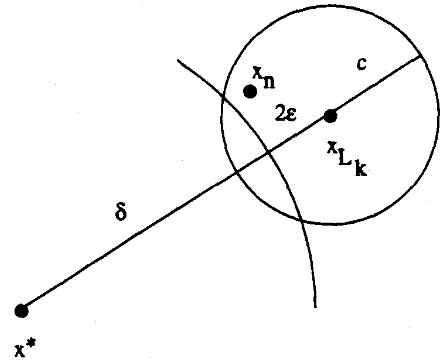

Fig. 2. Figure for Lemma 3.

Proof:

[Part a)]

$$\begin{aligned}\langle z, x\rangle &= \langle z, y\rangle + \langle z, x - y\rangle \ge h - |z||x - y|\\ &\ge h - (M/h)hc \ge (1-\eta)h.\end{aligned}$$

[Part b)]

$$\begin{aligned}&\langle y - x, y - x\rangle \le c^2\\ &\quad\Rightarrow 2\langle y, x\rangle \ge \langle x, x\rangle + \langle y, y\rangle - c^2 \ge 2|x|^2 - 4\epsilon|x| - c^2\\ &\quad\Rightarrow \left\langle y - x, \frac{x}{|x|}\right\rangle \ge |x| - 2\epsilon - \frac{c^2}{2\delta} - |x| = -\frac{c^2}{2\delta} - 2\epsilon.\end{aligned}$$ □

The third lemma basically states that if there is a long enough interval on which the movement is directed in magnitude away from x^*, then e_n does not satisfy (3) on that interval. Actually, the lemma also allows a small "slippage" of 2ϵ back toward x^* (refer to Fig. 2). The proof amounts to showing that the term $|\sum_{n\in I_k} a_n e_n|$ is bounded below by the sum of a term due to the restoring force of the function and a quadratic term in c coming from Lemma 2b). The quadratic relationship allows the existence of positive α and β for a fixed sufficiently small c and a correspondingly sufficiently small ϵ.

Lemma 3 (Detailed Balance): Assume A1) and B1) and let $c > 0$ and $\epsilon \ge 0$. Suppose $\exists \delta > 0$ and an interval $I_k = [L_k, M_k - 1]$ such that $c/8 \le K_f \sum_{n\in I_k} a_n$ and $\forall n \in I_k \cup \{M_k\}$, the inequalities $|x_n - x_{L_k}| < c, |x_n - x^*| \ge \delta$ and $|x_n - x^*| \ge |x_{L_k} - x^*| - 2\epsilon$ all hold. Then, $\exists c_0 > 0$ and $\epsilon_0(c) > 0$ such that if $c \in (0, c_0]$ and $\epsilon \in [0, \epsilon_0(c)]$, then $\exists \alpha, \beta > 0$ such that $|\sum_{n\in I_k} a_n e_n| \ge \alpha \sum_{n\in I_k} a_n + \beta$.

Proof: Let $c_0 = \min\{h_\delta/(2K_f), (\delta h_\delta)/(32K_f)\}$, and let $c \in (0, c_0]$. By assumption, $c \in (0, h_\delta/(2K_f)]$ so that Lemma 2a) applies (with $M = K_f, h = h_\delta$, and $\eta = 1/2$). Then by Assumption B1) and Lemma 2a) [with $x = x_{L_k} - x^*, y = x_n - x^*$, and $z = f(x_n)$]

$$\langle f(x_n), x_{L_k} - x^*\rangle \geq \tfrac{1}{2}\langle f(x_n), x_n - x^*\rangle \geq \tfrac{1}{2}h_\delta|x_n - x^*| \geq \tfrac{1}{2}h_\delta(|x_{L_k} - x^*| - 2\epsilon).$$

Therefore, $\Sigma_{n\in I_k}\ a_n\langle f(x_n), v_k\rangle \geq \frac{1}{2}(1 - 2\epsilon/\delta)h_\delta\ \Sigma_{n\in I_k}\ a_n$, where $v_k = (x_{L_k} - x^*)/|x_{L_k} - x^*|$. Also, by Lemma 2b) (with $x = x_{L_k} - x^*$ and $y = x_{M_k} - x^*$)

$$\langle x_{M_k} - x_{L_k}, v_k\rangle \geq -c^2/(2\delta) - 2\epsilon.$$

Therefore, by recalling that $x_{M_k} = x_{L_k} - \Sigma_{n\in I_k}\ a_n(f(x_n) + e_n)$, we have that

$$\sum_{n\in I_k} a_n\langle e_n, v_k\rangle \leq -\sum_{n\in I_k} a_n\langle f(x_n), v_k\rangle + c^2/(2\delta) + 2\epsilon.$$

