State Dependent Utility and Ambiguity

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Abstract

Models under uncertainty study choice behavior when outcomes depend on the realized state of the world. The typical assumption is that utilities of outcomes do not depend on the realized state and are state independent. Without this simplifying assumption, it is difficult to separately identify utilities and beliefs. This paper provides novel general foundations for models with state dependent utilities: once we depart from expected utility, it is often possible to uniquely identify utilities and beliefs. Specifically, we show that with general models of non-expected utility under ambiguity we have complete identification of utilities and probabilities under full-dimensional uncertainty. We offer a novel axiomatization for state dependent utility models. Finally, we provide two applications: First in social choice theory, we show the identification of fairness of the society and interpersonal utility comparisons. Second in intertemporal choice, we give foundations for evolving taste with a recursive time dependent expected utility.

1 Introduction

Decision making under uncertainty studies choice behavior when outcomes depend on the realized state of the world. Traditionally, it is assumed that the utilities of outcomes are state independent and do not depend on the realized state. This independence simplifies the identification of utilities and beliefs. However, as observed by Aumann (1971) in many situations outcomes and their utilities might be state dependent. Known examples are when the state of the world is the health of the decision maker, which is likely to affect the utility of outcomes, such as with health insurance (Arrow, 1970). Assuming state independent utilities in these situations may be inaccurate and lead to wrong identification and predictions.

This paper reconsiders state dependent utilities in the standard framework and shows that, under general conditions, it is possible to separately identify the state dependent utilities and

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probabilities whenever we have violations of the expected utility hypothesis. Additionally we provide a novel axiomatic characterization for models with state dependent utilities under ambiguity. We illustrate the usefulness of these results with two applications unrelated to choice under uncertainty.

Before moving on to the results, we highlight the importance of identifying utilities and probabilities with a concrete example. Consider a government that wants to change people’s behavior with a public health campaign but people find the change difficult or inconvenient e.g. reduce smoking or increase the use of seat belt. Here the choice of an effective campaign depends critically on if the lack of change reflects a taste based reason, quitting smoking is difficult, seat belts are uncomfortable, or if it reflects a belief based reason, only heavy smokers get cancer or only reckless drivers get into accidents. In the first case an effective campaign would make the change of behavior easier by increasing the availability of nicotine replacement products and making smoking socially less acceptable or redesigning seat belts to be more comfortable and convenient. In the second case an effective campaign would be an information campaign on the effects of behavior changes and risks associated with the current behavior. Here, it is crucial to separate tastes from beliefs in order to choose an effective campaign.

This paper characterizes and identifies general state dependent models under uncertainty. We focus on standard preferences under ambiguity, that is continuous, monotonic, and risk independent preferences. Our first characterization result shows that if these standard preferences have two unambiguous acts, in a sense that will be made precise, then they admit a dual-self expected utility representation (non-convex multiple prior preferences, Chandrasekher et al., 2020) with state dependent utilities.\footnote{This result extends Chandrasekher et al.’s (2020) characterization. However, this extension is not entirely straightforward since under state-dependence, constant acts may not longer be unambiguous so we need to find an alternative way to capture the characterizing properties of the representation. We will infer from the decision maker’s behavior which acts are unambiguous and use this to capture the characterizing properties.}

In this context, we show our first main identification result: when the uncertainty about states of the world is full-dimensional, the probabilities and the utilities are fully identified. That is, the probabilities are uniquely identified and the state dependent utilities are identified up to a common positive affine transformation. This shows that the impossibility of identification under expected utility is only a knife-edge case, due to the linearity of expected utility: once we depart from such linearity, we regain identification. Instead with any full-dimensional uncertainty, we can identify the utilities from the violations of the expected utility hypothesis, as we illustrate with an example.
in the next section. More in general, we show that the identification of utilities and probabilities between two states is characterized by having uncertainty about the relative likelihood between the states. This general identification result has many important special cases. It characterizes the identification of state dependent utilities for models such as maxmin expected utility (multiple prior preferences, Gilboa and Schmeidler, 1989), Choquet expected utility (Schmeidler, 1989), $\alpha$-maxmin expected utility (Ghirardato et al., 2004), and invariant biseparable preferences (Ghirardato et al., 2004; Amarante, 2009).

Next, we move on to a more general model under uncertainty. We show that if standard ambiguity preferences have two acts that share the same ambiguity and hedge ambiguity similarly, in a sense that will be made precise later on, then they admit a dual-self variational expected utility representation (non-convex variational preferences, Chandrasekher et al., 2020) with state dependent utilities. This generalizes the characterization from Chandrasekher et al. (2020) to state-dependent utilities. In this more general context, we show our second identification result: when the uncertainty about states is full-dimensional, then probabilities and intensities of preferences can be separated. However, in this case the levels of utilities are not identified.\footnote{Formally, the probabilities are uniquely identified and the changes of state dependent utilities are identified up to a common positive multiplication.} This general result characterizes the identification for state dependent variations of models such as monotone mean-variance preferences (Maccheroni et al., 2009), multiplier preferences (Hansen and Sargent, 2001; Strzalecki, 2011), variational preferences (Maccheroni et al., 2006), monotonic mean-dispersion preferences (Grant and Polak, 2013), and vector expected utility (Siniscalchi, 2009).

This new foundation for state dependent utilities has applications beyond ambiguity. Our first application is in social choice theory. Here, we interpret states to be members of a society and study the preferences of the society over distributions of goods. In this context, we show that the fairness of the society and interpersonal utility comparisons can be identified. In a general class of societies, utilitarian societies are only a literal knife-edge case where they are not identified. Instead, with any other society in this general class, the fairness and the interpersonal utilities can be identified.

Our second application is in intertemporal choice. Here, we interpret states to be time periods and study preferences over consumption streams with recursive preferences. In this context, we give foundations for evolving taste with a recursive time dependent expected utility.
This paper contributes to the literature studying the identification of state dependent utilities in models of non-expected utilities following Drèze (1987) and Chew and Wang (2020). This provides an alternative to the literature on using additional information (e.g. preferences conditional on signals with Bayesian updating, hypothetical lotteries of state-outcome pairs, or stochastic choice), for the identification of state dependent utilities as in Karni (2011a; 2011b), Karni and Schmeidler (2016), and Lu (2019).

The remainder of the paper proceeds as follows: We begin, in Section 1.1, by showing a simple example of the fundamental forces behind the identification result. Next, Section 2 studies state dependent dual-self expected utility. We axiomatize its existence in Sections 2.3 and 2.4 and characterize its identification in Section 2.5 with behavioral characterizations for the identification. Section 3 extends the existence and the identification results to state dependent dual-self variational expected utility. Section 4 discusses applications to social choice theory and intertemporal choice. Section 5 clarifies the underlying assumptions for our identification and discusses the limitations of the identification. Finally, Section 6 discusses related literature and Section 7 concludes. Proofs for all the results are in the Appendix.

### 1.1 An Example of Identification

We begin with a simple example illustrating that with state dependent expected utility the intensities of preferences and the probabilities cannot be separated. However, this is only an unidentified knife-edge case. In the second part of the example, we show that with state dependent maxmin expected utility these can be separated and identified from the violations of the expected utility hypothesis.

To make the problem of separating utilities and probabilities concrete, consider an unemployed person, Bob. Unemployed Bob is offered a choice between $1,000 if he is employed in six months and $1,000 if he is unemployed in six months. He chooses the unemployment insurance that pays in the case of unemployment. What can we infer from his choice in this situation? Does the choice mean that Bob considers it more likely that he is unemployed in six months than that he would find a job? Or does the choice reflect that without employment Bob could use the additional money to pay for overdue bills and would like to insure against this prospect?

This is the fundamental problem in identifying utilities and probabilities. With state independent utilities we can always infer which event is more likely by comparing the same pay-offs in
different states as above. However, as soon as we consider the possibility that utilities might be state dependent, the same pay-offs in different states are no longer the same in utilities and these two forces become inseparable. Utilities and probabilities are always only observed jointly and it is not possible to separate them using simple comparisons.

Next we formalize the above identification problem. Consider Anscombe-Aumann setup with two states of the world, 1 and 2, and outcomes that are lotteries on consequences $\Delta(X)$. Acts consist of consequences for each state, $(f_1, f_2)$. Assume that preferences over acts $(f_1, f_2)$ have a state dependent expected utility representation with an equal probability for both of the states

$$0.5u_1(f_1) + 0.5u_2(f_2)$$

where $u_1, u_2 : \Delta(X) \rightarrow \mathbb{R}$ are von Neumann-Morgenstern utilities.

Now these preferences have an alternative expected utility representation with any probability $p \in (0, 1)$ for state 1 since

$$0.5u_1(f_1) + 0.5u_2(f_2) = p\left(\frac{0.5}{p}u_1(f_1)\right) + (1 - p)\left(\frac{0.5}{1 - p}u_2(f_2)\right) = p\tilde{u}_1(f_1) + (1 - p)\tilde{u}_2(f_2),$$

where the terms inside the parentheses define new state dependent utility functions $\tilde{u}_1, \tilde{u}_2$.

In this alternative representation following the above example, we have replaced the probability for employment with an intensity of preference for money. This highlights the impossibility of identifying the state dependent expected utility since the intensities of preferences are inseparable from the probabilities.

However, this lack of identification is only due to the fact that we were using expected utility. Instead, we can identify the utilities across the states from the violations of the expected utility hypothesis. Going back to the example, assume this time that Bob is uncertain about the probability of employment and is uncertainty averse focusing on the worst case probability. Assume that we observe many pairwise choices and there is a violation of the expected utility hypothesis at an unemployment insurance of size $x$ in a sense that will be made precise later on. What should we infer from this violation? Since Bob is uncertain about the probability of employment, this violation reflects a change in the worst case probability. However, with two states of the world, there can be a change in the worst case probability only if the utility in both states is the same.\(^3\) So we are able to identify the utilities across states using changes in Bob’s uncertainty attitude.

\(^3\)The worst case probability always maximizes the probability of the state with a lower utility.
\[
\min_{p \in [p^*, p^*]} pu_1(\alpha x + (1 - \alpha)y) + (1-p)u_2(f_2)
\]

Figure 1. An example of identifying when the utilities across states are equal from non-linearities. \(x\) and \(y\) are such that \(u_1(0.5x + 0.5y) = u_2(f_2)\) and \(u_1(x) < u_2(y)\). The x-axis changes \(x\) to \(y\) with convex combinations. The y-axis is the state dependent maxmin expected utility for the act \((\alpha x + (1-\alpha)y, f_2)\). At \(\alpha = 0.5\) utilities across state are equal that is observable as a violation of the expected utility hypothesis.

Next we formalize this identification from changes in the uncertainty attitude. We move on to a state dependent maxmin expected utility over acts \((f_1, f_2)\) defined by two probabilities \(p^* < p^*\) for state 1 and affine von Neumann-Morgenstern utilities \(u_1, u_2\)

\[
\min_{p \in [p^*, p^*]} pu_1(f_1) + (1-p)u_2(f_2).
\]

We show that utilities are identifiable across states from non-linearities or violations of Independence axiom\(^4\). For this let \(f_2\) be a consequence for state 2 and let \(x, y\) be consequences such that \(u_1(0.5x + 0.5y) = 0.5u_1(x) + 0.5u_1(y) = u_2(f_2)\) and \(u_1(x) < u_1(y)\).\(^5\) Next we will show that there is a violation of Independence axiom at \((1/2x + 1/2y, f_2)\).

We focus on the maxmin expected utility for acts \((\alpha x + (1-\alpha)y, f_2)\) when \(\alpha\) changes from 0 to 1 as in Figure 1. First, between 0 and 0.5, \(\alpha x + (1-\alpha)y\) gives more weight to \(x\) than to \(y\) and the act \((\alpha x + (1-\alpha)y, f_2)\) gives lower utility in state 1 than in state 2. So the maxmin expected utility uses probability \(p^*\). Thus the maxmin expected utility for the act increases linearly at the rate \(p^*(u_1(y) - u_1(x))\). Second, between 0.5 and 1, \(\alpha x + (1-\alpha)y\) has lower utility in state 2 and so the maxmin expected utility for the act \((\alpha x + (1-\alpha)y, f_2)\) uses probability \(p^*\). This time the expected utility for the act increases linearly at the rate \(p^*(u_1(y) - u_1(x))\). Since the rate of

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\(^4\)Independence axiom from Anscombe and Aumann (1963) characterizes the linearity of the subjective expected utility. It states that for all acts \(f, g, h\) and \(\alpha \in (0,1)\)

\[
f \succ g \iff \alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h
\]

when consequences are gambles and mixtures of acts are defined statewise.

\(^5\)The first inequality follows from \(u_1\) being a von Neumann-Morgenstern utility. If \(u_1\) is unbounded, these \(x\) and \(y\) always exist.
increase changes at $\alpha = 0.5$, there is a non-linearity at that point. This represents a violation of Independence axiom at the act $(0.5x + 0.5y, f_2)$ that has the same utility for both states.

Finally, there can be violations of the independence axiom only if the utilities across states are the same. The only points where there can be non-linearities as in Figure 1 are points where the used probability changes. However, maxmin expected utility with two states always maximizes the probability for the state with lower utility. Thus the only points where there are changes of probabilities are points where the utility order of states changes. Especially, at that point the utilities across states are exactly the same.

In summary, the acts where the utilities across states are equal are characterized by the violations of Independence axiom and especially they are identifiable. This shows the identification of the utilities across states. Finally, this also shows the identification of probabilities since after identifying the utilities, we can apply the state-independent identification result directly.

This example illustrates that we can behaviorally observe when the utilities across states are equal from the violations of the expected utility hypothesis or Independence axiom. The identification came from the two different probabilities that is from the full dimensional set of probabilities.

This identification is generalized in our main result, Theorem 2, to finitely many states of the world and to non-convex dual-self expected utility. There we show that if there is full dimensional uncertainty, then we recover the state independent identification: the set of probabilities is unique and the utilities are unique up to a common positive affine transformation.

### 2 State Dependent Dual-Self Expected Utility

In order to study identification in the most general setup, we begin by studying a state dependent version of dual-self expected utility (Chandrasekher et al., 2020). The state-independent version is a general model that includes as special cases maxmin expected utility (Gilboa and Schmeidler, 1989), Choquet expected utility (Schmeidler, 1989), and $\alpha$-maxmin expected utility (Ghirardato et al., 2004) and that is an alternative representation for invariant biseparable preferences (Ghirardato et al., 2004; Amarante, 2009). Here we characterize the existence and the identification of the state dependent version. Our results thus encompass those for state dependent versions of all the special cases and alternative representations.
2.1 Preliminaries and Notation

Consider the finite Anscombe-Aumann (1963) framework with state dependent consequences. $S$ is a finite state space, for each $s \in S$, $X_s$ is a set of state dependent consequences and $\Delta(X_s)$ is the set of (simple) lotteries on $X_s$. Acts are mappings from states to state specific consequences and the set of acts is $H = \times_{s \in S} \Delta(X_s)$. Our primitive is a binary relation $\succeq$ on $H$. As usual, $\succ$ and $\sim$ denote the asymmetric and symmetric parts of $\succeq$ respectively.

The following notation will be useful. $\Delta(S)$ is the set of probability measures on $S$. We endow $\Delta(S)$ with the Euclidean topology. $\mathcal{K}(\Delta(S))$ is the set of all closed, convex, and non-empty subsets of $\Delta(S)$ endowed with the Hausdorff topology. For $P \subseteq \Delta(S)$, denote the convex closure of $P$ by $\overline{\text{co}} P$. For $S' \subseteq S$ and $P \subseteq \Delta(S)$, denote the projection of $P$ to $S'$ by $\text{pr}_{S'} P = \{(p_s)_{s \in S'} | p \in P\}$.

For $f \in H, s \in S, x_s \in \Delta(X_s)$, $f_s$ denotes the consequence of the act $f$ in the state $s$ and $(x_s, f_{-s})$ denotes the act where the consequence in the state $s$ is $x_s$ and in the states $s' \in S \setminus \{s\}$, $f_{s'}$. Mixtures of acts are defined statewise: for all $f, g \in H, \alpha \in [0, 1], s \in S$, $(\alpha f + (1-\alpha)g)_s = \alpha f_s + (1-\alpha)g_s$.

If consequences in some state do not affect the preferences, then the utility for these consequences is unobservable. Hence our focus is on proper states:

**Definition** A state $s \in S$ is proper if there exist $x_s, y_s \in \Delta(X_s)$ and $f \in H$ such that $(x_s, f_{-s}) \not\succeq (y_s, f_{-s})$. The collection of proper states is denoted $S^p$.

We infer the preferences on consequences within each state as follows:

**Definition** For each $s \in S$, define $\succsim_s$ on $\Delta(X_s)$ by for all $x_s, y_s \in \Delta(X_s)$

$$x_s \succsim_s y_s \Longleftrightarrow (x_s, f_{-s}) \succeq (y_s, f_{-s}) \text{ for all } f \in H.$$ 

Additionally, $\succ_s$ is the asymmetric part of $\succsim_s$.

2.2 State Dependent Dual-Self Expected Utility

The state independent dual-self expected utility was introduced by Chandrasekher et al. (2020) as a general model for preferences under ambiguity. This model generalizes the maxmin expected utility. This includes the standard state independent Anscombe-Aumann setting where $X_s = X$ for all $s$. This is the $|S| - 1$ dimensional Euclidean topology when $\Delta(S)$ is represented as $\{p \in \mathbb{R}_+^{[S]-1} | \sum_{i=1}^{[S]-1} p_i \leq 1\}$.
utility by allowing for multiple sets of beliefs that are aggregated optimistically. Axiomatically this model corresponds to relaxing the uncertainty aversion of the maxmin expected utility. We will use a state dependent variation of it:

**Definition** $(u, P)$ is a state dependent dual-self expected utility for $≿$ if $u = (u_s)_{s ∈ S}$ and for all $s ∈ S$, $u_s : Δ(X_s) → ℝ$ is affine and $P ⊆ K(Δ(S))$ is compact and non-empty such that for each $P ∈ P, p ∈ P, s /∈ S^p, p_s = 0$, and for all $f, g ∈ H,$

$$f ≿ g ⇐⇒ \max_{P ∈ P} \min_{p ∈ P} \sum_{s ∈ S} p_s u_s(f_s) ≥ \max_{P ∈ P} \min_{p ∈ P} \sum_{s ∈ S} p_s u_s(g_s).$$

This representation is the maxmin expected utility when $P$ is a singleton and the maxmax expected utility when each $P ∈ P$ is a singleton. Additionally since this representation is a non-convex generalization of maxmin expected utility, especially any uncertainty aversion and seeking of the preferences can be represented by the interplay of the sets of beliefs and the collection of sets of beliefs. That is by the interplay of the extreme concavity (min) and convexity (max).

As observed in Chandrasekher et al. (2020), the state independent dual-self expected utility is not unique. However, in the state independent case the convex and closed smallest set of probabilities used by the representation is unique which gives us tight dual-self expected utility. We will use this tight representation to study the identification of the state dependent dual-self expected utility.

**Definition** Let $≿ ⊆ H × H$. $(u, P)$ is a state dependent tight dual-self expected utility for $≿$ if $(u, P)$ is a state dependent dual-self expected utility of $≿$ and if $(u, \tilde{P})$ is another state dependent dual-self expected utility of $≿$, then $\overline{∩}_{P ∈ P} \tilde{P} ⊇ \overline{∩}_{P ∈ P} P$.

### 2.3 Axioms for Existence

This section introduces six axioms that characterize the existence of a state dependent dual-self expected utility representation. This axiomatization highlights the generality of the representation by showing that essentially the only standard preferences under ambiguity that do not have a dual-self expected utility representation are such that every act is ambiguous or every act contains different ambiguity. A reader interested only in the identification can skip this and the following existence section.
The first four axioms define standard preferences under ambiguity following Cerreia-Vioglio et al. (2011) in the state dependent context. The first two axioms are standard rationality assumptions that the preferences are a nontrivial weak order that satisfy continuity.

**Axiom 1** $\succ$ is complete, transitive, and non-trivial.

**Axiom 2** For all $f, g, h \in H$, the sets $\{\alpha \in [0, 1] | \alpha f + (1 - \alpha)g \succ h\}$ and $\{\alpha \in [0, 1] | h \succ \alpha f + (1 - \alpha)g\}$ are closed in $\mathbb{R}$.

The next assumption for standard preferences under ambiguity is Monotonicity.

**Axiom 3** For all $s \in S, x_s, y_s, z_s \in \Delta(X_s), f, g \in H$,

$$ (x_s, f_s) \succ (y_s, f_s) \implies (x_s, g_s) \succ (y_s, g_s). $$

This axiom states that the decision maker’s revealed statewise preferences are consistent with monotonicity. If the decision maker reveals to prefer $x_s$ over $y_s$ in some situation, then by monotonicity the decision maker cannot reveal to strictly prefer $y_s$ over $x_s$ in another situation. This axiom is a state dependent version of the standard monotonicity. It is assuming that the decision maker is a consequentialist that only cares about the outcomes received in each state. This axiom guarantees that $\succ_s$ is complete.

The next assumption for standard preferences under ambiguity is Risk Independence assuming Independence on lotteries. The next axiom assumes a weak version of Independence, ruling out strict preference reversals, for the lotteries within each state of the world. This Weak Independence was introduced in Einy (1989).

**Axiom 4** For all $s \in S, x_s, y_s, z_s \in \Delta(X_s), \alpha \in (0, 1)$,

$$ x_s \succ y_s \implies \alpha x_s + (1 - \alpha)z_s \succ \alpha y_s + (1 - \alpha)z_s. $$

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8The state independent monotonicity assumes that if $f, g \in H$ are such that for all $s \in S, f_s \succ g_s$, then $f \succ g$ where $f_s$ and $g_s$ are acts that give the consequences $f_s$ and $g_s$, respectively, in every state.

9Formally, this axiom is a weak version of Savage’s (1954) Sure Thing Principle applied only to a single state.

10This is usually assumed implicitly through Certainty Independence or Weak Certainty Independence axioms.
Under standard ambiguity preferences, the dual-self expected utility is characterized by Certainty Independence which assumes that constant acts are unambiguous. In order to discuss later on preferences that do not satisfy this axiom, we divide this assumption into two parts.

The first part of the Certainty Independence is the existence of a single constant act that is unambiguous and cannot be used for ambiguity hedging. The intuition for this part is that there is no uncertainty about the utility of constant acts since they give the same consequence with the same utility in every state of the world.

However, in the state dependent setting it is not known which of the acts are unambiguous. Instead, we will infer from the decision maker’s behavior which of the acts are unambiguous. The behavioral interpretation for these acts are that they never hedge ambiguity and Independence axiom is satisfied when mixing with them as defined in Ghirardato et al. (2004).

**Definition**  
$h \in H$ is a crisp act\(^{11}\) if for all $f, g \in H, \alpha \in (0, 1)$,

$$f \succsim g \iff \alpha f + (1 - \alpha)h \succsim \alpha g + (1 - \alpha)h.$$  

The Certainty Independence assumes that all constant acts are crisp. The next axiom relaxes this to assuming that there exist some crisp act.

**Axiom 5** There exists a crisp act $c \in H$.

Here, it is not necessary that the crisp act is unambiguous. Instead, it suffices that the crisp act is a least ambiguous act. That is the ambiguity of the crisp act is contained also in every other act and so it cannot be used for ambiguity hedging.

Our first part of Certainty Independence assumed the existence of only a single crisp act instead of a continuum of crisp acts as Certainty Independence assumes. Thus our second part of Certainty Independence captures this difference by assuming that all the constant acts contain exactly the same ambiguity since they are unambiguous.

In the state dependent setting we will need to infer from the decision maker’s behavior which acts contain the same ambiguity. Behaviorally this means that the acts always hedge ambiguity similarly and so trading one of the acts for another with the same ambiguity cannot change the ambiguity hedged or the preferences. This is formalized in the following property from Maccheroni et al.’s (2006) Weak Certainty Independence assumption.

\(^{11}\)Cerreia-Vioglio et al. (2011) use a different definition for crisp acts that does not extend to state dependent setting.
**Definition**  \( f, g \in H \) are *equally crisp acts* if for all \( h, h' \in H, \alpha \in (0, 1) \),

\[
\alpha h + (1 - \alpha)f \succeq \alpha h' + (1 - \alpha)f \iff \alpha h + (1 - \alpha)g \succeq \alpha h' + (1 - \alpha)g.
\]

In this definition \( f \) and \( g \) are equally crisp if trading \( f \) for \( g \) in any mixtures does not affect the preferences.

Under Certainty Independence all constant acts are equally crisp. We relax this by assuming that there exist some equally crisp acts that are statewise ordered.

**Axiom 6**  There exist \( f^*, g^* \in H \) such that \( f^* \) and \( g^* \) are equally crisp and for all \( s \in S^P \), \( f^*_s \succ_s g^*_s \).

The intuition for this axiom is that it captures the dispersional or relative nature of ambiguity: Ambiguity of an act comes from it having good outcomes in some states and bad outcomes in some other states and from the uncertainty if the realized state and outcome is good or bad. However, these bad states and outcomes are only bad in the relative sense that the realized outcome could have been better. Now, this axiom is assuming that it is always possible to make all the outcomes better and increase the level of the outcomes without changing the ambiguity of the act by using the two equally crisp act. Since we are making all the outcomes better, this does not change the relative comparisons of the outcomes. So this axiom is stating that the ambiguity of an act only depends on the relative comparisons of outcomes or the dispersion of outcomes. Finally, this axiom does not restrict how the levels of consequences are changed as long as the changes are strict improvements in all the proper states.

The assumption that the two equally crisp acts are statewise ordered is a weak restriction. Nonordered acts have additional possibilities for different ambiguity from the trade-offs between states. So essentially, the only preferences that do not satisfy this axiom are such that every act contains different ambiguity.

An assumption closer to Certainty Independence than Axioms 5 and 6 would be to assume that there exists two crisp acts that are statewise ordered. However, this is a stronger assumption than Axioms 5 and 6 which is formalized in the next remark.

**Remark 1**  If there exist crisp acts \( f, g \) such that for all \( s \in S^P \), \( f_s \succ_s g_s \), then \( \succeq \) satisfies Axioms 5 and 6.
2.4 Existence

The previous six axioms characterize the existence of a state dependent dual-self expected utility representation for preferences.

**Theorem 1 (Existence)** The following three conditions are equivalent:

1. $\succeq$ satisfies Axioms 1-6.
2. There exists $(u, \mathbb{P})$ that is a state dependent tight dual-self expected utility for $\succeq$ such that
   $$\bigcap_{s \in S} u_s(\Delta(X_s)) \neq \emptyset.$$
3. There exists $(u, \mathbb{P})$ that is a state dependent tight dual-self expected utility for $\succeq$ and $f \in H$ such that for all $p, q \in \bigcup_{\mathbb{P} \in \mathbb{P}} \mathbb{P}$, $\sum_{s \in S} p_s u_s(f_s) = \sum_{s \in S} q_s u_s(f_s)$.

This result shows that the Axioms 1-6 characterize the existence of a state dependent dual-self expected utility with the additional property that there exists an act without uncertainty about its expected utility. A representation with this additional property always has an alternative state dependent tight dual-self expected utility with a utility overlap as shown in the second condition. This additional property follows from Axiom 5.

This result shows the generality of the state dependent dual-self expected utility. Assume that the preferences satisfy Axioms 1-4 that are standard assumptions in the ambiguity literature. Then the preferences have a state dependent dual-self expected utility representation if there exist some well-behaving acts such that one of them is unambiguous and two of them share the same ambiguity and are statewise ordered. This statewise ordering is a weak restriction as discussed above. In other words, the only standard ambiguity preferences that do not have a dual-self representation are such that every act is ambiguous or, essentially, every act contains different ambiguity. We will study in Section 3 a generalization that might have uncertainty about every act.

On the other hand, this axiomatization shows how the state dependent dual-self expected utility generalizes the state independent dual-self expected utility by relaxing Monotonicity and Certainty Independence. First, Monotonicity and Risk Independence as a part of Certainty Independence is relaxed to weak statewise Monotonicity and weak Risk Independence on lotteries within each state of the world. These generalizations capture the original intuitions of Monotonicity and Risk Independence that preferences are well-behaving on lotteries and there is no taste uncertainty since the statewise comparisons are done on lotteries within each state.
Second, for the rest of Certainty Independence instead of assuming which acts do not have uncertainty, we infer from decision maker’s behavior which acts are crisp and equally crisp. Our generalization of Certainty Independence is assuming that there exist a crisp act and two statewise ordered equally crisp acts. These existential generalizations can be interpreted as an assumption on the decision maker’s behavior or on the structure of ambiguity under consideration as discussed in the previous section.

The next remark formalizes the connection of this result to the state independent case by considering what additional assumptions would be required for state independent utility.

**Remark 2** Assume a state independent setting where for all \( s \in S \), \( X_s = X \). Then there exists a (non-trivial) state independent dual-self expected utility for \( \succ \) if and only if the following conditions hold.

1. \( \succ \) satisfies Axioms 1-6
2. \( \succ \) satisfies monotonicity with between states comparisons: For all \( s, s' \in S, f, g \in H, x, y \in \Delta(X) \),
   \[
   (x_s, f_{-s}) \succ (y_s, f_{-s}) \implies (x_{s'}, g_{-s'}) \succ (y_{s'}, g_{-s'});
   \]
3. There exists a constant act \( z \in \Delta(X) \) such that \( z \) is a crisp act.
4. There exist constant acts \( x, y \in \Delta(X) \) such that \( x \succ y \) and \( x \) and \( y \) are equally crisp.

This result shows that the state dependent dual-self expected utility relaxes the monotonicity assumption to statewise monotonicity and does not restrict which acts are crisp or equally crisp. An example of a state dependent dual-self expected utility that satisfies the first two conditions but not necessarily the other ones is if all the state dependent utilities are positive affine transformations of each other.

### 2.5 Identification

This section provides the main result of the paper. We introduce a novel simple axiom stating that the decision maker has full dimensional uncertainty. This axiom characterizes the full identification of state dependent utilities and probabilities in the state dependent dual-self expected utility. Additionally, we characterize the identification of probabilities for a single state, the identification of a relative likelihood between two states, and the general partial identification of the representation when there does not exist uncertainty about every state of the world.
2.5.1 Full Identification

To make the underlying intuition for our identifying condition clear, we first derive it informally from the idea that the decision maker’s uncertainty about the proper states is full dimensional. Full dimensional uncertainty means that there is uncertainty about the likelihood ratio between any proper states.\(^{12}\) Then especially there is uncertainty about every trade-off between proper states. Now consider two acts \(f\) and \(g\) that have trade-offs between proper states that is the act \(f\) is better than \(g\) in some states and \(g\) is better than \(f\) in some other states. Then under full-dimensional uncertainty there is uncertainty about the trade-offs between \(f\) and \(g\) and uncertainty about their relative value. This uncertainty about their relative value can be observed as a difference in the ambiguity hedging of \(f\) and \(g\) when mixing with some act \(h\). This is formalized in the next identification axiom when \(f\) and \(g\) are indifferent to guarantee observability in the difference for ambiguity hedging.\(^{13}\)

**Axiom 7** If \(f,g \in H\) are such that \(f \sim g\) and there exist \(s,s' \in S\) such that \(f_s \succ g_s\) and \(g_{s'} \succ f_{s'}\), then there exist \(h \in H\) and \(\alpha \in (0,1)\) such that

\[
\alpha h + (1 - \alpha) f \not\succ \alpha h + (1 - \alpha) g.
\]

The main result of this paper is the following uniqueness result for the state dependent dual-self expected utility. The result states that Axiom 7 characterizes the separation and identification of the probabilities and state dependent utilities.

**Theorem 2 (Full Identification)** Let \((u,\mathbb{P})\) be a state dependent tight dual-self expected utility for \(\succsim\). The following four conditions are equivalent:

1. \(\succsim\) satisfies Axiom 7.
2. \(\text{pr}_{SP}\overline{\text{co}} \cup_{P \in \mathbb{P}} P\) has a non-empty interior in \(\Delta(S^P)\).
3. If \((\tilde{u},\overline{\mathbb{P}})\) is a state dependent tight dual-self expected utility for \(\succsim\), then there exist \(\alpha \in \mathbb{R}_+\) and \(\beta \in \mathbb{R}\) such that for all \(s \in S^P\)

\[
\tilde{u}_s = \alpha u_s + \beta.
\]

\(^{12}\)Formally full dimensional uncertainty states that the set of probabilities has a non-empty interior. So especially there is uncertainty about the likelihood ratio between any states.

\(^{13}\)This axiom is closely related to the notion of unambiguous preferences from Ghirardato et al. (2004). A decision maker is said to unambiguously prefer \(f\) to \(g\) if for all acts \(h\) and \(\alpha \in [0,1]\) \(\alpha f + (1 - \alpha) h \succsim \alpha g + (1 - \alpha) h\). Hence Axiom 7 states that if there is trade-offs between two acts across states and they are indifferent, then the acts are not unambiguously indifferent.
(4) If $(\tilde{u}, \tilde{P})$ is a state dependent tight dual-self expected utility for $\succcurlyeq$, then

$$\overline{\overline{\bigcup_{P \in \tilde{P}}} \tilde{P} = \overline{\bigcup_{P \in P}} P}.$$ 

The second condition shows that Axiom 7’s uncertainty about every trade-off between proper states is the behavioral characterization of full-dimensional uncertainty for the proper states.

The third and fourth conditions show that the full dimensional uncertainty characterizes the identification of the state dependent utilities for the proper states up to a common positive affine transformation and the identification of the convex closure of the probabilities. That is we have recovered the state independent identification. The equivalency between identifying the utilities and the probabilities shows they can be identified separately but identifying one of them means that the other one is also identified. This theorem with the existence theorem, Theorem 1, provides the foundations for state dependent utility under uncertainty. Next we discuss this result.

First, this identification result has important special cases. This result shows the identification for state dependent maxmin expected utility (Gilboa and Schmeidler, 1989), which will be discussed more in Section 4, for state dependent $\alpha$-maxmin expected utility (Ghirardato et al., 2004), and for state dependent Choquet expected utility (Schmeidler, 1989).

Second, the full dimensionality of the set of probabilities for the proper states is a generic property for a given utility function. Thus it is not a restrictive condition but rules out important special cases: If $\succcurlyeq$ satisfies Independence axiom, then Axiom 7 will not be satisfied. This is the observation from Section 1.1 that state dependent expected utility is not identified. More generally, if there are unambiguous events, that is each of the probabilities agrees on the probability for an event, then Axiom 7 will not be satisfied for these unambiguous events and the utilities and probabilities are not identified across the unambiguous event.

Third, under Axiom 7 the state dependent utilities are identified for all the non-null states and the tight set of probabilities is identified. However, the two-stage structure of probability set collections is not identified. This is symmetrical to the multiplicity of state independent dual-self expected utility where only the largest representing collection of probability sets is identified (Chandrasekher et al., 2020, Proposition S.1.1). Additionally, the utilities for null states are not identifiable since they have a zero probability and do not affect preferences.

\footnote{Generic property in the topological sense that the closure of all the probability set collections not satisfying this property has an empty interior in the space of compact sets of $\mathcal{K}(\Delta(S))$ that is they are nowhere dense.}

\footnote{This can be easily observed by the following: If $P \in \mathbb{P}$, $P' \supseteq P$, and $P'$ is convex and closed, then $\mathbb{P} \cup \{P'\}$ also gives a dual-self representation since $P'$ is dominated and never used.}
Fourth, Section 1.1 sketched the proof for this result. The intuition underlying that example generalizes to the state dependent dual-self expected utility and to any finite number of states.

2.5.2 Partial Identification

The previous result showed a characterization for the full identification. However, some states of the world might not be uncertain and in this case the full identification is not possible. The next two results study the identification of probabilities for a single state and the identification of a relative likelihood between two states.

The next result shows the identification of probabilities for a single state and how full dimensionality can be extended to a single state by applying Axiom 7 only to a single state. Additionally, it provides a formalization for a notion of full dimensionality of probabilities for a single state.

Proposition 3 (Probability Identification) Let \((u, \mathbb{P})\) be a state dependent tight dual-self expected utility for \(\succeq\) and \(s \in S^P\). The following three conditions are equivalent:

1. If \(f, g \in H\) are such that \(f \sim g\), \(f \succ_s g\), and there exist \(s' \in S\) such that \(g_{s'} \succ_{s'} f_{s'}\), then there exist \(h \in H\) and \(\alpha \in (0, 1)\) such that
   \[
   \alpha h + (1 - \alpha) f \not\sim \alpha h + (1 - \alpha) g.
   \]

2. \(S^P = \{s\}\) or there exists \(p, q \in \bigcup_{P \in \mathbb{P}} P\) such that \(p \neq q_s\) and for all \(\tilde{s} \in S \setminus \{s\}\),
   \[
   \frac{p_{\tilde{s}}}{1 - p_s} = \frac{q_{\tilde{s}}}{1 - q_s}.
   \]

3. If \((\tilde{u}, \tilde{\mathbb{P}})\) is a state dependent tight dual-self expected utility for \(\succeq\), then
   \[
   \left\{ p_s | p \in \bigcup_{\tilde{P} \in \tilde{\mathbb{P}}} \tilde{P} \right\} = \left\{ p_s | p \in \bigcup_{P \in \mathbb{P}} P \right\}.
   \]

Full dimensional uncertainty about a state \(s\) means behaviorally that there is uncertainty about any trade-offs involving state \(s\) as stated by the first condition. Our first result shows that a full dimensional uncertainty about a state \(s\) is formalized by the second condition: uncertainty about the probability of the state \(s\) while keeping all the probabilities conditional on the event \(S \setminus \{s\}\) as constant. In other words, there exist independent uncertainty about the state \(s\).

Our second result shows that full dimensional uncertainty about a state characterizes the identification of the probabilities for that state. However, this time the identification of utilities for a single state is not equivalent to identifying the probabilities for that state since utilities are always identified only up to a common positive affine transformation.
This result has an important corollary: if there exists an event $E$ such that Axiom 7 is satisfied for the event $E$: For all $f, g \in H$, $f \sim g$ and $s^* \in E, s \in S$ if $f_s \succ s^* g_s$, and $g_s \succ s f_s$, then there exist $h \in H$ and $\alpha \in (0, 1)$ such that

$$\alpha h + (1 - \alpha) f \not\sim \alpha h + (1 - \alpha) g.$$ 

Then this result shows that the tight probabilities for the states in $E$ are identified and the utilities in $E$ are identified up to a common positive affine transformation.\(^{16}\)

Next, we move on to the identification of relative likelihood of two states. When considering a gamble between two states the identification of the probabilities is not important since the gamble depends only on the relative likelihood of the states. The next result provides a behavioral condition for full dimensional uncertainty about a relative likelihood and shows how that characterizes the identification of utilities and the relative likelihoods between two states. Additionally, it provides a formalization for a notion of full dimensionality of relative likelihood for two states.

**Proposition 4 (Relative Likelihood Identification)** Let $(u, \mathbb{P})$ be a state dependent tight dual-self expected utility for $\succeq$ and $s, s' \in S^P, s \neq s'$. The following four conditions are equivalent:

1. If $f, g \in H$ are such that and $f_s \succ g_s$ and $g_{s'} \succ f_{s'}$, then there exist $h, h' \in H$ and $\alpha \in (0, 1)$ such that

$$\alpha h + (1 - \alpha) f \not\succ \alpha h + (1 - \alpha) g.$$ 

2. There exists $p, q \in \overline{\bigcup_{P \in \mathbb{P}}} P$ such that $\frac{p_s}{p_{s'}} \not= \frac{q_s}{q_{s'}}$ and for all $\bar{s} \in S \setminus \{s, s'\}$, $p_{\bar{s}} = q_{\bar{s}}$.

3. If $(\bar{u}, \bar{\mathbb{P}})$ is a state dependent tight dual-self expected utility for $\succeq$, then there exist $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$ such that

$$\bar{u}_s = \alpha u_s + \beta \quad \text{and} \quad \bar{u}_{s'} = \alpha u_{s'} + \beta.$$ 

4. If $(\bar{u}, \bar{\mathbb{P}})$ is a state dependent tight dual-self expected utility for $\succeq$, then

$$\left\{ \frac{p_s}{p_{s'}} \bigg| \bar{p} \in \overline{\bigcup_{P \in \mathbb{P}}} \bar{P} \right\} = \left\{ \frac{P_s}{P_{s'}} \bigg| p \in \overline{\bigcup_{P \in \mathbb{P}}} P \right\}.$$ 

The first condition is a generalization of Axiom 7 when applied to only two states. In Theorem 2, Axiom 7 is equivalent to the first condition when applied for any states $s, s'$ but there we could use the simpler formulation to capture all relevant trade-offs between states. However, when considering only two states $s$ and $s'$, this is not possible anymore since there can be trade-offs

\(^{16}\)The identification of utilities in $E$ is not a direct corollary but follows symmetrically.
where the two acts are not indifferent. Thus the interpretation for this condition is exactly the same as before that there is uncertainty about any trade-offs between $s$ and $s'$. However this time $f$ and $g$ are not indifferent so we need the stronger condition with two convex combinations to observe the uncertainty about the relative value of $f$ and $g$ as difference in hedging.

Our first result provides a formalization for the notion of full dimensional uncertainty about the relative likelihood of two states by the second condition: Uncertainty about the relative likelihood while keeping the probabilities for all the other states constant. That is there exist independent uncertainty about the relative likelihood of these states.

The second result shows that this this full dimensional uncertainty about states $s$ and $s'$ characterizes the identification of utilities between these states in the sense that they are identified up to a common positive affine transformation and the identification of the relative likelihoods between these states. As in Theorem 2 the identification of utilities and probabilities is equivalent.

Finally, we move on to the general partial identification of the state dependent dual-self expected utility. The next result shows that the value of the representation is unique up to a positive affine transformation, but there can be additional transformations for the utilities and probabilities as long as they do not affect the value of the representation: First, for the probabilities all state-wise multiplicative transformations for which the probabilities remain as probabilities are possible when we do the statewise reciprocal transformations for the utilities. Second, all statewise additive transformations for the utilities that do not affect the expected utility for any probabilities are possible. Finally, all common positive affine transformation are possible for the utilities.

Before stating the next result we introduce some notation. For all probabilities $p \in \Delta(S)$ and numbers for each state $x \in \mathbb{R}^S_+$, we define multiplications statewise, $xp = (x_sp_s)_{s \in S}$. This induces multiplications for sets of probabilities and collections of probability sets with $x \in \mathbb{R}^S_+$. Especially for $\mathcal{P} \in \mathcal{K}(\Delta(S))$, $x\mathcal{P} = \{xp|p \in P|P \in \mathcal{P}\}$.

**Theorem 5 (Partial Identification)** Let $(u, \mathcal{P})$ be a state dependent tight dual-self expected utility for $\succeq$, and $\tilde{u} = (\tilde{u}_s)_{s \in S}$ be such that for all $s \in S$, $\tilde{u}_s : \Delta(X_s) \rightarrow \mathbb{R}$ is affine and $\tilde{\mathcal{P}} \subseteq \mathcal{K}(\Delta(S))$ be compact and non-empty. Then $(\tilde{u}, \tilde{\mathcal{P}})$ is a state dependent dual-self expected utility for $\succeq$ if and only if there exist $x \in \mathbb{R}^S_+, y \in \mathbb{R}^S$ such that for all $p, q \in \bigcup_{P \in \mathcal{P}} P$

$$\sum_{s \in S} x_sp_s = \sum_{s \in S} x_sq_s =: \alpha \text{ and } \sum_{s \in S} y_sp_s = \sum_{s \in S} y_sq_s.$$
for all $s \in S^P$

$$\tilde{u}_s = \frac{1}{x_s}(u_s + y_s),$$

and for all $f \in H$

$$\max \min_{P \in \tilde{P}} \sum_{s \in S} \tilde{p}_s \tilde{u}_s(f_s) = \max \min_{P \in \tilde{P}} \sum_{s \in S} p_s u_s(f_s).$$

Especially, $(\tilde{u}, \tilde{x}P)$ is a state dependent tight dual-self expected utility for $\succsim$ and if $(\tilde{u}, \tilde{P})$ is a tight representation, then

$$\overline{\mathcal{U}} \bigcup_{\tilde{P} \in \tilde{P}} \tilde{P} = \overline{\mathcal{U}} \bigcup_{\alpha \in \alpha} x_\alpha P.$$

This theorem characterizes the general partial identification. For any other state dependent dual-self representation the values of the representation are identified up to a positive affine transformation and the utilities and the tight set of probabilities are partially identified. In the above formulation $\frac{1}{\alpha}$ is the common positive multiplicative transformations for utilities and $\frac{x}{\alpha}$ is the multiplicative transformation for the probabilities that keeps them as probabilities. These two multiplicative transformations are combined into a single term $x$. Similarly, the term $y$ combines all the additive transformations for the utilities.\(^{17}\)

This theorem shows how the size of the set of probabilities restricts the possible transformations. If there are more probabilities, then there are less transformations $x \in \mathbb{R}^S$ such that for all probabilities $p$ and $q$, $\sum_{s \in S} x_s p_s = \sum_{s \in S} x_s q_s$. Especially these transformations are restricted by how many linearly independent probabilities there are in $\bigcup_{P \in P} P$.\(^{18}\)

The rest of this section uses the definitions from Section 2.3 in order to provide an insight into the identification through the connection to the underlying preferences. A reader interested only in the identification can skip the rest of this section.

The next result shows the connection of possible transformations to crisp acts.

**Corollary 6** Let $(u, P)$ be a state dependent dual-self expected utility for $\succsim$ and $x \in \mathbb{R}^S_{++}, y \in \mathbb{R}^S$. The following two conditions are equivalent:

\(^{17}\)Denoting $p \in \bigcup_{P \in P} P$, the additive transformations that do not affect the expected utility are $y - \sum_{s \in S} y_s p_s$, where the subtraction is statewise, and the common additive transformations are $\sum_{s \in S} y_s p_s$.

\(^{18}\)This follow from the following observation. If $x \in \mathbb{R}^S$ and $\alpha \in \mathbb{R}$ are such that for all probabilities $p$, $\sum_{s \in S} x_s p_s = \alpha$, then we can write $x = \alpha + (x - \alpha)$ where for all probabilities $p$, $\sum_{s \in S} (x_s - \alpha)p_s = 0$. Thus we can decompose any transformation to a sum of a constant and a vector orthogonal (perpendicular) to the set of probabilities. Since $\mathbb{R}^S$ can be decomposed to the sum of the linear span of $\bigcup_{P \in P} P$ and its orthogonal complement, this shows the close connection between the size of the set of probabilities and the set of possible transformations.
(1) $(\tilde{u}, \tilde{P})$ is a state dependent dual-self expected utility for $\succsim$ such that

$$
\tilde{u}_s = \frac{1}{x_s}(u_s + y_s) \text{ for all } s \in S^p \text{ and } 0, 1 \in \bigcap_{s \in S^p} \tilde{u}_s(\Delta(X_s)).
$$

(2) There exist crisp acts $f^*, g^*$ such that for all $s \in S^p$, $f^*_s \succ g^*_s$, $u_s(g^*_s) = -y_s$ and $u_s(f^*_s) = -y_s + x_s$.

This corollary with Theorem 5 shows the idea of the partial identification. The multiplicity of representations comes from the freedom of choosing any two crisp acts that are statewise ordered to provide the constant utility of zero and one. However, Theorem 5 extends the partial identification to situations without utility overlap or crisp acts.

This corollary provides an insight into our main identification result Theorem 2. The full identification is only possible if all crisp acts have a constant utility. This is what Axiom 7 guarantees by assuming that for all pairs of crisp acts, one of the crisp acts (weakly) dominates the other one statewise.

3 State Dependent Dual-Self Variational Expected Utility

This section studies a more general setup with a state dependent version of dual-self variational expected utility (Chandrasekher et al., 2020). We characterize its existence and identification. The state-independent version is a general model that includes as special cases monotone mean-variance preferences (Maccheroni et al., 2009), multiplier preferences (Hansen and Sargent, 2001; Strzalecki, 2011), variational preferences (Maccheroni et al., 2006), monotonic mean-dispersion preferences (Grant and Polak, 2013), and vector expected utility (Siniscalchi, 2009). None of these representations is a special case of the previous dual-self expected utility. Our identification results in this section encompass those for state dependent versions of all the above special cases.

Dual-self variational expected utility generalizes variational expected utility. In a variational expected utility, the decision maker has an index of ambiguity attitude $c : \Delta(S) \to \mathbb{R} \cup \{\infty\}$ for each probability and the preferences are represented as with utility $u$

$$
f \succsim g \iff \min_{p \in \Delta(S)} \sum_{s \in S} p_s u(f_s) + c(p) \geq \min_{p \in \Delta(S)} \sum_{s \in S} p_s u(g_s) + c(p).
$$

This corresponds to the maxmin expected utility if $c$ is always 0 or $\infty$. Here the effective domain of $c$ denoted by $\text{dom } c = \{p \in \Delta(S) | c(p) \in \mathbb{R}\}$ captures the set of subjective probabilities that the decision maker uses.
The dual-self variational expected utility generalizes this representation by allowing for multiple cost functions that are aggregated optimistically. Axiomatically, this generalizes variational preferences by relaxing Uncertainty Aversion. We will use a state dependent variation of it.

**Definition** Let \( \succsim \subseteq H \times H \). \((u, C)\) is a state dependent dual-self variational expected utility for \( \succsim \) if \( u = (u_s)_{s \in S} \) and for all \( s \in S \), \( u_s : \Delta(X_s) \to \mathbb{R} \) is affine, \( C \subseteq \{ c : \Delta(S) \to \mathbb{R} \cup \{ \infty \} | c \text{ is convex} \} \) is such that \( \max_{c \in C} \min_{p \in \Delta(S)} c(p) = 0 \) and for all \( c \in C \) and \( p \in \text{dom} \, c, s \notin S^p, p_s = 0 \), and for all \( f, g \in H \)

\[
  f \succsim g \iff \max_{c \in C} \min_{p \in \Delta(S)} \sum_{s \in S} p_s u_s(f_s) + c(p) \geq \max_{c \in C} \min_{p \in \Delta(S)} \sum_{s \in S} p_s u_s(g_s) + c(p).
\]

This representation is the variational expected utility when \( C \) is a singleton. Additionally, since this representation is a non-convex generalization of variational preferences, especially any uncertainty aversion and seeking of the preferences can be represented by the interplay of cost function (min) and the set of cost functions (max).

Similarly to the dual-self expected utility, the dual-self variational expected utility is not unique. However, the smallest convex and closed effective domain of the cost functions is unique which gives us tight dual-self variational expected utility.

**Definition** Let \( \succsim \subseteq H \times H \). \((u, C)\) is a state dependent tight dual-self variational expected utility for \( \succsim \), if \((u, C)\) is a state dependent dual-self variational expected utility for \( \succsim \) and if \((u, \tilde{C})\) is another state dependent dual-self variational expected utility for \( \succsim \), then \( \overline{\text{co}} \bigcup_{c \in C} \text{dom} \, c \supseteq \overline{\text{co}} \bigcup_{c \in \tilde{C}} \text{dom} \, c \).