The right-hand side of this inequality consists of a restoring force term due to the function and a term with an opposite sign. If the restoring force term dominates, then the right-hand side is strictly negative. This would imply that

$$\begin{aligned}\left|\sum_{n\in I_k} a_n e_n\right| &\geq \left|\sum_{n\in I_k} a_n\langle e_n, v_k\rangle\right| \geq \frac{1}{2}\left(1 - \frac{2\epsilon}{\delta}\right)h_\delta \\ &\quad \cdot \sum_{n\in I_k} a_n - c^2/(2\delta) - 2\epsilon \\ &\geq \frac{1}{4}\left(1 - \frac{2\epsilon}{\delta}\right)h_\delta \sum_{n\in I_k} a_n + \frac{1}{4}\left(1 - \frac{2\epsilon}{\delta}\right)h_\delta \\ &\quad \cdot \sum_{n\in I_k} a_n - c^2/(2\delta) - 2\epsilon \\ &\geq \frac{1}{4}\left(1 - \frac{2\epsilon}{\delta}\right)h_\delta \sum_{n\in I_k} a_n + \beta\end{aligned}$$

where $\beta = \frac{1}{4}(1 - (2\epsilon/\delta))h_\delta(c/8K_f) - (c^2/2\delta) - 2\epsilon$. The important point is that β is quadratic in c and is in fact positive if c is fixed small enough (in particular, straightforward computations show that $c \in (0, c_0]$ is small enough), and then ϵ is taken sufficiently small (and again, straightforward computations show that $\epsilon \in [0, \epsilon_0(c)]$ is small enough, where $\epsilon_0(c) = [h_\delta c/(32K_f) - c^2/(2\delta)]/[4(1 + h_\delta c/(32\delta K_f))] > 0$). Therefore, for $c \in (0, c_0]$ and $\epsilon \in [0, \epsilon_0(c)], |\Sigma_{n\in I_k}\ a_n e_n| \geq \alpha\ \Sigma_{n\in I_k}\ a_n + \beta$, where $\alpha = \frac{1}{4}(1 - (2\epsilon/\delta))h_\delta$ and β as defined above are both positive. □

Proof of Theorem 1: Part a): ($\Leftarrow$) We shall prove the result by proving the contrapositive. Assume that $x_n \not\to x^*$ for some $f \in \mathcal{F}_1$ and some $x_1 \in \mathbb{H}$. We shall show that e_n does not satisfy (KC′). First, note that if $|a_n e_n| \not\to 0$, then certainly e_n does not satisfy (KC′). This is true since if e_n satisfies (KC′), then by choosing the intervals $I_k = \{k\}, k = 1, 2, \cdots$, we would obtain $\forall\alpha, \beta > 0, |a_n e_n| < \alpha a_n + \beta$ for all but a finite number of k's. Since $a_n \to 0$ by A2), this implies $|a_n e_n| \to 0$. Therefore, we need only consider the case $|a_n e_n| \to 0$. In this case, by A1) and A2), the movement per iteration goes to zero, i.e.,

$$|a_n(f(x_n) + e_n)| \leq a_n K_f + |a_n e_n| \to 0. \tag{4}$$

We will now consider the following three cases which characterize lack of convergence. In each of the three cases, we will show that there exist natural choices of I_k intervals in which either Lemma 1 or Lemma 3 applies. Note that the proof of the three cases involves choosing certain quantities small or large enough to satisfy conditions of the lemmas. To focus on the main points of the proof here, we will leave the exact choices of these quantities to the Appendix.

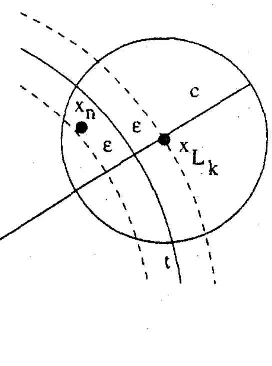

Fig. 3. Figure for Case 1.

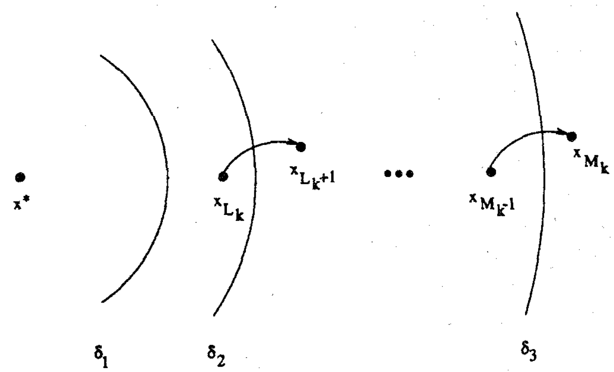

Fig. 4. Figure for Case 2.

Case 1 ($0 < \liminf|x_n - x^*| = \limsup|x_n - x^*| < \infty$): In this case, $|x_n - x^*|$ converges to a strictly positive number t. Fix a quantity c small enough and wait long enough so that the amount of movement per iteration is small enough in relation to this quantity. Also, wait long enough so that $|x_n - x^*|$ is at most some fixed ϵ away from its limiting value t. Now choose an infinite sequence of nonoverlapping intervals J_k, with $J_k = [L_k, M_k - 1]$, such that $c/8 \leq K_f\ \Sigma_{n\in J_k}\ a_n \leq c/4$, i.e., the intervals should be both long enough to apply Lemma 3 and small enough to apply Lemma 1.

Convergence of the quantity $|x_n - x^*|$ implies that x_n will "slip" at most 2ϵ back from x_{L_k} toward x^* (refer to Fig. 3). It can then be argued that either Lemma 1 can be applied on a subinterval of J_k or Lemma 3 can be applied on J_k. This is so because if $|x_{S_k} - x_{L_k}| \geq c$ for some S_k in the interval J_k (i.e., if some point lies in the 2ϵ annulus but outside the ball of radius c around x_{L_k}), then Lemma 1 applies on $I_k = [L_k, S_k - 1]$ (as the force due to the function would only allow movement within a $c/4$ ball). Otherwise, it is easy to verify the conditions so that Lemma 3 (with $\epsilon \neq 0$) applies on $I_k = J_k$.

Case 2 ($\liminf|x_n - x^*| \neq \limsup|x_n - x^*|$): In this case, infinitely often $|x_n - x^*|$ must alternate moving near the lim inf value and the lim sup value. Therefore, there exist constants $\delta_3 > \delta_2 > \delta_1 > 0$ such that $|x_n - x^*| \geq \delta_3$ infinitely often and $\delta_1 < |x_n - x^*| < \delta_2$ infinitely often. Accordingly, choose an infinite sequence of nonoverlapping intervals J_k, with $J_k = [L_k, M_k - 1]$, to denote the occurrences of movement from the lim inf value to the lim sup value (refer to Fig. 4). In other words, we start J_k at L_k which we use to represent the last time that $\{x_n\}$ is in the interval $\delta_1 < |x - x^*| < \delta_2$ before the kth exit of $|x - x^*| < \delta_3$, and we end J_k at $M_k - 1$ which we use to represent the last time that $\{x_n\}$ is in $|x - x^*| < \delta_3$ before the kth exit.

Fix a quantity c small enough in relation to the distances between δ_1, δ_2, and δ_3, and wait long enough so that the movement per iteration is small enough in relation to this quantity. Note that the movement in the J_k intervals is away (in magnitude) from x^*. It can then be argued that either Lemma 1 can be applied on J_k or a subinterval of J_k, or Lemma 3 can be applied on a subinterval of J_k.

We first ask whether $K_f \; \Sigma_{n\in J_k} \; a_n \leq c$ or not. If $K_f \; \Sigma_{n\in J_k} \; a_n \leq c$, then Lemma 1 applies on the interval $I_k = J_k$ (as $|x_{M_k} - x_{L_k}|$ is large in relation to c). If $K_f \; \Sigma_{n\in J_k} \; a_n > c$, then we consider a subinterval J_k' such that $c/8 \leq K_f \; \Sigma_{n\in J_k'} \; a_n \leq c/4$, i.e., as in Case 1, the intervals should be both long enough to apply Lemma 3 and small enough to apply Lemma 1. We can then proceed similarly to Case 1. If $|x_{S_k} - x_{L_k}| \geq c$ for some S_k in the interval J_k' (i.e., if some point lies more than a δ_2 distance away from x^*, but outside the ball of radius c around x_{L_k}), then Lemma 1 applies on $I_k = [L_k, S_k - 1]$ (as the force due to the function would only allow movement within a $c/4$ ball). Otherwise, it is easy to verify the conditions so that Lemma 3 (with $\epsilon = 0$) applies on $I_k = J_k'$.