### 3.1 Existence, Variational

We start off with characterizing the existence of the state dependent dual-self variational expected utility. This characterization highlights the generality of the state dependent dual-self variational expected utility by showing that the only standard ambiguity preferences that do not have this representation are such that, essentially, every act contains different ambiguity. This section uses the axioms from Section 2.3. A reader only interested in the identification can skip this section.

The next result shows that the existence of state dependent dual-self variational expected utility is characterized by the same axioms as state dependent dual-self expected utility with the exception that there might not be crisp acts and Axiom 5 might not be satisfied.
Theorem 7 (Existence, Variational) The following two conditions are equivalent:

1. $\succsim$ satisfies Axioms 1-4, and 6.
2. There exists $(u, C)$ that is a state dependent tight dual-self variational expected utility for $\succsim$.

This result shows that a state dependent dual-self variational expected utility is characterized by relatively weak axioms capturing the dispersional nature of ambiguity without imposing any additional restrictions on it. This representation has multiple special cases as discussed above and this theorem shows the state dependent foundations for these models.

This result highlights the generality of the dual-self variational expected utility. Under the standard assumptions under ambiguity Axioms 1-4, the only requirement for the state dependent dual-self variational expected utility is the existence of two acts that share the same ambiguity and that they are ordered statewise. As discussed in Section 2.3 the statewise ordering is a weak restriction. In other words, essentially the only standard preferences that do not have a state dependent dual-self variational representation are such that all the acts contain different ambiguity.

3.2 Identification, Variational

Next we move on to the identification of the state dependent dual-self variational expected utility. The following identification result shows the main result of this section: Under full dimensional uncertainty, the intensities of preferences and the probabilities can be separated.

Theorem 8 (Full Identification, Variational) Let $(u, C)$ be a state dependent tight dual-self variational expected utility for $\succsim$. The following four conditions are equivalent:

1. $\succsim$ satisfies Axiom 7.
2. $\text{pr}_{S^P} \overline{\text{co}} \bigcup_{c \in C} \text{dom } c$ has a non-empty interior in $\Delta(S^P)$.
3. If $(\tilde{u}, \tilde{C})$ is a state dependent tight dual-self variational expected utility for $\succsim$, then there exist $\alpha \in \mathbb{R}_+$ and $B \in \mathbb{R}^S$ such that for all $s \in S^P$
   \[ \tilde{u}_s = \alpha u_s + B_s. \]
4. If $(\tilde{u}, \tilde{C})$ is a state dependent tight dual-self variational expected utility for $\succsim$, then
   \[ \overline{\text{co}} \bigcup_{\tilde{c} \in \tilde{C}} \text{dom } \tilde{c} = \overline{\text{co}} \bigcup_{c \in C} \text{dom } c. \]
This result states that under Axiom 7 the changes of state dependent utilities are identified. However, the levels are not identified and instead any additive transformations are possible. This follows from the interchangeability of cost functions and additive constants as shown in the next result. Additionally, the convex closure of the tight set of probabilities is uniquely identified as with the state dependent dual-self expected utility. The symmetric result for the state dependent dual-self expected utility Theorem 2 was discussed extensively in Section 2.5 and the same discussion applies here. Especially the non-empty interior condition is a weak restriction for state dependent dual-self variational expected utility and Axiom 7 is not demanding but rules out important special cases.

This theorem shows that under the more general state dependent tight dual-self variational expected utility and Axiom 7, the utility differences between acts are still meaningful. Especially, it is possible to study the utility gains and losses from trading one act for another one. Additionally, states of the world have meaningful probabilities which is crucial for modeling decision making under uncertainty.

Next we move on to the general partial identification. Before stating the partial identification result, we introduce some notation. For all cost functions $c : \Delta(S) \to \mathbb{R} \cup \{\infty\}$ and numbers for each state $x \in \mathbb{R}^S_+$, we define the product $c \circ x : \Delta(S) \to \mathbb{R} \cup \{\infty\}$ as for all $p \in \Delta(S)$, $c \circ x(p) = c(xp)$ where $c(xp) = \infty$ if $xp \notin \Delta(S)$ and the product $xp$ is the elementwise product as in Section 2.5. This induces products of cost sets with $x \in \mathbb{R}^S_+$ as elementwise products. Second, for $B \in \mathbb{R}^S$ define $dB$, the linear cost function (or a measure) associated with $B$, by for all $p \in \Delta(S)$ $dB(p) = \sum_{s \in S} B_s p_s$.  

**Theorem 9 (Partial Identification, Variational)** Let $(u, C)$ be a state dependent tight dual-self variational expected utility for $\succsim$, and $\bar{u} = (\bar{u}_s)_{s \in S}$ be such that for all $s \in S$, $\bar{u}_s : \Delta(X_s) \to \mathbb{R}$ is affine and $\bar{C} \subseteq \{c : \Delta(S) \to \mathbb{R} \cup \{\infty\} | c \text{ is convex} \}$ be such that $\max_{c \in \bar{C}} \min_{p \in \Delta(S)} \bar{c}(p) = 0$. Then $(\bar{u}, \bar{C})$ is a state dependent dual-self variational expected utility for $\succsim$ if and only if there exist $x \in \mathbb{R}^S_+, B \in \mathbb{R}^S$ such that for all $p, q \in \bigcup_{c \in C} \text{dom } c$

$$\sum_{s \in S} x_s p_s = \sum_{s \in S} x_s q_s =: \alpha,$$

for all $s \in S^p$, 

$$\bar{u}_s = \frac{1}{x_s} u_s + B_s,$$
and by denoting $\beta = -\max_{c \in \mathbb{C}} \min_{p \in \Delta(S)} \alpha c(z/\alpha) - \sum_{s \in S} p_s B_s$ for all $f \in H$,
\[
\max_{\tilde{c} \in \tilde{\mathbb{C}}} \min_{p \in \Delta(S)} \sum_{s \in S} p_s \tilde{u}_s(f_s) + \tilde{c}(p) = \max_{c \in \mathbb{C}} \min_{p \in \Delta(S)} \sum_{s \in S} p_s u_s(f_s) + \alpha c(z/\alpha) - \sum_{s \in S} B_s p_s + \beta.
\]
Especially $(\tilde{u}, \alpha(\mathbb{C} \circ z/\alpha) - dB + \beta)$, where multiplication and subtractions are done elementwise for each cost function, is a state dependent tight dual-self variational expected utility for $\succeq$ and if $(\tilde{u}, \tilde{\mathbb{C}})$ is a tight representation, then
\[
\overline{\mathbb{C}} \cup \text{dom} \tilde{c} = \overline{\mathbb{C}} \cup \frac{x}{\alpha} \text{dom} c.
\]

This theorem shows that for any other state dependent dual-self variational representation the values of the representation are identified up to a positive affine transformation. All the additional transformations are the ones that do not affect the representation: For the tight set of probabilities all common statewise multiplicative transformations for which the probabilities still sum to a constant number are allowed when we do the statewise reciprocal transformation for the utilities and scale the probabilities to sum to 1. Additionally, additive transformation for the utilities are allowed when we do the negative of this transformation for all the cost functions and normalize the set of cost functions with $\beta$.

Symmetrical results to Propositions 3 and 4 for the identification of probabilities for a single state or for the identification of relative likelihoods between two states hold under the state dependent dual-self variational expected utility. The only difference is that the additive utility transformations are not identified anymore.

Finally, we connect the state dependent dual-self and dual-self variational expected utilities to each other by linear cost functions.

**Proposition 10 (Connecting Dual-Self and Dual-Self Variational)** The following two conditions are equivalent:

1. There exists $(u, \mathbb{C})$ that is a state dependent dual-self variational expected utility for $\succeq$ and there exists $B \in \mathbb{R}^S$ such that for all $c \in \mathbb{C}, p \in \text{dom}(c), c(p) = \sum_{s \in S} B_s p_s$.

2. There exists $(\tilde{u}, \mathbb{P})$ that is a state dependent dual-self expected utility for $\succeq$.

This proposition shows that state dependent dual-self variational expected utility with a linear cost function correspond to state dependent dual-self expected utility. This gives a new insight into the state dependent dual-self expected utility and into unambiguous acts since these linear
cost functions are underlying the intuition for unambiguous acts.\textsuperscript{19} It is important to contrast this result to the state independent case where only state independent dual-self variational expected utility with cost function that is constant at 0 correspond to state independent dual-self expected utility.

The rest of this section uses the definitions from Section 2.3 in order to provide an insight into the identification through the connection to the underlying preferences. A reader interested only in the identification can skip the rest of this section.

The next result connects the partial identification from Theorem 9 to equally crisp acts.

**Corollary 11** Let \((u, C)\) be a state dependent dual-self variational expected utility for \(\succeq\) and \(x \in \mathbb{R}^S_{++}, B \in \mathbb{R}^S\). The following two conditions are equivalent:

1. \((\tilde{u}, \tilde{C})\) is a state dependent dual-self variational expected utility for \(\succeq\) such that
   \[
   \tilde{u}_s = \frac{1}{x_s}(u_s + B_s) \quad \text{for all } s \in S^P \text{ and } 0, 1 \in \bigcap_{s \in S^P} \tilde{u}_s(\Delta(X_s)).
   \]

2. There exists equally crisp acts \(f^*, g^*\) such that for all \(s \in S^P\), \(f^*_s \succ_s g^*_s\), \(u_s(g^*_s) = -B_s\) and \(u_s(f^*_s) = -B_s + x_s\).

This corollary with Theorem 9 shows the idea of the partial identification. The multiplicity of representations comes from the freedom of choosing any act \(g^*\) to provide the constant utility of zero and choosing \(f^*\) such that \(g^*\) and \(f^*\) are equally crisp and statewise ordered to provide the constant utility of one.

This highlights the difference to the identification of state dependent dual-self expected utility. Corollary 6 showed that the partial identification of dual-self expected utility comes from crisp acts. However, if an act is equally crisp with a crisp act, then both of the acts are crisp acts. So the only difference between these two identifications is that with dual-self expected utility one of the acts has to be crisp, whereas with dual-self variational expected utility there might not exist crisp acts and we are free to choose one of the acts.

Symmetrically to the dual-self expected utility, this corollary provides an insight into the full identification of dual-self variational expected utility, Theorem 8. This identification is only possible if the utility differences between all equally crisp acts are constant. This is what Axiom 7\textsuperscript{19}This highlights how mixing with unambiguous or crisp acts changes ambiguity linearly that is the underlying property in the definition of crisp acts in Section 2.3.
guarantees by assuming that for all pairs of equally crisp acts, one of the acts (weakly) dominates the other one statewise.

4 Applications

This section presents two applications for state dependent utility models beyond uncertainty by changing the interpretations of the primitives of the models. The first application is in social choice theory: We show the identification of fairness of the society and interpersonal utility comparisons using state dependent maxmin expected utility. The second application is in intertemporal choice: We provide general yet simple foundations for evolving tastes over time with state dependent recursive maxmin and recursive dual-self expected utility.

We will use Uncertainty Aversion axiom from Gilboa and Schmeidler (1989).

Axiom 8  For all \( f, g \in H, \alpha \in [0, 1] \), if \( f \succeq g \), then \( \alpha f + (1 - \alpha)g \succeq g \).

First as a notation, a state dependent maxmin expected utility is a state dependent dual-self expected utility with a single probability set \((u, \{P\})\) and we omit the inner brackets in the following.

4.1 Social Choice Theory

We start off with an application to social choice theory. This application studies preferences of the society over distributions of goods to members of the society. Our purpose is to identify the fairness of the societal preferences and interpersonal utility comparisons across the members of the society.

We will interpret states of the world to be the members of the society, acts to be distributions of goods to each member, and \( \succeq \) to represent the preferences of the society. In this context, ambiguity reflects the fairness of the distribution. We focus on a state dependent maxmin expected utility \((u, P)\) with a symmetric set of probabilities that is for all permutations of states \(\pi \) if \((p_s)_{s \in S} \in P\), then \((p_{\pi(s)})_{s \in S} \in P\). This symmetry property captures anonymity of the societal preferences.

In the current context, this symmetric state dependent maxmin expected utility is a generalized Rawlsian social welfare function. Here, the set of probabilities captures the fairness of the society. The least unfairness aversive generalized Rawlsian function is the utilitarian social welfare function

\(\pi : S \to S\) is a permutation if it is a one-to-one (i.e. bijective) function.

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which is the average utility. This corresponds to a state dependent expected utility with a uniform probability. The most unfairness averse generalized Rawlsian function is the Rawlsian social welfare function which is the minimum utility. This corresponds to state dependent maxmin expected utility with a full set of probabilities $\Delta(S)$.

The main result of this section characterizes the existence of a generalized Rawlsian social welfare function and its identification. Especially we show that the utilitarian social welfare function is an unidentified knife-edge case of a generalized Rawlsian social welfare function: All representations except this knife-edge case are fully identified. First, we interpret shortly the axioms from Section 2.3 in the current context and introduce a new anonymity axiom.

The society has continuous preferences (Axioms 1 and 2) and is benevolent (Axiom 3) caring about the well-being of each person that is the society’s preferences respect individual Pareto improvements. Additionally, the society consists of rational expected utility maximizers (Axiom 4) and cares about the fairness of the distribution (Axiom 8).\footnote{This assumption can be also interpreted directly as an aversion to ambiguity under a Rawlsian interpretation: The preferences of the society are determined by a person behind the veil of ignorance. However, the person deciding the preferences is uncertain about their identity in the society. This uncertainty about the identity gives a rational for the uncertainty aversion.}

Axiom 5 assumes that there exists a fair distribution that does not hedge the fairness of any other distribution. Axiom 6 assumes that the unfairness is over the variation of well-being across the society. That is fairness depends only on what each person receives relative to others.

Finally we have a new anonymity axiom. This axiom states that there exists a better and a worse fair distributions such that the society is indifferent on the identity of the members who receive the better or the worse fair distribution.

**Axiom 9** There exists crisp acts $f^*, g^*$ such that $f^* \succ g^*$ and for all $\gamma \in \Delta(S)$ and permutations $\pi : S \rightarrow S$

$$
(\gamma_s f^*_s + (1 - \gamma_s) g^*_s)_{s \in S} \sim (\gamma_{\pi(s)} f^*_s + (1 - \gamma_{\pi(s)}) g^*_s)_{s \in S}.
$$

In this axiom $\gamma$ is a lottery across the members of the society and one of the members is awarded with the better fair consequence $f^*_s$ and everybody else receives the worse fair consequence $g^*_s$. That is $\gamma$ captures the distribution of the better and worse consequences in the society. Then the identity of the winning member does not matter for societal preferences but only the distribution of the consequences matters.
These axioms characterize the generalized Rawlsian social welfare function. That is the symmetric state dependent maxmin expected utility.

**Theorem 12 (Symmetric State Dependent MaxMin)** $\succeq$ satisfies Axioms 1-6, 8, and 9 if and only if (1) or (2) of the following conditions holds:

1. For all $p \in \text{int} \Delta(S)$, there exists $u = (u_s)_{s \in S}$ such that for all $s \in S$, $u_s : \Delta(X_s) \rightarrow \mathbb{R}$ is affine and $\text{int} u_s(\Delta(X_s)) \neq \emptyset$ and for all $f, g \in H$

   $$f \succeq g \iff \sum_{s \in S} p_s u_s(f_s) \geq \sum_{s \in S} p_s u_s(g_s).$$

2. There exists $(u, P)$ that is a state dependent maxmin expected utility for $\succeq$ such that $\text{int} P \neq \emptyset$, for all $p \in P$, permutations $\pi : S \rightarrow S$, $(p_{\pi(s)})_{s \in S} \in P$, and

   $$\text{int} \bigcap_{s \in S} u_s(\Delta(X_s)) \neq \emptyset.$$

Additionally, if (2) holds and $(\tilde{u}, \tilde{P})$ is another state dependent maxmin expected utility for $\succeq$, then there exists $\alpha > 0, \beta \in \mathbb{R}$ such that

$$\tilde{P} = P \text{ and } \tilde{u}_s = \alpha u_s + \beta \text{ for all } s \in S.$$  

First, this result highlights that utilitarianism is a knife-edge case of generalized Rawlsian social welfare functions where identification is not possible. There are only two possibilities: Either the preferences have a fully non-identified utilitarian representation or a fully identified generalized Rawlsian social welfare function with a non-singleton set of probabilities. In the ambiguity context, this result also highlights how state dependent expected utility is a non-identified knife-edge case of state dependent maxmin expected utility.

Second, this result provides the identification for the fairness of the society and interpersonal utility comparisons when the societal preferences are not utilitarian. The identification of fairness allows us to compare the fairness of different societies to each other independently of the members that form the society.\footnote{Under the Rawlsian interpretation the weights that society gives to its members are the probability assessments of the person behind the veil of ignorance for their identity.} Additionally, the identification of interpersonal utilities allows us to do comparisons within the society and compare the welfare of its members. As an important special case this result shows that the Rawlsian social welfare function is well identified if there is utility overlap.
Third, Theorem 2 generalizes this identification to non-symmetric generalized Rawlsian social welfare functions. In this context, the full dimensionality axiom, Axiom 7, assumes that any redistribution that involves redistributing from one person to another makes the distribution of well-being less fair in some situation. Essentially this assumes that starting from a fair (crisp) distribution, redistribution decreases the fairness of the distribution. Under this full dimensionality condition, we achieve the same identification of the fairness of the society and interpersonal utilities as above in the more general case without anonymity of the social welfare function. Especially, we are able to identify if the social welfare function is anonymous based on Axiom 9.

**Remark** Applying state dependent dual-self variational expected utility in this context would give the identification of fairness of the society independently of the members and the identification of interpersonal utility gain comparisons. This is an useful identification for example in the context of redistribution. In studying redistribution the focus is on people who gain and who lose from the redistribution but not necessarily on the absolute levels of utilities that are not identified here.

### 4.2 Intertemporal Choice

We move on to our second application in intertemporal choice. This application provides the foundations for evolving tastes over time with a recursive representation. We study a recursive version of state dependent maxmin expected utility that is characterized by pessimistic uncertainty about the discount factor.

This application models decision maker’s choices over future consumption streams. In this context states of the world are interpreted as time periods, $S = \mathcal{T} = \{0, \ldots, T\}$ where $T \in \mathbb{N}$, acts are consumption streams, and ambiguity of an act reflects a preference for consumption smoothing across time. For clarity, we assume that for each $t \in \mathcal{T}$ there are at least two different consequences, $|X_t| \geq 2$.

First, we define a state dependent recursive maxmin expected utility.

**Definition** Let $\succsim \subseteq H \times H$. $(u, (\delta_1^t, \delta_2^t)_{t \in \mathcal{T}})$ is a state dependent recursive maxmin expected utility for $\succsim$ if $u = (u_t)_{t \in \mathcal{T}}$, for all $t \in \mathcal{T}$, $u_t : \Delta(X_t) \to \mathbb{R}$ is affine, for all $t \in \mathcal{T}$, $0 \leq \delta_1^t \leq \delta_2^t \leq 1$ and by defining for all $t \in \mathcal{T}$, $V_t : H \to \mathbb{R}$ recursively as for all $f \in H$

$$V_t(f) = \begin{cases} \min_{\delta \in [\delta_1^t, \delta_2^t]} (1 - \delta)u_t(f_t) + \delta V_{t+1}(f), & \text{if } t \leq T - 1, \\ u_T(f_T), & \text{if } t = T, \end{cases}$$
the recursive solution $V_0$ represents $\succeq$: for all $f, g \in H$

$$f \succeq g \iff V_0(f) \geq V_0(g).$$

This representation is a state dependent variation of Wakai’s (2008) recursive maxmin expected utility and is discussed extensively in there. In this representation each $V_t$ captures the conditional preferences at time $t$ over current and future consumption and corresponds to the average future utility at time $t - 1$. The decision maker has uncertainty about the single-period discount factors and considers the interval $[\delta^1_t, \delta^2_t]$ possible. This uncertainty gives the preference for consumption smoothing. The uncertainty can reflect multiple sources such as uncertainty about the survival rate to the next period or uncertainty about time preferences.

Next we will provide an axiomatic foundation and identification for this representation. We begin with shortly interpreting the axioms from Section 2.3 in the current context and simplifying the identification axiom.

The decision maker is assumed to be within each time period an expected utility maximizer (Axiom 4), have continuous and monotonic preferences over future consumption streams (Axioms 1-3), and have a preference for smoothing consumption across time (Axiom 8). Additionally, the consumption smoothing is over the dispersion of consumption across time (Axiom 6).

Next we assume a slightly stronger version of Axiom 5.

**Axiom 5’** There exist a crisp act $c \in H$ and an act $f \in H$ such that for all $s \in S^P$, $c_s \succ_s f_s$ or for all $s \in S^P$, $c_s \prec_s f_s$.

This axiom has two parts. First there exists a smooth consumption stream that does not hedge the variance in other consumption streams. The second part is a weak simplification that the smooth consumption stream is not the worst consequence in some state and the best consequence in some other state.

Next, we have a new recursivity axiom. The key to the recursive formulation of preferences is the history independence of consumption. We assume a weak version of this. First we define some notation. For acts $a, f \in H$ and $t \in T$, we denote by $(a_{\{0,...,t-1\}}, f_{\{t,...,T\}})$ an act that gives in periods $0 \leq t' \leq t - 1$ consumption $a_{t'}$ and in periods $t \leq t' \leq T$ consumption $f_{t'}$.

**Axiom 10** For all $t \in T, f, g, a, b \in H$,

$$(a_{\{0,...,t-1\}}, f_{\{t,...,T\}}) \succ (a_{\{0,...,t-1\}}, g_{\{t,...,T\}}) \Rightarrow (b_{\{0,...,t-1\}}, f_{\{t,...,T\}}) \succeq (b_{\{0,...,t-1\}}, g_{\{t,...,T\}}).$$
This axiom states that common histories of consumption do not create strict preference reversals. However, after some histories future might be considered impossible which creates weak preference reversals.

Finally the full dimensionality axiom, Axiom 7, simplifies to stating that there is uncertainty between today and tomorrow at each time period.

**Axiom 7’** For all $t \leq T - 1$, if $f,g \in H$ are such that $f \sim g$, for all $\tilde{t} \notin \{t, t + 1\} f_{\tilde{t}} = g_{\tilde{t}}$, and $f_t \succ g_t$ and $g_{t+1} \succ f_{t+1}$, then there exist $h \in H$ and $\alpha \in (0, 1)$ such that

$$\alpha h + (1 - \alpha)f \not\sim \alpha h + (1 - \alpha)g.$$ 

The next result shows that the previous axioms characterize the state dependent recursive maxmin expected utility. Additionally if there exists uncertainty about each discount factor and Axiom 7’ is satisfied, the time dependent utilities and the discount factors are identified.

**Theorem 13 (State Dependent Recursive MaxMin)** The following two conditions are equivalent:

1. $\succsim$ satisfies Axioms 1-4, 5’, 6, 8, and 10.
2. There exists $(u, (\delta^1_t, \delta^2_t)_{t \in T})$ that is a state dependent recursive maxmin expected utility for $\succsim$ such that

$$\text{int} \bigcap_{t \in T} u_t \left( \Delta(X_t) \right) \neq \emptyset.$$ 

Additionally under the above conditions and if $S = S^P$, the following three conditions are equivalent:

1. $\succsim$ satisfies Axiom 7’.
2. For all $0 \leq t \leq T - 1$, $\delta^1_t \neq \delta^2_t$.
3. If $(\tilde{u}, (\tilde{\delta}^1_t, \tilde{\delta}^2_t)_{t \in T})$ is another state dependent recursive maxmin expected utility for $\succsim$, then there exist $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$ such that for all $t \in T$

$$u'_t = \alpha u_t + \beta$$

and for all $0 \leq t \leq T - 1$

$$\tilde{\delta}^1_t = \delta^1_t \text{ and } \tilde{\delta}^2_t = \delta^2_t.$$ 

First, this theorem provides foundations for evolving tastes over time. The first part characterizes the state dependent recursive maxmin expected utility. Additionally under Axiom 7’, time
dependent utilities and discount factors are identified. This identification is characterized by uncertainty about the discount factor at each time period. As discussed above, this uncertainty about discount factor can reflect uncertainty over multiple sources that affect the value of future utility and is not a restrictive assumption. However, it rules out important special cases such as exponential discounting.

Second, the first part of this theorem with the existence of a dual-self expected utility, Theorem 1, shows that the recursivity of the state dependent recursive maxmin expected utility is characterized by weak history independence, Axiom 10.

**Remark 3** Unlike previous identification results, Theorem 13 only provide identification under Axiom 5'. This simplification allows us to discuss about the identification without considering the tightness of the representation since under Axiom 5' all representations are tight. Without this assumption identification would include technical conditions that we have simplified for clarity.

**5 Discussion on Identification**

The example in Section 1.1 showed how the identification in our models comes from attributing the changes in uncertainty attitude to changes in the probabilities. Next, we formalize this underlying assumption and consider state dependent Bewley preferences that highlights the differences between state dependent and state independent identifications.

First, we consider the state dependent dual-self expected utility. Here, our identification reflects the observation that acts which are behaviorally revealed to be unambiguous, as crisp acts, have no uncertainty in the representation. Especially, under full-dimensional uncertainty, they must have a constant utility. The underlying assumption here is that the decision maker considers some acts as unambiguous: The condition that an act is behaviorally revealed to be unambiguous as a crisp act is only a necessary condition for the decision maker to consider it as unambiguous. As illustrated in Proposition 10, there exist alternative dual-self variational representations for these preferences with linear costs functions where these revealed unambiguous acts are not unambiguous in the representation.

Second, with state dependent dual-self variational expected utility, we have a similar underlying assumption. Our identification reflects the observation that acts which are behaviorally revealed

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23 Formally, this is shown in Corollary 6.
to share the same ambiguity, as equally crisp acts, are equally ambiguous in the representation.\footnote{Formally, this is shown in Corollary 11.} In this case, our underlying assumption is that the decision maker considers some acts as sharing the same ambiguity. Without this underlying assumption, these preferences have more general representations where equally ambiguous acts do not share the same ambiguity and in this case the probabilities and intensities of utilities would not be separated and identified anymore.

These underlying assumptions have a counterpart in the state independent representations. In the state independent case, the underlying assumption is that constant acts, giving the same consequence in every state, are unambiguous for every decision maker (Ghirardato and Marinacci, 2002). This gives the identification since the ambiguity of other acts can be measured relative to these known unambiguous constant acts. Our underlying assumptions relax this.

Finally, consider the possibility that the decision maker has also uncertainty about tastes. Then the above example’s change in uncertainty attitude could also be attributed to changes in tastes weakening the identification. In this case, crisp acts would not necessarily be unambiguous since they could also reflect certainty about tastes or utility for these acts. Karni (2020) has recently illustrated this lack of identification with rank dependent probabilities when state dependent utilities are also rank dependent.

### 5.1 State Dependent Bewley Expected Utility

With state independent utility, the decision maker’s uncertainty about states can be inferred from the Independence satisfying core of the preferences. These are incomplete preferences that have a Bewley representation. However, this does not extend to state dependent setting as we show next with state dependent Bewley representation.

**Definition** \((u, P)\) is a state dependent Bewley expected utility for \(\succsim\) if \(u = (u_s)_{s \in S}\) and for all \(s \in S, u_s : \Delta(X_s) \to \mathbb{R}\) is affine and \(P \subseteq \Delta(S)\) is closed, convex and, non-empty such that for each \(p \in P, s \notin S^P, p_s = 0\) and for all \(f, g \in H,\)

\[
f \succsim g \iff \sum_{s \in S} p_s u_s(f_s) \geq \sum_{s \in S} p_s u_s(g_s) \text{ for all } p \in P.
\]

The next result shows that state dependent Bewley preferences are not identified.

**Theorem 14 (Identification, Bewley)** Let \((u, P)\) be a state dependent Bewley representation for \(\succsim;\) \(\tilde{u} = (\tilde{u}_s)_{s \in S}\) be such that for all \(s \in S, \tilde{u}_s : \Delta(X_s) \to \mathbb{R}\) is affine, and \(\tilde{P} \subseteq \Delta(S)\) be closed,
convex and nonempty. Then $(\tilde{u}, \tilde{P})$ is a state dependent Bewley representation for $\succeq$ if and only if there exists $a \in \mathbb{R}_+^S$, $b \in \mathbb{R}^S$ such that

$$
\tilde{P} = \left\{ \left( \frac{a_s^{-1} p_s}{\sum_{s' \in S} a_{s'}^{-1} p_{s'}} \right) \left| p \in P \right. \right\}
$$

and for all $s \in S^P$,

$$
\tilde{u}_s = a_s u_s + b_s.
$$

In Appendix, Section 8.4, we also axiomatizally characterize the existence of this representation.

This result highlights the difference in identifications between state dependent and state independent representations. With state dependent utilities, the largest independence satisfying core gives the extent of ambiguity as given by the size of the set of probabilities and allows us to consider if the ambiguity is correlated across states. However, it does not allow us to compare the extent of ambiguity across states. That is we cannot compare the amount of uncertainty, that is the probabilities, across states. In the state independent case, this is avoided since we can compare uncertainty of acts relative to the known unambiguous constant acts. Instead, in a state dependent case, we need to infer from the decision maker’s behavior which acts are unambiguous that provides the basis for uncertainty comparisons. But these unambiguous acts are not revealed in the Independence satisfying core of the preferences.

### 6 Related Literature

State independent dual-self and dual-self variational expected utilities were introduced in Chandrasekher et al. (2020) building on Ghirardato et al.’s (2004) approach of using Clarke derivatives to capture the beliefs of the decision maker. These models are non-convex generalizations of multiple prior preferences (Gilboa and Schmeidler, 1989) and variational preferences (Maccheroni et al., 2006).

The previous literature on state dependent utility has mainly focused on axiomatizing the state dependent expected utility using some additional information: Karni (2007) assumes preferences on conditional acts which are acts conditional on a given event happening. Karni (2011a; 2011b) assumes preferences conditional on signals and uses updating of probabilities for identification. Karni et al. (1983) and Karni and Schmeidler (2016) assume preferences on hypothetical lotteries that are lotteries on state-consequence pairs. Lu (2019) achieves uniqueness for the utilities up to a
common positive multiplication and addition of any state specific constants by using two different random choice data based on updated random beliefs.

Chew and Wang (2020) exemplifies that state and rank dependent expected utility can be identified under two states of the world. State and rank dependent expected utility is a special case of state dependent dual-self expected utility. Karni (2020) shows that in state and rank dependent expected utility with rank-dependent probabilities the utilities and probabilities are not identified.

Drèze (1958; 1961; 1987; 2004) studies state dependent maxmax expected utility when the acts are lotteries of Anscombe-Aumann acts in the context of moral hazard. He characterizes the existence and the uniqueness of the state dependent maxmax expected utility when the intersection of utilities has a non-empty interior. In contrast, we characterize the uniqueness of any state dependent dual-self expected utility in the standard state dependent Anscombe-Aumann setting and characterize behaviorally when the utilities and the set of probabilities are fully or partially identified. Baccelli (2019) discusses Drèze’s contribution extensively.

Hill (2019) studies state dependent maxmin expected utility such that the best and the worst acts have a constant utility in the state independent Anscombe-Aumann setting without Risk Independence. Additionally, he focuses on representations with interior probabilities. He assumes that the best and the worst acts are crisp or unambiguous acts. In contrast under Risk Independence, we show the full generality of the dual-self and the maxmin expected utilities as assuming that there exists some act that is unambiguous and two acts that share the same ambiguity. Hill (2019) shows the identification of the representation when the best and the worst acts have a constant utility and the representation is linear between these best and worst act. Under these restrictions, we have the standard state independent identification. In contrast under Risk Independence, we show the full identification of the representation among any state dependent dual-self representations which were ruled out by the above restriction.

Wakai (2008) studies state independent intertemporal consumption with recursive maxmin preferences with finite and countable number of time periods. However, in this model primitives are preferences for each time period over current and future consumption. Our axiomatization and history independence axiom is closer to Montiel Olea and Strzalecki’s (2014) approach.

25This additional property is used in the proof but not stated in the theorem.
7 Conclusion

The assumption of state independent utilities has been the simplifying, but non-ideal, assumption in order to separate the subjective probabilities from the utilities. This paper provided the a novel foundation for state dependent utility by studying models of non-expected utilities. We showed that with a state dependent version of general dual-self expected utility (non-convex multiple priors preferences, Chandrasekher et al., 2020) state dependent utilities and probabilities can be identified. Additionally, with a state dependent version of more general dual-self variational expected utility (non-convex variational preferences, Chandrasekher et al., 2020) the intensities of preferences and probabilities can be separated; however the levels of utilities cannot be identified. These identification are characterized by full dimensional set of probabilities.

These identifications encompass those for state dependent versions of all the special cases and alternative representations of dual-self and dual-self variational expected utilities: maxmin expected utility (multiple prior preferences, Gilboa and Schmeidler, 1989), Choquet expected utility (Schmeidler, 1989), $\alpha$-maxmin expected utility (Ghirardato et al., 2004), invariant biseparable preferences (Ghirardato et al., 2004; Amarante, 2009), monotone mean-variance preferences (Maccheroni et al., 2009), multiplier preferences (Hansen and Sargent, 2001; Strzalecki, 2011), variational preferences (Maccheroni et al., 2006), monotonic mean-dispersion preferences (Grant and Polak, 2013), and vector expected utility (Siniscalchi, 2009).

We provided new characterizations for state dependent dual-self and dual-self variational expected utilities. First we generalized standard preferences under ambiguity to state dependent setting. Under these preferences, state dependent dual-self and dual-self variational expected utilities are characterized by assumptions on the structure of ambiguity: dual-self variational was characterized by an axiom stating that each act does not contain a different source of ambiguity but instead there exists two acts that are statewise ranked and share exactly the same sources of ambiguity. This was operationalized as the two acts hedging ambiguity similarly. Dual-self expected utility was characterized by an additional assumption stating that there exists a least ambiguous act that shares the sources of ambiguity with all the other acts. This was operationalized as the least ambiguous act not hedging ambiguity.

Finally we showed that these identifications have applications beyond ambiguity. First we applied it to social choice theory. There we provided the identification of fairness of the society
and for interpersonal utility comparisons and showed that a utilitarian social welfare function is a non-identified knife-edge case of Rawlsian type social welfare functions. Second we applied it to intertemporal choice and provided general yet simple foundations for evolving tastes with recursive preferences.
8 Appendix:

8.1 General Identification

This section shows the general identification results for any aggregator function $I : u(H) \to \mathbb{R}$. For this section we assume that $S = S^P$. The results extend to situations where not all states are proper and utilities are not identified on these states.

For $\psi, \phi \in \mathbb{R}^S$ we denote $\phi \cdot \psi = \sum_{s \in S} \phi_s \psi_s$. Let $A \subseteq \mathbb{R}^S$ be a convex set and $I : A \to \mathbb{R}$ be a function. We denote $\bar{1} \in \mathbb{R}^S$ such that for all $s \in S \bar{1}_s = 1$. We say that

- $I$ is C-additive if for all $\phi \in A, \alpha \geq 0$ such that $\phi + \alpha \bar{1} \in A$, $I(\phi + \alpha \bar{1}) = I(\phi) + \alpha$.
- $I$ is positive homogeneous if for all $\phi \in A, \alpha > 0$ such that $\alpha \phi \in A$, $I(\alpha \phi) = \alpha I(\phi)$.
- $I$ is monotonic if for all $\phi, \psi \in A$ such that for all $s \in S \phi_s \geq \psi_s$, $I(\phi) \geq I(\psi)$.

Let $A \subseteq \mathbb{R}^S$ be a convex set. For every $\phi \in \text{int} A, \xi \in \mathbb{R}^S$ the Clarke upper derivative of $I$ at $\phi$ in the direction $\xi$ is

$$I^\circ(\phi; \xi) = \limsup_{\psi \to \phi, t \downarrow 0} \frac{I(\psi + t\xi) - I(\psi)}{t}$$

and the Clarke lower derivative of $I$ at $\phi$ in the direction $\xi$ is

$$I^c(\phi; \xi) = \liminf_{\psi \to \phi, t \downarrow 0} \frac{I(\psi + t\xi) - I(\psi)}{t}.$$

The Clarke subdifferential of $I$ at $\phi$ is the set

$$\partial I(\phi) = \{ \chi \in \mathbb{R}^S | \forall \xi \in \mathbb{R}^S, \chi \cdot \xi \leq I^\circ(\phi; \xi) \}.$$

The following results show representation independent identification results.

**Proposition 15** Let $(u, I)$ and $(\tilde{u}, \tilde{I})$ be state dependent monotonic and C-additive representations for $\succsim$. Then there exist $x \in \left( \bigcup_{\phi \in \text{int} u(H)} \partial I(\phi) \right) \perp, \alpha > 0, B \in \mathbb{R}^S, \beta \in \mathbb{R}$ such that for all $f \in H$ such that $u(f) \in \text{int} u(H)$

$$\partial \tilde{I}(u(f)) = (1 + x) \partial I(\tilde{u}(f)),$$
especially
\[
\bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) = (1 + x) \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi),
\]
for all \( s \in S \)
\[
\tilde{u}_s = \frac{\alpha}{1 + x_s} u_s + B_s,
\]
and for all \( f \in H \)
\[
\tilde{I}(\tilde{u}(f)) = \alpha I(u(f)) + \beta.
\]

**Proposition 16** Let \((u, I)\) and \((\tilde{u}, \tilde{I})\) be state dependent monotonic, C-additive, and positive homogeneous representations for \(\succsim\). Then exists \( x, y \in \left( \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) \right)^\perp, \alpha > 0, \beta \in \mathbb{R} \) such that for all \( f \in H \) such that \( u(f) \in \text{int } u(H) \)
\[
\partial \tilde{I}(u(f)) = (1 + x)\partial I(\tilde{u}(f)),
\]
especially
\[
\bigcup_{\varphi \in \text{int } \tilde{u}(H)} \partial \tilde{I}(\varphi) = (1 + x) \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi),
\]
for all \( s \in S \)
\[
\tilde{u}_s = \frac{\alpha}{1 + x_s} (u_s + y_s) + \beta,
\]
and for all \( f \in H \)
\[
\tilde{I}(\tilde{u}(f)) = \alpha I(u(f)) + \beta.
\]

**Proposition 17** Let \((u, I)\) be a state dependent monotonic, C-additive representation for \(\succsim\). The following conditions are equivalent:

1. \(\succsim\) satisfies Axiom 7.
2. \( \left( \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) \right)^\perp = \emptyset. \)
3. \( \overline{\text{co}} \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) \) has a non-empty interior in \( \Delta(S) \).

**Proposition 18** Let \((u, I)\) be a state dependent monotonic, C-additive representation for \(\succsim\) and \( s \in S \). The following conditions are equivalent:

1. If \( f, g \in H \) are such that \( f \sim g, f_s \succ_s g_s \), and there exist \( s' \in S \) such that \( g_s' \succ s' f_s' \), then there exist \( h \in H \) and \( \alpha \in (0, 1) \) such that
\[
\alpha h + (1 - \alpha) f \not\sim \alpha h + (1 - \alpha) g.
\]
2. \( \text{pr}_s \left( \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) \right)^\perp = \{0\} \).
Proposition 19 Let $(u,I)$ be a state dependent monotonic, C-additive representation for $\succsim$ and $s,s' \in S$. The following conditions are equivalent:

1. If $f,g \in H$ are such that $f_s \succsim s g_s$ and $g_{s'} \succsim s' f_{s'}$, then there exist $h,h' \in H$ and $\alpha \in (0,1)$ such that
   \[ \alpha h + (1-\alpha)f \succsim \alpha h' + (1-\alpha)g. \]

2. Either
   \[ \text{pr}_{s,s'} \left( \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \right) = \{(0,0)\} \text{ or } \text{pr}_{s,s'} \left( \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \right) = \{(a,a) | a \in \mathbb{R}\}. \]

3. There exists $p,q \in \overline{\cup_{\varphi \in \text{int} u(H)} \partial I(\varphi)}$ such that $\frac{p_s}{1-p_s} \neq \frac{q_s}{1-q_s}$ and for all $\tilde{s} \in S \setminus \{s,s'\}$, $p_{\tilde{s}} = q_{\tilde{s}}$.

8.2 Proofs for the Representations

For acts $f,g \in H$, $\alpha \in [0,1]$, we will denote $f \alpha g := \alpha f + (1-\alpha)g$.

8.2.1 Weak Cardinal Representation

For this section we assume that $X$ is a nonempty set, $\Delta(X)$ is the set of all (simple) lotteries on $X$, $H = \Delta(X)$, and $\succsim \subseteq H \times H$. This section studies the following weak independence assumption.

Axiom 4′ For all $f,g,h \in H$, $\alpha \in (0,1)$,

\[ f \succsim g \implies \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h. \]

We will show that under this axiom with the usual completeness, transitivity and continuity assumption, $\succsim$ has a weak representation with an affine function.

Definition $u$ is weak representation for $\succsim$ if $u : \Delta(X) \to \mathbb{R}$ and for all $x,y \in \Delta(X)$

\[ x \succsim y \implies u(x) > u(y). \]

Lemma 20 Assume that $\succsim$ satisfies Axioms 1,2. If $x \succsim y$ then exists $\alpha^1, \alpha^2 \in (0,1)$ such that

\[ x \succsim x \alpha^1 y, x \alpha^2 y \succsim y. \]
and for all $\alpha \in (\alpha^1, 1], \alpha' \in [0, \alpha^2)$

$$x_0y \succ x_0^1y \text{ and } x_0^2y \succ x_0'y.$$ \hfill \Box

**Proof.** Let $x \succ y$. By Axiom 2 the set

$$\{\alpha \in [0, 1]| x \succ \alpha x + (1-\alpha)y\} \text{ is open and } \{\alpha \in [0, 1]| y \succ \alpha x + (1-\alpha)y\} \text{ is closed.}$$

Thus exists $\alpha^0 \in (0, 1)$ such that $\alpha^0 \in \{\alpha \in [0, 1]| x \succ \alpha x + (1-\alpha)y\}$ and $\alpha^0 \notin \{\alpha \in [0, 1]| y \succ \alpha x + (1-\alpha)y\}$. Thus $x \succ x_0^0y \succ y$. Next let

$$\alpha^1 = \max\{\alpha \in [0, 1]| (x_0^0y) \succ \alpha x + (1-\alpha)(x_0^0y)\}, \alpha^2 = \max\{\alpha \in [0, 1]| \alpha y + (1-\alpha)(x_0^0y) \succ (x_0^0y)\}$$

Now define $\alpha^3 = \alpha^1 + (1-\alpha^1)\alpha^0$ and $\alpha^4 = (1-\alpha^2)\alpha^0$. Then we have

$$x_0^0y \sim x_0^3y \sim x_0^4y$$

and for all $\alpha \in (\alpha^3, 1), \alpha' \in (0, \alpha^4)$

$$x_0y \succ x_0^3y \text{ and } x_0^4y \succ x_0'y.$$

\hfill \Box

**Lemma 21** Assume that $\succsim$ satisfies Axioms 1,2,4'. If $x \sim y$ then for all $\alpha \in (0, 1) x \sim \alpha x + (1-\alpha)y$.

**Proof.** Assume per contra exists $x, y \in \Delta(X), \alpha^0 \in (0, 1)$ such that $x \sim y$ and $x \nsuccsim \alpha^0 x + (1-\alpha^0)y$. Assume w.l.o.g. $\alpha^0 x + (1-\alpha^0)y \succ x$. By Lemma 20 exists $\alpha^1 \in (\alpha^0, 1)$ such that for all $\alpha \in (\alpha^1, 1)$

$$x_0^1y \succ x_0y. \hspace{1cm} (1)$$

Now $x_0^0y \succ y$ and thus by Weak Independence

$$x_0^1y \nsuccsim x_0^1(x_0^0y) \equiv x\left(\alpha^1 + (1-\alpha^1)\alpha^0\right)y \nsuccsim x_0^1y,$$

which contradicts transitivity. \hfill \Box

**Lemma 22** Assume that $\succsim$ satisfies Axioms 1,2,4'. If $x \succ z, y \succ z$ then for all $\alpha \in (0, 1) \alpha x + (1-\alpha)y \succ z$. If $z \succ x, z \succ y$ then for all $\alpha \in (0, 1) z \succ \alpha x + (1-\alpha)y$.

**Proof.** Assume per contra exists $x, y, z \in \Delta(X), \alpha^0 \in (0, 1)$ such that $x \nsuccsim z, y \nsuccsim z$ and $\alpha^0 x + (1-\alpha^0)y \nsuccsim z$. By Lemma 20 exists $\alpha^1 \in (\alpha^0, 1)$ such that for all $\alpha \in (\alpha^1, 1)$

$$x_0y \nsuccsim x_0^1y. \hspace{1cm} (2)$$
Now \( y \succ x^0 y \) and thus by Weak Independence
\[
x^1 y \succeq x^1 (x^0 y) \equiv x (\alpha^1 + (1 - \alpha^1) \alpha^0) y \overset{(2)}{\succ} x^1 y,
\]
which contradicts transitivity. The other case follows symmetrically. \( \square \)

**Proposition 23** Assume that \( \succeq \) satisfies Axioms 1, 2, 4'. Then exists \( u : \Delta(X) \to \mathbb{R} \) that is affine and a weak representation for \( \succeq \).

**Proof.** By nontriviality exists \( x^*, x_* \in \Delta(X) \) such that \( x^* \succ x_* \). By applying Lemma 20 twice exists \( \alpha^*, \alpha_* \in (0, 1) \) such that \( \alpha^* > \alpha_* \) and for all \( \alpha \in (\alpha^*, 1], \alpha' \in [0, \alpha_*) \)
\[
x^* \alpha x_* \succ x^* \alpha^* x_* \succ x^* \alpha_* x_* \succ x^* \alpha' x_*.
\]
Let us next define the representation. Let us consider cases. If \( x \succ x^* \alpha^* x_* \),
\[
\alpha^x = \min \{ \alpha \in [0, 1] | \alpha x + (1 - \alpha)(x^* \alpha_* x_*) \succ x^* \alpha^* x_* \}
\]
and define
\[
    u(x) := \frac{1}{\alpha^x}.
\]
Now especially \( u(x) > 1 \). If \( x^* \alpha_* x_* \succ x \),
\[
\alpha^x = \max \{ \alpha \in [0, 1] | x^* \alpha_* x_* \succeq \alpha(x^* \alpha^* x_*) + (1 - \alpha)x \}
\]
and define
\[
    u(x) := \frac{-\alpha^x}{1 - \alpha^x}.
\]
Now especially \( u(x) < -1 \). If \( x^* \alpha^* x_* \succeq x \succeq x^* \alpha_* x_* \),
\[
\alpha^x = \max \{ \alpha \in [0, 1] | x^* \alpha^* x_* \succeq \alpha x^* + (1 - \alpha)x \}
\]
and define
\[
    u(x) := \frac{1 - u(x^*) \alpha^x}{1 - \alpha^x}.
\]
Now especially \( 1 \leq u(x) \leq -1 \). Let us show that \( u \) is a weak representation for \( \succeq \). Let \( x, y \in \Delta(X) \) be such that \( u(x) \geq u(y) \). Assume per contra \( y \succ x \). If \( y \succ x^* \alpha^* x_* \succeq x \) or \( y \succeq x^* \alpha_* x_* \succ x \), then \( u(x) > u(y) \). So we have three cases to consider. 1) \( y \succ x \succ x^* \alpha^* x_* \). Now exists \( \alpha^0 \in (0, 1) \) such that \( y \alpha^0 (x^* \alpha_* x_*) \succ x \). Then by Weak Risk Independence and Lemma 21 for all \( \alpha \in (0, \alpha^x) \)
\[
\alpha(x^* \alpha_* x_*) + (1 - \alpha) y (\alpha^0 (x^* \alpha_* x_*)) \succeq \alpha(x^* \alpha_* x_*) + (1 - \alpha)x \succ x^* \alpha^* x_*.
\]

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Then the linearity follows from the definition of $u(x) \geq u(y)$. 2) $x^*\alpha^*x_+ \succ y \succ x$ follows symmetrically since exists $\alpha^0 \in (0, 1)$ such that $y \succ x\alpha^0(x^*\alpha^*x_+)$. 3) $x^*\alpha^*x_+ \succ y \succ x \succ x^*\alpha_+x_*$ follows symmetrically since exists $\alpha^0 \in (0, 1)$ such that $y \succ x\alpha^0x^*$.

Thus the utility is linear between $x^*$ and $x_+$. Let us show linearity of $u$ within each of the three above areas. 1) Let $y, x \succ x^*\alpha_+x_+, \alpha \in (0, 1)$ and $z := \alpha x + (1 - \alpha)y$. Now let us show that $\alpha^2 = \frac{\alpha x^2}{\alpha^2(1 - \alpha) + \alpha y^2}$. First by Lemma 21

\[
\frac{\alpha x^y}{\alpha^2(1 - \alpha) + \alpha y^2}(\alpha^x x + (1 - \alpha^x)(x^*\alpha_+x_+)) + \frac{\alpha^x(1 - \alpha)}{\alpha^2(1 - \alpha) + \alpha y^2} \left(\alpha^y y + (1 - \alpha^y)(x^*\alpha_+x_+)\right) = \frac{\alpha^x(1 - \alpha) + \alpha y^2}{\alpha^2(1 - \alpha) + \alpha y^2} \alpha^x \alpha^y (x^*\alpha_+x_+) \sim x^*\alpha_+x_+.
\]

Additionally for all $\alpha \succ \frac{\alpha^x\alpha^y}{\alpha^2(1 - \alpha) + \alpha y^2}$ by Lemma 22

\[
az + (1 - \alpha)(x^*\alpha_+x_+) \succ x^*\alpha_+x_+.
\]

Then the linearity follows from the definition of $u$. The cases $x^*\alpha^*x_+ \succeq x, y \succeq x^*\alpha_+x_+$ and $x^*\alpha^*x_+ \succ x, y$ follow similarly.