Case 3 ($\liminf|x_n - x^*| = \infty$): In this case, there exist constants $\Delta_{12}, \Delta_{23} > 0$ and an infinite sequence $\delta_{3,k} > \delta_{2,k} > \delta_{1,k} > 0$, with $\delta_{2,k} - \delta_{1,k} = \Delta_{12}$ and $\delta_{3,k} - \delta_{2,k} = \Delta_{23}$ for all k, such that x_n moves from $\delta_{1,k} < |x_n - x^*| < \delta_{2,k}$ to $|x_n - x^*| \geq \delta_{3,k}$. Now proceed similarly to Case 2.

Thus, in each of the three cases, we have shown that e_n does not satisfy (3) on an infinite set of nonoverlapping intervals. The constants α, β in each of the three cases only depend on whether Lemma 1 or Lemma 3 were applied and are otherwise independent of the particular intervals. Therefore, e_n does not satisfy (KC$'$).

($\Rightarrow$) Again, we will prove the contrapositive. Assume e_n does not satisfy (KC$'$). Then, from the proof of sufficiency for Part b) presented below, we will see that in fact $x_n \nrightarrow x^*$ for all $f \in \mathcal{F}_2$ and all $x_1 \in \mathbb{H}$. Since the set $\mathcal{F}_1 \cap \mathcal{F}_2$ is not empty, it follows that $x_n \nrightarrow x^*$ for some $f \in \mathcal{F}_1$ and some $x_1 \in \mathbb{H}$.

Part b): ($\Leftarrow$) We can easily show this direction using our alternative version of the Kushner–Clark condition. However, it is even slightly simpler to prove this direction using the decomposition condition, and so we do that here. We shall prove the contrapositive.

Assume $x_n \rightarrow x^*$ for some $f \in \mathcal{F}_2$ and some $x_1 \in \mathbb{H}$. Let $f_n = e_n + f(x_n) = -(x_{n+1} - x_n)/a_n$ and $g_n = -f(x_n)$. Clearly, $e_n = f_n + g_n$. Furthermore, $\Sigma_{n=1}^N a_n f_n = \Sigma_{n=1}^N -(x_{n+1} - x_n) = x_1 - x_{N+1} \rightarrow x_1 - x^*$ and $g_n \rightarrow 0$ by convergence of x_n and by C1). Therefore, e_n satisfies DC, and hence also satisfies (KC$'$).

($\Rightarrow$) Once more we will prove the contrapositive. Assume e_n satisfies (KC$'$). Then, from the proof of sufficiency for Part a) presented above, we saw that in fact $x_n \rightarrow x^*$ for all $f \in \mathcal{F}_1$ and all $x_1 \in \mathbb{H}$. Again, since the set $\mathcal{F}_1 \cap \mathcal{F}_2$ is not empty, it follows that $x_n \rightarrow x^*$ for some $f \in \mathcal{F}_2$ and some $x_1 \in \mathbb{H}$.

Part c): This follows directly from the application of Parts a) and b) to $\mathcal{F}_1 \cap \mathcal{F}_2$. □

IV. Conclusions

We have provided an alternative condition and proof for convergence of stochastic approximation algorithms to their desired value. We focused on the Robbins–Monro algorithm in this paper, but the approach developed here can also be applied to the Kiefer–Wolfowitz algorithm with only slightly added technical difficulty (for the 1-D version, see [5]). We introduced an alternative form of the Kushner–Clark condition and used this to provide an elementary proof of both the necessity and sufficiency of this condition. To our knowledge, our result is the first to prove necessity in a general setting. Our proof involves only elementary notions of convergence, is direct, and the approach provides a useful alternative to the standard embedding into a differential equation.

There are a number of possible directions for further work. It may be interesting to try to relax some of our assumptions, particularly the boundedness of f, as $|f(x)| < K(1 + |x|)$ is usually assumed instead. If we assume that the x_n sequence is bounded, then the boundedness assumption on f can be relaxed while still having (4) satisfied. It is desirable, however, to directly prove the result without first proving that the x_n sequence is bounded. Some general directions that may be worthwhile investigating include studying a number of other properties and variants to the standard stochastic approximation schemes using our approach (e.g., rates of convergence, accelerated schemes, global optimization, etc.), and seeing to what extent the methodology presented here is applicable to other areas, such as adaptive control.