Next let us show that $u$ is linear between $x^*$ and $x_+$. By the choice of $\alpha^*$ and $\alpha_+$ the following holds. We have for all $\alpha \in (\alpha^*, 1]$ $x^*\alpha x_+ \succ x^*\alpha^*x_+$ and

\[\alpha^x\alpha_+ = \frac{\alpha^* - \alpha_+}{\alpha - \alpha_+}.
\]

Thus

\[u(x^*\alpha x_+) = \frac{\alpha - \alpha_+}{\alpha^* - \alpha_+}.
\]

We have for all $\alpha \in [\alpha_+, \alpha^*] x^*\alpha^*x_+ \succeq x^*\alpha x_+ \succeq x^*\alpha_+x_+$ and

\[\alpha^x\alpha_+ = \frac{\alpha^* - \alpha}{1 - \alpha}.
\]

Thus

\[u(x^*\alpha x_+) = \frac{1 - \alpha_+}{\alpha^* - \alpha} \cdot \frac{\alpha^* - \alpha}{1 - \alpha} = \frac{\alpha - \alpha_+}{\alpha^* - \alpha_+}.
\]

We have for all $\alpha \in [0, \alpha_+] x^*\alpha x_+ \succ x^*\alpha x_+$ and

\[\alpha^x\alpha_+ = \frac{\alpha_+ - \alpha}{\alpha^* - \alpha}.
\]

Thus

\[u(x^*\alpha x_+) = \frac{\alpha_+ - \alpha}{\alpha^* - \alpha} = \frac{\alpha - \alpha_+}{\alpha^* - \alpha_+}.
\]

Thus the utility is linear between $x^*$ and $x_+$. Now let $x, y \in \Delta(X)$ be such that $x \succ x^*\alpha^*x_+ \succeq y \succeq x^*\alpha_+x_+$ and let us show that $u$ is affine between $x$ and $y$. Let $\alpha^1 \in (\alpha^*, 1]$ be such that

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$x \succ x^\ast x_\ast$. Then exists $\alpha^2 \in (0, 1)$ such that $u(x^2y) = u(x^\ast x_\ast)$ and $\alpha^3 \in [\alpha_\ast, \alpha^\ast]$ such that $u(y) = u(x^\ast x_\ast)$. Now since $u$ is a weak representation, we have by the linearity for all $z, z' \in \Delta(X)$ such that $z, z' \succ x^\ast x_\ast$ for all $\alpha \in (\frac{\alpha^2-\alpha^3}{\alpha_\ast - \alpha^3}, 1]$

$$u(x^2y) = u\left(x^\ast(\alpha^1 + \alpha^3(1 - \alpha))x_\ast\right).$$

And by the linearity for all $z, z' \in \Delta(X)$ such that $x^\ast x_\ast \succ z, z' \succ x^\ast x_\ast$ for all $\alpha \in [0, \frac{\alpha^2-\alpha^3}{\alpha_\ast - \alpha^3}]$

$$u(x^2y) = u\left(x^\ast(\alpha^1 + \alpha^3(1 - \alpha))x_\ast\right).$$

Thus by the of $u$ between $x^\ast$ and $x_\ast$, $u$ is linear between $x^2y$ and $y$. Additionally by the linearity for all $z, z' \in \Delta(X)$ such that $z, z' \succ x^\ast x_\ast$, $u$ is linear between $x$ and $x^2y$. Thus $u$ is linear between $x$ and $y$.

Symmetrically if $x, y \in \Delta(X)$ are such that $x^\ast x_\ast \succ x \succ x^\ast x_\ast \succ y$, then $u$ is affine between $x$ and $y$. Thus finally for all $x, y \in \Delta(X)$, $u$ is affine between $x$ and $y$. This shows the claim. □

**Lemma 24** Assume that $\succ$ satisfies Axioms 1,2. If exists $u : \Delta(X) \to \mathbb{R}$ that is affine and a weak representation for $\succ$, then $\succ$ satisfies Axiom 4’.

**Proof.** Assume, per contra, there exist $x, y, z \in \Delta(X), \alpha \in (0, 1)$ such that $x \succ y$ and $\alpha y + (1 - \alpha)z \succ \alpha x + (1 - \alpha)z$. Then since $u$ is a weak representation for $\succ$ and affine, we have

$$u(x) > u(y) \text{ and } \alpha u(y) + (1 - \alpha)u(z) = u(\alpha y + (1 - \alpha)z) > u(\alpha x + (1 - \alpha)z) = \alpha u(x) + (1 - \alpha)u(z)$$

which is a contradiction. □

**Lemma 25** Assume that $\succ$ satisfies Axioms 1,2,4’. If $u : \Delta(X) \to \mathbb{R}$ and $\tilde{u} : \Delta(X) \to \mathbb{R}$ are affine and weak representations for $\succ$, then exists $\alpha > 0, \beta \in \mathbb{R}$ such that

$$\tilde{u} = \alpha u + \beta.$$

**Proof.** Let us first show that for all $x, y \in \Delta(X)$

$$u(x) \geq u(y) \iff \tilde{u}(x) \geq \tilde{u}(y).$$

Assume, per contra, exists $\tilde{u}(x) > \tilde{u}(y)$ and $u(x) \leq u(y)$. By Axiom 1, exists $x^\ast, x_\ast \in \Delta(X)$ such that $x^\ast \succ x_\ast$. By applying Lemma 20 twice, exists $\alpha^\ast, \alpha_\ast \in (0, 1)$ such that $\alpha^\ast \succ \alpha_\ast$ and for all $\alpha \in (\alpha^\ast, 1], \alpha' \in [0, \alpha_\ast)$

$$x^\ast x_\ast \succ x^\ast x_\ast \succ x^\ast x_\ast \succ x^\ast x_\ast \succ x^\ast x_\ast.$$

(3)
Now since $u$ is a weak representation and so

$$u(x^*) > u(x^*_s), \quad (4)$$

and either $u(x^* \alpha^* x_*) \geq u(x)$ or $u(x^* \alpha_2 x_*) \leq u(y)$. Assume w.l.o.g. $u(x^* \alpha^* x_*) \geq u(x)$. By the affinity of $u$ and $\tilde{u}$ exists $\alpha^x, \alpha^y$ such that

$$\tilde{u}(x^* \alpha^x x_*) > \tilde{u}(y^0 x^*) \quad (5)$$

and thus $u(x^* \alpha^x x_*) < u(y^0 x^*)$. Denote $\bar{x} := x^* \alpha^x x_*$ and $\bar{y} := y^0 x^*$. By (4), exists $\alpha^0 \in (0, 1]$ such that $u(x^* \alpha^0 x_*) = u(\bar{x}^0 x^*)$. Thus especially by the affinity of $u$ we have $u(\bar{y}^0 x^*) > u(x^* \alpha^0 x_*)$. By affinity of $u$ exists $\alpha^1 \in (\alpha^0, 1]$ such that $u(\bar{y}^0 x^*) > u(x^* \alpha^1 x_*)$. Thus since $u$ is a weak representation for $\succcurlyeq$, we have

$$\bar{y}^0 x^* \succcurlyeq x^* \alpha^1 x_* \succcurlyeq x^* \alpha^0 x_* . \quad (3)$$

Thus especially since $\tilde{u}$ is a weak representation for $\succcurlyeq$

$$\alpha^0 \tilde{u}(\bar{y}) + (1 - \alpha^0)u(x^*) = \tilde{u}(\bar{y}^0 x^*) > \tilde{u}(\bar{x}^0 x^*) = \alpha^0 \tilde{u}(\bar{x}) + (1 - \alpha^0)u(x^*).$$

However, this contradicts (5).

Thus for all $x, y \in \Delta(X)$

$$u(x) \geq u(y) \iff \tilde{u}(x) \geq \tilde{u}(y).$$

Let us define $\succeq \subseteq \Delta(X) \times \Delta(X)$ by

$$x \succeq y \iff u(x) \geq u(y).$$

Now $u$ and $\tilde{u}$ are affine representations for $\succeq$ and hence by Herstein and Milnor (1953) and Fishburn (1970) there exist $\alpha > 0, \beta \in \mathbb{R}$ such that $\tilde{u} = \alpha u + \beta$. \hfill $\square$

### 8.2.2 Existence of Representation

For $x \in \mathbb{R}$, let $\bar{x}_S \in \mathbb{R}^S$ denote the constant vector of $x$. Let us denote $h^* := 1/2 f^* + 1/2 g^*$.

Let us first state and prove some basic observations.

**Lemma 26** Let $\succcurlyeq$ satisfy transitivity. If $f, g \in H$ s.t. for all $s \in S$, $f_s \succcurlyeq_s g_s$, then $f \succcurlyeq g$.

**Proof.** Since $S$ is finite we can enumerate $S = \{s_1, \ldots, s_{|S|}\}$. Now we have by transitivity of $\succcurlyeq_s$ and the definition of $\succcurlyeq_s$

$$f = (f_{s_1}, f_{s_2}, \ldots, f_{s_{|S|}}) \succcurlyeq (g_{s_1}, f_{s_2}, \ldots, f_{s_{|S|}}) \succcurlyeq \cdots$$

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\[ \preceq (g_{s_1}, \ldots, g_{s_{|s|-1}}, f_{s|s|}) \succeq (g_{s_1}, g_{s_2}, \ldots, g_{s_{|s|}}) \equiv g. \]

Thus by transitivity \( f \succeq g. \)

**Lemma 27** Let \( s \in S. \) If \( \succeq \) satisfies Axioms 1-3, then \( \succeq_s \) satisfies

1. completeness, i.e., for all \( x, y, z \in \Delta(X_s) \) \( x \succeq y \) or \( y \succeq x, \)
2. transitivity, i.e., for all \( x, y, z \in \Delta(X_s) \) if \( x \succeq y \) and \( y \succeq z, \) then \( x \succeq z, \)

**Proof.** By the definition of \( \succeq_s \) for \( x, y \in \Delta(X_s) \)

\[ x \succeq_s y \iff \forall f \in H, (x, f_{-s}) \succeq (y, f_{-s}) \]  

(6)  

and by Axiom 3

\[ x \succ s y \iff \exists f \in H, (x, f_{-s}) \succ (y, f_{-s}). \]

(7)

Complete: Let \( x, y \in \Delta(X_s) \). If there exists \( f \in H \) such that \( (x, f_{-s}) \succ (y, f_{-s}), \) then by (7), \( x \succ y \) and so especially by the definition of \( \succeq_s, x \succeq y. \) If the above \( f \in H \) does not exist, then by the completeness of \( \succeq \) for all \( f \in H \) \( (y, f_{-s}) \succeq (x, f_{-s}) \) and so by (6) \( y \succeq x. \)

Transitive: Let \( x, y, z \in \Delta(X_s) \) such that \( x \succeq y \succeq z, \) then \( \forall f \in H, (x, f_{-s}) \succeq (y, f_{-s}) \succeq (z, f_{-s}) \) and so by Axiom 1 \( \forall f \in H, (x, f_{-s}) \succeq (z, f_{-s}). \) Thus by (6) \( x \succeq z. \)

**Lemma 28** Let \( s \in S. \) If \( \succeq \) satisfies Axioms 1-3, then for all \( x, y, z \in \Delta(X_s), \) the sets \( \{ \alpha \in [0, 1]|\alpha x_s + (1 - \alpha)y_s \succeq z \} \) and \( \{ \alpha \in [0, 1]|z_s \succeq \alpha x_s + (1 - \alpha)y_s \} \) are closed in \( \mathbb{R}. \)

**Proof.** Let \( s \in S \) and \( x, y, z \in \Delta(X_s). \) Let us show that the sets \( \{ \alpha \in [0, 1]|\alpha x_s + (1 - \alpha)y_s \succeq z \} \) and \( \{ \alpha \in [0, 1]|z \succeq \alpha x_s + (1 - \alpha)y_s \} \) are open in \( \mathbb{R} \) which shows the claim since by Lemma 27 \( \succeq_s \) is complete. Let us show that \( \{ \alpha \in [0, 1]|\alpha x_s + (1 - \alpha)y_s \succeq z \} \) is open since \( \{ \alpha \in [0, 1]|z \succeq \alpha x_s + (1 - \alpha)y_s \} \) is open by a symmetric argument. Let \( \alpha^0 \in \{ \alpha \in [0, 1]|\alpha x_s + (1 - \alpha)y_s \succeq z \}. \) Thus \( \alpha^0 x_s + (1 - \alpha^0)y_s \succeq z. \) By the definition of \( \succeq_s \) exists \( f \in H \) such that \( (\alpha^0 x_s + (1 - \alpha^0)y_s, f_{-s}) \succeq (z, f_{-s}). \) Thus \( \alpha^0 \in \{ \alpha \in [0, 1]|\alpha x_s + (1 - \alpha)y_s \succeq z \}. \) By Axiom 2 exists a neighborhood \( V \) of \( \alpha^0 \) such that for all \( \alpha \in V \cap [0, 1] \) \( (\alpha x_s + (1 - \alpha)y_s, f_{-s}) \succeq (z, f_{-s}). \) Thus for all \( \alpha \in V \cap [0, 1] \) \( \alpha x_s + (1 - \alpha)y_s \succeq z. \) Hence \( V \cap [0, 1] \subseteq \{ \alpha \in [0, 1]|z \succeq \alpha x_s + (1 - \alpha)y_s \} \) and since \( V \) is a neighborhood of \( \alpha^0 \) this shows the claim.

**Definition** For \( s \in S, u_s \) is weak affine representation for \( \succeq_s \) if \( u_s : \Delta(X_s) \to \mathbb{R} \) is affine and for all \( x, y \in \Delta(X_s) \)

\[ x \succeq_s y \implies u_s(x) > u_s(y). \]
$(u_s)_{s \in S}$ is weak affine representation for $(\succcurlyeq_s)_{s \in S}$ if for all $s \in S$ $u_s$ is an weak affine representation for $\succcurlyeq_s$.

**Lemma 29** Let $s \in S$. If $\succcurlyeq_s$ satisfies Axioms 1-4, then exists an affine $u_s : \Delta(X_s) : \mathbb{R}$ such that $u_s$ is a weak affine representation for $\succcurlyeq_s$.

**Proof.** By Lemmas 27 and 28 $\succcurlyeq_s$ is a nontrivial weak order and satisfies continuity. By Axiom 4 $\succcurlyeq_s$ satisfies Weak Independence. By Proposition 23, there exists an affine $u_s : \Delta(X_s) : \mathbb{R}$ such that for all $x_s, y_s \in \Delta(X_s)$

$$x_s \succcurlyeq_s y_s \Rightarrow u_s(x_s) > u_s(y_s).$$

**Lemma 30** For each $s \in S$, let $(\succcurlyeq_s)_{s \in S}$ have a weak affine representation with $(u_s)_{s \in S}$. Then

1. For all $x_s, y_s \in \Delta(X_s)$ $u_s(x_s) \geq u_s(y_s) \implies x_s \succcurlyeq_s y_s$.
2. For all $f, g \in H$, if for all $s \in S^P$ $u_s(f_s) \geq u_s(g_s)$ then $f \succcurlyeq g$.
3. For $s \in S$, if $x_s \succ_m y_s$, then $u_s(x_s) > u_s(y_s)$.

**Proof.** The first one follows from the negation of the definition of weak representation. The second one follows from the first one and Lemma 26. The third one follows from the definition of weak representation and the affinity of $u_s$.

**Lemma 31** Let $\succcurlyeq_s$ satisfy Axioms 1-5. Let for all $s \in S^P$, $u_s : \Delta(X_s) : \mathbb{R}$ be an affine weak representation for $\succcurlyeq_s$. Let $f^*, g^*$ be equally crisp acts such that for all $s \in S^P$

$$u_s(f^*_s) > u_s(g^*_s).$$

(8)

Then for all $\alpha, \alpha' \in [0, 1]$

$$\alpha \geq \alpha' \iff f^* \alpha g^* \succcurlyeq f^* \alpha' g^*.$$  

(9)

**Proof.** Now $u(f^{1/2}g^*) \in \text{int } u(H)$. Assume per contra that exists $1 > \alpha^* > \alpha_s > 0$ such that

$$f^* \alpha^* g^* \sim f^* \alpha' g^*$$

By the nontriviality exists $f^0 \not\succ f^* \alpha g^*$. W.l.o.g. assume that $f^* \alpha g^* \not\succ f^0$. By continuity exists $\alpha^1 \in (0, 1)$ such that $f^1 = \alpha^1 f^0 + (1 - \alpha^1)(f^*/2g^*)$ and $f^* \alpha g^* \not\succ f^1$. Now $u(f^1) \in \text{int } u(H)$. Let

$$\alpha' = \inf \{\alpha \in [0, 1] | \alpha f^1 + (1 - \alpha) f^* \alpha g^* \succcurlyeq f^* \alpha g^*\}.$$  

(10)
By the continuity and the assumption $\alpha^* \in [0, 1]$ and $\alpha^f f^0 + (1 - \alpha^f) f^* \alpha^* g^* \sim f^* \alpha^* g^*$. Now by Corollary 59, $f^* \alpha^* g^*$ and $f^* \alpha^*_s g^*$ are equally crisp. Thus

$$\alpha^f f^0 + (1 - \alpha^f) f^* \alpha^*_s g^* \sim \alpha^f f^* \alpha^*_s g^* + (1 - \alpha^f) f^* \alpha^* g^*$$

and so

$$\alpha^f f^0 + (1 - \alpha^f) f^* \alpha^*_s g^* \sim \alpha^f f^* \alpha^*_s g^* + (1 - \alpha^f) f^* \alpha^* g^* = f^*(\alpha^f \alpha^* + (1 - \alpha^f) \alpha^*_s) g^*. \quad (11)$$

Let $0 < \varepsilon < 1 - \alpha^f$ and

$$0 < \varepsilon < \frac{(1 - \alpha^f)(\alpha^* - \alpha^*_s) \min_{s \in S^P} |u_s(f^*_s) - u(g^*_s)|}{\|u(f^0) - u(f^* \alpha^* g^*)\|_\infty}. \quad (12)$$

Then we have for all $s \in S^P$

$$u_s((\alpha^f + \varepsilon) f^0_s + (1 - \alpha^f - \varepsilon) f^*_s \alpha^*_s g^*_s) \quad (13)$$

$$= \alpha^f u_s(f^0_s) + (1 - \alpha^f) \left(u_s(g^*_s) + \alpha^*(u_s(f^*_s) - u_s(g^*_s))\right) + \varepsilon(u_s(f^0_s) - u_s(f^*_s \alpha^*_s g^*_s))$$

$$\geq \alpha^f u_s(f^0_s) + (1 - \alpha^f) u_s(g^*_s) + (1 - \alpha^f) \alpha^*(u_s(f^*_s) - u_s(g^*_s)) - \varepsilon \left\|u(f^0) - u(f^* \alpha^* g^*)\right\|$$

$$\geq \alpha^f u_s(f^0_s) + (1 - \alpha^f) u_s(g^*_s) + (1 - \alpha^f) \alpha^*(u_s(f^*_s) - u_s(g^*_s)) - (1 - \alpha^f) (\alpha^* - \alpha^*_s) \min_{\tilde{s} \in S^P} |u_{\tilde{s}}(f^*_s) - u(g^*_s)|$$

$$\geq \alpha^f u_s(f^0_s) + (1 - \alpha^f) u_s(g^*_s) + (1 - \alpha^f) \alpha^*(u_s(f^*_s) - u_s(g^*_s)) - (1 - \alpha^f) (\alpha^* - \alpha^*_s) (u_s(f^*_s) - u(g^*_s))$$

$$= \alpha^f u_s(f^0_s) + (1 - \alpha^f) u_s(g^*_s) + (1 - \alpha^f) \alpha^*_s (u_s(f^*_s) - u_s(g^*_s))$$

$$= u_s(\alpha^f f^0_s + (1 - \alpha^f) f^*_s \alpha^*_s g^*_s).$$

Thus by Lemma 30

$$(\alpha^f + \varepsilon) f^0 + (1 - \alpha^f - \varepsilon) f^* \alpha^*_s g^* \sim \alpha^f f^0 + (1 - \alpha^f) f^* \alpha^*_s g^* \quad (13)$$

$$\sim f^*(\alpha^f \alpha^* + (1 - \alpha^f) \alpha^*_s) g^* \sim f^* \alpha^*_s g^* \quad (8)$$

Thus for all $0 < \varepsilon < 1 - \alpha^f$ that satisfy (12) $(\alpha^f + \varepsilon) f^0 + (1 - \alpha^f - \varepsilon) f^* \alpha^*_s g^* \sim f^* \alpha^*_s g^*$. However this contradicts the definition of $\alpha^f$ in (10). Thus if $1 > \alpha^* > \alpha^*_s > 0$, then by (8) and Lemma 30 $f^* \alpha^*_s g^* \succ f^* \alpha^*_s g^*$. Finally if $1 \geq \alpha^* > \alpha^*_s \geq 0$, then by above (8) and Lemma 30

$$f^* \alpha^*_s g^* \succ f^*(\frac{3}{4} \alpha^* + \frac{1}{4} \alpha^*_s) g^* \sim f^*(\frac{1}{4} \alpha^* + \frac{3}{4} \alpha^*_s) g^* \sim f^* \alpha^*_s g^*.$$
Lemma 32 Let \( \succeq \) satisfy Axioms 1-5. Let \( f^*, g^* \) be equally crisp acts such that for all \( s \in \mathcal{S} \)
\( f^*_s \gtrsim_s g^*_s \). Then for all \( f \in H, \beta \in [0,1) \)
\[
\alpha \geq \alpha' \iff \beta f + (1 - \beta)(f^*\alpha g^*) \gtrsim \beta f + (1 - \beta)(f^*\alpha' g^*). \]

Proof. By Lemma 29, exists \((u_s)_{s \in S}\) that is a weak affine representation for \((\succeq_s)_{s \in S}\). By Lemma 30
for all \( s \in \mathcal{S} \) \( u_s(f^*_s) > u_s(g^*_s) \). Let \( f \in H \) and \( \beta \in (0,1) \). By Corollary 60, \( \beta f + (1 - \beta)f^*, \beta f + (1 - \beta)g^* \) are equally crisp acts and for all \( s \in \mathcal{S} \) by the affinity of \( u_s \) \( u_s(\beta f + (1 - \beta)f^*) > u_s(\beta f + (1 - \beta)g^*) \). Thus the claim follows from Lemma 31.

Lemma 33 Assume that \( \mathcal{S} = \mathcal{S}^* \). Let \( \succeq \) satisfy Axioms 1-4,6, and for all \( s \in \mathcal{S} \) there exists
affine \( u_s : \Delta(X_s) \rightarrow \mathbb{R} \) that is a weak representation for \( \succeq_s \) and
\[
u_s(f^*_s) - u_s(g^*_s) = 1. \tag{14} \]

Let \( \mathcal{A} \subseteq \{ (a_1, a_2) \subseteq \mathbb{R} | a_1, a_2 \in \mathbb{R} \cup \{-\infty, \infty\}, a_1 \leq a_2 \} \) be such that \( A \in \mathcal{A} \) if and only if
(1) exists \( \bar{I} \) such that \( \text{dom} \bar{I} \subseteq \text{int} u(H) \) and \( \text{Im} \bar{I} = A \)
(2) For all \( f \in H \) such that \( u(f) \in \text{int} u(H) \) if there exist \( f', g' \in H \) such that \( u(f'), u(g') \in \text{dom} \bar{I} \)
and \( f' \succeq f \succeq g' \), then \( u(f) \in \text{dom} \bar{I} \).
(3) For all \( \alpha \in (0,1) \), \( u(\alpha f^* + (1 - \alpha)g^*) \in \text{dom} \bar{I} \) and
\[
\bar{I} \circ u(\alpha f^* + (1 - \alpha)g^*) = \alpha
\]
(4) If \( \varphi \in \text{dom} \bar{I} \) and \( \alpha^1 > 0 \) are such that \( \varphi + \alpha^1 \bar{I} \in \text{dom} \bar{I} \), then \( \bar{I}(\varphi + \alpha^1 \bar{I}) = \bar{I}(\varphi) + \alpha^1 \).
(5) For all \( f, g \in \text{dom} \bar{I} \), \( P \succeq Q \) iff \( \bar{I}(u(f)) \geq \bar{I}(u(g)) \).

Let \( A_1, A_2 \in \mathcal{A}, A_1 \subseteq A_2 \) and \( I_1, I_2 \) be the associated functions. Then \( \text{dom} I_1 \subseteq \text{dom} I_2 \) and for all \( P \in \text{dom} I_1, I_1(P) = I_2(P) \).

Proof. Denote \( f^* \frac{1}{2} g^* = \frac{1}{2} f^* + \frac{1}{2} g^* \). We will show the claim for all \( \varphi \in \text{dom} I_1 \) such that \( I_1(\varphi) \geq \frac{1}{2} \).
The other case follows symmetrically. Denote
\[
m := \inf \{ a \in [\frac{1}{2}, \infty) | \exists \psi \in \text{dom} I_1, I_1(\psi) = a, [\psi \notin \text{dom} I_2 \text{ or } I_1(\psi) \neq I_2(\psi)] \}. \tag{15} \]
We will show that \( m = \infty \). We will first show that \( m > 0 \). Let \( \varphi \in \text{dom} I_1 \), be such that \( \frac{1}{2} \leq I_1(\varphi) < 1 \). Let \( f \in H \) be such that \( u(f) = \varphi \) and \( \alpha^\varphi \left( I_1(\varphi), 1 \right) \). Now we have by Condition 3,
\[
u \left( f^\frac{1}{2} g^* \right), u \left( \alpha^\varphi f^* + (1 - \alpha^\varphi)g^* \right), u \left( I_1(\varphi)f^* + (1 - I_1(\varphi))g^* \right) \in \text{dom} I_1, \text{dom} I_2.
\]
By Conditions 3 and 5 for $I_1$,
$$\alpha^* f^* + (1 - \alpha^*) g^* \succsim f \sim I_1(\varphi) f^* + (1 - I_1(\varphi)) g^* \succsim f^* \frac{1}{2} g^*.$$

By Condition 2 for $I_2$, $u(f) = \varphi \in \text{dom } I_2$. And by Conditions 3 and 5 for $I_2$,
$$I_2(\varphi) = I_2(u(f)) = I_2 \circ u \left( I_1(\varphi) f^* + (1 - I_1(\varphi)) g^* \right) = I_1(\varphi).$$

Thus $m > \frac{1}{2}$.

Assume per contra $m < \infty$. By Condition 1, there exist $\varphi \in \text{dom } I_1$ and $I_1 \circ u(\varphi) = m$. Since $A_1$ is open, there exist $f^1 \in H$ such that $u(f^1) \in \text{dom } I_1$ and $I_1 \circ u(f^m) < I_1 \circ u(f^1)$. Since $A_2 \supseteq A_1$, there exist $f^2 \in H$ such that $u(f^2) \in \text{dom } I_2$ and $I_2 \circ u(f^2) = I_1 \circ u(f^1)$. Since $\varphi \in \text{int } u(H)$, there exist $\alpha^* \in (0, 1)$ such that
$$\frac{1}{\alpha^*} \varphi - \frac{1 - \alpha^*}{\alpha^*} u(f^{*1/2}g^*) \in \text{int } u(H).$$

Thus there exist $f^m \in H$ such that
$$u(f^m) = \frac{1}{\alpha^*} \varphi - \frac{1 - \alpha^*}{\alpha^*} u(f^{*1/2}g^*).$$

Now
$$\alpha^* u(f^m) + (1 - \alpha^*) u(f^{*1/2}g^*) = \varphi.$$

By Condition 5 for $I_1$
$$f^1 \succsim \alpha^* u(f^m) + (1 - \alpha^*) u(f^{*1/2}g^*) \succsim f^{*1/2}g^*.$$

By Axiom 2, there exist $\alpha^* \in (0, 1/2)$ such that for all $\alpha \in (1/2 - \alpha^*, 1/2 + \alpha^*)$
$$f^1 \succsim \alpha^* f^m + (1 - \alpha^*) f^* g^* \succsim f^{*1/2}g^*.$$

By Condition 2 for all $\alpha \in (1/2 - \alpha^*, 1/2 + \alpha^*)$
$$\alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \in \text{dom } I_1.$$

By (14) and Condition 4, for all $\alpha \in (1/2 - \alpha^*, 1/2 + \alpha^*)$
$$I_1 \left( \alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \right) = I_1 \left( \alpha^* u(f^m) + (1 - \alpha^*) u(f^{*1/2}g^*) \right) + (1 - \alpha^*)(\alpha - 1/2).$$

Thus by (15) for all $\alpha \in (1/2 - \alpha^*, 1/2)$
$$\alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \text{ dom } I_2$$

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and
\[ I_2(\alpha^*u(f^m) + (1 - \alpha^*)u(f^*\alpha g^*)) = I_1(\alpha^*u(f^m) + (1 - \alpha^*)u(f^*\alpha g^*)) < I_1(\varphi) < I_2 \circ u(\bar{f}^2). \]

By Condition 5 for \( I_2 \) for all \( \alpha \in (\frac{1}{2} - \alpha^\dagger, \frac{1}{2}) \)
\[ \bar{f}^2 \succ \alpha^*f^m + (1 - \alpha^*)(f^*\alpha g^*) \]
and so by Axiom 2
\[ \bar{f}^2 \succsim \alpha^*f^m + (1 - \alpha^*)(f^*/2g^*). \]

By Condition 2,
\[ \alpha^*u(f^m) + (1 - \alpha^*)u(f^*/2g^*) \text{ dom } I_2 \]
and by Condition 4,
\[ I_2(\alpha^*u(f^m) + (1 - \alpha^*)u(f^*/2g^*)) = I_1(\alpha^*u(f^m) + (1 - \alpha^*)u(f^*/2g^*)). \tag{16} \]

By Condition 5, \( \bar{f}^2 \succ \alpha^*f^m + (1 - \alpha^*)(f^*/2g^*) \). By Axiom 2, there exist \( \alpha^\dagger \in (0, \alpha^\dagger) \) such that for all \( \alpha \in [\frac{1}{2}, \frac{1}{2} + \alpha^\dagger] \)
\[ \bar{f}^2 \succ \alpha^*f^m + (1 - \alpha^*)f^*\alpha g^*. \]

By Condition 2 for all \( \alpha \in [\frac{1}{2}, \frac{1}{2} + \alpha^\dagger] \)
\[ \alpha^*u(f^m) + (1 - \alpha^*)u(f^*\alpha g^*) \text{ dom } I_1 \cap \text{ dom } I_2. \tag{17} \]

By Condition 4 and (16) for all \( \alpha \in [\frac{1}{2}, \frac{1}{2} + \alpha^\dagger] \)
\[ I_1(\alpha^*u(f^m) + (1 - \alpha^*)u(f^*\alpha g^*)) = I_2(\alpha^*u(f^m) + (1 - \alpha^*)u(f^*\alpha g^*)). \tag{18} \]

Next, let \( h \in H \) and \( u(h) \in \text{ dom } I_1 \) be such that
\[ I_1(\alpha^*u(f^m) + (1 - \alpha^*)u(f^{\alpha^\dagger}g^*)) \geq I_1 \circ u(h) \geq I_1(\alpha^*u(f^m) + (1 - \alpha^*)u(f^*/2g^*)). \]

Let We show that \( u(h) \in \text{ dom } I_2 \) and \( I_2 \circ u(h) = I_1 \circ u(h) \). By Condition 5 for \( I_1 \),
\[ \alpha^*f^m + (1 - \alpha^*)(f^{\alpha^\dagger}g^*) \precsim h \precsim \alpha^*f^m + (1 - \alpha^*)(f^*/2g^*). \]

By Condition 2 and (17), \( u(h) \in \text{ dom } I_2 \). By Axiom 2, there exist \( \alpha^h \in [\frac{1}{2}, \alpha^\dagger] \) such that
\[ h \sim \alpha^*f^m + (1 - \alpha^*)(f^{\alpha^h}g^*). \]
By Condition 5 and (17,18)

\[ I_1(h) = I_1(\alpha^* u(f^m) + (1 - \alpha^*) u(f^* \alpha^h g^*)) = I_2(\alpha^* u(f^m) + (1 - \alpha^*) u(f^* \alpha^h g^*)) = I_2(h). \]

Thus for all \( \psi \in \text{dom} \ I_1 \) such that \( I_1(\alpha^* u(f^m) + (1 - \alpha^*) u(f^* \alpha^h g^*)) \geq I_1(\varphi) \geq I_1(\varphi) \), we have \( \psi \in \text{dom} \ I_2 \) and \( I_2(\psi) = I_1(\psi) \). This contradicts the choice of \( \varphi \) in (15). Thus \( m = \infty \). \( \square \)

**Lemma 34** Assume that \( S = S^p \). Let \( \succeq \) satisfy Axioms 1-4,6, and for all \( s \in S^p \) there exists affine \( u_s: \Delta(X_s) \to \mathbb{R} \) that is a weak representation for \( \succeq_s \) and

\[ u_s(f^*_s) - u_s(g^*_s) = 1. \]

Let \( \beta^* \in [0,1], f^\dagger \in H, c \in \mathbb{R} \) and denote

\[ B = \{ \varphi \in \text{int} \ u(H) \mid \exists f \in H, \alpha \in (0,1), u(f) = \varphi, f \sim \beta^* f^\dagger + (1 - \beta^*)(f^* \alpha g^*) \}. \]

Define \( I : B \to \mathbb{R} \) by the following. For each \( \varphi \in B \), let \( f^\varphi, \alpha^\varphi \in (0,1) \in H \) be such that \( u(f) = \varphi \) and \( f \sim \beta^* f^\dagger + (1 - \beta^*)(f^* \alpha g^*) \), then define

\[ I(\varphi) = (1 - \beta^*) \alpha + c. \quad (19) \]

If \( \varphi \in B \) and \( \alpha^1 > 0 \) are such that \( \varphi + \alpha^1 \bar{1} \in \text{dom} \ B \), then \( I(\varphi + \alpha^1 \bar{1}) = \bar{I}(\varphi) + \alpha^1 \).

**Proof.** First for all \( f, g \in H \) such that \( u(f), u(g) \in B \) and \( f \sim g \), we have by Lemma 32

\[ I(u(f)) = I(u(g)). \quad (20) \]

Let \( \varphi \in B \) and \( \alpha^0 > 0 \) such that \( \varphi + \alpha^0 \bar{1} \in B \). Let \( \varphi^{-1}, f^0 \in H \) be such that \( \varphi = u(\varphi^{-1}), \varphi + \alpha^0 \bar{1} = u(f^0) \). Let us consider the mapping \( J : [0, \alpha^0] \to \mathbb{R} \) defined by for all \( \alpha \in [0, \alpha^0] \), \( J(\alpha) = I(\varphi + \alpha \bar{1}) \). Let us show that this is an affine function with derivative 1. By Lemma 32 and the definition of \( B \),

\[ \beta^* f^\dagger + (1 - \beta^*) f^* \succ f^0, \varphi^{-1} \succ \beta^* f^\dagger + (1 - \beta^*) g^*. \quad (21) \]

Let \( \alpha^1 \in [0, \alpha^0] \). Since \( \varphi, \varphi + \alpha^1 \bar{0} \in \text{int} \ u(H), \varphi + \alpha^1 \bar{1} \in \text{int} \ u(H) \). Let \( f^1 \in H \) be such that \( u(f^1) = \varphi + \alpha^1 \bar{1} \). By Lemma 30,

\[ \beta^* f^\dagger + (1 - \beta^*) f^* \succ_{\varphi} f^1 \succ_{\varphi^{-1}} f^\dagger \succ \beta^* f^\dagger + (1 - \beta^*) g^*. \]

Thus exist \( \alpha^2 \in (0,1) \) such that \( \beta^* f^\dagger + (1 - \beta^*)(f^* \alpha^2 g^*) \sim f^1 \). Since \( u(\varphi + \alpha^1 \bar{1}) \in \text{int} \ u(H) \) there exists \( \varepsilon^\varphi > 0 \) such that \( B_{\infty}(\varphi + \alpha^1 \bar{1}, \varepsilon^\varphi) \subseteq u(H) \).
Let $\beta^* \in (\beta^*, 1)$, which exist since $\beta^* < 1$, be such that
\[
\frac{1 - \beta^*}{\beta^*} \left( \|\varphi + \alpha^1\|_\infty + \|uf^*\alpha^2g^*\|_\infty \right) < \varepsilon^*.
\]
Thus by the choice of $\beta^*$, there exists $f^* \in H$ such that
\[
u(f^*) = \varphi + \alpha^1\bar{1} + \frac{1 - \beta^*}{\beta^*} (\varphi + \alpha^1) - \frac{1 - \beta^*}{\beta^*} \nu(f^*\alpha^2g^*).\]
Especially
\[
\beta^*\nu(f^*) + (1 - \beta^*)\nu(f^*\alpha^2g^*) = \varphi + \alpha^1\bar{1} = \nu(f^1).
\]
Now by Lemma 30 and since $\beta^* > \beta^*$
\[
\beta^*\nu(f^*) + (1 - \beta^*)(f^*\alpha^2g^*) \sim f^1 \sim \beta^*\nu(f^1) + (1 - \beta^*)(f^*\alpha^2g^*)
\]
\[
\equiv \beta^*\left( \frac{\beta^*}{\beta^*} f^1 + \frac{\beta^* - \beta^*}{\beta^*} (f^*\alpha^2g^*) \right) + (1 - \beta^*)(f^*\alpha^2g^*).
\]
Since $f^*, g^*$ are equally crisp, we have for all $\gamma \in [0, 1]$ by Corollary 58
\[
\beta^*\nu(f^*) + (1 - \beta^*)(f^*\gamma g^*) \sim \beta^*\left( \frac{\beta^*}{\beta^*} f^1 + \frac{\beta^* - \beta^*}{\beta^*} (f^*\alpha^2g^*) \right) + (1 - \beta^*)(f^*\gamma g^*).
\]
By the linearity of $\nu$ we have
\[
u(f^1) + (1 - \beta^*)(\nu(f^*\gamma g^*) - \nu(f^*\alpha^2g^*)) = \varphi + (\alpha^1 + (1 - \beta^*)(\gamma - \alpha^2))\bar{1}
\]
and since $1 - \beta^* > 1 - \beta^*$
\[
\beta^*\nu(f^1) + (1 - \beta^*)(f^*\gamma g^*) = \beta^*\nu(f^1) + ((\beta^* - \beta^*)\alpha^2 + (1 - \beta^*)\gamma)\nu(f^*) + ((\beta^* - \beta^*)(1 - \alpha^2) + (1 - \beta^*)(1 - \gamma))\nu(g^*)
\]
\[
= \beta^*\nu(f^1) + (1 - \beta^*)\nu\left( f^*(\alpha^2 + \frac{1 - \beta^*}{1 - \beta^*}(\gamma - \alpha^2))g^* \right).
\]
We have for all $\gamma \in [0, 1]$
\[
J\left( \alpha^1 + (1 - \beta^*)(\gamma - \alpha^2) \right)
\]
\[
= I\left( \varphi + \left( \alpha^1 + (1 - \beta^*)(\gamma - \alpha^2) \right) \nu \right) \quad \text{by (23)}
\]
\[
= I\left( \beta^*\nu(f^1) + (1 - \beta^*)(f^*\gamma g^*) \right) \quad \text{by (20,22)}
\]
\[
= I\left( \beta^*\nu(f^1) + (1 - \beta^*)\nu\left( f^*(\alpha^2 + \frac{1 - \beta^*}{1 - \beta^*}(\gamma - \alpha^2))g^* \right) \right) \quad \text{by (24)}
\]
\[
= (1 - \beta^*)\left( \alpha^2 + \frac{1 - \beta^*}{1 - \beta^*}(\gamma - \alpha^2) \right) + c = (1 - \beta^*)\alpha^2 + (1 - \beta^*)(\gamma - \alpha^2) + c.
\]
Thus for all $x \in \left( \alpha^1 - (1 - \beta^e)\alpha^2, \alpha^1 + (1 - \beta^e)(1 - \alpha^2) \right) \cap [0, \alpha^1]$, first \( \frac{x - \alpha^1}{1 - \beta^e} + \alpha^2 \in (0, 1) \) and so by above

\[
J(x) = J\left( \alpha^1 + (1 - \beta^e)\left( \frac{x - \alpha^1}{1 - \beta^e} + \alpha^2 - \alpha^2 \right) \right) \\
= (1 - \beta^e)\alpha^2 + (1 - \beta^e)\left( \frac{x - \alpha^1}{1 - \beta^e} + \alpha^2 - \alpha^2 \right) + c = (1 - \beta^e)\alpha^2 + x + c.
\]

So \( J \) is affine in the neighborhood \( \left( \alpha^1 - (1 - \beta^e)\alpha^2, \alpha^1 + (1 - \beta^e)(1 - \alpha^2) \right) \cap [0, \alpha^1] \) of \( \alpha^1 \) and the derivative is 1. Since the point \( \alpha^1 \in [0, \alpha^0] \) was arbitrary, the derivative of \( J \) is locally constant at 1 and \( J \) is affine locally at 0 and \( \alpha^0 \). Since the interval \([0, \alpha^0]\) is connected, by (Viro et al., 2008, Problem 12.2x) the derivative of \( J \) is constant at 1 on the set \((0, \alpha^0)\). Thus \( J \) is affine function on the set \([0, \alpha^0]\). Thus by the Fundamental theorem of calculus and since \( J \) is locally affine at 0 and \( \alpha^0 \)

\[
J(\alpha^0) - J(0) = J(\alpha^0) - \lim_{\alpha \to \alpha^0} J(\alpha) - J(0) + \lim_{\alpha \to \alpha^0} J(\alpha) + \int_{\alpha^0}^{\alpha^0} \nabla J(\alpha)d\alpha = \int_{\alpha^0}^{\alpha^0} 1d\alpha = \alpha^0.
\]

Hence

\[
I(\varphi + \alpha^0 \tilde{1}) = J(\alpha^0) = J(0) + \alpha^0 = I(\varphi) + \alpha^0.
\]

\[\blacksquare\]

**Lemma 35** Assume that \( S = S^p \). Let \( \succeq \) satisfy Axioms 1-4,6, and for all \( s \in S^p \) there exists affine \( u_s : \Delta(X_s) \to \mathbb{R} \) that is a weak representation for \( \succeq_s \) and

\[
u_s(f^*_s) - u_s(g^*_s) = 1.
\]

Let \( \mathcal{A} \subseteq \left\{ (a_1, a_2) \subseteq \mathbb{R} | a_1, a_2 \in \mathbb{R} \cup \{-\infty, \infty\}, a_1 \leq a_2 \right\} \) be such that \( A \in \mathcal{A} \) if and only if

1. exists \( \tilde{1} \) such that dom \( \tilde{1} \subseteq \text{int} u(H) \) and Im \( \tilde{1} = A \)
2. For all \( f \in H \) such that \( u(f) \in \text{int} u(H) \) if there exist \( f', g' \in H \) such that \( u(f'), u(g') \in \text{dom} \tilde{1} \) and \( f' \succeq f \succeq g' \), then \( u(f) \in \text{dom} \tilde{1} \).
3. For all \( \alpha \in (0, 1) \), \( u(\alpha f^* + (1 - \alpha)g^*) \in \text{dom} \tilde{1} \) and

\[
\tilde{1} \circ u(\alpha f^* + (1 - \alpha)g^*) = \alpha
\]

4. If \( \varphi \in \text{dom} \tilde{1} \) and \( \alpha^1 > 0 \) are such that \( \varphi + \alpha^1 \tilde{1} \in \text{dom} \tilde{1} \), then \( \tilde{1}(\varphi + \alpha^1 \tilde{1}) = \tilde{1}(\varphi) + \alpha^1 \).
5. For all \( f, g \in \text{dom} \tilde{1}, P \succeq Q \text{ iff } \tilde{1}(u(f)) \geq \tilde{1}(u(g)) \).

Then \( \mathcal{A} \neq \varnothing \).
Proof. We show that \((0,1) \in \mathcal{A}\). Denote

\[ B = \{ \varphi \in \text{int} u(H) \mid \exists f \in H, u(f) = \varphi, f^* \succ f \succ g^* \}. \]

Define \(I^0 : B \to \mathbb{R}\) by the following. Let \( \varphi \in B \) and \( f \in H \) such that \( u(f) = \varphi, f^* \succ f \succ g^* \). By Axiom 2, there exist \( \alpha^f \in (0,1) \) such that \( f^* \alpha^f + (1 - \alpha^f) g^* \sim f \). Define \( I^0(\varphi) = \alpha^f \). By Lemmas 30 and 31, this is well-defined function. We show that \((0,1)\) and with the function \(I^0\) satisfies Conditions 1-5.

Conditions 1-3 follow from the definition of \(I^0\). Condition 4: Follows from Lemma 34. Condition 5: Since for all \( h \in \text{dom} I^0 \) \( f^* \succ h \succ g^* \), exists \( \alpha^f, \alpha^g \in (0,1) \) such that \( f \sim \alpha^f f^* + (1 - \alpha^f) g^* \) and \( g \sim \alpha^g f^* + (1 - \alpha^g) g^* \). By the definition of \(I^0\) and Lemma 32, we have

\[ I^0(u(f)) \geq I^0(u(g)) \Leftrightarrow \alpha^f \geq \alpha^g \Leftrightarrow f^* \alpha^f g^* \succeq f^* \alpha^g g^* \Leftrightarrow f \succsim g. \]

Thus \((0,1) \in \mathcal{A}\). \(\square\)

**Lemma 36** Assume that \( S = S^P \). Let \( \succsim \) satisfy Axioms 1-4,6, and for all \( s \in S^P \) there exists affine \( u_s : \Delta(X_s) \to \mathbb{R} \) that is a weak representation for \( \succsim_s \) and

\[ u_s(f^*_s) - u_s(g^*_s) = 1. \]

Let \( \mathcal{A} \subseteq \{(a_1, a_2) \subseteq \mathbb{R} \mid a_1, a_2 \in \mathbb{R} \cup \{-\infty, \infty\}, a_1 \leq a_2\} \) be such that \( A \in \mathcal{A} \) if and only if

1. exists \( \tilde{I} \) such that \( \text{dom} \tilde{I} \subseteq \text{int} u(H) \) and \( \text{Im} \tilde{I} = A \)
2. For all \( f \in H \) such that \( u(f) \in \text{int} u(H) \) if there exist \( f', g' \in H \) such that \( u(f'), u(g') \in \text{dom} \tilde{I} \) and \( f' \succsim f \succsim g' \), then \( u(f) \in \text{dom} \tilde{I} \).
3. For all \( \alpha \in (0,1) \), \( u\left(\alpha f^* + (1 - \alpha) g^*\right) \in \text{dom} \tilde{I} \) and \( \tilde{I} \circ u(\alpha f^* + (1 - \alpha) g^*) = \alpha \)
4. If \( \varphi \in \text{dom} \tilde{I} \) and \( \alpha^1 > 0 \) are such that \( \varphi + \alpha^1 \tilde{1} \in \text{dom} \tilde{I} \), then \( \tilde{I}(\varphi + \alpha^1 \tilde{1}) = \tilde{I}(\varphi) + \alpha^1 \).
5. For all \( f, g \in \text{dom} \tilde{I} \), \( P \succsim Q \) iff \( \tilde{I}(u(f)) \geq \tilde{I}(u(g)) \).

Let \( \mathcal{B} \subseteq \mathcal{A} \) be such that for each \( B_1, B_2 \in \mathcal{B} \) \( B_1 \subseteq B_2 \) or \( B_2 \subseteq B_1 \). Then \( \bigcup_{B \in \mathcal{B}} B \in \mathcal{A} \).

Proof. Denote \( B^* := \bigcup_{B \in \mathcal{B}} B \). For each \( B \in \mathcal{B} \) denote by \( I^B \) the associated function. Since each \( \mathcal{B} \) consists of nested open intervals, \( B^* \) is an open interval. Let us next define the associated extension \( I^* \). For all \( \varphi \in \text{int} u(H) \) if there exist \( B \in \mathcal{B} \) such that \( \varphi \in \text{dom} I^B \), define \( I^*(\varphi) = I^B(\varphi) \). Since \( \mathcal{B} \) is a chain by Lemma 33 \( I^* \) is well-defined. We will show that \( I^* \) and \( B^* \) satisfies Conditions 1-5.
Condition 1 follows from the definition of $I^*$.

Condition 2: Let $f \in H$ be such that $u(f) \in \text{int } u(H)$ and $f', g' \in H$ such that $u(f'), u(g') \in \text{dom } I^*$ and $f' \succ f \succ g'$. Now there exist $B_1, B_2$ such that $f' \in \text{dom } I^{B_1}, g' \in \text{dom } I^{B_2}$ and w.l.o.g. $B_1 \subseteq B_2$. By Lemma 33, $f' \in \text{dom } I^{B_2}$. By Condition 2 for $B_2$, $u(f) \in \text{dom } I^{B_2}$.

Condition 3 follows from the definition of $I^*$.

Conditions 4 and 5 follow similarly as 2 since $B$ is a chain.

\[\square\]

**Lemma 37** Assume that $S = S^\mathcal{P}$. Let $\succ$ satisfy Axioms 1-4,6, and for all $s \in S^\mathcal{P}$ there exists affine $u_s : \Delta(X_s) \to \mathbb{R}$ that is a weak representation for $\succ_s$ and

$$u_s(f^*_s) - u_s(g^*_s) = 1.$$

Let $\mathcal{A} \subseteq \{(a_1, a_2) \subseteq \mathbb{R} | a_1, a_2 \in \mathbb{R} \cup \{-\infty, \infty\}, a_1 \leq a_2\}$ be such that $A \in \mathcal{A}$ if and only if

1. exists $\bar{I}$ such that $\text{dom } \bar{I} \subseteq \text{int } u(H)$ and $\text{Im } \bar{I} = A$
2. For all $f \in H$ such that $u(f) \in \text{int } u(H)$ if there exist $f', g' \in H$ such that $u(f'), u(g') \in \text{dom } \bar{I}$ and $f' \succ f \succ g'$, then $u(f) \in \text{dom } \bar{I}$.
3. For all $\alpha \in (0, 1)$, $u(\alpha f^* + (1 - \alpha)g^*) \in \text{dom } \bar{I}$ and

$$\bar{I} \circ u(\alpha f^* + (1 - \alpha)g^*) = \alpha$$

4. If $\varphi \in \text{dom } \bar{I}$ and $\alpha^1 > 0$ are such that $\varphi + \alpha^1 \bar{I} \in \text{dom } \bar{I}$, then $\bar{I}(\varphi + \alpha^1 \bar{I}) = \bar{I}(\varphi) + \alpha^1$.
5. For all $f, g \in H$ with $u(f), u(g) \in \text{dom } \bar{I}$, $f \succ g$ iff $\bar{I}(u(f)) \geq \bar{I}(u(g))$.

If $A \in \mathcal{A}$ with an associated function $I^A$ and there exist $\varphi \in \text{int } u(H)$ such that $\varphi \notin \text{dom } I^A$, then there exist $A^* \supseteq A$ such that $A^* \in \mathcal{A}$.

**Proof.** Let $a_1, a_2 \in \mathbb{R} \cup \{\infty, -\infty\}$ be such that $A = (a_1, a_2)$. Assume that there exists $f^1 \in H$ such that $u(f^1) \in \text{int } u(H)$ and $u(f^1) \notin \text{dom } I^A$. Assume w.l.o.g. $f^1 \succ h^*$. Let

$$\alpha^2 := \sup\{\alpha \in [0, 1] | u(\alpha f^1 + (1 - \alpha)h^*) \notin \text{dom } I^A\}. \tag{25}$$

Since $u(f^1) \notin \text{dom } I^A$, $\alpha^2 \in [0, 1]$. Denote

$$f^2 := \alpha^2 f^1 + (1 - \alpha^2)h^*. \tag{26}$$

Now $u(f^2) \in \text{int } u(H)$ and thus there exists $\varepsilon > 0$ such that $B_\infty(u(f^2), \varepsilon) \subseteq u(H)$. Let $\beta^* \in (1/2, 1)$ be such that

$$\left(\frac{1 - \beta^*}{\beta^*}\right)\left(\|u(f^2)\|_\infty + \|u(h^*)\|_\infty\right) < \varepsilon.$$
By the choice of \( \beta^* \) there exists \( f^3 \in H \) such that
\[
    u(f^3) = u(f^2) + \left( \frac{1 - \beta^*}{\beta^*} \right) u(f^2) - \frac{1 - \beta^*}{\beta^*} u(h^*).
\]

Especially \( u(f^3) \in \text{int} u(H) \) and
\[
    \beta^* u(f^3) + (1 - \beta^*) u(h^*) = u(f^2). \tag{27}
\]

Next, we show that \( f^2 \vartriangleright f^* \): Assume, per contra, \( f^* \succ f^2 \). By applying Axiom 2 three times and by Lemma 31, there exist \( \alpha, \tilde{\alpha} \in (0, 1) \) such that for all \( \tilde{\alpha} \in [0, \alpha) \),
\[
    \tilde{\alpha} f^* + (1 - \tilde{\alpha}) g^* \succ \tilde{\alpha} f^1 + (1 - \tilde{\alpha}) f^2 = \left( \tilde{\alpha} + (1 - \tilde{\alpha}) \alpha^2 \right) f^1 + (1 - \alpha^2)(1 - \tilde{\alpha}) h^* \succ \alpha g^* + (1 - \tilde{\alpha}) f^*.
\]

Thus by Conditions 2 and 3 for all \( \tilde{\alpha} \in [0, \alpha) \),
\[
    u\left( \left( \tilde{\alpha} + (1 - \tilde{\alpha}) \alpha^2 \right) f^1 + (1 - \alpha^2)(1 - \tilde{\alpha}) h^* \right) \in \text{dom} I^A,
\]
which contradicts (25). Thus \( f^2 \vartriangleright f^* \). Especially, by Axiom 2
\[
    \alpha^2 > 0. \tag{28}
\]

Additionally, we have by Lemma 30
\[
    \beta^* f^3 + (1 - \beta^*) h^* \sim f^2 \vartriangleright f^* \succ h^*.
\]

Since by Lemma 57 \( h^* \) and \( g^* \) are equally crisp and by Lemma 32, we have for all \( \alpha \in [0, 1] \)
\[
    \beta^* f^3 + (1 - \beta^*) (f^* g^*) \succ \beta^* f^3 + (1 - \beta^*) g^* \succ \beta^* h^* + (1 - \beta^*) g^* \succ g^*. \tag{29}
\]

We show next that for all \( \alpha \in \left[ 0, \frac{1}{2} \right) \),
\[
    \beta^* u(f^3) + (1 - \beta^*) u(f^* g^*) \in \text{dom} I^A. \tag{30}
\]

Let \( \alpha \in \left[ 0, \frac{1}{2} \right) \). Let \( 0 < \varepsilon^\alpha < \alpha^2 \) be such that
\[
    0 < \varepsilon^\alpha < \frac{(1 - \beta^*)(1/2 - \alpha) \min_{s \in S^p} |u_s(f^*_s) - u(g^*_s)|}{\|u(h^*) - u(f^1)\|_\infty} \tag{31}
\]
and \( u((\alpha^2 - \varepsilon^\alpha)f^1 + (1 - \alpha^2 + \varepsilon^\alpha) h^*) \in \text{dom} I^A \) that exist by (25,28) and since \( u(h^*) \neq u(f^1) \) by Condition 3. Then we have for all \( s \in S^p \)
\[
    u_s((\alpha^2 - \varepsilon^\alpha)f^1_s + (1 - \alpha^2 + \varepsilon^\alpha) h^*_s)
    \geq \alpha^2 u_s(f^1_s) + (1 - \alpha^2) u_s(h^*_s) - \varepsilon^\alpha \|u(h^*) - u(f^1)\|_\infty
    \overset{(26,31)}{=} u_s(f^2) - \beta^*(1/2 - \alpha) \min_{s \in S^p} |u_s(f^*_s) - u_s(g^*_s)|
\]

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\[
\begin{align*}
&\geq u_s(\beta^* f_s^3 + (1 - \beta^*)h_s^*) - \beta^*\left(\frac{1}{2} - \alpha\right)(u_s(f_s^*) - u_s(g_s^*)) \\
&= \beta^* u_s(f_s^3) + (1 - \beta^*)\left(\frac{1}{2} u_s(f_s^*) + \frac{1}{2} u_s(g_s^*)\right) + \beta^*\left(\alpha - \frac{1}{2}\right)(u_s(f_s^*) - u_s(g_s^*)) \\
&= \beta^* u_s(f_s^3) + (1 - \beta^*)\left(\alpha u_s(f_s^*) + (1 - \alpha) u_s(g_s^*)\right) = u_s\left(\beta^* f_s^3 + (1 - \beta^*)(f_s^* \alpha g_s^*)\right).
\end{align*}
\]
By Lemma 30
\[
(\alpha^2 - \varepsilon^a)f^1 + (1 - \alpha^2 + \varepsilon^a)g^* \succeq \beta^* f^3 + (1 - \beta^*)(f^* \alpha g^*) \quad (29)
\]
By Condition 2,
\[
\beta^* u(f^3) + (1 - \beta^*)u(f^* \alpha g^*) \in \text{dom } I^A
\]
Next, we show that
\[
I^A(u(\beta^* f^3 + (1 - \beta^*)g^*)) + (1 - \beta^*)\frac{1}{2} = a_2 \quad (32)
\]
and so especially \(a_2 < \infty\). By Condition 5 and (30)
\[
I^A(u(\beta^* f^3 + (1 - \beta^*)g^*)) + (1 - \beta^*)\frac{1}{2} \leq a_2.
\]
Assume, per contra, that there exist \(k \in H, u(k) \in \text{dom } I^A\) such that
\[
I^A \circ u(k) > I^A(u(\beta^* f^3 + (1 - \beta^*)g^*)) + (1 - \beta^*)\frac{1}{2}.
\]
Then by (30) and Conditions 4 and 5 for all \(\alpha \in (0, 1/2)\)
\[
k \succeq \beta^* f^3 + (1 - \beta^*)(f^* \alpha g^*).
\]
Thus by Axiom 2
\[
k \succeq \beta^* f^3 + (1 - \beta^*)h^*.
\]
By Condition 2, \(\beta^* f^3 + (1 - \beta^*)h^* \in \text{dom } I^A\) which is a contradiction. This shows (32).

Denote
\[
B = \{ \varphi \in \text{int } u(H) | \exists f \in H, \alpha \in (0, 1), u(f) = \varphi, f \sim \beta^* f^3 + (1 - \beta^*)(f^* \alpha g^*) \}.
\]
We define an extension of \(I^A, I^*: B \cup \text{dom } I^A \to \mathbb{R}\) as follows: for all \(\varphi \in \text{dom } I^A, I^*(\varphi) = I^A(\varphi)\). For all \(f \in H\) such that \(u(f) \notin \text{dom } I^A, u(f) \in \text{int } u(H)\) and there exists \(\alpha \in [0, 1)\) \(f \sim \beta^* f^3 + (1 - \beta^*)(f^* \alpha g^*)\), define \(I^*(u(f)) = I^A\left(u(\beta^* f^3 + (1 - \beta^*)g^*)\right) + (1 - \beta^*)\alpha\).
First, by (32), $\text{Im } I^* \supseteq A$. Second, by Lemmas 30 and 32 and Conditions 4 and 5, $I^*$ is well-defined. Especially, we have for all $\alpha \in [0, 1/2)$, $I^A(u(\beta^* f^3 + (1 - \beta^*)(f^* a g^*))) = I^A(u(\beta^* f^3 + (1 - \beta^*)g^*)) + (1 - \beta^*)\alpha$ by Condition 4 and (30). Thus for all $\alpha \in [0, 1)\nabla$

$$I^*(u(\beta^* f^3 + (1 - \beta^*)(f^* a g^*))) = I^A(u(\beta^* f^3 + (1 - \beta^*)g^*)) + (1 - \beta^*)\alpha. \quad (33)$$

We show that $I^*$ satisfies Conditions 1-5 for the interval $(a_1, a_2 + \frac{1}{2}(1 - \beta^*))$.

Condition 1: For $a \in (a_1, a_2)$, there exist $\varphi \in \text{dom } I^*$ such that $I^*(\varphi) = a$ since $I^*$ is an extension of $I^A$. Let $a \in [a_2, a_2 + \frac{1}{2}(1 - \beta^*))$. We have $\frac{(a - a_2)}{(1 - \beta^*)} < \frac{1}{2}$ and so

$$I^* \circ u\left(\beta^* f^3 + (1 - \beta^*)\left[f^* \left(\frac{1}{2} + \frac{(a - a_2)}{(1 - \beta^*)}g^*\right)\right]\right) \equiv I^A\left(u(\beta^* f^3 + (1 - \beta^*)g^*)\right) + (1 - \beta^*)\left(\frac{1}{2} + \frac{(a - a_2)}{(1 - \beta^*)}\right) (32) \Rightarrow a_2 + a - a_2 = a.$$

Condition 2: Let $f \in H$ be such that $u(f) \in \text{int } u(H)$ and exists $f', g' \in H$ such that $u(f'), u(g') \in \text{dom } I^*$ and $f' \succ f \succ g'$. If $\beta^* f^3 + (1 - \beta^*)(f^*1/2g^*) \succ f$, then by (30) $u(f) \in \text{dom } I^A \subseteq \text{dom } I^*$. If $f \succ \beta^* f^3 + (1 - \beta^*)(f^*1/2g^*)$, then $u(f') \in B$. By Axiom 2, $u(f) \in B$.

Condition 3: Follows directly since $I^A$ satisfies Condition 3.

Condition 4: By Lemma 34 and Condition 4 for $I^A$, $I^*$ satisfies Condition 4 in $\text{dom } I^A$ and $B$. So the only case left by Lemma 30 is that $\varphi \in \text{dom } I^A \setminus B$ and $\varphi + \alpha^0 \in B \setminus \text{dom } I^A$. Let $\varphi^{-1}, f_0^0 \in H$ be such that $u(\varphi^{-1}) = \varphi$ and $u(f_0^0) = \varphi + \alpha^0 \widetilde{1}$. Since $u(f_0^0) \notin \text{dom } I^A, \varphi \notin B$, by (30) $f_0^0 \succ \beta^* f^3 + (1 - \beta^*)(f^*1/4g^*) \succ \varphi^{-1}$. Thus there exists $\alpha^1 \in (0, \alpha^0)$ such that $f_0^0 \alpha^1 \varphi^{-1} \sim \beta^* f^3 + (1 - \beta^*)(f^*1/4g^*)$. Then we have by Condition 4 in $\text{dom } I^A$ and $B$,

$$I^*(\varphi + \alpha^0 \widetilde{1}) - I^*(\varphi) = I^*(\varphi + \alpha^0 \widetilde{1}) - I^*(\varphi + \alpha^1 \alpha^0 \widetilde{1}) + I^*(\varphi + \alpha^1 \alpha^0 \widetilde{1}) - I^*(\varphi) = (1 - \alpha^1)\alpha^0 + I^A(\varphi + \alpha^1 \alpha^0 \widetilde{1}) - I^A(\varphi) = (1 - \alpha^1)\alpha^0 + \alpha^1 \alpha^0 = \alpha^0.$$

This show Condition 4.

Condition 5: Let $f, g \in H$ be such that $u(f), u(g) \in \text{dom } I^*$. If $u(f), u(g) \in \text{dom } I^A$ or $u(f), u(g) \in B$, then the condition holds. So assume that $u(f) \in \text{dom } I^* \setminus B$ and $u(g) \in \setminus B \setminus I^A$. Then by (30) and the definition of $B$,

$$g \succ \beta^* f^3 + (1 - \beta^*)(f^*1/4g^*) \succ f$$

and

$$I^*(u(g)) > I^*(\beta^* u(f^3) + (1 - \beta^*)u(f^*1/4g^*)) > I^*(u(f)).$$
Proposition 38 Assume that $S = S^P$. Let $\succeq$ satisfy Axioms 1-4,6, and for all $s \in S^P$ there exists affine $u_s : \Delta(X_s) \to \mathbb{R}$ that is a weak representation for $\succeq_s$ and $u_s(f_s^*) - u_s(g_s^*) = 1$. Then there exists $I : \text{int } u(H) \to \mathbb{R}$ such that $I$ is C-additive and for all $f, g \in H$ such that $u(f), u(g) \in \text{int } u(H)$

$$f \succeq g \iff I(u(f)) \geq I(u(g)).$$

Proof. Let $f^*, g^*$ be equally crisp acts. By Lemma 29, exists for all $s \in S^P$ an affine $u_s : \Delta(X_s) : \mathbb{R}$ and continuous and monotonic $H_s : \text{Im } u_s(\Delta(X_s)) \to \mathbb{R}$ such that for all $x_s, y_s \in \Delta(X_s)$

$$x_s \succeq_s y_s \iff H_s(u_s(x_s)) \geq H_s(u_s(y_s))$$

and $u_s(f_s^*) - u_s(g_s^*) = 1$. Let us construct $I$ that is C-additive and $I \circ u$ represents preferences. Let for all $\alpha \in [0, 1]$ $I^0 \circ u(\alpha f^* + (1 - \alpha)g^*) = \alpha$. For all $f \in H$ such that $u(f) \in \text{int } u(H)$ and exists $\alpha \in [0, 1]$ such that $f \sim \alpha f^* + (1 - \alpha)g^*$ define $I^0 \circ u(f) = \alpha$. Let us next do a transfinite induction on the collection $A \subseteq \{(a_1, a_2) \subseteq \mathbb{R}|a_1, a_2 \in \mathbb{R} \cup \{-\infty, \infty\}, a_1 \leq a_2\}$ such that $A \in A$ if and only if

1. exists $\bar{I}$ such that $\text{dom } \bar{I} \subseteq \text{int } u(H)$ and $\text{Im } \bar{I} = A$
2. For all $f \in H$ such that $u(f) \in \text{int } u(H)$ if there exist $f', g' \in H$ such that $u(f'), u(g') \in \text{dom } \bar{I}$ and $f' \succeq f \succeq g'$, then $u(f) \in \text{dom } \bar{I}$.
3. For all $\alpha \in (0, 1)$, $u(\alpha f^* + (1 - \alpha)g^*) \in \text{dom } \bar{I}$ and $I \circ u(\alpha f^* + (1 - \alpha)g^*) = \alpha$
4. If $\varphi \in \text{dom } \bar{I}$ and $\alpha^1 > 0$ are such that $\varphi + \alpha^1 \bar{I} \in \text{dom } \bar{I}$, then $\bar{I}(\varphi + \alpha^1 \bar{I}) = \bar{I}(\varphi) + \alpha^1$.
5. For all $f, g \in H$ with $u(f), u(g) \in \text{dom } \bar{I}$, $f \succ g$ iff $\bar{I}(u(f)) \geq \bar{I}(u(g))$.

We order $A$ by $\subseteq$. By Lemma 35, $A \neq \emptyset$. By Lemma 36, every chain has an upperbound. By Zorn’s lemma there exist a maximal element $A^* \in A$ and let $I^*$ be the associated function. By Lemma 37 and the maximality of $A^*$, $\text{dom } I^* = \text{int } u(H)$. This shows the claim.

Proposition 39 Assume that $S = S^P$. Let $\succeq$ satisfy Axioms 1-4,6, there exists crisp act $c \in H$ and for all $s \in S^P$ there exists affine $u_s : \Delta(X_s) \to \mathbb{R}$ that is a weak affine representation for $\succeq_s$, $u_s(f_s^*) - u_s(g_s^*) = 1$, and $u_s(c_s) = 0$. Then there exists $I : \text{int } u(H) \to \mathbb{R}$ such that $I$ is C-additive, positive homogeneous, and for all $f, g \in H$ such that $u(f), u(g) \in \text{int } u(H)$

$$f \succeq g \iff I(u(f)) \geq I(u(g)).$$
Proof. By Proposition 38 exists $I : \operatorname{int} u(H) \to \mathbb{R}$ such that $I$ is C-additive and for all $f, g \in H$ such that $u(f), u(g) \in \operatorname{int} u(H)$

$$f \succ g \iff I(u(f)) \geq I(u(g))$$

and after adding a constant since $u(c) = 0 \in u(H)$, $\lim_{\operatorname{int} u(H) \ni \varphi \to 0} I(\varphi) = 0$. Let us show that $I$ is positively homogeneous. Let $f \in H$ be such that $u(f) \in \operatorname{int} u(H)$. Now for all $\alpha \in (0, 1]$ $\alpha \varphi \in \operatorname{int} u(H)$. Let us denote the mapping $(0, 1) \ni \alpha \mapsto I(\alpha u(f))$. Since $I$ is 1-lipschitz, $J$ is lipschitz function and hence differentiable almost everywhere. Let the differentiability domain of $J$ be $\Omega \subseteq (0, 1)$ and $\Omega$ is dense in $(0, 1)$.

Let us show that $\nabla J$ is locally constant in $\Omega$. If $u(f) = 0$, then the claim follows from the normalization. Thus assume $u(f) \neq 0$. Let $x \in (0, 1)$. Then exists $y \in \Omega$ such that $x < y$. Let $z \in \Omega$ such that $z < y$. Let us show that $\nabla J(y) = \nabla J(z)$. Since $y > 0$ and so $u(yf + (1 - y)c) \in \operatorname{int} u(H)$ exists $\varepsilon > 0$ such that $B_\infty(u(yf + (1 - y)c), \varepsilon) \subseteq \operatorname{int} u(H)$. Let $\gamma \in (0, 1)$ be such that $|y - \gamma| < \frac{\varepsilon}{\|u(f)\|_\infty} < \infty$. Since $I$ is C-additive and hence especially 1-lipschitz, we have $|J(y) - J(\gamma)| \leq \|u(f)\|_\infty |y - \gamma| < \varepsilon$. Thus exists $g \in H$ such that $u(g) = u(yf + (1 - y)c) + \operatorname{I}(J(\gamma) - J(y)) \in \operatorname{int} u(H)$ by C-additivity, we have

$$I(u(g)) = I(yu(f) + (1 - y)u(c) + \operatorname{I}(J(\gamma) - J(y))) \overset{\text{C-add.}}{=} J(y) + J(\gamma) - J(y)$$

Thus by the representation

$$g \sim \gamma f + (1 - \gamma)c.$$ 

Thus by the crispness of $c$ for all $\alpha \in (0, 1)$

$$\alpha g + (1 - \alpha)c = \alpha(\gamma f + (1 - \gamma)c) + (1 - \alpha)c$$

Thus by the representation

$$I(\alpha u(g) + (1 - \alpha)u(c)) = I(\alpha yu(f) + \alpha \operatorname{I}(J(\gamma) - J(y))) \overset{\text{C-add.}}{=} J(\alpha y) + \alpha(J(\gamma) - J(y))$$

$$= I(\alpha(\gamma u(f) + (1 - \gamma)u(c)) + (1 - \alpha)u(c)) = I(\alpha \gamma u(f)) = J(\alpha \gamma).$$

Thus we have for all $\gamma \in (0, 1)$ such that $0 < |y - \gamma| < \frac{\varepsilon}{\|u(f)\|_\infty} < \infty$ and $\alpha \in (0, 1)$

$$J(\alpha \gamma) = J(\alpha y) + \alpha(J(\gamma) - J(y)) \iff \frac{J(\alpha \gamma) - J(\alpha y)}{\alpha \gamma - \alpha y} = \frac{J(\gamma) - J(y)}{\gamma - y}.$$
Thus by taking $\gamma \to y$ and $\alpha = z/y < 1$, we have since $y, z \in \Omega$

$$\nabla J(y) = \lim_{\gamma \to y} \frac{J(\gamma) - J(y)}{\gamma - y} = \lim_{\gamma \to y} \frac{J(z/\gamma) - J(z)}{z/\gamma - z} = \nabla J(z).$$

Since $z \in \Omega$ such that $z < y$ was arbitrary, $\nabla J$ is constant in the $(0, y)$ neighborhood of $x$. Since $x \in (0, 1)$ was arbitrary, $\nabla J$ is locally constant. Since the interval $(0, 1)$ is connected, by (Viro et al., 2008, Problem 12.2x) $\nabla J$ is constant on $\Omega$. Since $J$ is lipschitz function and by the normalization $\lim_{\alpha \to 0} J(\tilde{\alpha}) = 0$, we have for all $\alpha \in (0, 1)$

$$J(\alpha) - \lim_{\tilde{\alpha} \to 0} J(\tilde{\alpha}) = \int_{(0,\alpha) \cap \Omega} \nabla J(\alpha)d\alpha = \alpha \nabla J.$$

Thus $J$ is a linear function and by taking $\alpha = 1$, $\nabla J = J(1) = I(u(f))$. Thus for all $\alpha \in (0, 1]$

$$I(\alpha u(f)) = \alpha I(u(f)).$$

Since $f \in H$ such that $u(f) \in \text{int} u(H)$ was arbitrary, this shows the claim. \hfill $\Box$

**Proposition 40** Assume that $S = S^P$. If $\lesssim$ satisfies Axioms 1-6, then exists $(u, \mathbb{P})$ that is a dual-self representation for $\lesssim$.

**Proof.** By Lemma 29 and Proposition 39 exists for all $s \in S$ exists affine $u_s : \Delta(X_s) \to \mathbb{R}$ and $I : \text{int} u(H) \to \mathbb{R}$ such that $I \circ u$ represents $\lesssim$ in interior $u(H)$ and $I$ is C-additive and positive homogeneous. Thus by Chandrasekher et al.’s (2020) Lemma A.5 and proof of Theorem 1 exists $\mathbb{P} \in \mathcal{K}(\Delta(S))$ such that for all $\varphi \in \text{int} u(H)$ $I(\varphi) = \max_{p \in \mathbb{P}} \min_{p \in P} p \cdot \varphi$. Finally the representation can be extended to $u(H)$, which gives the claim. \hfill $\Box$

**Proposition 41** Assume that $S = S^P$. If $\lesssim$ satisfies Axioms 1-4, 6, then exists $(u, \mathbb{C})$ that is a dual-self variational representation for $\lesssim$.

**Proof.** By Lemma 29 and Proposition 38 exists for all $s \in S$ exists affine $u_s : \Delta(X_s) \to \mathbb{R}$ and $I : \text{int} u(H) \to \mathbb{R}$ such that $I \circ u$ represents $\lesssim$ in interior $u(H)$ and $I$ is C-additive and $u(f^*) = \overline{1}$, $u(g^*) = \overline{0}$. Thus by Chandrasekher et al.’s (2020) Lemma A.5 and Lemma S.3.2 exists $\mathbb{C} \subseteq \{c : \Delta(S) \to \mathbb{R} \cup \{\infty\} | c \text{ is convex} \}$ such that for all $\varphi \in \text{int} u(H)$ $I(\varphi) = \max_{c \in \mathbb{C}} \min_{p \in \Delta(S)} p \cdot \varphi + c(p)$. Finally the representation can be extended to $u(H)$, which gives the claim. \hfill $\Box$

**Proposition 42** If $\lesssim$ satisfies Axioms 1-6, then exists $(u, \mathbb{P})$ that is a dual-self representation for $\lesssim$.

**Proof.** Denote $\widehat{H} := \bigtimes_{s \in S^P} \Delta(X_s)$ and define $\lesssim \subseteq \widehat{H} \times \widehat{H}$ by for all $\widehat{f}, \widehat{g} \in \widehat{H}$

$$\widehat{f} \lesssim \widehat{g} \iff \forall h \in H(\widehat{f}, h_{S \setminus S^P}) \lesssim (\widehat{g}, h_{S \setminus S^P}).$$

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Now \( \sim \) satisfies Axioms 1-6 and it does not have null-states. Thus by Proposition 40 there exists \((\hat{u}, \hat{P})\) that is a dual-self representation for \( \sim \) and a crisp act \( c \in \hat{H} \) such that for all \( s, s' \in S^P \) \( u_s(c_s) = u_{s'}(c_{s'}) =: c^* \). For all \( s \in S \setminus S^P \) define \( u_s : \Delta(X_s) \to \mathbb{R} \) by for all \( x_s \in \Delta(X_s) \) \( u_s(x_s) = c^* \) and then \( u_s \) is affine. And for all \( s \in S^P \) define \( u_s = \hat{u}_s \). Additionally define
\[
P := \left\{ (p \in \Delta(S) | (p_s)_{s \in S^P} \in \hat{P}) | \hat{P} \in \tilde{P} \right\}.
\]
Now for all \( s \notin S^P \) \( p \in P \) \( p_s = 0 \). Additionally by the definition of null-states and \( \sim (u, \mathbb{P}) \) is a dual-self representation for \( \sim \). \( \square \)

**Proposition 43** If \( \sim \) satisfies Axioms 1-4, 6, then exists \((u, C)\) that is a dual-self variational representation for \( \sim \).

*Proof.* Symmetrically to Proposition 42. \( \square \)

### 8.3 Only if directions

**Lemma 44** Let \( \sim \) have a state dependent dual-self variational representation with \((u, C)\). Then \( u_s \) is a weak affine representation for \( \sim_s \).

*Proof.* Follows from the definition of \( \sim_s \) and the monotonicity of the dual-self representation. \( \square \)

**Lemma 45** Let \( \sim \) have a state dependent dual-self variational representation with \((u, C)\). Then \( \sim \) satisfies Axioms 1-4,6.

*Proof.* Let us define \( I : u(H) \to \mathbb{R} \) by for all \( f \in H \)
\[
I\left( u(f) \right) = \max_{c \in C} \min_{p \in \Delta(S)} p \cdot u(f) + c(p).
\]
Now \( I \) is \( C \)-additive on the non-null states and monotonic. Thus especially it is 1-lipschitz.

Completeness and transitivity follows from representation. Nontriviality follows from the definition of dual-self variational representation: Since \( C \) is grounded, especially exists \( c \in C \) and \( p \in \Delta(S) \) such that \( c(p) < \infty \). Since exists \( s \in S \) such that \( p_s > 0 \), by the definition of dual-self variational representation \( s \) is not null. Thus by the definition of null states exists \( x_s, y_s \in \Delta(X_s), h \in H \) such that \((x_s, h_{-s}) \not\sim (y_s, h_{-s})\).
Axiom 2: Let \( f, g, h \in H \). Define the mapping \([0, 1] \ni \alpha \mapsto J\left(\alpha u(f) + (1 - \alpha)u(g)\right)\). Now \( J \) is continuous since \( I \) is continuous and thus the following sets are closed as preimages of closed sets of a continuous function

\[
J^{-1}\left[I(u(h)), \infty\right) = \{\alpha \in [0, 1] | \alpha f + (1 - \alpha)g \gtrsim h\}
\]

and

\[
J^{-1}\left(-\infty, I(u(h))\right] = \{\alpha \in [0, 1] | h \gtrsim \alpha f + (1 - \alpha)g\}.
\]

Axiom 3: Follows from the monotonicity of \( I \).

Axiom 4: Follows from Lemmas 24 and 44.

Axiom 6: For all \( s \in S^p \), by above and Lemmas 20, 27, and 28 exists \( x^*_s, x_{ss} \in \Delta(X_s), \alpha^* \in (0, 1) \) such that

\[
x^*_s \succsim_s \alpha^* x^*_s + (1 - \alpha^*) x_{ss} \succsim_s x_{ss}
\]

and either for all \( \alpha \in (\alpha^*, 1] \)

\[
\alpha x^*_s + (1 - \alpha) x_{ss} \succsim_s \alpha^* x^*_s + (1 - \alpha^*) x_{ss}
\]

or for all \( \alpha \in [0, \alpha^*] \)

\[
\alpha^* x^*_s + (1 - \alpha^*) x_{ss} \succsim_s \alpha x^*_s + (1 - \alpha) x_{ss}.
\]

Since \( u_s \) is a weak representation for \( \succsim_s, \alpha^* x^*_s + (1 - \alpha^*) x_{ss} \in \text{int} u_s\left(\Delta(X_s)\right) \). Thus exists \( \alpha^* > 0 \) such that for all \( s \in S^p \) there exist \( f^*_s, g^*_s \) such that

\[
u_s(f^*_s) > u_s(\alpha^* x^*_s + (1 - \alpha^*) x_{ss}) > u_s(g^*_s)
\]

and \( u_s(f^*_s) - u_s(g^*_s) = \alpha^* \). Let us show that \( f^*_s \succ g^*_s \). Assume w.l.o.g. that for all \( \alpha \in (\alpha^*, 1] \)

\[
\alpha x^*_s + (1 - \alpha) x_{ss} \succsim_s \alpha^* x^*_s + (1 - \alpha^*) x_{ss}.
\]

Then by the affinity of \( u_s \) there exists \( \alpha^1 \in (\alpha^*, 1] \) such that \( u_s(f^*_s) > u_s(\alpha^1 x^*_s + (1 - \alpha^1) x_{ss}) \).

Since \( u_s \) is a weak representation for \( \succsim_s \), we have

\[
f^*_s \succsim \alpha^1 x^*_s + (1 - \alpha^1) x_{ss} \overset{(34)}{\succsim} \alpha^* x^*_s + (1 - \alpha^*) x_{ss} \succsim g^*_s.
\]
Let $\bar{h} \in H$ and define $\bar{f}^*, \bar{g}^* \in H$ by for all $s \in S^P, \bar{f}_s^* = f_s^*, \bar{g}_s^* = g_s^*$ and for all $s \notin S^P, \bar{f}_s^* = \bar{h}_s = \bar{g}_s^*$. Let us show that $f^*, g^*$ are equally crisp. Let $\alpha \in (0, 1], f, g \in H$, then we have since $I$ is C-additive on the non-null states

$$I(u(\alpha f + (1 - \alpha)g^*)) = I(u(\alpha f + (1 - \alpha)f^*)) - (1 - \alpha)\alpha^*$$

and

$$I(u(\alpha g + (1 - \alpha)\bar{g}^*)) = I(u(\alpha g + (1 - \alpha)f^*)) - (1 - \alpha)\alpha^*.$$ 

This shows that $\bar{f}^*, \bar{g}^*$ are equally crisp acts such that for all $s \in S^P \bar{f}^* \succ s \bar{g}^*$. \hfill $\square$

**Lemma 46** Let $\succsim$ have a state dependent dual-self representation with $(u, \mathbb{P})$ such that assume that there exists $c \in H$ such that for all $p, q \in \bigcup_{P \in \mathbb{P}} P, \sum_{s \in P} p_s u_s(c_s) = \sum_{s \in P} q_s u_s(c_s)$. Then $\succsim$ satisfies Axioms 1-6.

**Proof.** By Lemma 45, we need to only show non-triviality and Axiom 5. Non-triviality follows from that since $\mathbb{P}$ is not empty there exists $p \in P \in \mathbb{P}$, and by the definition of dual-self representation for all $s \in S^P p_s = 0$ and thus exists $s \in S$ such that $s$ is non-null. By the definition of null states $\succsim$ is non-trivial.

Let us show that $c$ from the antecedent is crisp. Denote Let $p \in \bigcup_{P \in \mathbb{P}} P$ and denote $c^* := p \cdot u(c)$. Let $f \in H, \alpha \in (0, 1)$. Then we have

$$\max_{P \in \mathbb{P}} \min_{p \in P} p \cdot u(\alpha f + (1 - \alpha)c) = \max_{P \in \mathbb{P}} \alpha p \cdot u(f) + (1 - \alpha)p \cdot u(c) = \alpha \max_{P \in \mathbb{P}} \min_{p \in P} p \cdot u(f) + (1 - \alpha)c^*.$$ 

This shows the crispness of $c$. \hfill $\square$

### 8.4 State Dependent Bewley Representation

This section axiomatizes the existence for a state dependent Bewley Representation and characterizes its uniqueness.

First we present the five characterizing axioms. The first axiom is the standard assumption that $\succsim$ is a nontrivial partial order.

**Axiom 1B** $\succsim$ is nontrivial, reflexive and transitive.

The next one is the continuity axiom as before.

**Axiom 2B** For all $f, g, h \in H$, the sets $\{ \alpha \in [0, 1] | \alpha f + (1 - \alpha)g \succsim h \}$ and $\{ \alpha \in [0, 1] | h \succsim \alpha f + (1 - \alpha)g \}$ are closed in $\mathbb{R}$. 
The next axiom assumes that there is no uncertainty about tastes for the consequences but instead when the acts differ only in one state they can always be compared. This also assumes that there is no ambiguity about a single state.

**Axiom 3** For all \( s \in S, f \in H, x_s, y_s \in \Delta(X_s) \),

\[(x_s, f_s) \succeq (y_s, f_s) \text{ or } (y_s, f_s) \succeq (x_s, f_s).\]

The next axiom is the second part of the assumption that there is no ambiguity about a single state. It assumes that preferences over consequences in a single state does not depend what common act is received in other states. This axiom guarantees that statewise preferences are well-defined.

**Axiom 4** For all \( s \in S, f, g \in H, x_s, y_s \in \Delta(X_s) \),

\[(x_s, f_s) \succeq (y_s, f_s) \implies (x_s, g_{-s}) \succeq (y_s, g_{-s}).\]

The last axiom is the standard independence axiom.

**Axiom 5** For all \( f, g \in H, \alpha \in (0, 1) \),

\[f \succeq g \iff \alpha f + (1 - \alpha)h \succeq \alpha g + (1 - \alpha)h.\]

The next result states that the previous five axioms characterize the existence of a state dependent Bewley representation and characterizes the uniqueness of the representation. The uniqueness part states that every statewise positive affine transformations for the utilities are allowed but for each utility function there is a unique set of probabilities.

**Theorem 47 (Bewley Representation)** \( \succeq \) satisfies Axioms 1-5 if and only if there exists a pair \((u, C)\) that is a state dependent Bewley representation for \( \succeq \).

Additionally, let \((u, C)\) be a state dependent Bewley representation for \( \succeq \), \( \bar{u} = (\bar{u}_s)_{s \in S} \) be such that for all \( s \in S \), \( \bar{u}_s : \Delta(X_s) \to \mathbb{R} \) is affine and \( \bar{C} \subseteq \Delta(S) \) be closed and convex. Then \((\bar{u}, \bar{C})\) is a state dependent Bewley representation for \( \succeq \) if and only if there exists \( a \in \mathbb{R}^S_{++}, b \in \mathbb{R}^S \) such that

\[
\bar{C} = \left[ \frac{a_s^{-1}p_s}{\sum_{s'} \sigma_{s'}^{-1}p_s'} \right]_{(p, s) \in C \times S}
\]

and for all \( s \in S \),

\[\bar{u}_s = a_su_s + b_s.\]

For all \( s \in S \) define \( \succeq_s \) and \( \succ_s \) as in Section 2.3.
Lemma 48  Let $\preceq$ satisfy Axioms 1B-5B. Then for each $s \in S$, there exist $u_s : \Delta(X_s) \to \mathbb{R}$ that is affine such that for all $x_s, y_s \in \Delta(X_s)$

$$x_s \preceq_s y_s \iff u_s(x_s) \geq u_s(y_s).$$

Proof. Let $s \in S$. We will show that $\preceq_s$ is complete, transitive, mixture continuous, and satisfy Independence. Let $x_s, y_s, z_s \in \Delta(X_s), f, g \in H$.

(1) Complete: By Axiom 3B, either $(x_s, f_{-s}) \succeq (y_s, f_{-s})$ or $(y_s, f_{-s}) \succeq (x_s, f_{-s})$. Hence, by Axiom 4B, respectively either for all $g \in H (x_s, g_{-s}) \succeq (y_s, g_{-s})$ or for all $g \in H (y_s, g_{-s}) \succeq (x_s, g_{-s})$. Thus by the definition either $x_s \preceq_s y_s$ or $y_s \preceq_s x_s$ respectively.

(2) Transitive: Let $x_s \preceq_s y_s$ and $y_s \preceq_s z_s$. By the definition for all $f \in H, (x_s, f_{-s}) \succeq (y_s, f_{-s})$ and $(y_s, f_{-s}) \succeq (z_s, f_{-s})$. Thus by Axiom 1B, for all $f \in H, (x_s, f_{-s}) \succeq (y_s, f_{-s})$ and hence $x_s \preceq_s y_s$.

(3) Mixture Continous: Let $x_s \preceq_s y_s$ and $y_s \preceq_s z_s$. By definition exists $f, g \in H$ such that $(x_s, f_{-s}) \succ (y_s, f_{-s}) \succeq (z_s, f_{-s})$ and $(x_s, g_{-s}) \succeq (y_s, g_{-s}) \succ (z_s, g_{-s})$. By Axiom 2B, the sets $\{\alpha \in [0, 1]|(y_s, f_{-s}) \succeq (\alpha x_s + (1-\alpha) z_s, f_{-s})\}$ and $\{\alpha \in [0, 1]|(\alpha x_s + (1-\alpha) z_s, g_{-s}) \succeq (y_s, g_{-s})\}$ are closed and hence by Axiom 3B the sets $\{\alpha \in [0, 1]|(\alpha x_s + (1-\alpha) z_s, f_{-s}) \succ (y_s, f_{-s})\}$ and $\{\alpha \in [0, 1]|(y_s, g_{-s}) \succ (\alpha x_s + (1-\alpha) z_s, g_{-s})\}$ are open as the complements of the previous sets. Additionally, they are nonempty since $1 \in \{\alpha \in [0, 1]|(\alpha x_s + (1-\alpha) z_s, f_{-s}) \succ (y_s, f_{-s})\}$ and $0 \in \{\alpha \in [0, 1]|(y_s, g_{-s}) \succ (\alpha x_s + (1-\alpha) z_s, g_{-s})\}$. Thus, exists $\alpha, \beta \in (0, 1)$ such that

$$(\alpha x_s + (1-\alpha) z_s, f_{-s}) \succ (y_s, f_{-s})$$

and

$$(y_s, g_{-s}) \succ (\beta x_s + (1-\beta) z_s, g_{-s}).$$

Additionally by, Axiom 4B for all $h \in H (\alpha x_s + (1-\alpha) z_s, h_{-s}) \succeq (y_s, h_{-s})$ and $(y_s, h_{-s}) \succeq (\beta x_s + (1-\beta) z_s, h_{-s})$. Thus by the definition of $\preceq_s$, $\alpha x_s + (1-\alpha) z_s \succ_s y_s$ and $y_s \succ_s \beta x_s + (1-\beta) z_s$.

(4) Independence: Let $\alpha \in (0, 1)$. For all $f \in H$ by Axiom 5B,

$$(x_s, f_{-s}) \succeq (y_s, f_{-s}) \iff (\alpha x_s + (1-\alpha) z_s, f_{-s}) \succeq (\alpha y_s + (1-\alpha) z_s, f_{-s}).$$

Thus by the definition of $\preceq_s$

$$x_s \preceq_s y_s \iff \alpha x_s + (1-\alpha) z_s \preceq_s \alpha y_s + (1-\alpha) z_s.$$
By Von Neumann–Morgenstern utility theorem (Fishburn, 1970), exists affine function \( u_s : \Delta(X_s) \to \mathbb{R} \) such that for all \( x_s, y_s \in \Delta(X_s) \)

\[
x_s \succeq_s y_s \iff u_s(x_s) \geq u_s(y_s).
\]

\[\square\]

Let \( h^* \in H \) be such that for all \( s \in S^p \) exists \( x_s, y_s \in \Delta(X_s) \) such that \( x_s \succeq_s h^*_s \succ_s y_s \). Define \( \sim \) on \( \mathbb{R}^{S^p} \) by for all \( \varphi, \psi \in \mathbb{R}^{S^p} \),

\[
\varphi \sim \psi \iff \exists \alpha \in (0, 1), f, g \in H, \alpha \varphi + (1 - \alpha)u(h^*) = u(f), \alpha \psi + (1 - \alpha)u(h^*) = u(g), f \succ g.
\]

**Lemma 49** Let \( \succ \) satisfy Axioms 1B-5B. For all \( \varphi, \psi \in \mathbb{R}^{S^p} \)

\[
\varphi \sim \psi \iff \forall \alpha \in (0, 1], f, g \in H, \alpha \varphi + (1 - \alpha)u(h^*) = u(f), \alpha \psi + (1 - \alpha)u(h^*) = u(g) \Rightarrow f \succ g
\]

**Proof.** First, we show the if direction. Let \( \varphi \sim \psi, \alpha \in (0, 1], f, g \in H \) be such that \( \alpha \varphi + (1 - \alpha)u(h^*) = u(f), \alpha \psi + (1 - \alpha)u(h^*) = u(g) \). We show that \( f \succ g \). Since \( \varphi \sim \psi \), there exist \( \alpha^* \in (0, 1), f^*, g^* \in H \) such that \( f^* \succ g^* \) and \( \alpha^* \varphi + (1 - \alpha^*)u(h^*) = u(f^*), \alpha^* \psi + (1 - \alpha^*)u(h^*) = u(g^*) \).

We consider two cases: 1) \( \alpha \geq \alpha^* \): Now by the linearity of \( u \), \( u\left(\alpha^*/\alpha f + (1 - \alpha^*/\alpha)h^*\right) = u(f^*) \) and \( \left(\alpha^*/\alpha g + (1 - \alpha^*/\alpha)h^*\right) = u(g^*) \). Since \( u_s \) represents \( \succ_s \) and by Lemma 26

\[
\alpha^*/\alpha f + (1 - \alpha^*/\alpha)h^* \sim f^* \succ g^* \sim \alpha^*/\alpha g + (1 - \alpha^*/\alpha)h^*.
\]

Thus by Axioms 1B and 5B, \( f \succ g \).

2) \( \alpha^* > \alpha \): Now by the linearity of \( u \), \( u\left(\alpha/\alpha^* f^*+(1-\alpha/\alpha^*)h^*\right) = u(f) \) and \( \left(\alpha/\alpha^* g^*+(1-\alpha/\alpha^*)h^*\right) = u(g) \). Since \( u_s \) represents \( \succ_s \) and by Lemma 26 and Axiom 5B

\[
f \sim \alpha/\alpha^* f^*+(1-\alpha/\alpha^*)h^* \succ \alpha/\alpha^* g^*+(1-\alpha/\alpha^*)h^* \sim g.
\]

Thus by Axiom 1B, \( f \succ g \).

Next, we show the only if direction. Now \( \emptyset \neq \{\alpha' \in (0, 1]|\alpha' \varphi + (1 - \alpha')u(h^*), \alpha' \psi + (1 - \alpha')u(h^*) \in H\} \) since \( u(h^*) \in \text{int} u(H) \), hence exists \( \alpha \in (0, 1) \) such that exists \( f, g \in H \) such that \( \alpha \varphi + (1 - \alpha)u(h^*) = u(f), \alpha \psi + (1 - \alpha)u(h^*) = u(g) \). By assumption \( f \succ g \) and hence by the definition \( \varphi \sim \psi \).

\[\square\]
Lemma 50 Let \( \succeq \) satisfy Axioms 1B-5B. \( \succeq \) is reflexive, nontrivial, transitive, continuous, Independence, and monotonic.

Proof. (1) Nontrivial: By Axiom 1B exists \( f, g \in H \) such that \( f \succ g \) and hence by Lemma 49 \( \text{pr}_{S^p} u(f) \preceq \text{pr}_{S^p} u(g) \).

(2) Reflexive: Let \( \varphi \in \mathbb{R}^{S^p} \). Since \( u(h^*) \in \text{int} u(H) \), there exist \( \alpha \in (0, 1) \), \( f \in H \) such that \( \alpha \varphi + (1 - \alpha)u(h^*) = u(f) \). Since \( \succeq \) is reflexive, \( f \succeq f \) and hence \( \varphi \succeq \varphi \).

(3) Transitive: Let \( \varphi \succeq \psi \) and \( \psi \succeq \theta \). Let \( \alpha, \alpha^2 \in (0, 1] \), \( f, g, g', h \in H \) be such that \( \alpha^1 \varphi + (1 - \alpha^1)u(h^*) = u(f) \), \( \alpha^1 \psi + (1 - \alpha^1)u(h^*) = u(g) \), \( \alpha^2 \psi + (1 - \alpha^2)u(h^*) = u(g') \), \( \alpha^2 \theta + (1 - \alpha^2)u(h^*) = u(h) \) and \( f \succeq g \), \( g' \succeq h \). Assume without loss of generality that \( \alpha^1 \geq \alpha^2 \). By Axiom 5B, \( \alpha^2/\alpha^1 f + (1 - \alpha^2/\alpha^1)h^* \succeq \alpha^2/\alpha^1 g + (1 - \alpha^2/\alpha^1)h^* \). Additionally by the linearity of \( u \), \( u(\alpha^2/\alpha^1 g + (1 - \alpha^2/\alpha^1)h^*) = u(g') \). Since \( u \) represents \( \succeq \) and by Lemma 26, \( \alpha^2/\alpha^1 g + (1 - \alpha^2/\alpha^1)h^* \sim g' \).

By Axioms 1B and 5B,

\[
\alpha^2/\alpha^1 f + (1 - \alpha^2/\alpha^1)h^* \succeq \alpha^2/\alpha^1 g + (1 - \alpha^2/\alpha^1)h^* \succeq g' \succeq h
\]

and \( \alpha^2 \varphi + (1 - \alpha^2)u(h^*) = u(\alpha^2 f + (1 - \alpha^2)h^*) \) and \( \alpha^2 \theta + (1 - \alpha^2)u(h^*) = u(h) \). Hence \( \varphi \succeq \theta \).

(4) Continuous: Let \( \varphi, \psi, \theta \in \mathbb{R}^{S^p} \). Let \( \alpha^* \in (0, 1) \), \( f, g, h \in H \) be such that \( \alpha^* \varphi + (1 - \alpha^*)u(h^*) = u(f) \), \( \alpha^* \psi + (1 - \alpha^*)u(h^*) = u(g) \), \( \alpha^* \theta + (1 - \alpha^*)u(h^*) = u(h) \). Now for all \( \alpha \in [0, 1] \),

\[
\alpha^* (\alpha \varphi + (1 - \alpha) \psi) + (1 - \alpha^*)u(h^*) = u(\alpha f + (1 - \alpha) g).
\]

By Lemma 49 for all \( \alpha \in [0, 1] \),

\[
\alpha \varphi + (1 - \alpha) \psi \succeq \theta \iff \alpha f + (1 - \alpha) g \succeq h
\]

and

\[
\theta \succeq \alpha \varphi + (1 - \alpha) \psi \iff h \succeq \alpha f + (1 - \alpha) g.
\]

Thus

\[
\{ \alpha \in [0, 1] | \alpha \varphi + (1 - \alpha) \psi \succeq \theta \} = \{ \alpha \in [0, 1] | \alpha f + (1 - \alpha) g \succeq h \}
\]

and

\[
\{ \alpha \in [0, 1] | \theta \succeq \alpha \varphi + (1 - \alpha) \psi \} = \{ \alpha \in [0, 1] | h \succeq \alpha f + (1 - \alpha) g \}.
\]

By Axiom 2B, \( \{ \alpha \in [0, 1] | h \succeq \alpha f + (1 - \alpha) g \} \) and \( \{ \alpha \in [0, 1] | \alpha f + (1 - \alpha) g \succeq h \} \) are closed and hence \( \{ \alpha \in [0, 1] | \alpha \varphi + (1 - \alpha) \psi \succeq \theta \} \) and \( \{ \alpha \in [0, 1] | \theta \succeq \alpha \varphi + (1 - \alpha) \psi \} \) are closed.
(5) Independence: Let \( \varphi, \psi, \theta \in \mathbb{R}^{S_P}, \alpha \in (0, 1) \). Let \( \alpha^* \in (0, 1), f, g, h \in H \) be such that \( \alpha^*\varphi + (1 - \alpha^*)u(h^*) = u(f), \alpha^*\psi + (1 - \alpha^*)u(h^*) = u(g), \alpha^*\theta + (1 - \alpha^*)u(h^*) = u(h) \). By Lemma 49 and Axiom 5B

\[
\varphi \succsim \psi \iff f \succ g \iff \alpha f + (1 - \alpha)h \succsim \alpha g + (1 - \alpha)h \iff \alpha\varphi + (1 - \alpha) \vartheta \succsim \alpha\psi + (1 - \alpha)\theta.
\]

(6) Monotonic: Let \( \varphi, \psi \in \mathbb{R}^{S_P} \) be such that for all \( s \in S_P, \varphi_s \geq \psi_s \). Let \( \alpha^* \in (0, 1), f, g \in H \) be such that \( \alpha^*\varphi + (1 - \alpha^*)u(h^*) = u(f), \alpha^*\psi + (1 - \alpha^*)u(h^*) = u(g) \). Now for all \( s \in S_P \)

\[
u_s(f) = \alpha^*\varphi_s + (1 - \alpha^*)u_s(h^*) \geq \alpha^*\psi_s + (1 - \alpha^*)u_s(h^*) = u_s(g).
\]

Since for all \( s \in S \), \( u_s \) represents \( \succsim_s \), we have for all \( s \in S \)

\[
f_s \succsim g_s.
\]

Thus by Lemma 26, \( f \succsim g \) and hence \( \varphi \succsim \psi \).

\( \square \)

**Proposition 51** \( \succsim \) satisfies Axioms 1B-5B iff. there exists a pair \((u, C)\) that give a Bewley representation for \( \succsim \).

**Proof.** By Ghirardato et al.’s (2004) Theorem A.2 and Gilboa et al.’s (2010) Lemma 3, exists closed and convex set \( C \subseteq \Delta(S) \) such that for all \( \varphi, \psi \in \mathbb{R}^{S_P} \)

\[
\varphi \succsim \psi \iff \forall p \in C, \varphi \cdot p \geq \psi \cdot p.
\]

Thus \((u, C)\) gives a Bewley representation for \( \succsim \) since for all \( f, g \in H \)

\[
f \succsim g \overset{\text{Lem. 49}}{\iff} u(f) \succsim u(g) \iff \forall p \in C, u(f) \cdot p \geq u(g) \cdot p.
\]

Let us now prove the only if direction.

(1) Nontrivial: Assume, per contra, for all \( f, g \in H \) \( f \sim h \). Now by the definition of null-states \( S_P = \emptyset \) and hence for all \( s \in S, p \in C, p_s = 0 \) which is a contradiction since \( C \neq \emptyset \).

(2) Reflexive: Let \( f \in H \). Now for all \( p \in C \),

\[
p \cdot u(f) = p \cdot u(f)
\]

and hence \( f \succsim f \).

(3) Transitive: Let \( f, g, h \in H, f \succsim g, \) and \( g \succsim h \). Now by the definition of \( \succsim \) for all \( p \in C \)

\[
p \cdot u(f) \geq p \cdot u(g) \text{ and } p \cdot u(g) \geq p \cdot u(h)
\]

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Thus \( f \succsim h \).

(4) Axiom 2B: Let \( f, g, h \in H \). Define for all \( p \in C \) an operator \( I_p : [0, 1] \to \mathbb{R} \),

\[
\alpha \mapsto \alpha \left( p \cdot (u(f) - u(g)) \right) + p \cdot u(g) - p \cdot u(h).
\]

As a linear function for all \( p \in C \), \( I_p \) is continuous. Now

\[
\{ \alpha \in [0, 1] | \alpha f + (1 - \alpha)g \succsim h \} = \{ \alpha \in [0, 1] | \forall p \in C p \cdot u(\alpha f + (1 - \alpha)g) \geq p \cdot u(h) \}
\]

\[
= \{ \alpha \in [0, 1] | \forall p \in C, \alpha \left( p \cdot (u(f) - u(g)) \right) + p \cdot u(g) - p \cdot u(h) \geq 0 \}
\]

\[
= \{ \alpha \in [0, 1] | \forall p \in C, I_p(\alpha) \geq 0 \} = \bigcap_{p \in C} I_p^{-1}[0, \infty)
\]

and symmetrically

\[
\{ \alpha \in [0, 1] | h \succsim \alpha f + (1 - \alpha)g \} = \bigcap_{p \in C} I_p^{-1}(-\infty, 0]
\]

are closed as the intersections of preimages of a continuous function over closed sets.