Appendix

Here we provide details on the choice of intervals, c and ϵ in the cases of the proof of Theorem 1.

Case 1 ($0 < \liminf|x_n - x^*| = \limsup|x_n - x^*| < \infty$): In this case, $t = \lim_{n\to\infty} |x_n - x^*|$ exists and is strictly positive. Choose $c = c_0$ and $\epsilon = \epsilon_0(c_0)$ (where for the definition of c_0 and $\epsilon_0(c_0)$ coming from the proof of Lemma 3, use $\delta = t/2$). Since the limit exists, $\exists N_1 < \infty$ s.t. $||x_n - x^*| - t| \leq \epsilon \; \forall n > N_1$. Now pick an infinite sequence of nonoverlapping intervals $\{J_k\}$ with $J_k = [L_k, M_k - 1]$ by letting $L_1 > N_1, L_k > M_{k-1}$, and M_k such that $c/8 \leq K_f \; \Sigma_{n\in J_k} \; a_n \leq c/4$. Note that Assumption A2) allows us to select the intervals J_k so that they satisfy the desired inequality.

Now either $|x_{S_k} - x_{L_k}| \geq c$ for some $S_k \in J_k \cup \{M_k\}$ or not. If $|x_{S_k} - x_{L_k}| \geq c$ for some $S_k \in J_k \cup \{M_k\}$, then let $I_k = [L_k, S_k - 1]$. In this case, since $\Sigma_{n\in I_k} \; a_n \leq \Sigma_{n\in J_k} \; a_n$, Lemma 1 implies that $|\Sigma_{n\in I_k} \; a_n e_n| \geq 2K_f \; \Sigma_{n\in I_k} \; a_n + c/4$. If $|x_{S_k} - x_{L_k}| < c$ for all $S_k \in J_k \cup \{M_k\}$, then let $I_k = J_k$. Then it is easy to verify that $\forall n \in I_k \cup \{M_k\}$, the inequalities $|x_n - x^*| \geq t - \epsilon, |x_n - x_{L_k}| < c$, and $|x_n - x^*| \geq |x_{L_k} - x^*| - 2\epsilon$ all hold. Therefore, by Lemma 3, $\exists \alpha, \beta > 0$ such that $|\Sigma_{n\in I_k} \; a_n e_n| \geq \alpha \; \Sigma_{n\in I_k} \; a_n + \beta$.

Case 2 ($\liminf|x_n - x^*| \neq \limsup|x_n - x^*|$): In this case, infinitely often $|x_n - x^*|$ must alternate moving near the $\liminf$ value and the $\limsup$ value. Therefore, there exist constants $\delta_3 > \delta_2 > \delta_1 > 0$ such that $|x_n - x^*| \geq \delta_3$ infinitely often and $\delta_1 < |x_n - x^*| < \delta_2$ infinitely often. Let $\Delta_{12} = \delta_2 - \delta_1$ and $\Delta_{23} = \delta_3 - \delta_2$. Now choose $c \in (0, \min\{\Delta_{12}, \Delta_{23}/4, c_0\})$ (where for the definition of c_0 coming from the proof of Lemma 3, use $\delta = \delta_1$). From (4) and A2), there exists a time such that movements from iteration to iteration are small. In particular, $\exists N_2 < \infty$ such that $|a_n(f(x_n) + e_n)| < c/4$ and $K_f a_n < c/8 \; \forall n > N_2$.

Let J_k be the time interval representing the kth occurrence (starting after time N_2) of exiting $\delta_1 < |x - x^*| < \delta_2$ and then exiting $|x - x^*| < \delta_3$ without ever reentering $|x - x^*| < \delta_2$. In other words, we start J_k at L_k which we use to represent the last time that $\{x_n\}$ is in the interval $\delta_1 < |x - x^*| < \delta_2$ before the kth exit of $|x - x^*| < \delta_3$ and we end J_k at $M_k - 1$, which we use to represent the last time that $\{x_n\}$ is in $|x - x^*| < \delta_3$ before the kth exit. By our definitions, $|x_{M_k} - x^*| \geq \delta_3$.