(5) Axiom 3B: Let \( s^* \in S, f \in H, x_{s^*}, y_{s^*} \in \Delta(X_{s^*}) \). Assume w.l.o.g. \( u_{s^*}(x_{s^*}) \geq u_{s^*}(y_{s^*}) \). Now for all \( p \in C \)

\[
p \cdot (x_{s^*}, f_{-s^*}) = \sum_{s \in S \setminus \{s^*\}} p_s u_s(f_s) + p_{s^*} u_{s^*}(x_{s^*}) \geq \sum_{s \in S \setminus \{s^*\}} p_s u_s(f_s) + p_{s^*} u_{s^*}(y_{s^*}) = p \cdot (y_{s^*}, f_{-s^*}).
\]

Thus \((x_{s^*}, f_{-s^*}) \succsim (y_{s^*}, f_{-s^*})\).

(6) Axiom 4B: Let \( s^* \in S, f, g \in H, x_{s^*}, y_{s^*} \in \Delta(X_{s^*}) \) and \((x_{s^*}, f_{-s^*}) \succsim (y_{s^*}, f_{-s^*})\). If \( s^* \notin S^p \), then by definition \((x_{s^*}, g_{-s^*}) \succsim (y_{s^*}, g_{-s^*})\). Thus assume \( s^* \in S^p \). Thus since \( C \neq \emptyset \), exists \( p^* \in C \) such that \( p^*_s \). Since \((x_{s^*}, f_{-s^*}) \succsim (y_{s^*}, f_{-s^*})\), especially

\[
p^* \cdot (x_{s^*}, f_{-s^*}) \geq p^* \cdot (y_{s^*}, f_{-s^*}).
\]

Hence

\[
\sum_{s \in S \setminus \{s^*\}} p^*_s u_s(f_s) + p^*_s u_{s^*}(x_{s^*}) \geq \sum_{s \in S \setminus \{s^*\}} p^*_s u_s(f_s) + p^*_s u_{s^*}(y_{s^*})
\]

\[
\iff p^*_s u_s(x_{s^*}) \geq p^*_s u_{s^*}(y_{s^*}) \iff u_{s^*}(x_{s^*}) \geq u_{s^*}(y_{s^*}).
\]
Thus by the above proof for Axiom 3B

$$(x_s^*, g_{-s}^*) \succsim (y_s^*, g_{-s}^*).$$

(7) Axiom 5B: Let $f, g, h \in H, \alpha \in (0, 1)$. By the linearity of $u$, we have

$$f \succsim g \iff \forall p \in C, \sum_{s \in S} p_s u_s(f_s) \geq \sum_{s \in S} p_s u_s(g_s)$$

$$\iff \forall p \in C, \sum_{s \in S} p_s \alpha u_s(f_s) + \sum_{s \in S} p_s (1 - \alpha) u_s(h_s) \geq \sum_{s \in S} p_s \alpha u_s(g_s) + \sum_{s \in S} p_s (1 - \alpha) u_s(h_s)$$

$$\iff \forall p \in C, \sum_{s \in S} p_s u_s(\alpha f_s + (1 - \alpha) h_s) \geq \sum_{s \in S} p_s u_s(\alpha g_s + (1 - \alpha) h_s)$$

$$\iff \alpha f_s + (1 - \alpha) h_s \succsim \alpha g_s + (1 - \alpha) h_s$$

Next we show the uniqueness of the representation. First some notation. First let $f^*, g^* \in H$ be such that for all $s \in S^P f_s^* \succsim s g_s^*$ which exist by Axiom 1B and $h^* := \frac{1}{2}f^* + \frac{1}{2}g^*$.

**Lemma 52** Let $\succsim$ be an order on $H$, $(u, C)$ be a state dependent Bewley representation for $\succsim$. $s^* \notin S^P$ if and only if for all $p \in C, p_{s^*} = 0$.

**Proof.** Let us first prove the if direction. Let $s^* \in S$ be such that for all $p \in C, p_{s^*} = 0$. Let $x_{s^*}, y_{s^*} \in \Delta(X_{s^*}), f \in H$. Now for all $p \in C$

$$p \cdot u((x_{s^*}, f_{-s^*})) = \sum_{s \in S \setminus \{s^*\}} p_s u_s(f_s) + p_{s^*} u_{s^*}(x_{s^*}) \overset{p_{s^*} = 0}{=} \sum_{s \in S \setminus \{s^*\}} p_s u_s(f_s) + p_{s^*} u_{s^*}(y_{s^*}) = p \cdot u((y_{s^*}, f_{-s^*})).$$

and thus by the representation

$$(x_{s^*}, f_{-s^*}) \sim (y_{s^*}, f_{-s^*}).$$

Hence by the definition $s^*$ is null.

The only if direction follows directly from the definition of a Bewley representation.

**Lemma 53** Let $\succsim$ be an order on $H$, $(u, C)$ be a state dependent Bewley representation for $\succsim$, $s^* \in S^P, x_{s^*}, y_{s^*} \in \Delta(X_{s^*})$ and $\succsim_{s^*}$ as defined in Section 2.3. Let $\succ$ and $\succsim_{s^*}$ be defined as in Section 2.1. If $x_{s^*} \succsim_{s^*} y_{s^*}$, then $u_{s^*}(x_{s^*}) > u_{s^*}(y_{s^*})$.

**Proof.** Assume, per contra, $u_{s^*}(y_{s^*}) \geq u_{s^*}(x_{s^*})$. Now for all $p \in C, f \in H$

$$\sum_{s \in S \setminus \{s^*\}} p_s u_s(f_s) + p_{s^*} u_{s^*}(y_{s^*}) \geq \sum_{s \in S \setminus \{s^*\}} p_s u_s(f_s) + p_{s^*} u_{s^*}(x_{s^*}).$$
Thus by the representation for all \( f \in H \)
\[
(y_s^*, f_{-s}^*) \preceq (x_s^*, f_{-s}^*).
\]

On the other hand, by the definition of \( \succeq_s \) since \( x_s^* \succ_s y_s^* \) i.e. \( x_s^* \succeq_s y_s^* \) and \( y_s^* \not\succeq_s x_s^* \), for all \( f \in H \) \((x_s^*, f_{-s}^*) \succeq (y_s^*, f_{-s}^*)\) and there exists \( f^0 \in H \) such that \((y_s^*, f^0_{-s}^*) \not\succeq (x_s^*, f^0_{-s}^*)\) which is a contradiction with the above.

\[\square\]

**Corollary 54** Let \( \succeq \) satisfy Axioms 1\(^B\)-5\(^B\) and \((u, C)\) be a Bewley representation for \( \succeq \). Then \( \pr_{SP} u(h^*) \in \mathrm{int} \pr_{SP} u(H) \).
\[
\begin{proof}
\text{Let} \ s^* \in SP, \text{Now by Lemma 48} \ f_s^* \succ_s h_s^* \succ_s g_s^*. \text{Hence by Lemma 53} \ u_{s^*}(f_s^*) > u_{s^*}(h_s^*) > u_{s^*}(g_s^*). \text{Thus by the linearity of} \ u_{s^*}, \text{we have} \ u_{s^*}(h_s^*) \in \mathrm{int} u_{s^*}(\Delta(X_s^*)). \text{Since} \ s^* \in SP \text{was arbitrary,} \ pr_{SP} u(h^*) \in \mathrm{int} pr_{SP} u(H).
\end{proof}
\]

**Lemma 55** Let \( \succeq \) be an order on \( H, (u, C) \) be a state dependent Bewley representation for \( \succeq, s^* \in SP, x_s^*, y_s^* \in \Delta(X_s^*) \) and \( \succeq_s \) as defined in Section 2.3. Let \( \succ \) and \( \succ_s \) be defined as in Section 2.1. If \( x_s^* \sim_s y_s^* \), then \( u_{s^*}(x_s^*) = u_{s^*}(y_s^*) \).
\[
\begin{proof}
\text{Assume, per contra without loss of generality,} \ u_{s^*}(y_s^*) > u_{s^*}(x_s^*). \text{Since} s^* \text{is not null, by Lemma 52 exists} p^0 \in C \text{such that} p^0_{s^*} > 0. \text{Let} f^0 \in H. \text{Now} \ p^0 \cdot u((x_s^*, f^0_{-s}^*)) = \sum_{s \in S \setminus \{s^*\}} p^0_{s^*} u_s(f^0_s) + p^0_{s^*} u_{s^*}(x_s^*) < \sum_{s \in S \setminus \{s^*\}} p^0_{s^*} u_s(f^0_s) + p^0_{s^*} u_{s^*}(y_s^*) = p^0 \cdot u((y_s^*, f^0_{-s}^*)).
\end{proof}
\]

Thus by the representation
\[(x_s^*, f^0_{-s}^*) \not\preceq (y_s^*, f^0_{-s}^*)\]
which is a contradiction since \( x_s^* \sim_s y_s^* \) and hence especially for all \( f \in H \)
\[(x_s^*, f_{-s}^*) \not\preceq (y_s^*, f_{-s}^*). \]

Thus \( u_{s^*}(x_s^*) = u_{s^*}(y_s^*) \).
\[\square\]

**Proposition 56** If \((u, C)\) and \((\bar{u}, \bar{C})\) are bewley representations for \( \succeq \), then exists \( a \in \mathbb{R}_+^S, b \in R^S \) such that
\[
\bar{C} = \{ \bar{p} \in \Delta(S) \mid \exists p \in C, \forall s \in S, \bar{p} = \frac{a_{s^*}^{-1} p_s}{\sum_{s \in S} a_{s^*}^{-1} p_s} \}
\]
and for all \( s \in SP \)
\[
\bar{u}_s = a_s u_s + b_s.
\]

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Proof. Denote

\[ u_{S^+} := pr_{S^+} u \quad \text{and} \quad \tilde{u}_{S^-} := pr_{S^+} \tilde{u} \]

and for all \( D \subseteq \Delta(S) \)

\[ D_{S^+} := pr_{S^+} D. \]

By Lemmas 53 and 55, for all \( s \in S^p \) \( u_s \) and \( \tilde{u}_s \) represents \( \succsim_s \). Additionally, by Lemma 48 \( \succsim_s \) satisfies, completeness, transitivity, mixture continuity, and independence and \( u_s \) and \( \tilde{u}_s \) are linear representations, hence by Von Neumann–Morgenstern utility theorem (Fishburn, 1970), exists \( a \in R^+_s \), \( b \in R^s \) such that when \( A \) denotes the diagonal matrix created by

\[ \tilde{u}_{S^-} = Au_{S^-} + b. \]

Let for all \( s \in S \setminus S^p \), \( a_s = 1 \). Define

\[ \tilde{C} := \left\{ \tilde{p} \in \mathbb{R}^S \mid \exists p \in C, \forall s \in S, \tilde{p}_s = \frac{a_s^{-1} p_s}{\sum_{s \in S} a_s^{-1} p_s} \right\}. \]

Now \( \tilde{C} \subseteq \Delta(S) \) since for all \( s \in S \) \( a_s > 0 \). Additionally, for all \( \tilde{p} \in \tilde{C}, s \notin S^p \),

\[ \tilde{p}_s = 0. \quad (35) \]

Now define \( \preceq \tilde{C} \) and \( \succeq \tilde{C} \) on \( R^S \) by for all \( \varphi, \psi \in R^S \),

\[ \varphi \preceq \tilde{C} \psi \iff \forall \tilde{p} \in \tilde{C}_{S^-}, p \cdot \varphi \geq p \cdot \psi \]

and

\[ \varphi \succeq \tilde{C} \psi \iff \forall \tilde{p} \in \tilde{C}_{S^-}, p \cdot \varphi \geq p \cdot \psi. \]

Let \( \alpha^* \in (0, 1] \) be such that \( \alpha^* A^{-1} \varphi + (1 - \alpha^*) u_{S^-}(h^*) \), \( \alpha^* A^{-1} \psi + (1 - \alpha^*) u_{S^-}(h^*) \) \( \in u_{S^-}(H) \),

which exists by Corollary 54 and let \( f, g \in H \) be such that

\[ \alpha^* A^{-1} \varphi + (1 - \alpha^*) u_{S^-}(h^*) = u(f) \quad \text{and} \quad \alpha^* A^{-1} \psi + (1 - \alpha^*) u_{S^-}(h^*) = u(g). \]

Now we have

\[ \varphi \preceq \tilde{C} \psi \iff \forall p \in \tilde{C}_{S^-}, p \cdot \varphi \geq p \cdot \psi \]

\[ \iff \forall \tilde{p} \in \tilde{C}_{S^-}, \alpha^* \tilde{p} \cdot \varphi + (1 - \alpha^*) \tilde{p} \cdot Au_{S^-}(h^*) \geq \alpha^* \tilde{p} \cdot \psi + (1 - \alpha^*) \tilde{p} \cdot Au_{S^-}(h^*) \]

\[ \iff \forall \tilde{p} \in \tilde{C}_{S^-}, \tilde{p} \cdot A \left( \alpha^* A^{-1} \varphi + (1 - \alpha^*) u_{S^-}(h^*) \right) \geq \tilde{p} \cdot \left( \alpha^* A^{-1} \psi + (1 - \alpha^*) u_{S^-}(h^*) \right) \]

\[ \iff \forall p \in C_{S^-}, \frac{A^{-1} p}{\sum_{s \in S} p_s a_s^{-1}} \cdot A \left( \alpha^* A^{-1} \varphi + (1 - \alpha^*) u_{S^-}(h^*) \right) \]
\[ \geq \frac{A^{-1}p}{\sum_{s \in S} p_s a_s^{-1}} \cdot A \left( \alpha^* A^{-1} \psi + (1 - \alpha^*) u_{S^-}(h^*) \right) \]

\[ \iff \forall p \in C_{S^-}, p \cdot \left( \alpha^* A^{-1} \varphi + (1 - \alpha^*) u_{S^-}(h^*) \right) \geq p \cdot \left( \alpha^* A^{-1} \psi + (1 - \alpha^*) u_{S^-}(h^*) \right) \]

\[ \iff \forall p \in C_{S^-}, p \cdot u_{S^-}(f) \geq p \cdot u_{S^-}(g) \iff \forall p \in C, p \cdot u(f) \geq p \cdot u(g) \iff f \succsim g \]

\[ \iff \forall \tilde{p} \in \tilde{C}, \tilde{p} \cdot \tilde{u}(f) \geq \tilde{p} \cdot \tilde{u}(g) \iff \varphi \succsim \tilde{\psi} \]

where the second part last equivalence follows symmetrically to the first part.

Thus \( \succsim \tilde{C} \) has a bewley representation with \((\text{Id}, \tilde{C}_{S^-})\) and \((\text{Id}, \hat{C}_{S^-})\) and hence by Ghirardato et al.’s (2004) Theorem A.2

\[ \tilde{C}_{S^-} = \hat{C}_{S^-}. \]

Finally by Lemma 52 and (35)

\[ \tilde{C} = \hat{C}. \]

This proves the uniqueness. \(\square\)

**Theorem 47 (Bewley Representation)** \( \succsim \) satisfies Axioms 1\(^B\)-5\(^B\) if and only if there exists a pair \((u, C)\) that is a state dependent Bewley representation for \( \succsim \).

Additionally, let \((u, C)\) be a state dependent Bewley representation for \( \succsim \), \( \tilde{u} = (\tilde{u}_s)_{s \in S} \) be such that for all \( s \in S \), \( \tilde{u}_s : \Delta(X_s) \to \mathbb{R} \) is affine and \( \tilde{C} \subseteq \Delta(S) \) be closed and convex. Then \((\tilde{u}, \tilde{C})\) is a state dependent Bewley representation for \( \succsim \) if and only if there exists \( a \in \mathbb{R}_{++}^S \), \( b \in \mathbb{R}^S \) such that

\[ \tilde{C} = \left[ \frac{a_s^{-1} p_s}{\sum_{s' \in S} a_{s'}^{-1} p_{s'}} \right]_{(p,s) \in C \times S} \]

and for all \( s \in S^P \),

\[ \tilde{u}_s = a_s u_s + b_s. \]

**Proof.** Follows from Propositions 51 and 56. \(\square\)

### 8.5 Dual-Self Identification

For this section we assume that \( S = S^P \) since by the definition of dual-self and dual-self variational representations probabilities for null states are always identified and the utilities are not. Thus this is without loss of generality.

With an abuse of definitions we define following.
Definition Let $\succsim \subseteq H \times H$. $(u, I)$ is a state dependent dual-self representation for $\succsim$ if $u = (u_s)_{s \in S}$ and for all $s \in S$ $u_s : \Delta(X_s) \to \mathbb{R}$ is affine, $I : u(H) \to \mathbb{R}$ is C-additive, positive homogeneous, and monotonic and for all $f, g \in H$,

$$f \succsim g \iff I(u(f)) \geq I(u(g)).$$

Definition Let $\succsim \subseteq H \times H$. $(u, I)$ is a state dependent variational dual-self representation for $\succsim$ if $u = (u_s)_{s \in S}$ and for all $s \in S$ $u_s : \Delta(X_s) \to \mathbb{R}$ is affine, $I : u(H) \to \mathbb{R}$ is C-additive, and monotonic and for all $f, g \in H$,

$$f \succsim g \iff I(u(f)) \geq I(u(g)).$$

Since the state space is assumed to be finite $\mathbb{R}^S$ is metrizable space and hence convergence is characterized by the convergence of sequences. Let $f^*, g^*$ be equally crisp acts such that for all $s \in S^P f^*_s \succsim_s g^*_s$ and denote $h^* := \frac{1}{2} f^* + \frac{1}{2} g^*$.

Definition For $f, g \in H, \beta \in (0, 1)$, define for all $\alpha \in (0, 1)$

$$A_{f, \beta}^{\alpha, \alpha} := \left\{ \hat{\alpha} \in [-1/2\alpha, 1/2\alpha]| \beta f + (1 - \beta)\left((\alpha\hat{\alpha} + 1/2)f^* + (1/2 - \alpha\hat{\alpha})g^*\right) \sim \alpha g + (1 - \alpha)\left(\beta f + (1 - \beta)h^*\right) \right\}$$

and

$$C_{f, \beta}^{\alpha, \alpha} = \begin{cases} \arg\min \left\{ |\hat{\alpha}| \mid \hat{\alpha} \in A_{f, \beta}^{\alpha, \alpha} \right\}, & \text{if } A_{f, \beta}^{\alpha, \alpha} \neq \emptyset \\ \infty, & \text{if } A_{f, \beta}^{\alpha, \alpha} = \emptyset \end{cases}$$

where $\sup \emptyset = \infty = \inf \emptyset$.

Lemma 57 Let $f^*, g^*$ and $\tilde{f}^*, \tilde{g}^*$ be equally crisp acts and $\alpha \in [0, 1]$ Then $\alpha f^* + (1-\alpha)\tilde{f}^*, \alpha g^* + (1-\alpha)\tilde{g}^*$ are equally crisp acts.

Proof. If $\alpha \in \{0, 1\}$ then the claim follows directly. So assume $\alpha \in (0, 1)$. Let $f, g \in H, \beta \in (0, 1)$. Now for $h \in \{f, g\}, h^* \in \{f^*, g^*\}, \tilde{h}^* \in \{\tilde{f}^*, \tilde{g}^*\}$

$$ah + (1 - \alpha)(h^* \beta \tilde{h}^*) \equiv (\alpha + \beta - \alpha\beta)\left(\frac{\alpha}{\alpha + \beta - \alpha\beta} h + \frac{\beta - \alpha\beta}{\alpha + \beta - \alpha\beta} h^*\right) + (1 - \alpha - \beta + \alpha\beta)\tilde{h}^* \quad (36)$$

and

$$ah + (1 - \alpha)(h^* \beta \tilde{h}^*) \equiv (1 - \beta + \alpha\beta)\left(\frac{\alpha}{1 - \beta + \alpha\beta} h + \frac{1 - \alpha - \beta + \alpha\beta}{1 - \beta + \alpha\beta} h^*\right) + (1 - (1 - \beta + \alpha\beta))h^*. \quad (37)$$
Thus where equivalences (⋆) follow from the assumptions that \( f^*, g^* \) and \( \tilde{f}^*, \tilde{g}^* \) are equally crisp.

\[
\alpha f + (1 - \alpha)(f^* \beta \tilde{f}^*) \succsim \alpha g + (1 - \alpha)(f^* \beta \tilde{f}^*)
\]

\[
\iff (\alpha + \beta - \alpha \beta)(\frac{\alpha}{\alpha + \beta - \alpha \beta} f + \frac{\beta - \alpha \beta}{\alpha + \beta - \alpha \beta} f^*) + (1 - \alpha - \beta + \alpha \beta)\tilde{f}^*
\]

\[
\succsim (\alpha + \beta - \alpha \beta)(\frac{\alpha}{\alpha + \beta - \alpha \beta} g + \frac{\beta - \alpha \beta}{\alpha + \beta - \alpha \beta} f^*) + (1 - \alpha - \beta + \alpha \beta)\tilde{f}^*
\]

\[
\iff (\alpha + \beta - \alpha \beta)(\frac{\alpha}{\alpha + \beta - \alpha \beta} f + \frac{\beta - \alpha \beta}{\alpha + \beta - \alpha \beta} f^*) + (1 - \alpha - \beta + \alpha \beta)\tilde{g}^*
\]

\[
\succsim (\alpha + \beta - \alpha \beta)(\frac{\alpha}{\alpha + \beta - \alpha \beta} g + \frac{\beta - \alpha \beta}{\alpha + \beta - \alpha \beta} f^*) + (1 - \alpha - \beta + \alpha \beta)\tilde{g}^*
\]

\[
\iff (\alpha + \beta - \alpha \beta)(\frac{\alpha}{1 - \beta + \alpha \beta} f + \frac{1 - \alpha - \beta + \alpha \beta}{1 - \beta + \alpha \beta} g^*) + (1 - (1 - \beta + \alpha \beta)) f^*
\]

\[
\succsim (1 - \beta + \alpha \beta)(\frac{\alpha}{1 - \beta + \alpha \beta} g + \frac{1 - \alpha - \beta + \alpha \beta}{1 - \beta + \alpha \beta} g^*) + (1 - (1 - \beta + \alpha \beta)) f^*
\]

\[
\iff (1 - \beta + \alpha \beta)(\frac{\alpha}{1 - \beta + \alpha \beta} f + \frac{1 - \alpha - \beta + \alpha \beta}{1 - \beta + \alpha \beta} \tilde{g}^*) + (1 - (1 - \beta + \alpha \beta)) g^*
\]

\[
\succsim (1 - \beta + \alpha \beta)(\frac{\alpha}{1 - \beta + \alpha \beta} g + \frac{1 - \alpha - \beta + \alpha \beta}{1 - \beta + \alpha \beta} \tilde{g}^*) + (1 - (1 - \beta + \alpha \beta)) g^*
\]

\[
\iff (\alpha + \beta - \alpha \beta)(\frac{\alpha}{\alpha + \beta - \alpha \beta} f + \frac{\beta - \alpha \beta}{\alpha + \beta - \alpha \beta} f^*) + (1 - \alpha - \beta + \alpha \beta)\tilde{g}^*
\]

\[
\succsim (\alpha + \beta - \alpha \beta)(\frac{\alpha}{\alpha + \beta - \alpha \beta} g + \frac{\beta - \alpha \beta}{\alpha + \beta - \alpha \beta} f^*) + (1 - \alpha - \beta + \alpha \beta)\tilde{g}^*.
\]

\[\square\]

**Corollary 58** Let \( f^*, g^* \) be such that for all \( f, g \in H \) and \( \alpha \in (0, 1) \)

\[
\alpha f + (1 - \alpha)f^* \succsim \alpha g + (1 - \alpha)f^* \iff \alpha f + (1 - \alpha)g^* \succeq \alpha g + (1 - \alpha)g^*.
\]

Then for all \( \beta \in (0, 1) \)

\[
\alpha f + (1 - \alpha)f^* \succsim \alpha g + (1 - \alpha)f^* \iff \alpha f + (1 - \alpha)(f^* \beta g^*) \succsim \alpha g + (1 - \alpha)(f^* \beta g^*).
\]

**Proof.** Now \( f^*, g^* \) are equally crisp and \( f^* \) and \( f^* \) are equally crisp thus for \( \beta \in (0, 1) \) by Lemma 57 \( f^*, \beta f^* + (1 - \beta)g^* \) are equally crisp which shows the claim. \[\square\]

**Corollary 59** Let \( f^*, g^* \) be equally crisp acts and \( \alpha, \beta \in [0, 1] \) Then \( \alpha f^* + (1 - \alpha)g^*, \beta f^* + (1 - \beta)g^* \) are equally crisp acts.
Proof. Assume w.l.o.g. \( \alpha \geq \beta \). Now \((f^*, g^*)\) and \((g^*, g^*)\) are equally crisp by Lemma 57 \( (g^*, \frac{\alpha - \beta}{1 - \beta} f^* + (1 - \frac{\alpha - \beta}{1 - \beta}) g^*) \) are equally crisp. Now since \((f^*, f^*)\) and \((g^*, \frac{\alpha - \beta}{1 - \beta} f^* + (1 - \frac{\alpha - \beta}{1 - \beta}) g^*)\) are equally crisp by Lemma 57 \( (\beta f^* + (1 - \beta) g^*, \alpha f^* + (1 - \alpha) g^*) \) are equally crisp since 
\[
\beta f^* + (1 - \beta) \left( \frac{\alpha - \beta}{1 - \beta} f^* + \left( 1 - \frac{\alpha - \beta}{1 - \beta} \right) g^* \right) \equiv \alpha f^* + (1 - \alpha) g^*.
\]
\[
\square
\]

**Corollary 60** Let \( f^*, g^* \) be equally crisp acts and \( f \in H, \beta \in (0, 1) \) Then \( \beta f + (1 - \beta)f^*, \beta f + (1 - \beta)g^* \) are equally crisp acts.

**Proof.** Now \( f^*, g^* \) are equally crisp and \( f \) and \( f \) are equally crisp thus for \( \beta \in (0, 1) \) by Lemma 57 \( \beta f + (1 - \beta)f^*, \beta f + (1 - \beta)g^* \) are equally crisp which shows the claim. \( \square \)

**Lemma 61** Let \((u, I)\) be a dual-self variational representation for \( \succeq \) and \( f^*, g^* \) equally crisp acts. Let \( f \in H, \beta \in (0, 1) \) be such that \( f \in \text{int} u(H) \). Then the mapping \( (0, 1) \ni \alpha \mapsto I \circ u \left( \beta f + (1 - \beta)(f^* g^*) \right) \) has locally constant derivative at the differentiability points.

**Proof.** Let \( \Omega \) be the differentiability points of the mapping \( J \). Let us assume per contra that exists a point \( x \in (0, 1) \) such that \( x \) does not have a neighborhood \( U \) such that for all \( y, z \in U \cap \Omega \) \( \nabla J(y) = \nabla J(z) \). First since \( u(f) \in \text{int} u(H) \), exists \( \varepsilon > 0 \) such that \( B_\varepsilon(u(f), \varepsilon) \subseteq u(H) \). Second, since \( J \) is lipschitz, by the counter assumption exists \( y, z \in (0, 1) \) such that \( |y - z| < \frac{\min(x,1-x)}{3} \), \( \nabla J(y) > \nabla J(z) \). Since \( I \) is C-additive and hence especially 1-lipschitz, we have \( \frac{1}{\beta} |J(y) - J(z)| \leq \frac{1}{\beta} \|u(f^*) - u(g^*)\|_\infty \leq |y - z| < \varepsilon \). Thus exists \( g \in H \) such that 
\[
u(g) = u(f) + \frac{1}{\beta} I(J(z) - J(y))\]
and by C-additivity, we have since \( \frac{1}{2}(1 + \beta) = \beta + (1 - \beta)/2 \) and \( 1 - \frac{1}{2}(1 + \beta) = 1/2(1 - \beta) \)
\[
I \left( \frac{1}{2}(1 + \beta) \frac{\beta}{1/2(1 + \beta)} u(g) + \frac{(1 - \beta)/2}{1/2(1 + \beta)} u(f^* g^*) \right) + \left( 1 - \frac{1}{2}(1 + \beta) \right) u(f^* g^*)
\]
\[
= I \left( \beta u(f) + \frac{1}{2}(1 - \beta) u(f^* g^*) + \frac{1}{2}(1 - \beta) u(f^* g^*) \right) + \frac{1}{2}(1 - \beta) u(f^* g^*)
\]
\[
\overset{\text{C-add.}}{=} J(y) + J(z) - J(y) = J(z)
\]
Additionally since \( 1 > 2z - y > 0 \), we have
\[
I \left( \frac{1}{2}(1 + \beta) \frac{\beta}{1/2(1 + \beta)} u(f) + \frac{(1 - \beta)/2}{1/2(1 + \beta)} u(f^*(2z - y)g^*) \right) + \left( 1 - \frac{1}{2}(1 + \beta) \right) u(f^* g^*)
\]
\[
= I \left( \beta u(f) + \frac{1}{2}(1 - \beta) u(f^*(2z - y)g^*) + \frac{1}{2}(1 - \beta) u(f^* g^*) \right) = J(z).
\]
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Thus by the representation
\[
1/2(1 + \beta)\left(\frac{\beta}{1/2(1 + \beta)}g + \frac{(1-\beta/2)}{1/2(1 + \beta)}(f^*yg^*)\right) + \left(1 - 1/2(1 + \beta)\right)(f^*yg^*)
\] (38)
\[
\sim 1/2(1 + \beta)\left(\frac{\beta}{1/2(1 + \beta)}f + \frac{(1-\beta/2)}{1/2(1 + \beta)}(f^*(2z - y)g^*)\right) + \left(1 - 1/2(1 + \beta)\right)(f^*yg^*).
\]

But since \(\nabla J(y) > \nabla J(z)\), exists \(0 < \delta < \frac{1-x}{3}\) such that
\[
\frac{J(y + \delta) - J(y)}{\delta} > \frac{J(z + \delta) - J(z)}{\delta} \Rightarrow J(y + \delta) + J(z) - J(y) > J(z + \delta).
\] (39)

Thus since \(y + 2\delta < 1\)
\[
I\left(\frac{1}{2}(1 + \beta)\left(\frac{\beta}{1/2(1 + \beta)}u(g) + \frac{(1-\beta/2)}{1/2(1 + \beta)}u(f^*yg^*)\right) + \left(1 - 1/2(1 + \beta)\right)u(f^*(y + 2\delta)g^*)\right)
\]
\[
= I\left(\beta u(f) + \frac{1/2(1 - \beta)}{1/2(1 -\beta)}u(f^*yg^*) + \frac{1/2(1 - \beta)}{1/2(1 -\beta)}u(f^*(y + 2\delta)g^*) + \beta(1/\beta) - 1 \times (J(z) - J(y))\right)
\]
\[
\Rightarrow J(y + \delta) + J(z) - J(y)(39) > J(z + \delta)
\]
\[
= I\left(\beta u(f) + \frac{1/2(1 - \beta)}{1/2(1 -\beta)}u(f^*(2z - y)g^*) + \frac{1/2(1 - \beta)}{1/2(1 -\beta)}u(f^*(y + 2\delta)g^*)\right)
\]
\[
= I\left(\frac{1}{2}(1 + \beta)\left(\frac{\beta}{1/2(1 + \beta)}u(f) + \frac{(1-\beta/2)}{1/2(1 + \beta)}u(f^*(2z - y)g^*)\right) + \left(1 - 1/2(1 + \beta)\right)u(f^*(y + 2\delta)g^*)\right).
\]

Thus by the representation
\[
1/2(1 + \beta)\left(\frac{\beta}{1/2(1 + \beta)}g + \frac{(1-\beta/2)}{1/2(1 + \beta)}(f^*yg^*)\right) + \left(1 - 1/2(1 + \beta)\right)(f^*(y + 2\delta)g^*)
\]
\[
\sim 1/2(1 + \beta)\left(\frac{\beta}{1/2(1 + \beta)}f + \frac{(1-\beta/2)}{1/2(1 + \beta)}(f^*(2z - y)g^*)\right) + \left(1 - 1/2(1 + \beta)\right)(f^*(y + 2\delta)g^*).
\]

but this contradicts Corollary 58 and (38). \(\square\)

Lemma 62 Let \((u, I)\) be a variational representation for \(\succeq\) and \(f^*, g^*\) equally crisp acts. Let \(f \in H, \beta \in (0, 1)\) be such that \(f \in \text{int } u(H)\). Then the mapping \([0, 1] \ni \alpha \mapsto I \circ u\left(\beta f + (1 - \beta)(f^*\alpha g^*)\right)\) is affine.

Proof. Since \(I\) is C-additive and hence 1-lipschitz, especially \(J\) is lipschitz and it is differentiable almost everywhere. Let the set of differentiability points be \(\Omega\). Define
\[
\tilde{J}(\alpha) = \begin{cases} 
  \nabla J(\alpha), & \text{if } J \text{ is differentiable at } \alpha \\
  \liminf_{\alpha \to \alpha} \nabla J(\tilde{\alpha}), & \text{if } J \text{ is not differentiable at } \alpha
\end{cases}
\]
Since \(J\) is differentiable almost everywhere \(\tilde{J}\) is well-defined. Let \(0 < \varepsilon < 1/2\). Let us show that \(\tilde{J}\) is constant in the set \((0, 1)\). By Lemma 61, for all \(\alpha \in (0, 1)\) exists \(\varepsilon_\alpha > 0, c \in \mathbb{R}\) such that for all \(x \in B_\infty(\alpha, \varepsilon_\alpha) \cap \Omega \nabla J(x) = c\). Thus by the definition of \(\tilde{J}\), for all \(x \in B_\infty(\alpha, \varepsilon_\alpha) \tilde{J}(x) = c\).
Thus $\bar{J}$ is locally constant function in the set $(0,1)$. Since the interval $(0,1)$ is connected, by (Viro et al., 2008, Problem 12.2x) $J'$ is constant on the set $(0,1)$. Thus exists $c \in \mathbb{R}$ such that for all $x \in (0,1) \cap \Omega \ \nabla J(x) = c$. Thus since $J$ is lipschitz, we have for all $x \in [0,1]$

$$J(x) - J(0) = \int_{[0,x] \cap \Omega} \nabla J(y) \, dy = \int_{[0,x] \cap \Omega} \nabla J(y) \, dy = \int_{[0,x] \cap \Omega} c \, dy = cx.$$ 

Thus especially the mapping $J$ is affine.

**Lemma 63** Let $(u, I)$ be a dual-self variational representation for $\preceq$ and $f^*, g^*$ equally crisp acts and denote $h^* := f^{*1/2}g^*$. Let $f, g \in H, \beta, \alpha \in (0,1)$ be such that $f, g \in \text{int} \ u(H)$ and $\alpha f + (1 - \alpha)h^* \sim \beta g + (1 - \beta)h^*$. Then the affine mappings $[0,1] \ni \gamma \mapsto I \circ u \left( \alpha f + (1 - \alpha)(f^* g^*) \right)$ and $[0,1] \ni \gamma \mapsto I \circ u \left( \beta g + (1 - \beta)(f^* g^*) \right)$ with derivatives $\nabla J^f$ and $\nabla J^g$ respectively has $\nabla J^f = \frac{1-\alpha}{1-\beta} \nabla J^g$.

**Proof.** Assume w.l.o.g. $\alpha \geq \beta$. Then

$$\alpha \left( \frac{\beta}{\alpha} g + \frac{\alpha - \beta}{\alpha} h^* \right) + (1 - \alpha)h^* \equiv \beta g + (1 - \beta)h^*.$$

Let us define the mapping

$$[0,1] \ni \gamma \mapsto I \circ u \left( \alpha \left( \frac{\beta}{\alpha} g + \frac{\alpha - \beta}{\alpha} h^* \right) + (1 - \alpha)(f^* g^*) \right)$$

Thus since $f^*$ and $g^*$ are equally crisp acts and by Corollary 58, we have by the representation for all $\gamma \in [0,1]$

$$J^{gh^*}(\gamma) = I \circ u \left( \alpha \left( \frac{\beta}{\alpha} g + \frac{\alpha - \beta}{\alpha} h^* \right) + (1 - \alpha)(f^* g^*) \right) = I \circ u \left( \alpha f + (1 - \alpha)(f^* g^*) \right) = J^f(\gamma).$$

Thus especially $J^{gh^*}$ is affine with derivative $\nabla J^{gh^*} = \nabla J^f$. But on the other hand for all $\gamma \in [0,1]$ we have

$$\alpha \left( \frac{\beta}{\alpha} g + \frac{\alpha - \beta}{\alpha} h^* \right) + (1 - \alpha)(f^* g^*) \equiv \beta g + (1 - \beta) \left( f^* \frac{(\alpha - \beta)/2 + (1 - \alpha)}{1 - \beta} g^* \right)$$

and so

$$J^{gh^*}(\gamma) = I \circ u \left( \alpha \left( \frac{\beta}{\alpha} g + \frac{\alpha - \beta}{\alpha} h^* \right) + (1 - \alpha)(f^* g^*) \right) = I \circ u \left( \beta g + (1 - \beta) \left( f^* \frac{(\alpha - \beta)/2 + (1 - \alpha)}{1 - \beta} g^* \right) \right) = J^g \left( \frac{(\alpha - \beta)/2 + (1 - \alpha)}{1 - \beta} g^* \right).$$

Thus especially by the chain rule of derivatives $\nabla J^{gh^*} = \nabla J^g \frac{1-\alpha}{1-\beta}$ which shows the claim.
Lemma 64 Let \((u, I)\) be a dual-self variational representation for \(\preceq\) and \(f^*, g^*\) equally crisp acts and denote \(h^* := f^{*1/2}g^*\). For all \(f, \beta \in H\), define the mapping \([0,1] \ni \gamma \mapsto I \circ u(\beta f + (1 - \beta)(f^*g^*))\) with derivative \(\nabla J^I_\beta\). Define the mappings \(K^* : (\inf I(u(H)), \sup I(u(H))) \to \mathbb{R} \cup \{\infty\}\) and \(K_* : (\inf I(u(H)), \sup I(u(H))) \to \mathbb{R} \cup \{\infty\}\) by for all \(v \in (\inf I(u(H)), \sup I(u(H)))\)

\[
K^*(v) = \sup \left\{ \frac{1}{1 - \beta} \nabla J^I_\beta \big| f \in H, \beta \in (0,1), u(f) \in \text{int} u(H), I \circ u(\beta f + (1 - \beta)h^*) = v \right\}
\]

and

\[
K_*(v) = \inf \left\{ \frac{1}{1 - \beta} \nabla J^I_\beta \big| f \in H, \beta \in (0,1), u(f) \in \text{int} u(H), I \circ u(\beta f + (1 - \beta)h^*) = v \right\}
\]

where \(\sup \emptyset = \inf \emptyset = \infty\). Then \(K^*\) is constant and finite and \(K^* = K_*\).

**Proof.** Let us show first that locally \(K^*\) is constant and finite and \(K^* = K_*\). Let \(v \in (\inf I(u(H)), \sup I(u(H)))\).

First exists \(f^1, f^2 \in H\) such that \(I(u(f^1)) > v > I(u(f^2))\). Since \(S\) is finite exists \(f^3 \in H\) such that \(u(f^3) \in \text{int} u(H)\). By mixture continuity exists \(\alpha^1, \alpha^2, \alpha^3 \in (0,1)\) such that \(\alpha^1 + \alpha^2 + \alpha^3 = 1\) and \(I \circ u(\alpha^1 f^1 + \alpha^2 f^2 + \alpha^3 f^3) = v\). Denote \(f^4 := \alpha^1 f^1 + \alpha^2 f^2 + \alpha^3 f^3\). Since \(\alpha^3 > 0\) and \(u(f^3) \in \text{int} u(H)\) and \(u(H)\) is a convex set \(u(f^4) \in \text{int} u(H)\). Thus exists \(\varepsilon > 0\) such that \(B_\infty(u(f^4), \varepsilon) \subseteq u(H)\). Let \(\beta \in (1/2, 1)\) be such that

\[
\left(\frac{1}{\beta} - 1\right) \|u(f^4)\|_\infty + \frac{1 - \beta}{\beta} \|u(h^*)\|_\infty < \varepsilon.
\]

Thus by the choice of \(\beta\) exists \(f \in H\) such that

\[
u(f) = u(f^4) + \left(\frac{1}{\beta} - 1\right) u(f^4) - \frac{1 - \beta}{\beta} u(h^*).\]

Thus especially \(u(f) \in \text{int} u(H)\) and

\[
\beta u(f) + (1 - \beta) u(h^*) = u(f^4)
\]

and so \(I(\beta u(f) + (1 - \beta) u(h^*)) = v\). First for all \(\tilde{f} \in H, \tilde{\beta} \in (0,1)\) such that \(u(\tilde{f}) \in \text{int} u(H), I \circ u(\tilde{\beta} \tilde{f} + (1 - \tilde{\beta})h^*) = v\), then \(\tilde{\beta} \tilde{f} + (1 - \tilde{\beta})h^* \sim (1 - \beta)f + (1 - \beta)h^*\) and by Lemma 63

\[
\frac{1}{1 - \beta} \nabla J^\tilde{f}_{\tilde{\beta}} = \frac{1}{(1 - \beta)} \nabla J^f_{\beta} \quad (40)
\]

Thus since \(\tilde{f} \in H, \tilde{\beta} \in (0,1)\) were arbitrary such that \(u(\tilde{f}) \in \text{int} u(H), I \circ u(\tilde{\beta} \tilde{f} + (1 - \tilde{\beta})h^*) = \tilde{v}\)

\[
K^*(v) \overset{(40)}{=} \frac{1}{(1 - \beta)} \nabla J^f_{\beta} \overset{(40)}{=} K_*(v). \quad (41)
\]

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Let us consider two cases: 1) exists $\alpha^0 \in (0, 1) \setminus \{1/2\}$ $\beta f + (1 - \beta)h^* \sim \beta f + (1 - \beta)f^*\alpha^0 g^*$. Now $J^{\beta}_f(1/2) = J^\beta_2(\alpha^0)$ and by Lemma 62 $J^{\beta}_f$ is affine and hence for all $\alpha \in (0, 1)$

$$J^{\beta}_f(\alpha) = J^{\beta}_f(1/2).$$

(42)

Thus $K^*(v) = 0$. Let us show that exists a neighborhood $A$ of $v$ such that for all $\tilde{v} \in A K^*(\tilde{v}) = 0 = K_*(\tilde{v})$. Assume per contra that such a neighborhood $A$ of $v$ does not exist. Since $u(f) \in \text{int } u(H)$, exists $\epsilon > 0$ such that $B_\epsilon(u(f)) \subseteq u(H)$. Now exists $\tilde{v} \in B(v, 1/2\beta \epsilon)$ such that $\tilde{v} K^*(\tilde{v}) \neq 0$ or $K_*(\tilde{v}) \neq 0$. Thus exists $\tilde{f} \in H, \tilde{\beta} \in (0, 1)$ such that $u(\tilde{f}) \in \text{int } u(H)$, $I \circ u(\beta \tilde{f} + (1 - \beta)h^*) = \tilde{v}$, and $\nabla J^{\tilde{f}}_\beta \neq 0$. Now $u(f) + (\tilde{v} - v) \in u(H)$ and hence exists $f^5 \in H$ such that $u(f^5) = u(f) + 1/\beta(\tilde{v} - v)\tilde{I}$. Thus $u(\beta f^5 + (1 - \beta)h^*) = u(\beta f + (1 - \beta)h^*) + (\tilde{v} - v)\tilde{I}$ and so by C-additivity $I \circ u(\beta f^5 + (1 - \beta)h^*) = \tilde{v}$. By Lemma 63 $0 \neq \nabla J^{\tilde{f}}_\beta = \nabla J^{f^5}_\beta$ and thus especially exists $\alpha^1 \in (0, 1)$ such that $I \circ u(\beta f^5 + (1 - \beta)(f^*\alpha^1 g^*)) \neq \tilde{v}$. However, then by C-additivity

$$I\left(u(\beta f + (1 - \beta)(f^*\alpha^1 g^*))\right) = I\left(u(\beta f + (1 - \beta)(f^*\alpha^1 g^*))\right) + (\tilde{v} - v) - (\tilde{v} - v)$$

C-add.$\equiv I\left(u(\beta f + (1 - \beta)(f^*\alpha^1 g^*)) + 1/\beta(\tilde{v} - v)\right) - (\tilde{v} - v)$

$$= I\left(u(\beta f^5 + (1 - \beta)(f^*\alpha^1 g^*))\right) - (\tilde{v} - v) \neq I\left(u(\beta f^5 + (1 - \beta)h^*)\right) - (\tilde{v} - v)$$

$$= I\left(u(\beta f + (1 - \beta)h^*) + 1/\beta(\tilde{v} - v)\right) - (\tilde{v} - v) \equiv I\left(u(\beta f + (1 - \beta)h^*)\right)$$

However, this contradicts (42). Thus exists a neighborhood $A$ of $v$ such that for all $\tilde{v} \in A K^*(\tilde{v}) = K^*(v) = K_*(\tilde{v})$.

2) Exists $\alpha^0 \in (0, 1)$ $\beta f + (1 - \beta)h^* \not\sim \beta f + (1 - \beta)f^*\alpha^0 g^*$. Thus $J^{\beta}_f(\alpha^0) \neq J^{\beta}_f(1/2)$. Since $J^{\beta}_f$ is affine, we have

$$v^* := \max\{J^{\beta}_f(1/4), J^{\beta}_f(3/4)\} > J^{\beta}_f(1/2) > \min\{J^{\beta}_f(1/4), J^{\beta}_f(3/4)\} =: v_*.$$

Assume w.l.o.g. $v^* = I\left(\beta u(f) + (1 - \beta)u(f^*3/4g^*)\right)$ and $v_* = I\left(\beta u(f) + (1 - \beta)u(f^*1/4g^*)\right)$ since the other case follows symmetrically. Let $\tilde{v} \in [v_*, v^*]$. Then by Lemma 62

$$I\left(\beta u(f) + (1 - \beta)u\left[f^*\left(1/4 + 1/2\frac{\tilde{v} - v_*}{v^* - v_*}\right)g^*\right]\right) = \tilde{v}$$

Now since $1/2(1 + \beta) = \beta + (1 - \beta)/2$ and $1 - 1/2(1 + \beta) = 1/2(1 - \beta)$

$$1/2(1 + \beta)\left(\frac{\beta}{1/2(1 + \beta)}f + \frac{1/2(1 - \beta)\tilde{v} - v_*}{1/2(1 + \beta)}f^* + \frac{1/2(1 - \beta)\tilde{v} - v_*}{1/2(1 + \beta)}g^*\right) + \left(1 - 1/2(1 + \beta)\right)h^*$$

$$\equiv \beta f + (1 - \beta)f^*\left(1/4 + 1/2\frac{\tilde{v} - v_*}{v^* - v_*}\right)g^*.$$

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Let us denote
\[ f^6 := \frac{\beta}{1/2(1 + \beta)} f + \frac{1/2(1 - \beta)}{1/2(1 + \beta)} \bar{v} - v_* f^* + \frac{1/2(1 - \beta)}{1/2(1 + \beta)} v^* - \bar{v} g^*. \]

Then we have for all \( \gamma \in (0, 1) \), as above,
\[ \frac{1/2(1 + \beta)}{1} f^6 + \left( 1 - \frac{1}{2}(1 + \beta) \right) (f^* g^*) \equiv \beta f + (1 - \beta) f^* \left( \frac{\bar{v} - v_*}{2(v^* - v_*)} + \frac{\gamma}{2} \right) g^*. \]

Thus for all \( \gamma \in (0, 1) \)
\[ J^f_{\beta, \bar{\beta}}(\gamma) = I \circ u \left( \frac{1}{2}(1 + \beta) f^6 + \left( 1 - \frac{1}{2}(1 + \beta) \right) (f^* g^*) \right) \]
\[ = I \circ u \left( \beta f + (1 - \beta) f^* \left( \frac{\bar{v} - v_*}{2(v^* - v_*)} + \frac{\gamma}{2} \right) g^* \right) = J^f_{\beta} \left( \frac{\bar{v} - v_*}{2(v^* - v_*)} + \frac{\gamma}{2} \right). \]

Thus by the chain rule of derivatives and by the affinity of \( J \) functions from Lemma 62 we have
\[ \nabla J^f_{\beta, \bar{\beta}}(\gamma) = \frac{1}{2} \nabla J^f_{\beta} \]

and thus
\[ \frac{1}{1 - \frac{1}{2}(1 + \beta)} \nabla J^f_{\beta, \bar{\beta}} = \frac{2}{1 - \beta} \nabla J^f_{\beta, \bar{\beta}} = \frac{1}{1 - \beta} \nabla J^f_{\beta}. \]  \( (43) \)

And finally we have for all \( \bar{f} \in H, \bar{\beta} \in (0, 1) \) such that \( u(\bar{f}) \in \text{int } u(H), I \circ u \left( \beta \bar{f} + (1 - \beta) h^* \right) = \bar{v}, \) then \( \beta \bar{f} + (1 - \beta) h^* \sim \frac{1}{2}(1 + \beta) f^6 + \left( 1 - \frac{1}{2}(1 + \beta) \right) h^* \) and by Lemma 63
\[ \frac{1}{1 - \beta} \nabla J^f_{\beta} = \frac{1}{1 - \frac{1}{2}(1 + \beta)} \nabla J^f_{\beta, \bar{\beta}} \stackrel{(43)}{=} \frac{1}{1 - \beta} \nabla J^f_{\beta} \stackrel{(41)}{=} K^*(v) = K_*(v) \]

Thus since \( \bar{f} \in H, \bar{\beta} \in (0, 1) \) were arbitrary such that \( u(\bar{f}) \in \text{int } u(H), I \circ u \left( \beta \bar{f} + (1 - \beta) h^* \right) = \bar{v} \)
\[ K^*(\bar{v}) \stackrel{(44)}{=} K^*(v) = K_*(v) \stackrel{(44)}{=} K_*(\bar{v}). \]

Thus exists a neighborhood of \( v \) such that \( K^* = K_* \) and \( K^* \) is constant and finite. Since \( v \in (\inf I(u(H)), \sup I(u(H))) \) was arbitrary, locally \( K^* = K_* \) and \( K^* \) is constant and finite. Since the interval \( (\inf I(u(H)), \sup I(u(H))) \) is connected, by (Viro et al., 2008, Problem 12.2x) \( K^* \) and \( K_* \) are constants and finite on the set \( (\inf I(u(H)), \sup I(u(H))) \) and \( K^* = K_* \). \( \square \)

**Lemma 65** Let \( (u, I) \) be a variational representation for \( \zeta \). \( f^*, g^* \) are equally crisp acts iff for all \( \varphi \in u(H), \alpha \in (0, 1) \) such that \( \varphi + \alpha (u(f^*) - u(g^*)) \in u(H), \)
\[ I(\varphi + \alpha (u(f^*) - u(g^*)) = I(\varphi) + \alpha (I(u(f^*)) - I(u(g^*)). \]

**Proof.** Let first \( f^*, g^* \) be equally crisp acts. Let \( \varphi \in \text{int } u(H), \alpha \in \mathbb{R}_{++} \) such that \( \varphi + \alpha (u(f^*) - u(g^*)) \in \text{int } u(H). \) Define the mapping \( [0, \alpha] \ni \beta \mapsto I(\varphi + \beta (u(f^*) - u(g^*))). \) Now let \( \beta \in (0, \alpha). \)
Thus we have for all $\alpha$.

Let $h \in \partial u(\alpha)$. Then $\phi$.

Define $\{0, 1/2, 1\}$ be such that

Thus by the choice of $\gamma$ exists $f \in H$ such that

Thus $u(\gamma f + (1 - \gamma)h^*) = \varphi + \beta(u(f^*) - u(g^*))$. Now for all $\zeta \in (0, 1)$ such that $\beta + (1 - \gamma)(\zeta - 1/2) \geq 0$. Let $\zeta_* := \max\{0, 1/2 - \frac{\beta}{1 - \gamma}\}$ and $\zeta^* := \min\{1, 1/2 + \frac{\alpha - \beta}{1 - \gamma}\}$

Define $[\zeta_*, \zeta^*] \ni \zeta \mapsto J(\beta + (1 - \gamma)(\zeta - 1/2))$ where $\zeta^* > 1/2 > \zeta_*$. Thus by Lemma 64 $J^3$ is affine and $\nabla J^3 = I(\varphi) = I(f^*) - I(g^*)$. Thus by the chain rule and (45)

Thus finally since $I$ is 1-lipschitz we have

Thus we have for all $\varphi \in \text{int} u(H), \alpha \in (0, 1]$ such that $\varphi + \alpha(u(f^*) - u(g^*)) \in \text{int} u(H),$

Finally, since the mapping $I$ is continuous, we have for all $\varphi \in u(H), \alpha \in (0, 1]$ such that $\varphi + \alpha(u(f^*) - u(g^*)) \in u(H),$

Let us next assume that $f^*, g^* \in H$ are such that for all for all $\varphi \in u(H), \alpha \in (0, 1]$ such that $\varphi + \alpha(u(f^*) - u(g^*)) \in u(H),$

and let us show that $f^*$ and $g^*$ are equally crisp. Let $f, g \in H, \alpha \in (0, 1)$. Then we have for $h \in \{f, g\}$

\begin{align*}
I(\alpha u(h) + (1 - \alpha)g^*) &= I(\alpha u(h) + (1 - \alpha)f^*) - I(u(f^*)) + I(u(g^*)).
\end{align*}

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Thus
\[ I(\alpha u(f) + (1 - \alpha)g^*) \geq I(\alpha u(g) + (1 - \alpha)g^*) \Leftrightarrow I(\alpha u(f) + (1 - \alpha)f^*) \geq I(\alpha u(g) + (1 - \alpha)f^*), \]
which proves the claim since \( I \circ u \) represents \( \succcurlyeq \).

\textbf{Lemma 66} Let \( \succcurlyeq \) have a state dependent dual-self variational representation. Let \( f, g \in H \) and \( \beta \in (0, 1) \) Let \( (\alpha_i)_{i=1}^{\infty} \) and \( \lim_{i \to \infty} \alpha_i = 0 \). Then exists \( i^1 \) such that for all \( i > i^1 \ C_{f,\beta}^{g,\alpha_i} \in \mathbb{R} \).

\textit{Proof.} Let us assume that \( \alpha^0 \in (0, 1/2) \) since the other case follows symmetrically. First exists \( i^1 \in \mathbb{N} \) such that for all \( i > i^1 \ C_{f,\beta}^{g,\alpha_i} \in \mathbb{R} \). Since the sets
\[
\left\{ \alpha \in [0, 1] \mid \alpha g + (1 - \alpha)(\beta f + (1 - \beta)h^*) \succ \beta f + (1 - \beta)g^* \right\}
\]
and
\[
\left\{ \alpha \in [0, 1] \mid \beta f + (1 - \beta)g^* \succ (1 - \alpha)(\beta f + (1 - \beta)h^*) \right\}
\]
are open by Axiom 2 and include 0, exists \( \varepsilon > 0 \) such that for all \( \alpha \in (0, \varepsilon) \)
\[
\beta f + (1 - \beta)g^* \succ \alpha g + (1 - \alpha)(\beta f + (1 - \beta)h^*) \succ \beta f + (1 - \beta)g^*.
\]
Thus especially for all \( \alpha \in (0, \varepsilon) \ C_{f,\beta}^{g,\alpha_i} \in \mathbb{R} \). Since \( \lim_{i \to \infty} \alpha_i = 0 \), exists \( i^1 \in \mathbb{N} \) such that for all \( i > i^1 \ \alpha_i \in [0, \varepsilon] \).

\textbf{Lemma 67} Let \( \succcurlyeq \) have a state dependent dual-self variational representation. Let \( f, g \in H \) and \( \beta, \alpha \in (0, 1) \). If \( C_{f,\beta}^{g,\alpha} \in \mathbb{R} \), then
\[
\beta f + (1 - \beta)(\alpha C_{f,\beta}^{g,\alpha} + 1/2)f^* + (1/2 - \alpha C_{f,\beta}^{g,\alpha})g^* \sim \alpha g + (1 - \alpha)(\beta f + (1 - \beta)h^*)
\]

\textit{Proof.} Assume, per contra,
\[
\beta f + (1 - \beta)(\alpha C_{f,\beta}^{g,\alpha} + 1/2)f^* + (1/2 - \alpha C_{f,\beta}^{g,\alpha})g^* \not\succ \alpha g + (1 - \alpha)(\beta f + (1 - \beta)h^*).
\]
Assume w.l.o.g.
\[
\beta f + (1 - \beta)(\alpha C_{f,\beta}^{g,\alpha} + 1/2)f^* + (1/2 - \alpha C_{f,\beta}^{g,\alpha})g^* \succ \alpha g + (1 - \alpha)(\beta f + (1 - \beta)h^*)
\]
since the other case is symmetric. Now the set
\[
A := \left\{ \alpha \in [0, 1] \mid \alpha (\beta f + (1 - \beta)f^*) + (1 - \alpha)(\beta f + (1 - \beta)g^*) \succ \alpha g + (1 - \alpha)(\beta f + (1 - \beta)h^*) \right\}
\]
is open by Axiom 2 and includes \( \alpha C_{f,\beta}^{g,\alpha} + 1/2 + \gamma \). Thus exists \( \varepsilon > 0 \) such that for all \( \gamma \in (-\varepsilon, \varepsilon) \)
\[
\alpha C_{f,\beta}^{g,\alpha} + 1/2 + \gamma \in A. \text{ Thus for all } \gamma \in (-\varepsilon, \varepsilon) \frac{\varepsilon}{\alpha} \alpha C_{f,\beta}^{g,\alpha} + 1/2 + \gamma \in A. \text{ Thus for all } \gamma \in (-\varepsilon, \varepsilon) \frac{\varepsilon}{\alpha} \frac{\varepsilon}{\alpha} \alpha C_{f,\beta}^{g,\alpha} + 1/2 + \gamma \in A.
\]
\[
\beta f + (1 - \beta)\left( (\alpha(C_{f,\beta}^{g,\alpha} + \gamma) + 1/2)f^* + \left(1/2 - \alpha(C_{f,\beta}^{g,\alpha} + \gamma) \right)g^* \right)
\]

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\begin{align*}
\& \equiv (\alpha C_{f,\beta}^{\alpha,\alpha} + 1/2 + \alpha \gamma) \left( \beta f + (1 - \beta) f^* \right) + (1 - \alpha C_{f,\beta}^{\alpha,\alpha} + 1/2 + \alpha \gamma) \left( \beta f + (1 - \beta) g^* \right) \\
\& \succ \alpha g + (1 - \alpha) \left( \beta f + (1 - \beta) h^* \right).
\end{align*}

However, this contradicts the definition of $C_{f,\beta}^{\alpha,\alpha}$ since exists $\hat{\alpha} \in [-1/2\alpha, 1/2\alpha]$ such that $|C_{f,\beta}^{\alpha,\alpha} - \hat{\alpha}| < \varepsilon$ and

$$
\beta f + (1 - \beta) \left( (\alpha \hat{\alpha} + 1/2) f^* + (1/2 - \alpha \hat{\alpha}) g^* \right) \sim \alpha g + (1 - \alpha) \left( \beta f + (1 - \beta) h^* \right)
$$

\hfill \Box

**Lemma 68** Let $(u, I)$ be a dual-self representation and $f, \beta, \alpha \in (0, 1)$. Then

$$
I \left( \beta u(f) + (1 - \beta) (\alpha u(f^*) + (1 - \alpha) u(g^*)) \right) - I \left( \beta u(f) + (1 - \beta) u(h^*) \right) = (1 - \beta)(\alpha - 1/2) \left( I(u(f^*)) - I(u(g^*)) \right)
$$

*Proof.* Let first $\alpha \in (0, 1)$. First we have that for all $\beta \in (0, 1)$, $J_\beta^{h^*}$ is affine, and hence especially the mapping $(0, 1) \ni \alpha \mapsto I \left( \alpha u(f^*) + (1 - \alpha) u(g^*) \right)$. Since $I$ is continuous, the mapping $[0, 1] \ni \alpha \mapsto I \left( \alpha u(f^*) + (1 - \alpha) u(g^*) \right)$ is affine. Thus for all $\beta \in (0, 1)$ and $\gamma(0, 1)$, we have

$$
J_\beta^{h^*}(\gamma) = I \left( \left( \frac{\beta}{2} + (1 - \beta) \gamma \right) u(f^*) + \left( \frac{\beta}{2} + (1 - \beta)(1 - \gamma) \right) u(g^*) \right) = J_0^{h^*} \left( \frac{\beta}{2} + (1 - \beta) \gamma \right).
$$

By affinity of $J_0^{h^*}$ we have

$$
\nabla J_0^{h^*} = J_0^{h^*}(1) - J_0^{h^*}(0) = I \left( u(f^*) \right) - I \left( u(g^*) \right).
$$

And by the chain rule of derivatives for all $\beta \in (0, 1)$

$$
\nabla J_\beta^{h^*} = (1 - \beta) \nabla J_0^{h^*} = (1 - \beta) \left( I(u(f^*)) - I(u(g^*)) \right).
$$

Thus finally by Lemma 64

$$
(1 - \beta) \left( I(u(f^*)) - I(u(g^*)) \right) = \nabla J_\beta^{h^*} = \nabla J_\beta^{f^*} \tag{48}
$$

and by the affinity of $J_\beta^{f^*}$

$$
J_\beta^{f^*}(\alpha) = J_\beta^{f^*}(1/2) + J_\beta^{f^*}(\alpha) - J_\beta^{f^*}(1/2) = J_\beta^{f^*}(1/2) + \nabla J_\beta^{f^*}(\alpha - 1/2) \tag{49}
$$

Thus

$$
I \left( \beta u(f) + (1 - \beta) (\alpha u(f^*) + (1 - \alpha) u(g^*)) \right) - I \left( \beta u(f) + (1 - \beta) u(h^*) \right) = J_\beta^{f^*}(\alpha) - J_\beta^{f^*}(1/2) \overset{(49)}{=} \nabla J_\beta^{f^*}(\alpha - 1/2) \overset{(48)}{=} (1 - \beta) \left( I(u(f^*)) - I(u(g^*)) \right)
$$

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Lemma 69  Let \((u, I)\) be a dual-self representation and \(f, g \in H, \beta \in (0, 1)\). Let \((\alpha_i)_{i=1}^\infty \subset (0, 1)\) be such that \(\lim_{i \to \infty} \alpha_i = 0\). Then exists \(i^1\) such that for all \(i > i^1\)
\[
\frac{I\left(\alpha_i u(g) + (1 - \alpha_i) u(f, g + (1 - \beta) h^*)\right) - I(\beta u(f) + (1 - \beta)(f, g + (1 - \beta) h^*))}{\alpha_i} = (1 - \beta)\tilde{C}_{f, \beta}^{g, \alpha_i}\left(I\left(u\left(f^*\right)\right) - I\left(u\left(g^*\right)\right)\right).
\]

Proof. By Lemmas 66 and 67 exists \(i^1 \in \mathbb{N}\) such that for all \(i > i^1\)
\[
I\left(\alpha_i u(g) + (1 - \alpha_i) u(f, g + (1 - \beta) h^*)\right) = I\left((\alpha_i C_{f, \beta}^{g, \alpha_i} + 1/2) u(f^*) + (1/2 - \alpha_i C_{f, \beta}^{g, \alpha_i}) u(g^*)\right).
\]
Let \(i > i^1\). Now by Lemma 68
\[
\frac{I\left(\alpha_i u(g) + (1 - \alpha_i) u(f, g + (1 - \beta) h^*)\right) - I(\beta u(f) + (1 - \beta)(f, g + (1 - \beta) h^*))}{\alpha_i} = (1 - \beta)(\alpha_i C_{f, \beta}^{g, \alpha_i} + 1/2 - 1/2)\left(I\left(u\left(f^*\right)\right) - I\left(u\left(g^*\right)\right)\right).
\]
Thus the claim follows by dividing both sides by \(\alpha_i\).

Lemma 70  Let \(S\) be finite and \((u, I)\) be a dual-self representation \(f, g \in H, \beta \in (0, 1)\), and \((\alpha_i)_{i=1}^\infty \subset (0, 1)\) be such that \(\lim_{i \to \infty} \alpha_i = 0\). If
\[
\lim_{i \to \infty} C_{f, \beta}^{g, \alpha_i} \in \mathbb{R} \quad \text{or} \quad \lim_{i \to \infty} \frac{I\left(\alpha_i u(g) + (1 - \alpha_i) u(f, g + (1 - \beta) h^*)\right) - I(\beta u(f) + (1 - \beta)(f, g + (1 - \beta) h^*))}{\alpha_i} \in \mathbb{R},
\]
then both limits exist and
\[
\lim_{i \to \infty} \frac{I\left(\alpha_i u(g) + (1 - \alpha_i) u(f, g + (1 - \beta) h^*)\right) - I(\beta u(f) + (1 - \beta)(f, g + (1 - \beta) h^*))}{\alpha_i} = (1 - \beta)\left(I\left(u\left(f^*\right)\right) - I\left(u\left(g^*\right)\right)\right)\lim_{i \to \infty} C_{f, \beta}^{g, \alpha_i}.
\]

Proof. By Lemma 69 exists \(i^1\) such that for all \(i > i^1\)
\[
\frac{I\left(\alpha_i u(g) + (1 - \alpha_i) u(f, g + (1 - \beta) h^*)\right) - I(\beta u(f) + (1 - \beta)(f, g + (1 - \beta) h^*))}{\alpha_i} = (1 - \beta)\tilde{C}_{f, \beta}^{g, \alpha_i}\left(I\left(u\left(f^*\right)\right) - I\left(u\left(g^*\right)\right)\right).
\]
Thus taking the limit of \(i \to \infty\) shows the claim since either the limit on the left or the right hand side exists.

Lemma 71  Let \((u, I)\) and \((\tilde{u}, \tilde{I})\) be a dual-self representations for \(\simeq\). Let \(f \in H\). Then \(u(f) \in \text{int } u(H)\) iff \(\tilde{u}(f) \in \text{int } \tilde{u}(H)\)

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Proof. Let us show w.l.o.g. that if \( u(f) \in \text{int} \ u(H) \), then \( \overline{u}(f) \in \text{int} \ \overline{u}(H) \). By Lemmas 25 and 44 for all \( s \in S^p \) exists \( A_s > 0, B_s \in \mathbb{R} \) such that \( \overline{u}_s = A_s u_s + B_s \). Thus if \( u(f) \in \text{int} \ u(H) \) exists \( \varepsilon > 0 \) such that \( B_\infty(u(f), \varepsilon) \subseteq \text{int} \ u(H) \) and so \( B_\infty(u(f), \varepsilon) \subseteq \text{int} \ u(H) \). Thus by above \( B_\infty(\overline{u}(f), \min_{A_s \in S^p} \varepsilon) \subseteq \text{int} \ \overline{u}(H) \), which shows the claim.

\[ \square \]

Lemma 72 Let \((u, I)\) and \((\overline{u}, \overline{I})\) be a dual-self representations for \( \overline{\mathcal{C}} \). Let \( A \in \mathbb{R}_{++}^S, B \in \mathbb{R}^s \) be such that for all \( s \in S \overline{u}_s = A_s u_s + B_s \). Denote \( A^{-1} := (A_s^{-1})_{s \in S} \). Let \( f \in H \) be such that \( u(f) \in \text{int} \ u(H) \). If \( I \) is differentiable at \( u(f) \) with derivative \( \nabla I(u(f)) \), then \( \overline{I} \) is differentiable at \( \overline{u}(f) \) with derivative

\[
\frac{\overline{I}(\overline{u}(f^*)) - \overline{I}(\overline{u}(g^*))}{\overline{I}(u(f^*)) - \overline{I}(u(g^*))} A^{-1} \nabla I(u(f)).
\]

Proof. Let the derivative of \( I \) at \( u(f) \) be \( \nabla I(u(f)) \). By Lemmas 25 and 44, for all \( s \in S^p \) exists \( A_s > 0, B_s \in \mathbb{R} \) such that

\[
\overline{u}_s = A_s u_s + B_s.
\]

By Lemma 71, \( \overline{u}(f) \in \text{int} \ \overline{u}(H) \). Let \( v \in \mathbb{R}^S \). Then exists \( \alpha^* > 0 \) such that \( \overline{u}(f) + \alpha^* v, \overline{u}(f) - \alpha^* v \in \overline{u}(H) \). Let \( g, \overline{g} \in H \) be such that \( \overline{u}(g) = \overline{u}(f) + \alpha^* v \) and \( \overline{u}(\overline{g}) = \overline{u}(f) - \alpha^* v \). Let \( \beta \in (0, 1) \) be such that \( \frac{1}{\beta} \overline{u}(f) - \frac{1 - \beta}{\beta} w(h^*) \in \overline{u}(H) \). Let \( \overline{f} \in H \) be such that \( u(\overline{f}) = \frac{1}{\beta} \overline{u}(f) - \frac{1 - \beta}{\beta} w(h^*) \). Then

\[
\overline{u}(\beta \overline{f} + (1 - \beta) h^*) = \overline{u}(f).
\]

Thus by (50) \( u(\beta \overline{f} + (1 - \beta) h^*) = u(f) \). Let \( (\alpha_i)_{i=1}^\infty \subseteq (0, 1) \) be such that \( \alpha_i \to 0 \). Since \( I \) is differentiable at \( u(\beta \overline{f} + (1 - \beta) h^*) = u(f) \) and so

\[
\lim_{i \to \infty} \frac{I(\alpha_i u(g) + (1 - \alpha_i) u(f)) - I(u(f))}{\alpha_i} = \nabla I(u(f)) \cdot (u(g) - u(f)) \in \mathbb{R}.
\]

Thus by Lemma 70 \( \lim_{i \to \infty} C_{f, \beta}^{g, \alpha_i} \) exists and

\[
(1 - \beta) \left( I(u(f^*)) - I(u(g^*)) \right) \lim_{i \to \infty} C_{f, \beta}^{g, \alpha_i} = \nabla I(u(f)) \cdot (u(g) - u(f)).
\]

Thus by Lemma 70 and (51)

\[
\lim_{i \to \infty} \frac{I(\alpha_i \overline{u}(g) + (1 - \alpha_i) \overline{u}(f)) - I(\overline{u}(f))}{\alpha_i} = (1 - \beta) \left( \overline{I}(\overline{u}(f^*)) - \overline{I}(\overline{u}(g^*)) \right) \lim_{i \to \infty} C_{f, \beta}^{g, \alpha_i}
\]

\[
= \frac{\overline{I}(\overline{u}(f^*)) - \overline{I}(\overline{u}(g^*))}{\overline{I}(u(f^*)) - \overline{I}(u(g^*))} \nabla I(u(f)) \cdot (u(g) - u(f)).
\]
Symmetrically for $\tilde{g}$ we have
\[
\lim_{i \to \infty} \frac{\bar{I}(\alpha_i \tilde{u}(\tilde{g}) + (1 - \alpha_i) \tilde{u}(f)) - I(\tilde{u}(f))}{\alpha_i} = (1 - \beta) \left( \frac{\bar{I}(\tilde{u}(f^*)) - \bar{I}(\tilde{u}(g^*)))}{I(u(f^*)) - I(u(g^*))} \right) \lim_{i \to \infty} \tilde{C}_{f, \beta}^{\tilde{g}, \alpha_i}
\]
(53)

Next by (50), we have
\[
\nabla I(u(f)) \cdot (u(g) - u(f)) = \sum_{s \in S} \nabla I(u(f))_s \tilde{A}^{-1}(\tilde{u}(g) - \tilde{u}(f))
\]
\[
= \sum_{s \in S} \tilde{A}^{-1} \nabla I(u(f))_s (\alpha^* v_s) = \alpha^* \nabla I(u(f)) \cdot (\alpha^* v).
\]

Since $\tilde{u}(g) - \tilde{u}(f) = - (\tilde{u}(\tilde{g}) - \tilde{u}(f))$, by (50) we have $u(g) - u(f) = -(u(\tilde{g}) - u(f))$. Thus by (52, 53) and the choice of $g$ we have
\[
\lim_{i \to \infty} \frac{\bar{I}(\tilde{u}(f) + \alpha_i \alpha^* v) - I(\tilde{u}(f))}{\alpha_i} = \frac{1}{\alpha^*} \left( \frac{\bar{I}(\tilde{u}(f^*)) - \bar{I}(\tilde{u}(g^*)))}{I(u(f^*)) - I(u(g^*))} \right) \tilde{A}^{-1} \nabla I(u(f)) \cdot (\alpha^* v)
\]
\[
= \lim_{i \to \infty} \frac{\bar{I}(\tilde{u}(f) - \alpha_i \alpha^* v) - I(\tilde{u}(f))}{-\alpha_i}
\]

Since the sequence $(\alpha_i)_{i=1}^\infty \subseteq (0, 1)$ such that $\alpha_i \to 0$ was arbitrary, we have that
\[
\lim_{\alpha \to 0} \frac{\bar{I}(\tilde{u}(f) + \alpha v) - I(\tilde{u}(f))}{\alpha} = \frac{\bar{I}(\tilde{u}(f^*)) - \bar{I}(\tilde{u}(g^*)))}{I(u(f^*)) - I(u(g^*))} \tilde{A}^{-1} \nabla I(u(f)) \cdot v.
\]

Since $v \in \mathbb{R}^s$ was arbitrary, $\bar{I}$ is differentiable at $\tilde{u}(f)$ with derivative
\[
\frac{\bar{I}(\tilde{u}(f^*)) - \bar{I}(\tilde{u}(g^*)))}{I(u(f^*)) - I(u(g^*))} \tilde{A}^{-1} \nabla I(u(f))
\]

\[
\square
\]

**Lemma 73** Let $S$ be finite and $(u, I)$ be dual-self representation for $\zeta$. Then for all $f \in H$ such that $u(f) \in \text{int} u(H)$,
\[
\partial I(u(f)) = \text{convex} \{ \lim_{i \to \infty} \nabla I(\varphi_i)|(\varphi_i)_{i=1}^\infty \subseteq u(H), \lim_{i \to \infty} \varphi_i = u(f), \forall i \in \mathbb{N}, I \text{ differentiable at } \varphi_i \}
\]

**Proof.** Since $I$ is lipschitz on $u(H)$, this follows directly from Clarke’s (1983) Theorem 2.5.1. \[
\square
\]

**Lemma 74** Let $S$ be finite and $(u, I)$ be dual-self representation for $\zeta$. Let $f \in H$ be such that $u(f) \in \text{int}_\infty u(H)$. If $I$ is differentiable at $u(f)$ with derivative $p$ then $I(u(f)) = p \cdot u(f)$.
Proof. Since $u(f) \in \text{int}_\infty u(H)$, exists $\alpha^0 > 0$ such that $(1 + \alpha^0)u(f), (1 - \alpha^0)u(f) \in u(H)$. Since $u(H)$ is convex and $I$ is positive homogeneous we have for all $\alpha \in (-\alpha^0, \alpha^0) \setminus \{0\}$

$$I((1 + \alpha)u(f)) = (1 + \alpha)I(u(f)) \Rightarrow \frac{I(\alpha u(f)) + u(f) - I(u(f))}{\alpha} = I(u(f)).$$

Thus by taking the limit $\alpha \to 0$ we have by the differentiability of $I$ at $u(f)$.

$$I(u(f)) = \lim_{\alpha \to 0} \frac{I(\alpha u(f)) + u(f) - I(u(f))}{\alpha} = p \cdot u(f).$$

\[\square\]

**Lemma 75** Let $S$ be countable and $C \subseteq \Delta_{ca}(S)$. If exists $x \in \mathbb{R}_++^S$ and $\beta \in \mathbb{R}_+$ such that for all $p \in C$

$$\sum_{s \in S} x_sp_s = \beta$$

then exists $y \in C^\perp$ such that

$$x = I_S \beta + y$$

Proof. Let $x \in \mathbb{R}_++^S$ be as in the antecedent. First, let us solve for all $z \in \mathbb{R}^S$ such that for all $p \in \tilde{C}$

$$\sum_{s \in S} z_sp_s = \beta.$$ 

However since $\sum_{s \in S} p_s = 1$ this is equivalent to finding all $y := z - \beta I_S$ such that for all $p \in \tilde{C}$

$$\sum_{s \in S} y_sp_s = 0$$

and all the solution to this are $\tilde{C}^\perp := \{ y \in \mathbb{R}^S | \forall p \in \tilde{C}, \sum_{s \in S} y_sp_s = 0 \}$. Thus exists $y \in \tilde{C}^\perp$ such that

$$\beta I_S + y = x.$$

\[\square\]

### 8.5.1 Partial Identification

**Lemma 76** Let $(u, I)$ and $(\tilde{u}, \tilde{I})$ be state dependent dual-self variational representations for $\succsim$. Then exists $x \in \left( \bigcup_{\varphi \in u(H)} \partial I(\varphi) \right)^\perp, \alpha > 0, B \in \mathbb{R}^S \beta \in \mathbb{R}$ such that when we denote the mapping for all $\mu \in \Delta(S)$

$$(1 + x)(\mu) := \left( (1 + x_s)\mu_s \right)_{s \in S}$$

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we have
\[
\bigcup_{\varphi \in \text{int} \tilde{u}(H)} \partial \tilde{I}(\varphi) = (1 + x) \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi).
\]
and for all $S^p$
\[
(\tilde{u}_s)_{s \in S^p} = \left( \frac{\alpha}{1 + x_s} u_s + B_s \right)_{s \in S^p}.
\]

**Proof.** By Lemmas 25 and 44 exists $A \in \mathbb{R}^{S^+}_+$, $B \in \mathbb{R}^{S^+}_+$ such that for all $s \in S^p$
\[
\tilde{u}_s = A_s u_s + B_s.
\]

Let us denote
\[
\alpha := \frac{\tilde{I}(\tilde{u}(f^*)) - \tilde{I}(\tilde{u}(g^*))}{I(u(f^*)) - I(u(g^*))} > 0.
\]

By Lemmas 71 and 72 $I$ is differentiable at $\varphi \in \text{int} u(H)$ with derivative $\nabla I(\varphi)$ if and only if $\tilde{I}$ is differentiable at $A\varphi + B \in \text{int} \tilde{u}(H)$ with derivative
\[
\alpha A^{-1} \nabla I(u(f^*)).
\]

Thus for all $\varphi \in \text{int} u(H)$ by Lemma 73
\[
\partial \tilde{I}(A\varphi + B) = \alpha A^{-1} \partial I(\varphi).
\]

Thus by Lemma 71
\[
\bigcup_{\varphi \in \text{int} \tilde{u}(H)} \partial \tilde{I}(\varphi) = \bigcup_{\varphi \in \text{int} u(H)} \alpha A^{-1} \partial I(\varphi).
\]

By continuity of $A^{-1}$ operator we have
\[
\bigcup_{\varphi \in \text{int} \tilde{u}(H)} \partial \tilde{I}(\varphi) = \alpha A^{-1} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi).
\]

Thus for all $p \in \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi)$ we have
\[
\alpha A^{-1} p \in \Delta(S).
\]

Thus by Lemma 75 exists $x \in \left( \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \right) \perp$ such that
\[
1 + x = \alpha A^{-1}.
\]

Thus especially for all $s \in S$
\[
A_s = \frac{\alpha}{1 + x_s}.
\]

This proves the claim. 

Lemma 77  Let \((u, I)\) and \((\bar{u}, \bar{I})\) be state dependent dual-self variational representations for \(\preceq\). Then exists \(\alpha > 0, \beta \in \mathbb{R}\) such that for all \(f \in H\)

\[
\bar{I}(\bar{u}(f)) = \alpha I(u(f)) + \beta
\]

where

\[
\alpha = \frac{\bar{I}(\bar{u}(f^*)) - \bar{I}(\bar{u}(g^*))}{I(u(f^*)) - I(u(g^*))} \text{ and } \beta = \bar{I}(\bar{u}(g^*)) - \alpha I(u(g^*)).
\]

Proof. Let \(\alpha\) and \(\beta\) be as above. Let us define the mapping \(K : (\inf I(u(H)), \sup I(u(H))) \rightarrow \mathbb{R} \cup \{\infty\}\)

\[K(v) = \sup \left\{ |\bar{I}(\bar{u}(f)) - \alpha I(u(f)) + \beta f| \in H, I \circ u(f) = v \right\} \]

where \(\sup \nothing = \infty\). Let us show that \(K\) is locally constant. Let \(v \in (\inf I(u(H)), \sup I(u(H)))\).

First exists \(f^1, f^2 \in H\) such that \(I(u(f^1)) > v > I(u(f^2))\). Since \(S\) is finite exists \(f^3 \in H\) such that \(u(f^3) \in \text{int} u(H)\). By mixture continuity exists \(\alpha^1, \alpha^2, \alpha^3 \in (0, 1)\) such that \(\alpha^1 + \alpha^2 + \alpha^3 = 1\) and \(I \circ u(\alpha^1 f^1 + \alpha^2 f^2 + \alpha^3 f^3) = v\). Denote \(f^4 := \alpha^1 f^1 + \alpha^2 f^2 + \alpha^3 f^3\). Since \(\alpha^3 > 0\) and \(u(f^3) \in \text{int} u(H)\)

and \(u(H)\) is a convex set \(u(f^4) \in \text{int} u(H)\). Thus exists \(\varepsilon > 0\) such that \(B_\infty(u(f^4), \varepsilon) \subseteq u(H)\). Let \(\beta \in (1/2, 1)\) be such that

\[
\left(\frac{1}{\beta} - 1\right)\|u(f^4)\|_\infty + \frac{1-\beta}{\beta} \|u(h^*)\|_\infty < \varepsilon.
\]

Thus by the choice of \(\beta\) exists \(f \in H\) such that

\[
u(f) = u(f^4) + \left(\frac{1}{\beta} - 1\right)u(f^4) - \frac{1-\beta}{\beta}u(h^*).
\]

Thus especially \(u(f) \in \text{int} u(H)\) and

\[
\beta u(f) + (1 - \beta)u(h^*) = u(f^4)
\]

and so \(I(\beta u(f) + (1 - \beta)u(h^*)) = v\). By Lemma 64 for all \(\alpha \in (0, 1)\) we have

\[
I(\beta u(f) + (1 - \beta)u(f^* \alpha g^*)) = I(\beta u(f) + (1 - \beta)u(h^*)) + (1 - \beta)(\alpha - 1/2)\left(I(u(f^*)) - I(u(g^*))\right) \tag{54}
\]

and

\[
\bar{I}(\beta \bar{u}(f) + (1 - \beta)\bar{u}(f^* \alpha g^*)) = \bar{I}(\beta \bar{u}(f) + (1 - \beta)\bar{u}(h^*)) + (1 - \beta)(\alpha - 1/2)\left(\bar{I}(\bar{u}(f^*)) - \bar{I}(\bar{u}(g^*))\right). \tag{55}
\]

Thus let

\[
\bar{v} \in \left( v - \frac{1}{2}(1 - \beta)\left(I(u(f^*)) - I(u(g^*))\right), v + \frac{1}{2}(1 - \beta)\left(I(u(f^*)) - I(u(g^*))\right) \right).
\]
and let $\tilde{\alpha} \in (0,1)$ be such
\[
\tilde{v} = I(\beta u(f) + (1 - \beta)u(h^*)) + (1 - \beta)(\tilde{\alpha} - 1/2) \left(I(u(f^*)) - I(u(g^*))\right).
\]
Let $\tilde{f} \in H$ be such that $I \circ u(\tilde{f}) = \tilde{v}$. Then $\tilde{f} \sim \beta \tilde{u}(f) + (1 - \beta)\tilde{u}(f^*\tilde{g}^*)$. Thus by (54,55)
\[
I(u(\tilde{f})) = \tilde{I}(\beta \tilde{u}(f) + (1 - \beta)u(f^*\tilde{g}^*)) = I(\beta u(f) + (1 - \beta)u(h^*)) + (1 - \beta)(\tilde{\alpha} - 1/2) \left(I(u(f^*)) - I(u(g^*))\right)
\]
and
\[
\tilde{I}(\tilde{u}(f)) = \tilde{I}(\beta \tilde{u}(f) + (1 - \beta)\tilde{u}(f^*\tilde{g}^*)) = \tilde{I}(\beta \tilde{u}(f) + (1 - \beta)\tilde{u}(h^*)) + (1 - \beta)(\tilde{\alpha} - 1/2) \left(\tilde{I}(u(f^*)) - \tilde{I}(u(g^*))\right).
\]
Hence
\[
\begin{align*}
\tilde{I}(\tilde{u}(f)) - \alpha I(u(\tilde{f})) - \beta \\
&= \tilde{I}(\beta \tilde{u}(f) + (1 - \beta)\tilde{u}(h^*)) + (1 - \beta)(\tilde{\alpha} - 1/2) \left(\tilde{I}(u(f^*)) - \tilde{I}(u(g^*))\right) \\
&\quad - \frac{\tilde{I}(\tilde{u}(f^*)) - \tilde{I}(\tilde{u}(g^*))}{\tilde{I}(u(f^*)) - \tilde{I}(u(g^*))} \left(I(\beta u(f) + (1 - \beta)u(h^*)) + (1 - \beta)(\tilde{\alpha} - 1/2) \left(I(u(f^*)) - I(u(g^*))\right)\right) - \beta \\
&= \tilde{I}(\beta \tilde{u}(f) + (1 - \beta)\tilde{u}(h^*)) - \alpha I(\beta u(f) + (1 - \beta)u(h^*)) - \beta.
\end{align*}
\]
Since the last line does not depend on $\tilde{f}, \tilde{\alpha}, \tilde{v}$, we have for all
\[
\tilde{v} \in \left(v - 1/2(1 - \beta) \left(I(u(f^*)) - I(u(g^*))\right), v + 1/2(1 - \beta) \left(I(u(f^*)) - I(u(g^*))\right)\right)
\]
\[
K(\tilde{v}) = \left|\tilde{I}(\beta \tilde{u}(f) + (1 - \beta)\tilde{u}(h^*)) - \alpha I(\beta u(f) + (1 - \beta)u(h^*)) - \beta\right|.
\]
Thus $K$ is constant in a neighborhood of $v$. Since $v \in (\inf I(u(H)), \sup I(u(H)))$ was arbitrary, $K$ is locally constant function. Since the interval $(\inf I(u(H)), \sup I(u(H)))$ is connected, by (Viro et al., 2008, Problem 12.2x) $K$ is constant function. Finally let $f \in H$ be such that $I \circ u(f) = I \circ u(g^*)$. Then we have $f \sim g^*$ and so
\[
\tilde{I}(\tilde{u}(f)) - \alpha I(u(f)) - \beta = \tilde{I}(\tilde{u}(g^*)) - \alpha I(u(g^*)) - \tilde{I}(\tilde{u}(g^*)) + \alpha I(u(g^*)) = 0.
\]
Since $f$ such that $I \circ u(f) = I \circ u(g^*)$ was arbitrary, we have $K(I \circ u(g^*)) = 0$. Thus $K$ is a constant function at 0. Finally by the continuity of $I$ and $\tilde{I}$ we have for all $f \in H$
\[
\tilde{I}(\tilde{u}(f)) = \alpha I(u(f)) + \beta.
\]
Lemma 78  Let \((u, I)\) and \((\tilde{u}, \tilde{I})\) be state dependent dual-self representations for \(\preceq\). Then exists \(x, y \in \left(\bigcup_{\varphi \in u(H)} \partial I(\varphi)\right)^\perp\), \(\alpha > 0\), \(\beta \in \mathbb{R}\) such that when we denote the mapping for all \(\mu \in \Delta(S)\)

\[
(1 + x)(\mu) := \left((1 + x_s)\mu_s\right)_{s \in S}
\]
we have

\[
\bigcup_{\varphi \in \text{int } u(H)} \partial \tilde{I}(\varphi) = (1 + x) \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi).
\]
and for all \(S^p\)

\[
(\tilde{u}_s)_{s \in S^p} = \left(\frac{\alpha}{1 + x_s}(u_s + y_s) + \beta\right)_{s \in S^p}.
\]

Proof. By the proof of Lemma 76 there exist \(x \in \left(\bigcup_{\varphi \in u(H)} \partial I(\varphi)\right)^\perp\), \(\alpha > 0\), \(B \in \mathbb{R}\) \(\beta \in \mathbb{R}\) such that

\[
(\tilde{u}_s)_{s \in S^p} = \left(\frac{\alpha}{1 + x_s}u_s + B_s\right)_{s \in S^p}
\]
where

\[
\alpha = \frac{\tilde{I}(\tilde{u}(f^*)) - I(\tilde{u}(g^*))}{I(u(f^*)) - I(u(g^*))}.
\]
Let \(\beta := \tilde{I}(\tilde{u}(g^*)) - \alpha I(u(g^*))\).

By Lemmas 71 and 72 \(I\) is differentiable at \(\varphi \in \text{int } u(H)\) with derivative \(\nabla I(\varphi)\) if and only if \(\tilde{I}\) is differentiable at \(A\varphi + B \in \text{int } \tilde{u}(H)\) with derivative

\[
(1 + x)\nabla I(u(f)).
\]

Let \(f \in H\) be such that \(u(f) \in \text{int } u(H)\) and \(u(f)\) is a differentiability point of \(I\). Then by Lemma 74 \(I(u(f)) = \nabla I(u(f)) \cdot u(f)\) and \(\tilde{I}(\tilde{u}(f)) = (1 + x)\nabla I(u(f)) \cdot \tilde{u}(f)\). Thus we have by (56)

\[
\tilde{I}(\tilde{u}(f)) = (1 + x)\nabla I(u(f)) \cdot \frac{\alpha}{1 + x}u(f) + B
\]
\[
= \alpha \nabla I(u(f)) \cdot u(f) + (1 + x)\nabla I(u(f)) \cdot B = \alpha I(u(f)) + (1 + x)\nabla I(u(f)) \cdot B.
\]
Thus by Lemma 77

\[
(1 + x)\nabla I(u(f)) \cdot B = \tilde{I}(\tilde{u}(f)) - \alpha I(u(f)) = \beta
\]
and so

\[
(1 + x)\nabla I(u(f)) \cdot B = \tilde{I}(\tilde{u}(f)) - \alpha I(u(f)) = \beta
\]
Since \( f \) was arbitrary such that \( u(f) \in \text{int} \ u(H) \) and \( u(f) \) is a differentiability point of \( I \), we have by Lemma 73 for all \( p \in \bigcup_{\varphi \in \text{int} \ u(H)} \partial I(\varphi) \)

\[
\beta = (1 + x)p \cdot B = \sum_{s \in S} ((1 + x_s)B_s)p_s
\]

and since \((1 + x)p \in \Delta(S)\)

\[
0 = \sum_{s \in S} ((1 + x_s)B_s - \beta(1 + x_s))p_s.
\]

Thus by Lemma 75 exists \( \alpha y \in \left( \bigcup_{\varphi \in \text{int} \ u(H)} \partial I(\varphi) \right)^\perp \) such that

\[
\alpha y = (1 + x)B - (1 + x)\beta \iff B = \frac{\alpha y}{1 + x} + \beta.
\]

Thus since \( \left( \bigcup_{\varphi \in \text{int} \ u(H)} \partial I(\varphi) \right)^\perp \) is a linear space \( y \in \left( \bigcup_{\varphi \in \text{int} \ u(H)} \partial I(\varphi) \right)^\perp \) and for all \( s \in S^P \)

\[
(\check{u}_s)_{s \in S^P} = \left( \frac{\alpha}{1 + x_s}(u_s + y_s) + \beta \right)_{s \in S^P}.
\]

Now for all differentiability points \( p \cdot u(f) = I(u(f)) \) and \((1 + x)p \cdot \check{u}(f) = \check{I}(\check{u}(f))\). Thus

\[
(1 + x)p \cdot (\frac{\alpha}{1 + x}(1 + x)u(f) + B) = \check{I}(\check{u}(f))
\]

and so

\[
(1 + x)p \cdot B = \check{I}(\check{u}(f)) - \alpha I(u(f))
\]

and since this works for all differentiability points we are done. \( \square \)

**Corollary 79** Let \((u, \mathbb{P})\) be a state dependent tight dual-self representation for \( \preceq \), and \((\check{u}, \check{\mathbb{P}})\) be a utility-probability set collection pair. Then \((\check{u}, \check{\mathbb{P}})\) is a state dependent tight dual-self representation for \( \preceq \) if and only if there exist \( x \in \mathbb{R}^S_{++}, y \in \mathbb{R}^S \) such that for all \( p, q \in \bigcup_{P \in \mathbb{P}} P \)

\[
\sum_{s \in S} x_sp_s = \sum_{s \in S} x_sq_s =: \alpha \text{ and } \sum_{s \in S} y_sp_s = \sum_{s \in S} y_sq_s
\]

for all \( s \in S^P \),

\[
\check{u}_s = \frac{1}{x_s}(u_s + y_s),
\]

and for all \( f \in H \)

\[
\max_{\check{P} \in \check{\mathbb{P}}} \min_{\check{P} \in \check{\mathbb{P}}} \sum_{s \in S} \check{p}_s\check{u}_s(f_s) = \max_{P \in \mathbb{P}/\alpha} \min_{P \in \mathbb{P}/\alpha} \sum_{s \in S} p_s u_s(f_s).
\]

Especially \((\check{u}, x/\alpha \mathbb{P})\) is another state dependent tight dual-self representation for \( \preceq \) and

\[
\bigcup_{\check{P} \in \check{\mathbb{P}}} \check{P} = \bigcup_{P \in \mathbb{P}/\alpha} \frac{x}{\alpha} P.
\]

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Proof. Let \((\tilde{u}, \tilde{P})\) be a state dependent tight dual-self representation for \(\succsim\). Define \(I : u(H) \rightarrow \mathbb{R}\) and \(\tilde{I} : \tilde{u}(H) \rightarrow \mathbb{R}\) by for all \(f \in H\)

\[
I(u(f)) = \max_{P \in \tilde{P}} \min_{p \in P} p \cdot u(f) \quad \text{and} \quad \tilde{I}(\tilde{u}(f)) = \max_{\tilde{P} \in \tilde{P}} \min_{\tilde{p} \in \tilde{P}} \tilde{p} \cdot \tilde{u}(f).
\]

Now \((u, I)\) and \((\tilde{u}, \tilde{I})\) are state dependent dual-self representations for \(\succsim\). By Chandrasekher et al.’s (2020) Lemma B.3

\[
\varnothing \bigcup_{\varphi \in u(H)} \partial I(\varphi) = \varnothing \bigcup_{P \in \tilde{P}} P \quad \text{and} \quad \varnothing \bigcup_{\tilde{\varphi} \in \tilde{u}(H)} \partial \tilde{I}(\tilde{\varphi}) = \varnothing \bigcup_{\tilde{P} \in \tilde{P}} \tilde{P}.
\]

Thus by Lemma 78 there exist \(\bar{x}, \bar{y} \in \left(\bigcup_{P \in \tilde{P}} P\right)^{\perp}\), \(\tilde{\alpha} > 0, \beta \in \mathbb{R}\) such that

\[
\bigcup_{\varphi \in \text{int} u(H)} \partial \tilde{I}(\varphi) = (1 + \bar{x}) \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi).
\]

and for all \(S^{P}\)

\[
(u_s)_{s \in S^{P}} = \left(\frac{\alpha}{1 + \bar{x}} (u_s + \bar{y}_s) + \beta\right)_{s \in S^{P}}.
\]

Thus define \(y = \bar{y} + \tilde{\alpha}^{-1} \beta, x = \tilde{\alpha}^{-1} (1 + \bar{x}) \alpha = \tilde{\alpha}^{-1}\) which gives the utility identification. Additionally we have for all \(f \in H\)

\[
\max_{P \in \tilde{P}} \min_{p \in P} \tilde{p} \cdot \tilde{u}(f) = \alpha \max_{P \in \tilde{P}} \min_{p \in P} p \cdot u(f) + \beta.
\]

And \(x/\alpha \in \mathcal{K}(\Delta(S))\) since \(\|x/\alpha\|_{\infty} < \infty\). Thus \((\tilde{u}, x/\alpha \tilde{P})\) is a dual-self representation for \(\succsim\) and thus by Lemma 77 for all \(f \in H\)

\[
\max_{P \in \tilde{P}} \min_{p \in P} \sum_{s \in S} \tilde{p}_s u_s(f_s) = \max_{P \in x/\alpha \tilde{P}} \min_{p \in P} \sum_{s \in S} p_s \tilde{u}_s(f_s).
\]

\(\square\)

**Corollary 80** Let \((u, C)\) be a state dependent tight dual-self variational representation for \(\succsim\), and \((\tilde{u}, \tilde{C})\) be a utility-cost set pair. Then \((\tilde{u}, \tilde{C})\) is a state dependent tight dual-self variational representation for \(\succsim\) if and only if there exist \(x \in \mathbb{R}_{++}^{S}, B \in \mathbb{R}^{S}, \beta \in \mathbb{R}\) such that for all \(p, q \in \bigcup_{c \in C} \text{dom} c\)

\[
\sum_{s \in S} x_s p_s = \sum_{s \in S} x_s q_s =: \alpha,
\]

for all \(s \in S^{P}\),

\[
\tilde{u}_s = \frac{1}{x_s} u_s + B_s.
\]
and for all \( f \in H \)
\[
\max_{c \in C} \min_{p \in \Delta(S)} \sum_{s \in S} p_s \bar{u}_s(f_s) + \bar{c}(p) = \max_{c \in C} \min_{p \in \Delta(S)} \sum_{s \in S} p_s \bar{u}_s(f_s) + \alpha c(z/\alpha p) - \sum_{s \in S} B_s p_s + \beta.
\]

Especially \((\bar{u}, \alpha(\mathcal{C} \circ z/\alpha) - dB + \beta)\), where multiplication and subtractions are done elementwise, is another state dependent tight dual-self variational representation for \(\succeq\) and
\[
\text{co} \bigcup_{c \in C} \text{dom} \bar{c} = \text{co} \bigcup_{c \in C} \frac{X}{c} \text{dom} c.
\]

**Proof.** Follows symmetrically to Corollary 79 since for \( I : u(H) \to \mathbb{R} \) by defining
\[
I(u(f)) = \max_{c \in C} \min_{p \in \Delta(S)} p \cdot u(f) + c(p)
\]
we have by Chandrasekher et al.’s (2020) Section S.3
\[
\text{co} \bigcup_{\varphi \in u(H)} \partial I(\varphi) = \text{co} \bigcup_{c \in C} \text{dom} c.
\]

\[8.5.2\] Full Identification

**Lemma 81** Let \(\succeq\) have a state dependent dual-self variational representation with \((u, I)\). Let \(f, g \in H\) be equally crisp acts such that exists \(s, s' \in S^P\) such that \(u_s(f_s) > u_s(g_s)\) and \(u_{s'}(g_{s'}) > u_{s'}(f_{s'})\). Then exist equally crisp acts \(f^1, g^1\) such that \(f^1 \sim g^1\) and \(s^1, s^2 \in S^P\) \(f_{s^1} \succ s^1\) \(g_{s^1}^1\) and \(g_{s^2}^1 \succ s^2\) \(f_{s^2}^1\).

**Proof.** Assume that exists \(\tilde{f}^*, \tilde{g}^*, s, s' \in S\) such that \(u_s(\tilde{f}^*) > u_s(\tilde{g}^*), s\) and \(u_{s'}(\tilde{g}^*) > u_{s'}(\tilde{f}^*)\) and \(\tilde{f}^*, \tilde{g}^*\) are equally crisp. Assume without loss of generality \(\tilde{f}^* \succeq \tilde{g}^*\). By Lemma 65 for all \(\varphi \in u(H), \alpha \in (0, 1]\) such that \(\varphi + \alpha(u(\tilde{f}^*) - u(\tilde{g}^*)) \in u(H),\)
\[
I(\varphi + \alpha(u(\tilde{f}^*) - u(\tilde{g}^*))) = I(\varphi) + \alpha\left(I(u(\tilde{f}^*)) - I(u(\tilde{g}^*))\right).
\]

Let \(g^0 \in H\) be such that \(u(f^0) \in \text{int} u(H)\). Now exists \(\alpha^* > 0\) such that \(u(f^0) + \alpha^*(u(\tilde{f}^*) - u(\tilde{g}^*)), u(f^0) - \alpha^*\left(I(u(\tilde{f}^*)) - I(u(\tilde{g}^*))\right) \in u(H)\). Let \(f^0, f^1 \in H\) be such that
\[
u(f^0) = u(f^0) + \alpha^*(u(\tilde{f}^*) - u(\tilde{g}^*))
\]
and
\[
u(f^1) = u(f^0) - \alpha^*\left(I(u(\tilde{f}^*)) - I(u(\tilde{g}^*))\right).
\]

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By Equation (57) and C-additivity of $I$, $(g^0, f^0)$ and $(g^0, f^1)$ are equally crisp acts. Then by Lemma 57 $(g^0, \frac{1}{2}f^0 + \frac{1}{2}f^1)$ are equally crisp acts. Denote $f^2 := \frac{1}{2}f^0 + \frac{1}{2}f^1$. Now we have since $u(H)$ is a convex set

$$I(u(f^2)) = I\left( u(f^0) + \frac{1}{2}2\alpha^*(u(\tilde{f}^*) - u(\tilde{g}^*)) - \frac{1}{2}2\alpha^*(I(u(\tilde{f}^*)) - I(u(\tilde{g}^*))) \right)$$

Thus $f^2 \sim f^0$. First since $u_s'(\tilde{g}^*_s) > u_s'(\tilde{f}^*_s)$. Thus

$$u_s'(f^2_s) = u_s'(f^0_s) + \frac{1}{2}2\alpha^*(u_s'(\tilde{f}^*_s) - u_s'(\tilde{g}^*_s)) + \frac{1}{2}(I(u(\tilde{g}^*_s)) - I(u(\tilde{f}^*_s))) < u_s'(f^0_s).$$

Since $s' \in S^P$, by Lemma 73 exists $\varphi \in \text{int} u(H)$ such that $I$ is differentiable at $\varphi$ and $\nabla I(\varphi)_s' > 0$. By Lemma 65 we have

$$\nabla I(\varphi) \cdot (u(f^2) - u(f^0)) = 0.$$ 

Thus exists $s''$ such that $u_{s''}(f^2_{s''}) > u_{s''}(g^0_{s''})$ and $\nabla I(\varphi)_{s''} > 0$. Let $g_3, f_3, \alpha^3$ be such that $u(g^3) = \varphi$, $u(f^3) = \varphi + \alpha^3(u(f^2) - u(g^0))$. By Lemma 65 $(g^3, f^3)$ are equally crisp acts and by (58) $g^3 \sim f^3$ and

$$u_{s''}(f^3_{s''}) - u_{s''}(g^3_{s''}) > 0 > u_{s'}(f^3_{s'}) - u_{s'}(g^3_{s'}).$$

Additionally, by the differentiability of $I$ at $u(g^3)$ exists

$$0 < \varepsilon < \min\{u_{s''}(f^3_{s''}) - u_{s''}(g^3_{s''}), u_{s'}(g^3_{s'}) - u_{s'}(f^3_{s'})\}$$

such that

$$I(u(g^3) + \text{pr}_{s'} \varepsilon) > I(u(g^3)) + \nabla(I(\varphi))_{s'} \frac{\varepsilon}{2}$$ and $I(u(g^3) - \text{pr}_{s'} \varepsilon) < I(u(g^3)) - \nabla(I(\varphi))_{s'} \frac{\varepsilon}{2}$.