Now either $K_f \; \Sigma_{n\in J_k} \; a_n \leq c$ or not. If $K_f \; \Sigma_{n\in J_k} \; a_n \leq c$, then let $I_k = J_k$. In this case, since $c < \Delta_{23}/4$, Lemma 1 implies that $|\Sigma_{n\in I_k} \; a_n e_n| \geq 2K_f \Sigma_{n\in I_k} \; a_n + \Delta_{23}/4$. If $K_f \; \Sigma_{n\in J_k} \; a_n > c$, then let $J_k' = [L_k, R_k - 1]$ where R_k is chosen so that $c/8 \leq K_f \; \Sigma_{n\in J_k'} \; a_n \leq c/4$. Now either $|x_{S_k} - x_{L_k}| \geq c$ for some $S_k \in J_k' \cup \{R_k\}$ or not. If $|x_{S_k} - x_{L_k}| \geq c$ for some $S_k \in J_k' \cup \{R_k\}$, then let $I_k = [L_k, S_k - 1]$. In this case, since $\Sigma_{n\in I_k} \; a_n \leq \Sigma_{n\in J_k'} \; a_n$, Lemma 1 implies that $|\Sigma_{n\in I_k} \; a_n e_n| \geq 2K_f \; \Sigma_{n\in I_k} \; a_n + c/4$. Finally, if $|x_{S_k} - x_{L_k}| < c$ for all $S_k \in J_k' \cup \{R_k\}$, then let $I_k = J_k'$.

By our definitions of L_k and M_k, we have $|x_n - x^*| \geq \delta_2$ for $n \in I_k \cup \{R_k\} - \{L_k\}, \delta_2 > |x_{L_k} - x^*| > \delta_1$, and $|x_n - x_{L_k}| < c$ for all $n \in I_k \cup \{R_k\}$. Therefore, by Lemma 3 (with $\epsilon = 0$), $\exists \alpha, \beta > 0$ such that $|\Sigma_{n \in I_k} a_n e_n| \geq \alpha \Sigma_{n \in I_k} a_n + \beta$.

Case 3 ($\liminf |x_n - x^*| = \infty$): With the choice of Δ_{12} and Δ_{23}, proceed exactly as in Case 2 above.

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Embedding Adaptive JLQG into LQ Martingale Control with a Completely Observable Stochastic Control Matrix

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Abstract—**With jump linear quadratic Gaussian (JLQG) control, one refers to the control under a quadratic performance criterion of a linear Gaussian system, the coefficients of which are completely observable, while they are jumping according to a finite-state Markov process. With adaptive JLQG, one refers to the more complicated situation that the finite-state process is only partially observable. Although many practically applicable results have been developed, JLQG and adaptive JLQG control are lagging behind those for linear quadratic Gaussian (LQG) and adaptive LQG. The aim of this paper is to help improve the situation by introducing an exact transformation which embeds adaptive JLQG control into LQM (linear quadratic Martingale) control with a completely observable stochastic control matrix. By LQM control, we mean the control of a martingale driven linear system under a quadratic performance criterion. With the LQM transformation, the adaptive JLQG control can be studied within the framework of robust or minimax control without the need for the usual approach of averaging or approximating the adaptive JLQG dynamics. To show the effectiveness of our transformation, it is used to characterize the open-loop-optimal feedback (OLOF) policy for adaptive JLQG control.**

I. INTRODUCTION

With jump linear quadratic Gaussian (JLQG) control, one refers to the control under a quadratic performance criterion of a linear Gaussian system, the coefficients of which are completely observable, while they are jumping according to a finite-state Markov process $\{\theta_t\}$. With adaptive JLQG, one refers to the more complicated situation that the process $\{\theta_t\}$ is only partially observable. Both JLQG and adaptive JLQG control studies have led to many practically applicable results. A good overview of these results can be found in [1]. In spite of these practical results, the developments in JLQG and adaptive JLQG control are lagging behind those for linear quadratic Gaussian (LQG) and adaptive LQG. This is best explained at the hand of the formal status of JLQG and adaptive JLQG achievements:

- Controllability and stabilizability results for JLQG control have been developed during the last decade (e.g., [2]–[4]).
- The conditional evolution of the hybrid state in adaptive JLQG control has been characterized [5]–[7].
- In contrast with adaptive LQG control [8], for adaptive JLQG control a verification theorem has only been derived in case the system state is completely observable [9].
- Similarly recent is a complete derivation of the well-known JLQG control policy, [10], [11].
- The study of robust JLQG control has only recently started [12].

There obviously is a large gap between JLQG and LQG developments. A common approach to bridging this gap is to modify JLQG control problems into approximated simpler control problems. Examples of this approach are:

- Derivation of the averaging JLQG policy as the exact solution of adaptive JLQG control under a modified performance criterion [13];
- Robust control through averaging the dynamics of adaptive JLQG control (e.g., [14]);

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