Thus

$$I(u(f^3_{s''}, g^3_{-s''})) \leq I(u(g^3) - \text{pr}_{s'} \varepsilon) < I(u(g^3))$$ and $I(u(f^3_{s'}, g^3_{-s'})) \geq I(u(g^3) + \text{pr}_{s'} \varepsilon) > I(u(g^3)).$

Thus $f^3_{s''} \succ_{s'} g^3_{s''}$ and $g^3_{s'} \succ_{s'} f^3_{s'}$ and $(g^3, f^3)$ are equally crisp.

\[\square\]

**Lemma 82** Let $\succ$ have a state dependent dual-self variational representation with $(u, I)$ then the following are equivalent:

(I) If $f, g$ are equally crisp acts, then for all $s \in S$ $f_s \succ_s g_s$ or for all $s \in S$ $g_s \succ_s f_s$. 

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(II) For all \( f, g \sim g \) and exists \( s, s' \in S \) \( f_s >_s g_s \) and \( f_{s'} >_{s'} g_{s'} \), exists \( h \in H, \alpha \in (0, 1) \) such that \( \alpha h + (1 - \alpha)f \not\sim \alpha h + (1 - \alpha)g \).

**Proof.** Assume that (II) does not hold and let us show that (I) does not hold. By the counter assumption there exist \( f, g \in H \) such that \( f \sim g \) and there exist \( s, s' \in S \) \( f_s >_s g_s \) and \( f_{s'} >_{s'} g_{s'} \) and for all \( h \in H, \alpha \in (0, 1) \) \( \alpha h + (1 - \alpha)f \not\sim \alpha h + (1 - \alpha)g \). Now we have for all \( h, h', \alpha \in (0, 1) \)

\[
\alpha h + (1 - \alpha)f \sim ah + (1 - \alpha)g \text{ and } ah' + (1 - \alpha)f \sim ah' + (1 - \alpha)g
\]

Thus

\[
\alpha h + (1 - \alpha)f \succ ah' + (1 - \alpha)f \iff \alpha h + (1 - \alpha)g \succ ah' + (1 - \alpha)g.
\]

Hence \( f, g \) are equally crisp and I) does not hold.

Assume that (I) does not hold and let us show that (II) does not hold. Assume that exists \( \tilde{f}^*, \tilde{g}^*, s, s' \in S \) such that \( \tilde{f}^*_s >_s \tilde{g}^*_s \) and \( \tilde{g}^*_{s'} >_{s'} \tilde{f}^*_{s'} \) and \( \tilde{f}^*, \tilde{g}^* \) are equally crisp. First, by Lemma 44 \( u_s(\tilde{f}^*)_s > u_s(\tilde{g}^*)_s \) and \( u_{s'}(\tilde{g}^*)_{s'} > u_{s'}(\tilde{f}^*_{s'}) \). Thus by Lemma 81 exist equally crisp acts \( f^1, g^1 \) such that \( f^1 \sim g^1 \) and \( s^1, s^2 \in S^P \) \( f^1_{s^1} >_{s^1} g^1_{s^1} \) and \( g^1_{s^2} >_{s^2} f^1_{s^2} \). However, for all \( h \in H, \alpha \in (0, 1) \) by Lemma 65

\[
I(\alpha u(h)+(1-\alpha)u(f^1)) = I(\alpha u(h)+(1-\alpha)u(g^1)) + (1-\alpha)(I(u(f^1)) - I(u(g^1))) = I(\alpha u(h) + (1 - \alpha)u(g^1))
\]

and hence \( \alpha h + (1 - \alpha)f^1 \sim \alpha h + (1 - \alpha)g^1 \). This contradicts (II). \( \square \)

**Lemma 83** Let \( \succsim \) have a state dependent dual-self variational representation with \((u, I)\) then the following are equivalent:

(I) \( \succsim \) satisfies Axiom 7.

(II) \( \text{pr}_{S^P} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \) has a non-empty interior in \( \Delta(S^P) \).

**Proof.** Let us enumerate \( S^P = \{s_1, \ldots, s_n\} \) where \( n = |S^P| \). Assume that (I) holds. Then if \( f, g \) are equally crisp acts, by Lemmas 81 and 82 for all \( s, s' \in S^P \) \( u_s(f_s) - u_s(g_s) = u_{s'}(f_{s'}) - u_{s'}(g_{s'}) \). Thus \( \left( \text{pr}_{S^P} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \right)^\perp = \{0\} \). Hence \( \text{pr}_{S^P} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \) contains \( |S^P| = n \) linearly independent probabilities \( \{p^1, \ldots, p^n\} \). Let us consider the set \( \text{pr}_{S^P} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) - p^n \) where the subtraction is vector subtraction in \( \mathbb{R}^{S^P} \). Now the set \( \text{pr}_{S^P} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) - p^n \supset \{p^1 - p^n, \ldots, p^{n-1} - p^n\} \) is linearly independent. Let us consider matrix \( P = [p^i - p^n]_{1 \leq i \leq n-1} \). First, the rank of \( P \) is \( n - 1 \) by above. Second, we have for all \( 1 \leq i \leq n - 1 \sum_{j=1}^{n} p^i_j - p^n_j = 0 \). Third, since the row rank of \( P \) is \( n - 1 \) and the last row \( \tilde{s}_n \) is linearly dependent on rows \( \{\tilde{s}_1, \ldots, \tilde{s}_{n-1}\} \), the row vectors of \( P \) for rows \( \{\tilde{s}_1, \ldots, \tilde{s}_{n-1}\} \) are linearly independent. Next, let us consider the matrix
\[ \hat{P} = [pr_{S^P \setminus \{\tilde{s}_n\}} p^i - p^n]_{1 \leq i \leq n-1}. \] This is collecting row vectors of \( P \) for rows \( \{\tilde{s}_1, \ldots, \tilde{s}_{n-1}\} \) and thus by above the rank of \( \hat{P} \) is \( n - 1 \). Thus especially \( \{pr_{S^P \setminus \{\tilde{s}_n\}} p^1 - p^n, \ldots, pr_{S^P \setminus \{\tilde{s}_n\}} p^{n-1} - p^n\} \) are linearly independent. Thus the set \( pr_{S^P \setminus \{\tilde{s}_n\}} pr_{S^P} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) - p^n \) contains the zero vector and \( n - 1 \) linearly independent vectors and is a convex set. Hence the set \( pr_{S^P \setminus \{\tilde{s}_n\}} pr_{S^P} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) - p^n \) cannot be contained in any \( n - 2 \)-dimensional set and thus it has a non-empty interior in \( \mathbb{R}^{S^P \setminus \{\tilde{s}_n\}} \) (Boyd and Vandenberghe, 2004, Section 2.5.2). Thus especially the set \( pr_{S^P \setminus \{\tilde{s}_n\}} pr_{S^P} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \) has a non-empty interior in \( \mathbb{R}^{S^P \setminus \{\tilde{s}_n\}} \) as a translation. Thus by the choice of topology for \( \Delta(S^P) \), \( pr_{S^P} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \) has a non-empty interior in \( \Delta(S^P) \).

Assume that (II) holds. Then \( pr_{S^P} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \) contains \( |S^P| \) linearly independent probabilities. Let \( f, g \) be equally crisp acts. Assume w.l.o.g. that exists \( s \in S^P \) such that \( u_s(f_s) > u_s(g_s) \) or for all \( u_s(f_s) = u_s(g_s) \). Then by Lemmas 65 and 73 for all \( p, q \in pr_{S^P} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \)

\[
\alpha + y = u(f) - u(g).
\]

By Lemma 75, exists \( y \in \left( pr_{S^P} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \right)^{\perp} \) such that for all \( s \in S \)

\[
\alpha + y = u(f) - u(g).
\]

Since \( \left( pr_{S^P} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \right)^{\perp} = \{ \tilde{0} \} \) we have for all \( s \in S^P \) \( u_s(f_s) - u_s(g_s) = \alpha \). Now \( \alpha \geq 0 \) by the above assumption. Thus for all \( s \in S^P \) \( u_s(f_s) \geq u_s(g_s) \). Thus by Lemma 44 for all \( s \in S^P \) \( f_s \succ_s g_s \). This shows (I) by Lemma 82.

\[ \square \]

### 8.6 Partial Identification Characterizations

We first show lemmas.

**Lemma 84** Let \((u, I)\) and be a state dependent dual-self representations for \( \succsim \). If \( c \in H \) is crisp, \( \alpha \in (0, 1) \) and \( h \in H \) is such that \( u(h) \in \text{int} u(H) \) and \( I \) is differentiable at \( u(h) \), then \( I \) is differentiable at \( \alpha u(c) + (1 - \alpha) u(h) \) with

\[
\nabla I \left( \alpha u(c) + (1 - \alpha) u(h) \right) = \nabla I \left( u(h) \right).
\]

**Proof.** Let \( \alpha^* \in (0, 1) \). As in the proof of Lemma 45, there exist equally crisp \( f^*, g^* \in H, a > 0 \) such that for all \( s \in S^P \) \( u_s(f_s^*) - u_s(g_s^*) = a \) and \( f_s^* \succ_s g_s^* \). Denote \( h^* = \frac{1}{2} f^* + \frac{1}{2} g^* \).
We use notation from Section 8.5 from using equally crisp acts $f^*, g^*$. Since $u(h) ∈ \text{int } u(H)$, there exist $\hat{h} ∈ H, β ∈ (0, 1)$ such that for all $s ∈ S^p$, $u_s(\beta \hat{h} + (1 - β)h^*) = u_s(h)$.

Let $g ∈ H$. Denote $\hat{h} = \frac{α^*}{α^*(1-α^*)β}c + \frac{(1-α^*)β}{α^*(1-α^*)β} h$, $\tilde{β} = α^* + (1 - α^*)β$, $\tilde{g} = α^*c + (1 - α^*)g$. We will show that for all $α ∈ (0, 1)$,

$$C_{h, β}^{α, α} = C_{\hat{h}, β}^{\tilde{g}, α}$$  \hspace{1cm} (59)$$

This follows from the definition of $C_{h, β}^{α, α}$ and the observation that for all $\tilde{α} ∈ [-1/2α, 1/2α]$, we have

$$\beta \hat{h} + (1 - β)((α\hat{α} + 1/2)f^* + (1/2 - α\hat{α})g^*) ∼ αg + (1 - α)(\beta \hat{h} + (1 - β)h^*)$$

$$⇔ α^*c + (1 - α^*)\left(β \hat{h} + (1 - β)((α\hat{α} + 1/2)f^* + (1/2 - α\hat{α})g^*)\right) ∼ (αg + (1 - α)(\beta \hat{h} + (1 - β)h^*))$$

$$⇔ \tilde{β} \hat{h} + (1 - \tilde{β})((α\tilde{α} + 1/2)f^* + (1/2 - α\tilde{α})g^*) ∼ (αg + (1 - α)(\beta \hat{h} + (1 - β)h^*))$$

where the second equivalence follows as an identity.

Since $I$ is differentiable at $u(h)$, the differentiability follows from Lemma 70 as in the proof of Lemma 72 since for all $v ∈ R^{S^p}$, there exist $g ∈ H, a^* > 0$ such that for all $s ∈ S^p$ $u_s(g) = u_s(h) + a^*v$ and so

$$u_s(α^*c + (1 - α^*)g) = u_s(\tilde{β} \hat{h} + (1 - \tilde{β})h^*) + (1 - α^*)a^*v.$$  

Additionally, the derivative being the same follows from Lemma 70.

\hspace{1cm} \Box

**Lemma 85** Let $(u, I)$ be a state dependent dual-self representations for $\succcurlyeq$. If $c ∈ H$ is crisp, then for all $p ∈ \bigcup_{φ ∈ u(H)} \partial I(φ)$

$$p \cdot u(c) = I(u(c)).$$

*Proof.* Let $h ∈ H$ be such that $u(h) ∈ \text{int } u(H)$ and $I$ is differentiable at $u(h)$. Then by Lemmas 74 and 84 for all $α ∈ [0, 1]$

$$I(u(αc + (1 - α)h)) = \nabla I(u(h)) \cdot u(αc + (1 - α)h).$$

Thus by the continuity of $I$ by taking limit $α → 1$

$$I(u(c)) = \nabla I(u(h)) \cdot u(c).$$

Finally, the claim follows from Lemma 73.  \hspace{1cm} \Box
Lemma 86 Let \( \preceq \) have a state dependent dual-self variational representation with \((u,I)\). If \( s \in S^P \), then there exist \( p \in \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) \) such that \( p_s > 0 \).

Proof. Follows from the definition of \( S^P \) and the definition of Clarke derivatives.

Lemma 87 Let \( \preceq \) have a state dependent dual-self variational representation with \((u,I)\). Let \( x \in \mathbb{R}^s \) and \( s', s'' \in S^P \) be such that \( x_s > 0 > x_{s''} \) and for all \( \tilde{s} \notin S^P \), \( x_{\tilde{s}} = 0 \). Then there exist \( f, g \in H, a > 0 \) such that \( f_s \succ s \, g_s \) and \( g_{s'} \succ s' \, f_{s'} \), \( u(f), u(g) \in \text{int } u(H) \), and \( u(f) - u(g) = ax \).

Proof. Since \( s, s' \in S^P \) by Lemmas 73 and 86, there exist \( f^s, f^{s'} \in H \) such that \( u(f^s), u(f^{s'}) \in \text{int } u(H) \), \( I \) is differentiable at \( u(f^s), u(f^{s'}) \) and

\[
\nabla I(u(f^s))_s, \nabla I(u(f^{s'}))_{s'} > 0.
\]

Let \( f \in H \) be such that \( u(f) \in \text{int } u(H) \), \( u_s(f_s) = \varphi^s_s \) and \( u_{s'}(f_{s'}) = \varphi^{s'}_{s'} \). Since \( u(f) \in \text{int } u(H) \) and for all \( \tilde{s} \notin S^P \), \( x_{\tilde{s}} = 0 \), there exist \( a > 0, g \in H \) such that \( u(f) - u(g) = ax \). Additionally, since \( I \) is monotonic by (60), we have

\[
(f_s, f^{s}_{-s}) \succ (g_s, f^{s}_{-s}) \text{ and } (g_{s'}, f^{s'}_{-s'}) \succ (f_{s'}, f^{s'}_{-s'}).
\]

Lemma 88 Let \( \preceq \) have a state dependent dual-self variational representation with \((u,I)\). Let \( x \in \mathbb{R}^s \) and \( c \in \mathbb{R} \) be such that

\[
(x - \overline{1}c) \in \left( \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) \right)^\perp.
\]

Then for all \( \varphi \in \text{int } u(H) \) such that \( \varphi + x \in \text{int } u(H) \), we have

\[
I(\varphi + x) = I(\varphi) + c.
\]

Proof. By Chandrasekher et al.’s (2020) Section S.3

\[
\bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) \subseteq \Delta(S).
\]

Since \( \text{int } u(H) \) is convex, for all \( \alpha \in [0, 1], \varphi + \alpha x \in \text{int } u(H) \) Let us define a mapping \([0, 1] \ni \alpha \rightarrow I(\varphi + \alpha x) \). Since \( I \) is C-additive, it is 1-lipschitz. So especially \( J \) is lipschitz and so differentiable almost everywhere. Now if \( J \) is differentiable at \( \alpha \in (0, 1) \) we have by Lemma 73

\[
J'(\alpha) \leq \sup_{\psi \in \partial I(\varphi + \alpha x)} \psi \cdot x = \sup_{\psi \in \partial I(\varphi + \alpha x)} \psi \cdot (x - \overline{1}c) + \psi \cdot \overline{1}c = \sup_{\psi \in \partial I(\varphi + \alpha x)} \psi \cdot \overline{1}c = c
\]

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and symmetrically
\[ J'(\alpha) \geq \inf_{\psi \in \partial I(\psi+\alpha x)} \psi \cdot x = c. \]

Let \( \Omega \subseteq (0, 1) \) be the set of differentiability points for \( J \). By the Fundamental theorem of calculus, we have
\[ I(\varphi + x) = J(0) + J(1) - J(0) = I(\varphi) + \int_{\Omega} J'(x) \, dx = I(\varphi) + c. \]

\[ \square \]

8.6.1 Identification with Crisp and Equally Crisp Acts

**Corollary 89** Let \((u, P)\) be a state dependent dual-self representation for \( \succcurlyeq \) and \( x \in \mathbb{R}^S_{++}, y \in \mathbb{R}^S \). The following two conditions are equivalent:

1. \((\tilde{u}, \tilde{P})\) is a state dependent dual-self representation for \( \succcurlyeq \) such that for all \( s \in S^P \)
\[ \tilde{u}_s = \frac{1}{x_s}(u_s + y_s) \]
and
\[ 0, 1 \in \bigcap_{s \in S^P} \tilde{u}_s\left(\Delta(X_s)\right). \]
2. There exist crisp acts \( f^*, g^* \) such that for all \( s \in S^P \), \( f^*_s \succcurlyeq g^*_s \), \( u_s(g^*_s) = -y_s \) and \( u_s(f^*_s) = -y_s + x_s \).

*Proof.* Follows from Lemma 85 and Proposition 3

\[ \square \]

**Corollary 90** Let \((u, C)\) be a state dependent dual-self variational representation for \( \succcurlyeq \) and \( x \in \mathbb{R}^S_{++}, B \in \mathbb{R}^S \). The following two conditions are equivalent:

1. \((\tilde{u}, \tilde{C})\) is a state dependent dual-self variational representation for \( \succcurlyeq \) such that for all \( s \in S^P \),
\[ \tilde{u}_s = \frac{1}{x_s} u_s + \frac{B_s}{x_s} \]
and
\[ 0, 1 \in \bigcap_{s \in S^P} \tilde{u}_s\left(\Delta(X_s)\right). \]
2. There exists equally crisp acts \( f^*, g^* \) such that for all \( s \in S^P \), \( f^*_s \succcurlyeq g^*_s \), \( u_s(g^*_s) = -B_s \) and \( u_s(f^*_s) = -B_s + x_s \).

*Proof.* Follows from Lemma 65 and Theorem 9.

\[ \square \]
Lemma 91  Let \((u, I)\) be state dependent dual-self variational representations for \(\succeq\) and \(s, s' \in S^p\), \(s \neq s'\). The following two are equivalent:

1. For all \(f, g \in H\) such that and \(f_s \succ_s g_s\) and \(g_{s'} \succ_{s'} f_{s'}\), there exist \(h, h' \in H\) and \(\alpha \in (0, 1)\) such that
   \[\alpha h + (1 - \alpha) f \succ \alpha h' + (1 - \alpha) f\]
   \[\alpha h + (1 - \alpha) g \prec \alpha h' + (1 - \alpha) g.\]

2. For all \(x \in \left(\bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi)\right)^\perp\), \(x_s = x_{s'}\).

Proof. We will first show that (1) \(\Rightarrow\) (2). Assume, per contra, there exist \(x \in \left(\bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi)\right)^\perp\), 
\(x_s \neq x_{s'}\). Assume w.l.o.g. \(x_s > x_{s'}\). Let \(c = -\frac{1}{2}x_s - \frac{1}{2}x_{s'}\). By (87), there exist \(f, g \in H, a > 0\) such that \(f_s \succ_s g_s\) and \(g_{s'} \succ_{s'} f_{s'}\) and \(u(f) - u(g) = a(x+c)\). Let \(h \in H\), \(\alpha \in (0, 1)\) with \(u(h) \in \text{int} u(H)\). By Lemma 88, we have
   \[I(\alpha u(h) + (1 - \alpha) u(f)) = I(\alpha u(h) + (1 - \alpha) u(g) + (1 - \alpha) (u(f) - u(g)))\]
   \[= I(\alpha u(h) + (1 - \alpha) u(g) + (1 - \alpha) a(x+c)) = I(\alpha u(h) + (1 - \alpha) u(g)) + (1 - \alpha) a c.\]

Since \(I\) is continuous we have for all \(h \in H\), \(\alpha \in (0, 1)\)
\[I(\alpha u(h) + (1 - \alpha) u(f)) = I(\alpha u(h) + (1 - \alpha) u(g)) + (1 - \alpha) a c.\]

Thus we have for all \(h, h' \in H\),
\[\alpha h + (1 - \alpha) f \succeq \alpha h' + (1 - \alpha) f \iff I(\alpha u(h) + (1 - \alpha) u(f)) \geq I(\alpha u(h') + (1 - \alpha) u(f))\]
\[\iff I(\alpha u(h) + (1 - \alpha) u(f)) + (1 - \alpha) a c \geq I(\alpha u(h') + (1 - \alpha) u(f)) + (1 - \alpha) a c\]
\[\iff I(\alpha u(h) + (1 - \alpha) u(g)) \geq I(\alpha u(h') + (1 - \alpha) u(g)) \iff \alpha h + (1 - \alpha) f \succeq \alpha h' + (1 - \alpha) f.\]

This contradicts (1).

Second, we show that (2) \(\Rightarrow\) (1). Assume, per contra, that there exist \(f, g \in H\) such that \(f_s \succ_s g_s\) and \(g_{s'} \succ_{s'} f_{s'}\) and for all \(h, h', \alpha \in (0, 1)\)
\[\alpha h + (1 - \alpha) f \succ \alpha h' + (1 - \alpha) f \Rightarrow \alpha h + (1 - \alpha) g \succeq \alpha h' + (1 - \alpha) g.\]

We show first that for all \(h, h', \in H\) with \(u(h), u(h') \in \text{int} u(H), \alpha \in (0, 1)\)
\[\alpha h + (1 - \alpha) f \succ \alpha h' + (1 - \alpha) f \Rightarrow \alpha h + (1 - \alpha) g \succ \alpha h' + (1 - \alpha) g.\]
By the continuity of $I$ and since $h \in \text{int } u(H)$, there exist $h^\dagger \in H, a > 0$ such that $u(h^\dagger) = u(h) - 1a$ and $\alpha h^\dagger + (1 - \alpha)f \succ \alpha h' + (1 - \alpha)f$. So by assumption and C-additivity of $I$ we have

$$\alpha h + (1 - \alpha)g \succ \alpha h^\dagger + (1 - \alpha)g \succ \alpha h' + (1 - \alpha)g.$$ Symmetrically by the negation of the assumption we have for all $h, h', \alpha \in (0, 1)$

$$\alpha h + (1 - \alpha)g \succ \alpha h' + (1 - \alpha)g \Rightarrow \alpha h + (1 - \alpha)f \succ \alpha h' + (1 - \alpha)f$$ and so for all $h, h', \in H$ with $u(h), u(h') \in \text{int } u(H), \alpha \in (0, 1)$

$$\alpha h + (1 - \alpha)g \succ \alpha h' + (1 - \alpha)g \Rightarrow \alpha h + (1 - \alpha)f \succ \alpha h' + (1 - \alpha)f.$$ So for all $h, h', \in H$ with $u(h), u(h') \in \text{int } u(H), \alpha \in (0, 1)$

$$\alpha h + (1 - \alpha)f \succ \alpha h' + (1 - \alpha)f \iff \alpha h + (1 - \alpha)g \succ \alpha h' + (1 - \alpha)g.$$ (61)

Thus $f$ and $g$ are equally crisp acts in interior of $u(H)$. By Lemma 65, we have for all $\varphi \in \text{int } u(H), \alpha \in (0, 1)$ such that $\varphi + \alpha(u(f) - u(g)) \in \text{int } u(H),$

$$I(\varphi + \alpha(u(f) - u(g))) = I(\varphi) + \alpha(I(u(f)) - I(u(g))).$$

Thus for all differentiability points $\varphi$ of $I$ in $\text{int } u(H)$, we have

$$\nabla I(\varphi) \cdot (u(f) - u(g)) = I(u(f)) - I(u(g)).$$

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$$\bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) \subseteq \Delta(S).$$

Thus we have for all differentiability points $\varphi$ of $I$ in $\text{int } u(H)$, we have

$$\nabla I(\varphi) \cdot \left(u(f) - u(g) - \bar{1}(I(u(f)) - I(u(g)))\right) = I(u(f)) - I(u(g)) - \left(I(u(f)) - I(u(g))\right) = 0.$$ Additionally, by Lemma 44

$$u_s(f_s) > u_s(g_s) \text{ and } u_{s'}(g_{s'}) > u_{s'}(f_{s'})$$

and so

$$u_s(f_s) - u_s(g_s) - \left(I(u(f)) - I(u(g))\right) \neq u_{s'}(f_{s'}) - u_{s'}(g_{s'}) - \left(I(u(f)) - I(u(g))\right).$$

By Lemma 73,

$$u(f) - u(g) - \bar{1}(I(u(f)) - I(u(g))) \in \left(\bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi)\right)^\perp.$$
Lemma 92 Let $P \subseteq \Delta(S)$, $P \neq \emptyset$, $s, s' \in S, s \neq s'$. The following two are equivalent:

1. For all $x \in P^\perp$, $x_s = x_{s'}$.
2. There exist $p, q \in P$ such that for all $\bar{s} \in S \setminus \{s, s'\}$, $p_{\bar{s}} = q_{\bar{s}}$ and $\frac{p_s}{p_{s'}} \neq \frac{q_s}{q_{s'}}$.

Proof. We will first show that (2) $\Rightarrow$ (1). Assume per contra that there exist $x \in P^\perp$ such that $x_s \neq x_{s'}$. Let $p, q$ be as in (2). Then

$$\sum_{s \in S} x_sp_{\bar{s}} = 0 = \sum_{s \in S} x_qq_{\bar{s}}$$

Since for all $\bar{s} \in S \setminus \{s, s'\}$, $p_{\bar{s}} = q_{\bar{s}}$ and so $p_s + p_{s'} = q_s + q_{s'}$,

$$(x_s - x_{s'})p_s + x_s(p_s + p_{s'}) = x_sp_s + x_{s'}p_{s'} = x_sp_s + x_{s'}q_{s'} = (x_s - x_{s'})q_s + x_s(q_s + q_{s'})$$

Hence,

$$(x_s - x_{s'})(p_s - q_s) = 0.$$

Since $x_s \neq x_{s'}$ by the counter assumption, we have $p_s = q_s$ which is a contradiction.

Next we show that (1) $\Rightarrow$ (2). Denote $n^\perp = \dim P^\perp$ and $n = \dim P$. Now $n^\perp + n = |S|$. There exist $(c^s)_{i=1}^{n^\perp} \subseteq P^\perp$ linearly independent vectors and $(p^i)_{i=1}^n \subseteq P$ linearly independent vectors.

First we show that $n \geq 2$. Since $P$ is non-empty, $n \geq 1$. Assume, per contra, $n = 1$. Thus $P$ is a singleton $P = \{p\}$. If $p_s = 0$, then $(1_s, 0_{-s}) \in P^\perp$ which is a contradiction. Similarly $p_s \neq 0$. Now $\{(1_{p_{s}}, s), (-1_{p_{s}}), 0_{-s}, s'\} \in P^\perp$ that is a contradiction. So $n \geq 2$.

We consider two cases. First assume that for all $i \in \{1, \ldots, n^\perp\}$,

$$c^s_i = 0.$$

(62)

Now we can consider the matrix $C$ formed by row vectors $(c^s)_{i=1}^{n^\perp}$ and each column in $\bar{s} \in S$. By (62), we can reduce $C$ into Smith normal form $\tilde{C}$ formed by row vectors $(\tilde{c}^s)_{i=1}^{n^\perp}$ such that for each $i \in \{1, \ldots, n^\perp\}$ there exist $s^i \in S \setminus \{s, s'\}$, such that $\tilde{c}^s_{s^i} = -1$, for all $j \in \{1, \ldots, n^\perp\} \setminus \{i\}$, $\tilde{c}^s_{s^i} = 0$, and for all $k, l \in \{1, \ldots, n^\perp\}, k \neq l$, $s^k \neq s^l$. Now $(\tilde{c}^s)_{i=1}^{n^\perp} \subseteq P^\perp$ are linearly independent and for all $i \in \{1, \ldots, n^\perp\}$,

$$\tilde{c}^s_i = \tilde{c}^s_{s^i} = 0.$$

(63)

Denote $S^\perp = \{s^1, \ldots, s^{n^\perp}\}$. Now we have for all $i \in \{1, \ldots, n^\perp\}$ and $p \in P$,

$$p_{s^i} = \sum_{\bar{s} \in S \setminus S^\perp} \tilde{c}^s_{s^i}p_{\bar{s}}.$$

(64)
Let \( \pi : \{1, \ldots, n - 2\} \to S \setminus (S^\perp \cup \{s, s'\}) \) be a one-to-one function. We show by induction that for each \( i \in \{0, \ldots, n - 2\} \) there exist linearly independent collection of probabilities \((\tilde{p}^{j,i})_{j=1}^{n-i}\) such that for all \( j, k \in \{1, \ldots, n - i\}, 1 \leq l \leq i, \)

\[
p^{j,i}_{\pi(l)} = p^{k,i}_{\pi(l)}.
\] (65)

For the first step \( i = 0 \), define for all \( j \in \{1, \ldots, n\} \) \( \tilde{p}^{j,i} = p^{j} \). For the induction step, assume that for \( 0 \leq i < n - 2 \) there exist \((\tilde{p}^{j,i})_{j=1}^{n-i}\) that satisfy (65). First we show that

\[
\min \{p^{j,i}_{\pi(i+1)} \mid j \in \{1, \ldots, n - i\} \} \neq \max \{p^{j,i}_{\pi(i+1)} \mid j \in \{1, \ldots, n - i\} \}.
\] (66)

Assume, per contra, that for all \( j, k \in \{1, \ldots, n - i\}, \) \( p^{j,i}_{\pi(i+1)} = p^{k,i}_{\pi(i+1)} \). We collect \((\tilde{p}^{j,i})_{j=1}^{n-i}\) into a matrix \( P^i \) where each \( \tilde{p}^{j,i} \) is a column vector. The column rank of \( P^i \) is \( n - i \) since each column is linearly independent. Thus the row rank of \( P^i \) is \( n - i \). However each row \( \tilde{s} \in S^\perp \) is linearly dependent on rows \( S \setminus S^\perp \) by (64) and each row \( \tilde{s} \in \{\pi(1), \ldots, \pi(i), \pi(i+1)\} \) is constant and so since the rows sum to 1, linearly dependent on the rows \( S \setminus \{\pi(1), \ldots, \pi(i), \pi(i+1)\} \). Thus the maximum row rank for \( P^i \) is \( |S| - n^+ - i - 1 = n - i - 1 \) which is a contradiction. This shows (66).

Let

\[
j^* \in \text{arg max} \{p^{j,i}_{\pi(i+1)} \mid j \in \{1, \ldots, n - i\} \} \text{ and } j_* \in \text{arg min} \{p^{j,i}_{\pi(i+1)} \mid j \in \{1, \ldots, n - i\} \}.
\]

By (66), there exist \( \beta^i \in (0, 1) \) such that for all \( j \in \{1, \ldots, n - i\} \) \( \setminus j_* \)

\[
\frac{1}{2}p^{j,i}_{\pi(i+1)} + \frac{1}{2}p^{j^*,i}_{\pi(i+1)} > \beta^i > p^{j_*}_i.
\]

Thus for each \( j \in \{1, \ldots, n - i\} \setminus j_* \), there exist \( \alpha^j \in (0, 1) \) such that

\[
\alpha^j \left( \frac{1}{2}p^{j,i}_{\pi(i+1)} + \frac{1}{2}p^{j^*,i}_{\pi(i+1)} \right) + (1 - \alpha^j)p^{j^*,i}_{\pi(i+1)} = \beta^i.
\]

Denote for \( j \in \{1, \ldots, n - i - 1\} \setminus j_* \)

\[
\tilde{p}^{j-1(j > j_*), i+1} = \alpha^j \left( \frac{1}{2}p^{j,i}_i + \frac{1}{2}p^{j^*,i}_i \right) + (1 - \alpha^j)p^{j^*,i}_i,
\]

where \( 1(j > j_* ) \) is an indicator function for \( j > j_* \). Now \((\tilde{p}^{j,i+1})_{j=1}^{n-i-1}\) are linearly independent since they have been created using elementary column operations using the above \( P^i \). Additionally, it satisfies (65). This shows the induction step and concludes the induction.

Since \( n \geq 2 \), by the induction, there exist \( p^*, p^j \in P \) that are linearly independent and for all \( \tilde{s} \in S \setminus (S^\perp \cup \{s, s'\}) \), \( p^*_\tilde{s} = p^j_{\tilde{s}} \).
By (63,64) for all \( \tilde{s} \in S \setminus \{s, s'\} \)
\[
p_{\tilde{s}}^{n+} = p_{\tilde{s}}^{1+}.
\]
This shows the claim since \( p^{1+} \) and \( p^{n+} \) are linearly independent.

Second we consider the case that there exist \( i \in \{1, \ldots, n^+\} \) such that \( c_i^i \neq 0 \). First, if a vector \( \tilde{c} \in \mathbb{R}^S \) is such that \( \tilde{c}_s = \tilde{c}_{s'} \neq 0 \) and for all \( \tilde{s} \in S \setminus \{s, s'\} \), \( \tilde{c}_s = 0 \), then
\[
\tilde{c} \notin P^\perp
\] (67)
since each \( p \in P \) is non-negative.

Now we can consider the matrix \( C \) formed by row vectors \( (c^i)_{i=1}^{n+} \) and each column in \( \tilde{s} \in S \). Denote \( s^{n^+} = s, s^{n^++1} = s' \). By (67), we can reduce \( C \) into Smith normal form \( \tilde{C} \) formed by row vectors \( (\tilde{c}^i)_{i=1}^{n+} \) such that for each \( i \in \{1, \ldots, n^+-1\} \) there exist \( s^i \in S \setminus \{s, s'\} \) such that for each \( \tilde{i} \in \{1, \ldots, n^+\} \) \( \tilde{c}^i_{\tilde{s}} = -1 \), for all \( j \in \{1, \ldots, n^+\} \setminus \{i\} \), \( \tilde{c}^i_{\tilde{s}} = 0 \), and for all \( k, l \in \{1, \ldots, n^+\} \), \( k \neq l \), \( s^k \neq s^l \). Now \( (\tilde{c}^i)_{i=1}^{n^+} \subseteq P^\perp \) are linearly independent and for all \( i \in \{1, \ldots, n^+-1\} \)
\[
\tilde{c}_s^i = \tilde{c}_{s'}^i = 0 \text{ and } \tilde{c}_s^{n+} = \tilde{c}_{s'}^{n+} = -1.
\]
Denote \( S^\perp = \{s^1, \ldots, s^{n^+}, s^{n^++1}\} \). Now we have for all \( i \in \{1, \ldots, n^+-1\} \) and \( p \in P \)
\[
p_{s^i} = \sum_{\tilde{s} \in S \setminus S^\perp} \tilde{c}_{s}^i p_{\tilde{s}} \text{ and } p_{s^{n^+}} + p_{s^{n^++1}} = \sum_{\tilde{s} \in S \setminus S^\perp} \tilde{c}_{s}^{n^+} p_{\tilde{s}}.
\] (68)

Next, we show that there exist \( s^i \in S \setminus S^\perp \) such that
\[
\sum_{i=1}^{n^+} \tilde{c}_{s^i}^i \neq -1.
\] (69)
Assume, per contra, for all \( \tilde{s} \in S \setminus S^\perp \)
\[
\sum_{i=1}^{n^+} \tilde{c}_{s^i}^i = -1.
\]
Then we have for \( p \in P \)
\[
1 = \sum_{\tilde{s} \in S} p_{\tilde{s}}^{(68)} = \sum_{\tilde{s} \in S \setminus (S^\perp \cup \{s^1\})} \left(1 + \sum_{i=1}^{n^+} \tilde{c}_{s^i}^i\right) p_{\tilde{s}} = 0
\]
that is a contradiction which shows (69).

Let \( \pi : \{1, \ldots, n-2\} \to S \setminus (S^\perp \cup \{s^1\}) \) be a one-to-one function. By an induction as in the previous case for each \( i \in \{0, \ldots, n-2\} \) there exist linearly independent collection of probabilities \( (\hat{p}_{\pi(i)}^{j,i})_{j=1}^{n-i} \) such that for all \( j, k \in \{1, \ldots, n-i\}, 1 \leq l \leq i \),
\[
\hat{p}_{\pi(i)}^{j,i} = p_{\pi(i)}^{k,i}.
\] (70)
Since \( n \geq 2 \), by the induction, there exist \( p^*, p^\dagger \in P \) that are linearly independent and for all \( \tilde{s} \in S \setminus (S^\perp \cup \{s^f\}) \), \( p^*_s = p^\dagger_s \).

Now we have for \( p \in \{p^*, p^\dagger\} \)

\[
1 = \sum_{\tilde{s} \in S} p^\dagger_{\tilde{s}} = \sum_{\tilde{s} \in S \setminus (S^\perp)} \left( 1 + \sum_{i=1}^{n^\perp} c^i_{\tilde{s}} \right) p^\dagger_{\tilde{s}} \quad \Rightarrow \quad p^\dagger_s = \frac{1 - \sum_{\tilde{s} \in S \setminus (S^\perp \cup \{s^f\}))} {1 + \sum_{i=1}^{n^\perp} c^i_{s^f}} p^\dagger_{\tilde{s}}.
\]

Thus \( p^*_s = p^\dagger_s \).

Now for all \( \tilde{s} \in S \setminus S^\perp \), \( p^*_s = p^\dagger_s \). By (68) for all \( \tilde{s} \in S \setminus \{s, s'\} \)

\[
p^*_s = p^\dagger_s \quad \text{and} \quad p^*_s + p^*_s' = p^\dagger_s + p^\dagger_{s'}.
\]

This shows the claim since \( p^\dagger_s \) and \( p^s \) are linearly independent.

\[\Box\]

**Proposition 93 (Relative Likelihood Identification)** Let \((u, \mathbb{P})\) be a state dependent tight dual-self representation for \( \succcurlyeq \) and \( s, s' \in S^P \). The following four conditions are equivalent:

1. If \( f, g \in H \) are such that \( f_s \succcurlyeq g_s \) and \( g_{s'} \succcurlyeq f_{s'} \), then there exist \( h, h' \in H \) and \( \alpha \in (0, 1) \) such that

\[
\alpha h + (1 - \alpha) f \succcurlyeq \alpha h + (1 - \alpha) g \quad \text{and} \quad \alpha h' + (1 - \alpha) f \prec \alpha h' + (1 - \alpha) g.
\]

2. There exists \( p, q \in \mathbb{R} \cup \bigcup_{P \in \mathbb{P}} P \) such that \( \frac{p_s}{p_{s'}} \neq \frac{q_s}{q_{s'}} \) and for all \( \tilde{s} \in S \setminus \{s, s'\} \), \( p_{\tilde{s}} = q_{\tilde{s}} \).

3. If \((\tilde{u}, \tilde{\mathbb{P}})\) is a state dependent tight dual-self representation for \( \succcurlyeq \), then there exist \( \alpha \in \mathbb{R}_+ \) and \( \beta \in \mathbb{R} \) such that

\[
\tilde{u}_s = \alpha u_s + \beta \quad \text{and} \quad \tilde{u}_{s'} = \alpha u_{s'} + \beta.
\]

4. If \((\tilde{u}, \tilde{\mathbb{P}})\) is a state dependent tight dual-self representation for \( \succcurlyeq \), then

\[
\left\{ \frac{\tilde{p}_s}{\tilde{p}_{s'}} \left| \tilde{p} \in \mathbb{R} \bigcup \bigcup_{P \in \tilde{\mathbb{P}}} \tilde{P} \right. \right\} = \left\{ \frac{p_s}{p_{s'}} \left| p \in \mathbb{R} \bigcup \bigcup_{P \in \mathbb{P}} P \right. \right\}.
\]

**Proof.** (1) \(\iff\) (2) follows from Lemmas 91 and 92. (2) \(\iff\) (3) and (2) \(\iff\) (4) follows from Corollary 79 and Lemma 92. \(\Box\)

### 8.6.3 Probability Identification

**Lemma 94** Let \((u, I)\) be state dependent dual-self variational representations for \( \succcurlyeq \) and \( s \in S^P \). The following two are equivalent:
(1) For all \( f, g \in H \) such that \( f \sim g \), \( f_s \succ_s g_s \), and for some \( s' \in S \), \( g_{s'} \succ_{s'} f_{s'} \), there exist \( h \in H \) and \( \alpha \in (0, 1) \) such that

\[
\alpha h + (1 - \alpha) \sim f \not\sim \alpha h + (1 - \alpha) g.
\]

(2) For all \( x \in \left( \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) \right)^\perp \), \( x_s = 0 \).

**Proof.** We will first show that (1) \( \Rightarrow \) (2). Assume, per contra, there exist \( \tilde{x} \in \left( \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) \right)^\perp \), \( \tilde{x}_s \neq 0 \). Assume w.l.o.g. \( \tilde{x}_s > 0 \). Since for all \( p \in \left( \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) \right), s \notin S^P \), we have \( p_s = 0 \) so \( x = (\tilde{x}_{S^p}, 0_{-S^p}) \in \left( \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) \right)^\perp \). We show that there exist \( s' \in S^P \) such that \( x_{s'} < 0 \). Assume, per contra, for all \( s' \in S^P x_{s'} \geq 0 \). Since \( s \in S^P \), by Lemma 86, there exist \( p \in \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) \) such that \( p_s > 0 \). So by the counterassumption since \( p \) is non-negative

\[
\sum_{s \in S} x_sp_s \geq x_sp_s > 0
\]

which is a contradiction since \( x \in \left( \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) \right)^\perp \).

By Lemma 87, there exist \( f, g \in H \), \( \alpha > 0 \) such that \( f_s \succ_s g_s \) and \( g_{s'} \succ_{s'} f_{s'} \), \( u(f), u(g) \in \text{int } u(H) \), and \( u(f) - u(g) = ax \). Let \( h \in H \) and \( \alpha^* \in [0, 1) \). Consider a mapping \( J : [0, 1] \to \mathbb{R} \), for \( \alpha \in [0, 1] \),

\[
J(\alpha) = I \circ u\left(\alpha^* h + (1 - \alpha^*) \left(\alpha f + (1 - \alpha) g\right)\right) = I\left(\alpha^* u(h) + (1 - \alpha^* u(g) + (1 - \alpha^*) ax\right).
\]

Since \( u(f), u(g) \in \text{int } u(H) \), we have by the definition of Clarke derivative and since \( x \in \left( \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi) \right)^\perp \) for all differentiability points \( \alpha \) of \( J \)

\[
J'(\alpha) = 0.
\]

Since \( J \) is differentiable almost everywhere, we alve by the fundamental theorem of calculus

\[
I \circ u\left(\alpha^* h + (1 - \alpha^*)g\right) = J(0) = J(1) = I \circ u\left(\alpha^* h + (1 - \alpha^*) f\right)
\]

and so \( \alpha^* h + (1 - \alpha^*) g \sim \alpha^* h + (1 - \alpha^*) f \). Especially \( f \sim g \) and this contradicts (1).

Second, we show that (2) \( \Rightarrow \) (1). Assume, per contra, that there exist \( f, g \in H, s' \in S \) such that \( f \sim g \), \( f_s \succ_s g_s \) and \( g_{s'} \succ_{s'} f_{s'} \) and for all \( h \alpha \in (0, 1) \)

\[
\alpha h + (1 - \alpha) f \not\sim \alpha h + (1 - \alpha) g.
\]

Let \( \varphi \in \text{int } u(H) \) be a differentiability point of \( I \). Now there exist \( h \in H \) with \( u(h) \in \text{int } u(H) \) and \( \alpha^* \in (0, 1) \) such that

\[
u(\alpha^* h + (1 - \alpha^*)g) = \varphi.
\]
Now by the counter assumption for all $\alpha \in [0, 1]$
\[\alpha^* h + (1 - \alpha^*) g \sim \alpha^* h + (1 - \alpha^*) (\alpha f + (1 - \alpha) g).\]

Thus
\[
(1 - \alpha^*) \Delta I(\varphi) \cdot (u(f) - u(g)) = \lim_{\alpha \to 0} \frac{I[u(\alpha^* h + (1 - \alpha^*) g) + \alpha(1 - \alpha^*)(u(f) - u(g))] - I[u(\alpha^* h + (1 - \alpha^*) g)]}{\alpha} = 0.
\]

By Lemma 73, $u(f) - u(g) \in \left( \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \right)^\perp$. But by Lemma 44 $u_s(f_s) - u_s(g_s) \neq 0$ which is a contradiction with (2).

**Lemma 95** Let $P \subseteq \Delta(S), P \neq \emptyset, s \in S$. The following two are equivalent:

1. For all $x \in P^\perp, x_s = 0$ and there exists $s' \in S, s \neq s', p \in P$ such that $p_{s'} > 0$.
2. There exist $p, q \in P$ such that $p_s \neq q_s$ and for all $\tilde{s} \in S \setminus \{s\}$,
\[
\frac{p_{\tilde{s}}}{1 - p_s} = \frac{q_{\tilde{s}}}{1 - q_s}.
\]

**Proof.** We show first that (2) $\Rightarrow$ (1). First, by (2) $|P| \geq 2$ and so there exists $s' \in S, s \neq s', p \in P$ such that $p_{s'} > 0$. Assume, per contra, that there exist $x \in P^\perp$ such that $x_s \neq 0$. Let $p, q$ be as in (2). Since $p_s \neq 1$ or $q_s \neq 1$, we have by (71) since $|S| \geq 2$,
\[
p_s \neq 1 \text{ and } q_s \neq 1.
\]

Since $p, q \in P^\perp$, we have
\[
\sum_{\tilde{s} \in S} x_{\tilde{s}} p_{\tilde{s}} = 0 \text{ and } \sum_{\tilde{s} \in S} x_{\tilde{s}} q_{\tilde{s}} = 0.
\]

By (72) and by multiplying the first equation by $(1 - p_s)^{-1}$ and the second equation by $(1 - q_s)^{-1}$, we have
\[
\sum_{\tilde{s} \in S \setminus \{s\}} x_{\tilde{s}} \frac{p_{\tilde{s}}}{1 - p_s} + x_s \frac{p_s}{1 - p_s} = 0 = \sum_{\tilde{s} \in S \setminus \{s\}} x_{\tilde{s}} \frac{q_{\tilde{s}}}{1 - q_s} + x_s \frac{q_s}{1 - q_s}.
\]

By (71), we have
\[
x_s \frac{p_s}{1 - p_s} = x_s \frac{q_s}{1 - q_s} \xrightarrow{x_s \neq 0} p_s \frac{p_s}{1 - p_s} = q_s \frac{q_s}{1 - q_s}.
\]

Now $x \mapsto \frac{x}{1-x}$ is strictly increasing function for $x \in [0, 1)$ and so by (72),
\[
\frac{p_s}{1 - p_s} = \frac{q_s}{1 - q_s} \Rightarrow p_s = q_s.
\]
This contradicts (2).

Next we show that (1)⇒(2). Denote \( n^+ = \dim P^+ \) and \( n = \dim P \). Now \( n^+ + n = |S| \). There exist \( (c_i)_{i=1}^n \subseteq P^+ \) linearly independent vectors and \( (p^j)_{i=1}^n \subseteq P \) linearly independent vectors.

First we show that \( n \geq 2 \). Since \( P \) is non-empty, \( n \geq 1 \). Assume, per contra, \( n = 1 \). Thus \( P \) is a singleton \( P = \{p\} \). If \( p_s = 0 \), then \((1_s, 0_{-s}) \in P^+\) which is a contradiction. By assumption there exist \( s' \in S, s' \neq s \) such that \( p_{s'} > 0 \). Now \( (\frac{1}{p_s}s, (-\frac{1}{p_s})s', 0_{-s, s'}) \in P^+ \) that is a contradiction. So \( n \geq 2 \).

Now we can consider the matrix \( C \) formed by row vectors \((c_i)_{i=1}^n \) and each column in \( \tilde{s} \in S \).
We can reduce \( C \) into Smith normal form \( \tilde{C} \) formed by row vectors \((\tilde{c}_i)_{i=1}^n \) such that for each \( i \in \{1, \ldots, n\} \) there exist \( s^i \in S \setminus \{s\} \), such that \( \tilde{c}_s = -1 \), for all \( j \in \{1, \ldots, n\} \) \( \setminus \{i\} \), \( \tilde{c}_s = 0 \), and for all \( k, l \in \{1, \ldots, n\} \), \( k \neq l \), \( s^k \neq s^l \). Now \( (\tilde{c}_i)_{i=1}^n \subseteq P^+ \) are linearly independent and for all \( i \in \{1, \ldots, n\} \), \( \tilde{c}_s = 0 \).

Denote \( S^+ = \{s^1, \ldots, s^n\} \). Now we have for all \( i \in \{1, \ldots, n\} \) and \( p \in P \),

\[
p_{s^i} = \sum_{\tilde{s} \in S \setminus (S^+ \cup \{s\})} \tilde{c}_{\tilde{s}}^{i} p^i_{\tilde{s}}.
\]

(73)

Let \( \pi : \{1, \ldots, n-1\} \to S \setminus (S^+ \setminus \{s\}) \) be a one-to-one function such that \( \pi(n) = s \) and \( \pi(n-1) = s^i \). We show by induction that for each \( i \in \{0, \ldots, n\} \) there exist linearly independent collection of probabilities \((\tilde{p}^{j,i})_{j=1}^n\) such that for all \( 1 \leq m \leq i \), and \( j, k \in \{1, \ldots, n\} \setminus \{m\} \),

\[
p^{j,i}_{\pi(m)} = \frac{p^{k,i}_{m,i}}{p^{m,i}_{\pi(m)}} > p^{m,i}_{\pi(m)} \cdot \]

(74)

This follows symmetrically to the proof in Lemma 92 since at each step for \( i < n - 1 \),

\[
\min \{p^{j,i}_{\pi(i+1)} \mid j \in \{i + 1, \ldots, n\}\} \neq \max \{p^{j,i}_{\pi(i+1)} \mid j \in \{i + 1, \ldots, n\}\}
\]

by linear independence and since for all \( k \leq i \) and \( j, l \in \{i + 1, \ldots, n\} \) \( p^{j,i}_{\pi(k)} = p^{j,i}_{\pi(l)} \).

Since \( n \geq 2 \), by the induction by considering a convex combination of \((\tilde{p}_{i,n-1})_{j=1}^{n-1}\), there exist \( a > 1 \) and \( p^*, p^t \in P \) that are linearly independent and for all \( \tilde{s} \in S \setminus (S^+ \cup \{s\}) \), \( p^*_{\tilde{s}} = ap_{\tilde{s}}^t > 0 \). Let \( s^i \in S \setminus (S^+ \cup \{s\}) \). Now we have for all \( \tilde{s} \in S \setminus (S^+ \cup \{s\}) \)

\[
\frac{p^*_{s^i}}{p^*_{\tilde{s}^i}} = \frac{ap_{s^i}^t}{ap_{\tilde{s}^i}^t} = \frac{p_{s^i}^t}{p_{\tilde{s}^i}^t}.
\]

By (73), we have for all \( i \in \{1, \ldots, n\} \) and \( p \in P \),

\[
\frac{p_{s^i}}{p_{\tilde{s}^i}} = \sum_{\tilde{s} \in S \setminus (S^+ \cup \{s\})} \tilde{c}_{\tilde{s}}^{i} p_{\tilde{s}} p_{s^i}.
\]

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Thus for all $\tilde{s} \in S \setminus \{s\}$

\[
\frac{p_{s}^{\ast}}{p_{s}^{\dagger}} = \frac{p_{\tilde{s}}^{\dagger}}{p_{\tilde{s}}^{\ast}}. \tag{75}
\]

By taking a sum over $\tilde{s}$, we have

\[
\frac{1 - p_{s}}{p_{s}^{\ast}} = \frac{1 - p_{\tilde{s}}}{p_{\tilde{s}}^{\ast}}.
\]

Thus by multiplying (75) by $\frac{p_{s}^{\dagger}}{1 - p_{s}}$, we have for all $\tilde{s} \in S \setminus \{s\}$

\[
\frac{p_{s}^{\ast}}{1 - p_{s}} = \frac{p_{\tilde{s}}^{\dagger}}{1 - p_{\tilde{s}}}. \tag{76}
\]

Finally, we show that $p_{s}^{\ast} \neq p_{s}^{\dagger}$. Assume, per contra, $p_{s}^{\ast} = p_{s}^{\dagger}$. Then by (76) for all $\tilde{s} \in S$, $p_{s}^{\ast} = p_{\tilde{s}}^{\dagger}$, which contradicts, that $p^{\ast}$ and $p^{\dagger}$ are linearly independent. Thus $p_{s}^{\ast} \neq p_{s}^{\dagger}$ which shows the claim.

\[\square\]

**Proposition 96 (Probability Identification)** Let $(\mathbf{u}, \mathbb{P})$ be a state dependent tight dual-self representation for $\succeq$ and $s \in S^{P}$. The following three conditions are equivalent:

1. If $f, g \in H$ are such that $f \sim g, f_{s} \succeq_{s} g_{s}$, and there exist $s' \in S$ such that $g_{s'} \succeq_{s'} f_{s'}$, then there exist $h \in H$ and $\alpha \in (0, 1)$ such that

   \[
   \alpha h + (1 - \alpha)f \nsim \alpha h + (1 - \alpha)g.
   \]

2. $S^{P} = \{s\}$ or there exists $p, q \in \varnothing \cup P \in \mathbb{P}$ such that $p_{s} \neq q_{s}$ and for all $\tilde{s} \in S \setminus \{s\}$,

   \[
   \frac{p_{s}}{1 - p_{s}} = \frac{q_{\tilde{s}}}{1 - q_{\tilde{s}}}.
   \]

3. If $(\mathbf{\tilde{u}}, \mathbb{\tilde{P}})$ is a state dependent tight dual-self representation for $\succeq$, then

   \[
   \left\{ \tilde{p}_{s} \mid \tilde{p} \in \varnothing \bigcup_{P \in \mathbb{P}} \tilde{P} \right\} = \left\{ p_{s} \mid p \in \varnothing \bigcup_{P \in \mathbb{P}} P \right\}.
   \]

**Proof.** (1) $\iff$ (2): We have two cases. If $S^{P} = \{s\}$, then $\bigcup_{P \in \mathbb{P}} P = \{\delta_{s}\}$ and so the claim follows Lemma 94 and since for all $x \in \mathbb{R}^{S} \delta_{s} x = 0$ iff $x_{s} = 0$. If $S^{P} \neq \{s\}$, then there exists $s' \in S, s \neq s'$ and $p \in \bigcup_{P \in \mathbb{P}} P$ such that $p_{s'} > 0$. Thus the equivalence follows from Lemmas 94 and 95. (2) $\iff$ (3) follows from Corollary 79 and Lemma 95.

\[\square\]
8.7 Social Choice Theory: Symmetric State Dependent MaxMin

Lemma 97 Let $P \subseteq \mathbb{R}^S_+$ be non-empty and $\bar{0} \notin P$ such that if $p, q \in P$ are such that $p \neq q$, then for all $a > 0$ $ap \neq q$ and for all permutations $\pi : S \to S$ and $p \in P$,

$$(p_{\pi(s)})_{s \in S} \in P.$$  \hspace{1cm} (77)

Then there exist $c \in R_{++}$ such that $P = \{(c)_{s \in S}\}$ or $P^\perp = \{\bar{0}\}$.

Proof. We will show that there does not exist $p \in P, q \in P^\perp, s_1, s_2, s_3, s_4 \in S$ such that $p_{s_1} \neq p_{s_2}$ and $q_{s_3} \neq q_{s_4}$. Let $\pi^1 : S \to S$ and $\pi^2 : S \to S$ be two permutations such that $\pi^1(s_1) = s_3$, $\pi^1(s_2) = s_4$, $\pi^2(s_1) = s_4$, $\pi^2(s_2) = s_3$, and for all $s \notin \{s_1, s_2\}$ $\pi^1(s) = \pi^2(s)$. Now by (77) and $q \in P^\perp$, we have

$$\sum_{s \in S} p_{(\pi^1)^{-1}(s)}q_s = 0 = \sum_{s \in S} p_{(\pi^2)^{-1}(s)}q_s.$$ 

Since for all $s \notin \{s_3, s_4\}$, $(\pi^1)^{-1}(s) = (\pi^2)^{-1}(s)$, we have

$$p_{s_1}q_{s_3} + p_{s_2}q_{s_4} = p_{s_2}q_{s_3} + p_{s_1}q_{s_4} \Rightarrow (p_{s_1} - p_{s_2})(q_{s_4} - q_{s_3}) = 0.$$ 

Thus $p_{s_1} = p_{s_2}$ or $q_{s_4} = q_{s_3}$.

First, assume that $P$ is a singleton with $p \in P$. Then by (77) and since $P$ is a singleton, for all $s, s' \in S, p_s = p_{s'}$. This shows the claim since $\bar{0} \notin P$.

Assume next that $P$ is not a singleton. Thus there exist $p, q \in P$ such that $p \neq q$. Now for all $a \in \mathbb{R}_{++}$ $ap \neq q$. Especially, there exist $p \in P, s_1, s_2 \in S$ such that $p_{s_1} \neq p_{s_2}$. Thus by above for all $q \in P^\perp, s_3, s_4 \in S, q_{s_3} = q_{s_4}$. Since for all $s \in S p_s \geq 0$ and for some $s' \in S, p_{s'} > 0$, this is possible only if $P^\perp = \{\bar{0}\}$. This shows the claim. \hfill \Box

Proposition 98 (Symmetric State Dependent MaxMin) $\succeq$ satisfies Axioms 1-6, 8, 9 if and only if (1) or (2) of the following conditions holds:

(1) For all $p \in \text{int} \Delta(S)$, there exists $u = (u_s)_{s \in S}$ such that for all $s \in S, u_s : \Delta(X_s) \to \mathbb{R}$ is affine and $\text{int} u_s(\Delta(X_s)) \neq \emptyset$ and for all $f, g \in H$

$$f \succeq g \iff \sum_{s \in S} p_su_s(f_s) \geq \sum_{s \in S} p_su_s(g_s).$$

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(2) There exists \((u, P)\) that is a state dependent maxmin representation for \(\succsim\) such that \(\int P \neq \emptyset\), for all \(p \in P\), permutations \(\pi: S \to S\), \((p_{\pi(s)})_{s \in S} \in P\), and
\[
\int \bigcap_{s \in S} u_s\left(\Delta(X_s)\right) \neq \emptyset.
\]
Additionally, if (2) holds and \((\bar{u}, \bar{P})\) is another state dependent maxmin representation for \(\succsim\), then there exists \(\alpha > 0, \beta \in \mathbb{R}\) such that
\[
\bar{P} = P \text{ and for all } s \in S, \bar{u}_s = \alpha u_s + \beta.
\]

Proof. We show first the if direction.

By Axioms \(\succsim\) satisfies Axioms 1-6,8, there exists a concave state dependent Boolean representation for \(\succsim\). This corresponds to a state dependent maxmin representation \((P, u)\) with probabilities \(P \subseteq \Delta(S)\) and affine utility \(u = (u_s)_{s \in S}\).

If \(|S| = 1\), then by non-triviality, (1) holds. So assume that \(|S| > 1\).

Since \(f^* \succ g^*\), there exist \(s^* \in S^P\) such that \(u_{s^*}(f^*_{s^*}) > u_{s^*}(g^*_{s^*})\). By the maxmin representation, \(s^* \in S^P\), and Lemma 85, since the set of crisp acts is convex, we have
\[
\frac{1}{|S| - 1} f^* + \frac{|S| - 2}{|S| - 1} g^* > \frac{1}{|S| - 1} (g^*_{s^*}, f^*_{s^*}) + \frac{|S| - 2}{|S| - 1} g^* = \left(g^*_{s^*}, \frac{1}{|S| - 1} f^*_{s^*}, \frac{|S| - 2}{|S| - 1} g^*_{s^*}\right).
\]

By Axiom 9, for all \(s \in S\)
\[
\frac{1}{|S| - 1} f^* + \frac{|S| - 2}{|S| - 1} g^* > \left(g^*_{s^*}, \frac{1}{|S| - 1} f^*_{s^*}, \frac{|S| - 2}{|S| - 1} g^*_{s^*}\right).
\]
Thus \(S = S^P\) and for all \(s \in S\) \(u_s(f^*_{s^*}) > u_s(g^*_{s^*})\). Denote for each \(s \in S\),
\[
\tau_s = u_s(f^*_{s^*}) - u_s(g^*_{s^*}) \in \mathbb{R}_{++}.
\] (78)

Next assume that \(h \in H, u(h) \in \text{int } u(H)\) is such that \(I\) is differentiable at \(u(h)\). Let \(\pi: S \to S\) be a permutation. We will show that \(\left(\nabla I(u(h))_{\pi(s)}\right)_{s \in S} \in P\).

First since \(|S| > 1\), there exist \(\alpha^1 \in (0, 1)\) such that for all \(s \in S\),
\[
u_s(f^*_{s^*}) > \alpha^1 u_s\left(\frac{3}{4}f^*_{s^*} + \frac{1}{4}g^*_{s^*}\right) + (1 - \alpha^1) u_s(h_s) > \frac{1}{|S|} u_s(f^*_{s^*}) + \frac{|S| - 1}{|S|} u_s(g^*_{s^*}).
\]

Thus there exist \(\theta \in (0, 1)^S\) such that for each \(s \in S\),
\[
u_s(\theta_s f^*_{s^*} + (1 - \theta_s) g^*_{s^*}) = \alpha^1 u_s\left(\frac{3}{4}f^*_{s^*} + \frac{1}{4}g^*_{s^*}\right) + (1 - \alpha^1) u_s(h_s).
\]

Now especially \(\sum_{s \in S} \theta_s > 1\) and so \(\alpha^2 = \left(\sum_{s \in S} \theta_s\right)^{-1} \in (0, 1)\). Denote
\[
\hat{h} = \alpha^2 \alpha^1 \left(\frac{3}{4}f^* + \frac{1}{4}g^*\right) + (1 - \alpha^1) u_s(h_s) + (1 - \alpha^2) g^*.
\]
By Lemma 84 $I$ is differentiable at $u(\hat{h})$ and $\nabla I(\hat{h}) = \nabla I(h)$. Denote for all $s \in S$, $\hat{\gamma}_s = \theta_s \alpha^2$. Now $\sum_{s \in S} \gamma_s = 1$. Especially, $I$ is differentiable at $(\gamma_u u_s(f_s^*) + (1 - \gamma_u u_s(g_s^*))_{s \in S}$ with derivative $\nabla I(h)$.

We will show that $I$ is differentiable at $(\gamma_u u_s(f_s^*) + (1 - \gamma_u u_s(g_s^*))_{s \in S}$ with a derivative $(\nabla I(u(h))\pi(s))_{s \in S}$.

Let $v \in \mathbb{R}^S$ and $a > 0$ be such that for all $s \in S$,

$$u_s(f_s^*) > u_s(\hat{h}_s) + av_s > u_s(g_s^*).$$

Thus there exist $\hat{\theta} \in (0, 1)^S$ such that for each $s \in S$,

$$u_s(\hat{\theta}_s f_s^* + (1 - \hat{\theta}_s)g_s^*) = u_s(\hat{h}_s) + av_s.$$

Now for each $s \in S$

$$\hat{\theta}_s - \gamma_s = av_s(\tau_s^{-1}).$$

Assume w.l.o.g. $\sum_{s \in S} v_s \geq 0$ since the other case follows similarly. Now $\sum_{s \in S} \hat{\theta}_s \geq 1$. Denote $\alpha = (\sum_{s \in S} \hat{\theta}_s)^{-1}$. Now $\alpha \hat{\theta} \in \Delta(S)$. Thus by Axiom 9

$$\alpha(\hat{\theta}_s f_s^* + (1 - \hat{\theta}_s)g_s^*)_{s \in S} + (1 - \alpha)g^* = (\alpha \hat{\theta}_s f_s^* + (1 - \alpha \hat{\theta}_s)g_s^*)_{s \in S}$$

$$\sim (\alpha \hat{\theta}_s f_s^* + (1 - \alpha \hat{\theta}_s)g_s^*)_{s \in S} = \alpha(\hat{\theta}_s f_s^* + (1 - \hat{\theta}_s)g_s^*)_{s \in S} + (1 - \alpha)g^*.$$

Since $g^*$ is crisp, we have

$$(\hat{\theta}_s f_s^* + (1 - \hat{\theta}_s)g_s^*)_{s \in S} \sim (\hat{\theta}_s f_s^* + (1 - \hat{\theta}_s)g_s^*)_{s \in S}.$$

Now

$$\left(\hat{\theta}_s f_s^* + (1 - \hat{\theta}_s)g_s^*\right)_{s \in S} = \left(\gamma_u u_s(f_s^*) + (1 - \gamma_u u_s(g_s^*))_{s \in S} + (av_\pi(\tau_\pi^{-1})_{s \in S}. $$

By the representation,

$$I(\hat{u}(h) + av) = I\left(\left(\gamma_u u_s(f_s^*) + (1 - \gamma_u u_s(g_s^*))_{s \in S} + (av_\pi(\tau_\pi^{-1})_{s \in S}\right)$$

Thus especially, by subtracting $I(\hat{u}(h)) = I\left[\left(\gamma_u u_s(f_s^*) + (1 - \gamma_u u_s(g_s^*))_{s \in S}\right]$ from both hand sides, dividing by $a$ and taking $a \to 0$, we have

$$\nabla I(u(h)) \cdot v$$

$$= \lim_{a \to 0} \frac{I\left(\gamma_u u_s(f_s^*) + (1 - \gamma_u u_s(g_s^*) + av_\pi(\tau_\pi^{-1})_{s \in S}\right) - I\left(\gamma_u u_s(f_s^*) + (1 - \gamma_u u_s(g_s^*))_{s \in S}\right)}{a}. $$

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Since \( v \) was arbitrary, \( I \) is differentiable at \( (\gamma_{\pi(s)}u_s(f_s^*) + (1 - \gamma_{\pi(s)})u_s(g_s^*))_{s \in S} \) and
\[
\nabla I \left( (\gamma_{\pi(s)}u_s(f_s^*) + (1 - \gamma_{\pi(s)})u_s(g_s^*))_{s \in S} \right) = \left( \frac{\tau_{\pi(s)}^{-1}}{\tau_s} \nabla I(u(h))_{\pi^{-1}(s)} \right)_{s \in S}.
\]
That is for all \( p \in P \)
\[
p = \left( \frac{\tau_{\pi(s)}}{\tau_s} p_{\pi(s)} \right)_{s \in S} \iff (\tau_s p_s)_{s \in S} = (\tau_{\pi(s)} p_{\pi(s)})_{s \in S}. \tag{79}
\]
Define the set
\[
\tau P = \left\{ (\tau_s p_s)_{s \in S} \mid p \in P \right\}.
\]
By (78,79), \( \tau P \subseteq \mathbb{R}_{++} \) and for all permutations \( \pi : S \to S \) and \( p \in P \), \( (\tau_{\pi(s)} p_{\pi(s)})_{s \in S} \in \tau P \).
Additionally, let \( p, q \in P, p \neq q \) and assume, per contra, that there exist \( a > 0 \) such that \( (\tau_s p_s)_{s \in S} = (a \tau_s q_s)_{s \in S} \). Then \( p = aq \), which is a contradiction since \( p \) and \( q \) are different probabilities that sum to 1.
By Lemma 97, there exist \( c \in \mathbb{R}_{++} \) such that \( \tau P = \{(c)_{s \in S}\} \) or \( (\tau P)\perp = \{0\} \).
We first consider the case that there exist \( c \in \mathbb{R}_{++} \) such that \( \tau P = \{(c)_{s \in S}\} \) and show that in this case \( (1) \) holds. This follows from the observation that since \( \tau P \) is a singleton, \( P \) is a singleton. Thus the claim follows from \( \tau \in \mathbb{R}_{++}^S \) and Theorem 5.
Second, we consider the case that \( (\tau P)\perp = \{0\} \) and show that \( (2) \) holds. First \(|S| > 1 \) and so the first case does not hold. First, we show that \( P\perp = \{0\} \). Assume, per contra there exist \( x \in P\perp, x \neq 0 \). Now for all \( p \in P \) since \( \tau \in \mathbb{R}_{++}^S \),
\[
0 = x \cdot p = \sum_{s \in S} x_s \tau_s p_s.
\]
So \( (\frac{x_s}{\tau_s})_{s \in S} \in (\tau P)\perp \) and by (78) \( (\frac{x_s}{\tau_s})_{s \in S} \neq 0 \) which is a contradiction. Thus \( P\perp = \{0\} \). By Lemma 85, for all \( s \in S \),
\[
u_s(f_s^*) = I\left(u(f^*)\right) \text{ and } u_s(g_s^*) = I\left(u(g^*)\right).
\]
Thus especially for all \( s \in S \), \( \tau_s = I\left(u(f^*)\right) - I\left(u(g^*)\right) \). By (79), for all \( p \in P \) and permutations \( \pi : S \to S \), \( (p_{\pi(s)})_{s \in S} \in P \). By the proof of Lemma 83, \( P \neq \emptyset \). This shows \( (2) \) and the additional claim follows from Theorem 2 since if \( \text{int} \bigcap_{s \in S} u_s(\Delta(X_s)) \neq \emptyset \), then every state dependent maxmin representation is tight.
Next, we show the only if direction. By Theorem 1, we only need to show Axiom 9. First, assume that \( (1) \) holds. Then \( \succsim \) satisfies independence axiom and every act is crisp. Let \( p \in \text{int} \Delta(S) \) and let \( (u_s)_{s \in S} \) be the corresponding utility functions. Since for all \( s \in S \), \( u_s(\Delta(X_s)) \neq \emptyset \) there
exist $g^*, f^* \in H, a > 0$ such that for all $s \in S$, $u_s(f^*_s) = u_s(g^*_s) + \frac{a}{p_s}$. Let $\gamma \in \Delta(S)$ and $\pi : S \to S$ be a permutation. Now we have

$$\sum_{s \in S} p_s u_s(\gamma_s f^*_s + (1 - \gamma_s) g^*_s) = \sum_{s \in S} p_s \left( u_s(g^*_s) + \frac{\gamma_s a}{p_s} \right) = \sum_{s \in S} p_s u_s(g^*_s) + a \sum_{s \in S} \gamma_{s'}$$

$$= \sum_{s \in S} p_s u_s(g^*_s) + a \sum_{s' \in S} \gamma_{\pi(s')} = \sum_{s \in S} p_s \left( u_s(g^*_s) + \frac{a \gamma_{\pi(s)}}{p_s} \right) = \sum_{s \in S} p_s u_s(\gamma_{\pi(s)} f^*_s + (1 - \gamma_{\pi(s)}) g^*_s).$$

This shows Axiom 9.

Second, assume that (2) holds. Since $\text{int} \cap_{s \in S} u_s\left(\Delta(X_s)\right) \neq \emptyset$, there exist $f^*, g^* \in H, c^f, c^g \in \mathbb{R}, c^f > c^g$ such that for all $s \in S$, $u_s(f^*_s) = c^f$ and $u_s(g^*_s) = c^g$. As constant utility acts, $f^*, g^*$ are crisp acts. First for all permutations $\pi : S \to S$ and $\gamma \in \Delta(S)$, we have

$$p \cdot u\left(\left(\gamma_s f^*_s + (1 - \gamma_s) g^*_s\right)_{s \in S}\right) = \sum_{s \in S} p_s \cdot c^g + \sum_{s \in S} p_s \gamma_s (c^f - c^g)$$

$$= \sum_{s \in S} p_{\pi(s)} \cdot c^g + \sum_{s \in S} p_{\pi(s)} \gamma_{\pi(s)} (c^f - c^g) = (p_{\pi(s)})_{s \in S} \cdot u\left(\left(\gamma_{\pi(s)} f^*_s + (1 - \gamma_{\pi(s)}) g^*_s\right)_{s \in S}\right).$$

Let $\gamma \in \Delta(S)$ and permutation $\pi : S \to S$. First we show that

$$p \in \arg \min_{q \in P} q \cdot u\left(\left(\gamma_s f^*_s + (1 - \gamma_s) g^*_s\right)_{s \in S}\right) \Rightarrow (p_{\pi(s)})_{s \in S} \in \arg \min_{q \in P} q \cdot u\left(\left(\gamma_{\pi(s)} f^*_s + (1 - \gamma_{\pi(s)}) g^*_s\right)_{s \in S}\right).$$

(81)

Assume, per contra, that there exist $q \in P$ such that

$$q \cdot u\left(\left(\gamma_{\pi(s)} f^*_s + (1 - \gamma_{\pi(s)}) g^*_s\right)_{s \in S}\right) < (p_{\pi(s)})_{s \in S} \cdot u\left(\left(\gamma_{\pi(s)} f^*_s + (1 - \gamma_{\pi(s)}) g^*_s\right)_{s \in S}\right).$$

Then by Equation (80) using the permutation $\pi^{-1}$

$$(q_{\pi^{-1}(s)})_{s \in S} \cdot u\left(\left(\gamma_s f^*_s + (1 - \gamma_s) g^*_s\right)_{s \in S}\right) < p \cdot u\left(\left(\gamma_s f^*_s + (1 - \gamma_s) g^*_s\right)_{s \in S}\right)$$

which is a contradiction by the symmetry of $P$.

Now we have by (80,81)

$$\min_{p \in P} p \cdot u\left(\left(\gamma_s f^*_s + (1 - \gamma_s) g^*_s\right)_{s \in S}\right) = \min_{p \in P} p \cdot u\left(\left(\gamma_{\pi(s)} f^*_s + (1 - \gamma_{\pi(s)}) g^*_s\right)_{s \in S}\right)$$

which shows Axiom 9.

□
8.8 Recursive Dual-Self

Assume that $S = T = \{0, \ldots, T\}, T \in \mathbb{N}$. For all $t \in T$ define for all $f, g \in \mathcal{X}_{t+1}^T \Delta(X_t)$

$$f \succeq_t g \iff \forall h \in H, (h_{[0,\ldots,t-1]}, f) \succ (h_{[0,\ldots,t-1]}, g).$$

**Lemma 99** Let $\succ$ satisfies Axioms 1-4,5',6, and 10, $1 \leq t \leq T - 1$, $(u, I)$ be a state dependent C-additive, positive homogeneous, and monotonic representation for $u$ such that

$$\text{int} \bigcap_{t \in T} u_t \left( \Delta(X_t) \right) \neq \emptyset,$$

and $V_{t+1} \circ u$ be a weak repesentation for $\succeq_{t+1}$ such that $V_{t+1}$ is C-additive, monotonic, and positively homogeneous. Then exists $V_t : u(H) \to \mathbb{R}$ that is C-additive, monotonic, and positively homogeneous and $V_t \circ u$ is a weak representation for $\succ$ and exists $\delta_1, \delta_2 \in [0,1]$ such that for all $h \in H$

$$V_t(u(h)) = \begin{cases} \min_{\delta \in [\delta_1,\delta_2]} (1 - \delta)u_t(h) + \delta V_{t+1}(u(h)), & \text{if } \delta_1 \leq \delta_2 \\ \max_{\delta \in [\delta_1,\delta_2]} (1 - \delta)u_t(h) + \delta V_{t+1}(u(h)), & \text{if } \delta_1 > \delta_2. \end{cases}$$

**Proof.** Now

$$\text{int} \bigcap_{t \in T} u_t \left( \Delta(X_t) \right) \neq \emptyset.$$

Thus let $c^*, c_\star \in H$ be such that for all $t', t'' \in T$ $u_{t'}(c^*) = u_{t''}(c^*)$, $u_{t'}(c_\star) = u_{t''}(c_\star)$ and $u_{t'}(c^*) > u_{t'}(c_\star)$. Now especially $c^*$ and $c_\star$ are crisp acts and $V_{t+1}(u(c^*)) = V_{t+1}(u(c_\star)) = u_t(h)$. Let $f, g \in H$ be such that $u(f), u(g) \in \text{int} u(H)$ and $I$ is differentiable at $u(f), u(g)$ with derivative $p^f, p^g$ such that $\sum_{t'=1}^T p^f_{t'}, \sum_{t'=1}^T p^g_{t'} > 0$ and $u_t(f) > V_{t+1}(u(f))$ and $u_t(g) > V_{t+1}(u(g))$. Let us show that

$$\frac{\sum_{t'=1}^T p^f_{t'}}{\sum_{t'=1}^T p^g_{t'}} = \frac{\sum_{t'=1}^T p^g_{t'}}{\sum_{t'=1}^T p^f_{t'}}. \quad (82)$$

Now since $V_{t+1}$ is C-additive and positive homogeneous and hence normalized exists $\alpha_1^f, \alpha_2^f, \alpha_3^f, \alpha_1^g, \alpha_2^g, \alpha_3^g \in (0,1)$ such that $\alpha_1^f + \alpha_2^f + \alpha_3^f = \alpha_1^g + \alpha_2^g + \alpha_3^g = 1$ and

$$u_t(\alpha_1^f c_\star + \alpha_2^f c_\star + \alpha_3^f c^*) = u_t(\alpha_1^g c_\star + \alpha_2^g c_\star + \alpha_3^g c^*) \text{ and } V_{t+1}(u(\alpha_1^f c_\star + \alpha_2^f c_\star + \alpha_3^f c^*)) = V_{t+1}(u(\alpha_1^g c_\star + \alpha_2^g c_\star + \alpha_3^g c^*)).$$

Let us denote $\tilde{f} := \alpha_1^f f + \alpha_2^f c_\star + \alpha_3^f c^*$ and $\tilde{g} := \alpha_1^g f + \alpha_2^g c_\star + \alpha_3^g c^*$. By the linearity of $I$ $I$ is differentiable at $u(\tilde{f})$ with derivative $p^f$ and at $u(\tilde{g})$ with derivative $p^g$ and $u(\tilde{f}), u(\tilde{g}) \in \text{int} u(H)$. Thus let $\varepsilon > 0$ be such that for all $B_\infty(u(\tilde{f}), \varepsilon), B_\infty(u(\tilde{g}), \varepsilon) \subseteq \text{int} u(H)$ and for $\alpha, \alpha' \in (-\varepsilon, \varepsilon)$

$$I(u(\tilde{f}) + \text{pr}_{[\varepsilon]} \theta(\alpha)) \geq I(u(\tilde{f}) + \text{pr}_{[\varepsilon]} \theta(\alpha')) \iff \alpha \geq \alpha'. \quad (83)$$
which exists since $\sum_{t=1}^{T} p^f_t > 0$.

By C-additivity and monotonicity of $I$ for all $\alpha \in (0, \varepsilon)$ exists $\theta(\alpha) \in (0, \varepsilon)$ such that

$$I\left(u(\tilde{f}) + pr_t \alpha\right) = I\left(u(\tilde{f}) + pr_{[t]} \theta(\alpha)\right).$$

(84)

Let us show that

$$I\left(u(\tilde{g}) + pr_t \alpha\right) = I\left(u(\tilde{g}) + pr_{[t]} \theta(\alpha)\right).$$

Assume, per contra,

$$I\left(u(\tilde{g}) + pr_t \alpha\right) > I\left(u(\tilde{g}) + pr_{[t]} \theta(\alpha)\right).$$

Then exists $\varepsilon' > 0$ such that

$$I\left(u(\tilde{g}) + pr_t \alpha\right) > I\left(u(\tilde{g}) + pr_{[t]} \theta(\alpha) + pr_{[t]} \varepsilon'\right).$$

Thus by Axiom 10

$$I\left(u(\tilde{f}_{1,...,t-1}, \tilde{g}_{[t]} + pr_t \alpha\right) \geq I\left(u(\tilde{f}_{1,...,t-1}, \tilde{g}_{[t]}\right) + pr_{[t]} \theta(\alpha) + pr_{[t]} \varepsilon').$$

Since $u_t(\tilde{f}) = u_t(\tilde{g})$ we have

$$I\left(u(\tilde{f}_{1,...,t}, \tilde{g}_{[t+1]} + pr_t \alpha\right) \geq I\left(u(\tilde{f}_{1,...,t}, \tilde{g}_{[t+1]}\right) + pr_{[t]} \theta(\alpha) + pr_{[t]} \varepsilon').$$

Finally let $h^{f,1}, h^{f,2}, h^{g,1}, h^{g,2} \in H$ be such that

$$u(h^{f,1}) = u(\tilde{f}) + pr_t \alpha \text{ and } u(h^{f,2}) = u(\tilde{f}) + pr_{[t]} \theta(\alpha) + pr_{[t]} \varepsilon'$$

and

$$u(h^{g,1}) = u(\tilde{f}_{1,...,t}, \tilde{g}_{[t+1]} + pr_t \alpha \text{ and } u(h^{g,2}) = u(\tilde{f}_{1,...,t}, \tilde{g}_{[t+1]} + pr_{[t]} \theta(\alpha) + pr_{[t]} \varepsilon').$$

Then by the C-additivity of $V_{t+1}$ we have $V_{t+1}(u(h^{f,1})) = V_{t+1}(u(h^{g,1}))$ and $V_{t+1}(u(h^{f,2})) = V_{t+1}(u(h^{g,2}))$. Thus since $V_{t+1}$ weakly represents $\succeq_t$ we have

$$h^{f,1} \sim_{t+1} h^{g,1} \text{ and } h^{f,1} \sim_{t+1} h^{g,1}$$

and since for all $t' \leq t$ $u_{t'}(h^{f,1}) = u_{t'}(h^{g,1})$ and $u_{t'}(h^{f,2}) = u_{t'}(h^{g,2})$ and thus $h^{f,1} \sim h^{g,1}$ and $h^{f,2} \sim h^{g,2}$. Thus

$$I\left(u(\tilde{f}) + pr_t \alpha\right) \geq I\left(u(\tilde{f}) + pr_{[t]} \theta(\alpha) + pr_{[t]} \varepsilon'\right).$$

However, this contradicts (83,84). The other case

$$I\left(u(\tilde{g}) + pr_t \alpha\right) < I\left(u(\tilde{g}) + pr_{[t]} \theta(\alpha)\right).$$
follows symmetrically by subtracting \(0 < \varepsilon'\). Thus for all \(\alpha \in (0, \varepsilon)\).

\[
I\left(u(\tilde{g}) + pr_1 \alpha\right) < I\left(u(\tilde{g}) + pr_{[t]} \theta(\alpha)\right). \tag{85}
\]

First, in (84) by subtracting \(I\left(u(\bar{f})\right)\) from both sides, dividing by \(\alpha > 0\) and taking \(\alpha \to 0\) we have by the differentiability of \(I\) with derivative \(p^f\)

\[
p^f_t = \lim_{\alpha \to 0} \theta(\alpha) \sum_{t' = t}^T p^f_{t'}. \tag{86}
\]

Symmetrically, in (85) by subtracting \(I\left(u(\tilde{g})\right)\) from both sides, dividing by \(\alpha > 0\) and taking \(\alpha \to 0\) we have by the differentiability of \(I\) with derivative \(p^g\)

\[
p^g_t = \lim_{\alpha \to 0} \theta(\alpha) \sum_{t' = t}^T p^g_{t'}. \tag{87}
\]

Thus

\[
\frac{p^f_t}{\sum_{t' = t}^T p^f_{t'}} = \frac{p^g_t}{\sum_{t' = t}^T p^g_{t'}}.
\]

Now let us define \(V_t\) by considering cases:

(1) Exists \(f^0 \in H\) such that \(I\) is differentiable at \(u(f^0) \in \text{int} u(H)\) with derivative \(p^1\) and \(u_t(f^0) > V_{t+1}\left(u(f^0)\right)\) and \(\sum_{t' = t}^T p^1_{t'} > 0\).

(2) Exists \(g^0 \in H\) such that \(I\) is differentiable at \(u(f^0) \in \text{int} u(H)\) with derivative \(p^2\) and \(u_t(f^0) < V_{t+1}\left(u(f^0)\right)\) and \(\sum_{t' = t}^T p^2_{t'} > 0\).

Then for all \(h \in H\) define

\[
V_t(u(h)) = \begin{cases} 
\frac{p^1_t}{\sum_{t' = t}^T p^1_{t'}} u_t(h) + \left(1 - \frac{p^1_t}{\sum_{t' = t}^T p^1_{t'}}\right)V_{t+1}\left(u(h)\right), & \text{if } (1), u_t(h) \geq V_{t+1}\left(u(h)\right), \\
\frac{p^2_t}{\sum_{t' = t}^T p^2_{t'}} u_t(h) + \left(1 - \frac{p^2_t}{\sum_{t' = t}^T p^2_{t'}}\right)V_{t+1}\left(u(h)\right), & \text{if } (2), u_t(h) < V_{t+1}\left(u(h)\right), \\
V_{t+1}\left(u(h)\right), & \text{if } \neg(1), u_t(h) \geq V_{t+1}\left(u(h)\right), \\
V_{t+1}\left(u(h)\right), & \text{if } \neg(2), u_t(h) < V_{t+1}\left(u(h)\right).
\end{cases}
\]

First, since \(V_{t+1}\) is \(C\)-additive, positive homogeneous, and monotonic, \(V_t\) will inherit these properties by its definition. Second, let \(f \in H_t\) and \(a \in H\) be such that for all \(t' \geq t\), \(u_{t'}(c_a) \leq u_{t'}(f) \leq u_{t'}(c^*)\) and \(u_t(f) = V_{t+1}\left(u(f)\right)\). Let us show that exists \(\alpha \in [0, 1]\) such that

\[
(a_{\{0, \ldots, t-1\}}, f) = (a_{\{0, \ldots, t-1\}}, c^* \alpha c_a) \text{ and } V_t\left(u(f)\right) = V_t\left(u(c^* \alpha c_a)\right). \tag{86}
\]
Then by the monotonicity, C-additivity and positive homogeneity of $V_{t+1}$ exists $\alpha \in [0,1]$ such that
\[
u_t(f) = V_{t+1}(u(f)) = V_{t+1}(u(c^*\alpha c_*)) = u_t(c^*\alpha c_*).
\]
Thus especially by the definition of $V_t$, $V_t(u(f)) = V_t(u(c^*\alpha c_*)$. Additionally, since $V_{t+1}$ weakly represents $\succeq_{t+1}$, $c^*\alpha c_* \sim_{t+1} f$. Hence especially
\[(a_{\{0,\ldots,t-1\}}, f_t, c^*\alpha c_{t+1},\ldots,T) \sim (a_{\{0,\ldots,t-1\}}, f_{t}, T).
\]
Finally by the monotonicity of $I$ and since $u_t(c^*\alpha c_*) = u_t(f)$ we have
\[(a_{\{0,\ldots,t-1\}}, c^*\alpha c_{t},\ldots,T) \sim (a_{\{0,\ldots,t-1\}}, f_{t}, T).
\]
Next, let $f \in H_t$ and $a \in H$ be such that $u_t(f) \neq V_{t+1}(u(f))$ and for all $t' \geq t$
\[
|u_{t'}(f) - 1/2(u_{t'}(c^*) + u_{t'}(c_*))| < 1/4(u_{t'}(c^*) - u_{t'}(c_*)).
\]
(87)
Let us show that exists $\alpha \in [0,1]$ such that $(a_{\{0,\ldots,t-1\}}, f_{t}, \alpha c_{t+1},\ldots,T) \sim (a_{\{0,\ldots,t-1\}}, c^*\alpha c_{t+1},\ldots,T)$ and $V_t(u(f)) = V_t(u(c^*\alpha c_*)$. Let first consider the case $u_t(f) > V_{t+1}(u(f))$ and (1) is true. Now $p_t^1 \neq 0$ or $\sum_{t'=1}^{T} p_t^1 \neq 0$. Assume that $p_t^1 \neq 0$ since the other case follows symmetrically.
\[
\gamma := \frac{\sum_{t'=1}^{T} p_t^1}{p_t^1}
\]
Now by (87) for all $\varepsilon \in [0, \frac{1}{1+\gamma} (u_t(f) - V_{t+1}(u(f)))]$
\[
u((a_{\{0,\ldots,t-1\}}, f)) - \gamma pr_t \varepsilon + pr_{\{t+1,\ldots,T\}} \varepsilon \in u(H)
\]
since $u_t(f) - V_{t+1}(u(f)) < 1/4(u_{t'}(c^*) - u_{t'}(c_*))$. Let us define the mapping
\[
\big[0, \frac{1}{1+\gamma} (u_t(f) - V_{t+1}(u(f))\big] \ni \varepsilon \mapsto J \bigg(u((a_{\{0,\ldots,t-1\}}, f)) - \gamma pr_t \varepsilon + pr_{\{t+1,\ldots,T\}} \varepsilon \bigg).
\]
Since $I$ is 1-lipschitz, $J$ is lipschitz and differentiable almost everywhere. Let us show that for all $\varepsilon \in \big(0, \frac{1}{1+\gamma} (u_t(f) - V_{t+1}(u(f))\big)$ if $J$ is differentiable at $\varepsilon$, then $\nabla J(\varepsilon) = 0$. Let $\varepsilon \in \big(0, \frac{1}{1+\gamma} (u_t(f) - V_{t+1}(u(f))\big)$ be such that $J$ is differentiable at $\varepsilon$. Let $\psi^\varepsilon \in u(H)$ be such that
\[
\psi^\varepsilon = u((a_{\{0,\ldots,t-1\}}, f)) - \gamma pr_t \varepsilon + pr_{\{t+1,\ldots,T\}} \varepsilon.
\]
Now especially by the C-additivity of $V_{t+1}$ $V_{t+1}(\psi^\varepsilon) = V_{t+1}(u(f)) + \varepsilon$. Thus
\[
\psi^\varepsilon - V_{t+1}(\psi^\varepsilon) = u_t(f) - V_{t+1}(u(f)) - (1 + \gamma)\varepsilon > 0.
\]
(88)
Let \( \varphi \in B_\infty \left( \psi^\varepsilon, \frac{1}{2}(u_t(f) - V_{t+1}(u(f))) - (1 + \gamma)\varepsilon \right) \subseteq u(H) \) be such that \( I \) is differentiable at \( \varphi \) with derivative \( p^\varphi \). By C-additivity of \( V_{t+1} \)

\[
V_{t+1}(\varphi) < V_{t+1}(\psi^\varepsilon) + \frac{1}{2} \left( u_t(f) - V_{t+1}(u(f)) - (1 + \gamma)\varepsilon \right) = \psi^\varepsilon - \frac{1}{2} \left( u_t(f) - V_{t+1}(u(f)) - (1 + \gamma)\varepsilon \right) < \varphi_t.
\]

Thus by (82),

\[
\sum_{t'=t}^T p_{t'}^\varphi = 0 \quad \text{or} \quad \frac{p_t^1}{\sum_{t'=t}^T p_{t'}^1} = \frac{p_t^\varphi}{\sum_{t'=t}^T p_{t'}^\varphi}.
\]

Thus by Lemma 73, for all \( p \in \partial I(\psi^\varepsilon) \),

\[
\sum_{t'=t}^T p_{t'} = 0 \quad \text{or} \quad \frac{p_t^1}{\sum_{t'=t}^T p_{t'}^1} = \frac{p_t}{\sum_{t'=t}^T p_{t'}}.
\]

Thus for all \( p \in \partial I(\psi^\varepsilon) \), such that \( \sum_{t'=t}^T p_{t'} = 0 \)

\[
\sum_{t'=t+1}^T p_{t'}^I = 1 - \frac{p_t^1}{\sum_{t'=t}^T p_{t'}^1} = 1 - \frac{p_t}{\sum_{t'=t}^T p_{t'}} = \sum_{t'=t+1}^T p_{t'}.
\]

Thus finally since \( p_t^1 \neq 0 \), we have, for all \( p \in \partial I(\psi^\varepsilon) \),

\[
\sum_{t'=t}^T p_{t'} = 0 \quad \text{or} \quad \sum_{t'=t+1}^T p_{t'}^I = \sum_{t'=t+1}^T p_{t'}.
\]

Now we have by the definitions of upper and lower directional clarke derivatives (Clarke, 1983, Proposition 2.1.2)

\[
\nabla J(\varepsilon) = \lim_{\alpha \to 0} \frac{I \left( \psi^\varepsilon + \alpha \left( -\gamma \text{pr}_t \varepsilon + \text{pr}_{(t+1, T]} \varepsilon \right) \right) - I(\psi^\varepsilon)}{\alpha} \leq I^o \left( \psi^\varepsilon; -\gamma \text{pr}_t \varepsilon + \text{pr}_{(t+1, T]} \varepsilon \right)
\]

\[
= \max \left\{ -\frac{\sum_{t'=t+1}^T p_{t'}^I}{p_t^1} \varepsilon p_t + \sum_{t'=t+1}^T p_{t'} \varepsilon \bigg| p \in \partial I(\psi^\varepsilon) \right\} \geq I_o \left( \psi^\varepsilon; -\gamma \text{pr}_t \varepsilon + \text{pr}_{(t+1, T]} \varepsilon \right)
\]

and (Bednařík and Pastor, 2013, Proposition 2)

\[
\nabla J(\varepsilon) = \lim_{\alpha \to 0} \frac{I \left( \psi^\varepsilon + \alpha \left( -\gamma \text{pr}_t \varepsilon + \text{pr}_{(t+1, T]} \varepsilon \right) \right) - I(\psi^\varepsilon)}{\alpha} \geq I_o \left( \psi^\varepsilon; -\gamma \text{pr}_t \varepsilon + \text{pr}_{(t+1, T]} \varepsilon \right)
\]

\[
= \min \left\{ -\frac{\sum_{t'=t+1}^T p_{t'}^I}{p_t^1} \varepsilon p_t + \sum_{t'=t+1}^T p_{t'} \varepsilon \bigg| p \in \partial I(\psi^\varepsilon) \right\} \geq 0,
\]

Thus \( \nabla J(\varepsilon) = 0 \). Since \( \varepsilon \in \left( 0, \frac{1}{1 + \gamma} \left( u_t(f) - V_{t+1}(u(f)) \right) \right) \) was arbitrary differentiability point of \( J \), we have by the Fundamental theorem of calculus since \( J \) is continuous at the end points.

\[
J \left( \frac{1}{1 + \gamma} \left( u_t(f) - V_{t+1}(u(f)) \right) \right) - J(0) = \int_0^{\frac{1}{1 + \gamma}} \left( u_t(f) - V_{t+1}(u(f)) \right) \nabla J(\varepsilon) d\varepsilon = 0.
\]
Thus
\[
I\left(u(\{a_{\{0,\ldots,t-1\}}, f\})\right) = I\left(u(\{a_{\{0,\ldots,t-1\}}, f\}) - \frac{1}{1+\gamma} \left(u_t(f) - V_{t+1}(u(f))\right) \left(\gamma \text{pr}_t 1 + \text{pr}_{t+1} \right)\right).
\] (90)

Let \(\bar{f} \in H_t\) be such that \(u_t(\bar{f}) = u_t(f) - \frac{\gamma}{1+\gamma} \left(u_t(f) - V_{t+1}(u(f))\right)\) and for \(t' \geq t + 1\) \(u_{t'}(\bar{f}) = u_{t'}(f) + \frac{1}{1+\gamma} \left(u_t(f) - V_{t+1}(u(f))\right)\). Then by C-additivity of \(V_{t+1}\) we have
\[
\begin{align*}
&u_t(\bar{f}) - V_{t+1}(u(\bar{f})) \\
&= u_t(f) - V_{t+1}(u(f)) - \frac{\gamma}{1+\gamma} \left(u_t(f) - V_{t+1}(u(f))\right) - \frac{1}{1+\gamma} \left(u_t(f) - V_{t+1}(u(f))\right) = 0.
\end{align*}
\]

By the definition of \(V_t\) and \(\gamma\)
\[
V_t(u(\bar{f})) = u_t(f) - \frac{\gamma}{1+\gamma} \left(u_t(f) - V_{t+1}(u(f))\right) = u_t(f) - \left(1 - \frac{p_t^1}{\sum_{t' = t} p_{t'}^1} \right) \left(u_t(f) - V_{t+1}(u(f))\right)
\]
\[
= \frac{p_t^1}{\sum_{t' = t} p_{t'}^1} u_t(h) + \left(1 - \frac{p_t^1}{\sum_{t' = t} p_{t'}^1} \right) V_{t+1}(u(f)) = V_t(u(f)).
\]

And by (90) \((a_{\{0,\ldots,t-1\}}, f) \sim (a_{\{0,\ldots,t-1\}}, \bar{f})\) and by (87) for all \(t' \geq t\) \(u_{t'}(c_*) \leq u_{t'}(\bar{f}) \leq u_{t'}(c^*)\). Thus by (86) exists \(\alpha \in (0,1)\) such that
\[
(a_{\{0,\ldots,t-1\}}, f) \sim (a_{\{0,\ldots,t-1\}}, \bar{f}) \sim (a_{\{0,\ldots,t-1\}}, c^* \alpha c_*)\text{ and } V_t(u(\bar{f})) = V_t(u(\bar{f})) = V_t(u(c^* \alpha c_*)).
\]

Second, let us consider the case \(u_t(f) > V_{t+1}(u(f))\) and (1) is false. Now by (87) for all \(\varepsilon \in \left[0, u_t(f) - V_{t+1}(u(f))\right]\), \(u((a_{\{0,\ldots,t-1\}}, f)) - \text{pr}_t \varepsilon \in u(H)\) since \(u_t(f) - V_{t+1}(u(f)) < 1/\varepsilon(u_{t'}(c^*) - u_{t'}(c_*)).\) Let us define the mapping
\[
\left[0, u_t(f) - V_{t+1}(u(f))\right] \ni \varepsilon \mapsto I\left(u((a_{\{0,\ldots,t-1\}}, f)) - \text{pr}_t \varepsilon\right).
\]

Since \(I\) is 1-lipschitz, \(J\) is lipschitz and hence differentiable almost everywhere. Let us show that if \(\varepsilon \in \left(0, u_t(f) - V_{t+1}(u(f))\right)\) \(J\) is differentiable at \(\varepsilon\), then \(\nabla J(\varepsilon) = 0\). Thus let \(\varepsilon \in \left(0, u_t(f) - V_{t+1}(u(f))\right)\) be such that \(J\) is differentiable at \(\varepsilon\). Let \(\psi^\varepsilon \in u(H)\) be such that
\[
\psi^\varepsilon = u((a_{\{0,\ldots,t-1\}}, f)) - \text{pr}_t \varepsilon.
\]

Now since for all \(t' \geq t + 1\) \(\psi^\varepsilon_{t'} = u_{t'}(f), V_{t+1}(\psi^\varepsilon) = V_{t+1}(u(f))\) and so
\[
\psi^\varepsilon_t - V_{t+1}(\psi^\varepsilon) = u_t(f) - V_{t+1}(u(f)) - \varepsilon > 0.
\] (91)
Let \( \varphi \in B_\infty \left( \psi^\varepsilon, \frac{1}{2}(u_t(f) - V_{t+1}(u(f)) - \varepsilon) \right) \subseteq u(H) \) be such that \( I \) is differentiable at \( \varphi \) with derivative \( p^\varphi \). By C-additivity of \( V_{t+1} \)
\[
V_{t+1}(\varphi) < V_{t+1}(\psi^\varepsilon) + \frac{1}{2}\left( u_t(f) - V_{t+1}(u(f)) - \varepsilon \right) \tag{91} = \frac{\psi^\varepsilon}{2} - \frac{1}{2}\left( u_t(f) - V_{t+1}(u(f)) - \varepsilon \right) < \varphi_t.
\]
Thus since (1) is false, \( \sum_{t'=t}^T p^\varphi_{t'} = 0 \). Thus by Lemma 73, for all \( p \in \partial I(\psi^\varepsilon) \),
\[
\sum_{t'=t}^T p_{t'} = 0. \tag{92}
\]
Now we have by the definitions of upper and lower directional clark derivatives (Clarke, 1983, Proposition 2.1.2)
\[
\nabla J(\varepsilon) \leq I^c(\psi^\varepsilon; -pr\varepsilon) = \max \left\{ -\varepsilon p \middle| p \in \partial I(\psi^\varepsilon) \right\} \tag{92} = 0
\]
and (Bednařík and Pastor, 2013, Proposition 2)
\[
\nabla J(\varepsilon) \geq I^c(\psi^\varepsilon; -pr\varepsilon) = \min \left\{ -\varepsilon p \middle| p \in \partial I(\psi^\varepsilon) \right\} \tag{92} = 0.
\]
Thus \( \nabla J(\varepsilon) = 0 \). Since \( \varepsilon \in \left( 0, u_t(f) - V_{t+1}(u(f)) \right) \) was arbitrary differentiability point of \( J \), we have by the Fundamental theorem of calculus since \( J \) is continuous at the end points.
\[
J\left( u_t(f) - V_{t+1}(u(f)) \right) - J(0) = \int_0^{u_t(f) - V_{t+1}(u(f))} \nabla J(\varepsilon)d\varepsilon = 0.
\]
Thus
\[
I\left( u((a_{0,\ldots,t-1}, f)) \right) = I\left( u((a_{0,\ldots,t-1}, f)) - (u_t(f) - V_{t+1}(u(f))) \right) + 1. \tag{93}
\]
Let \( \tilde{f} \in H_t \) be such that \( u_t(\tilde{f}) = u_t(f) - \left( u_t(f) - V_{t+1}(u(f)) \right) \) and for \( t' \geq t + 1 \ u_{t'}(\tilde{f}) = u_{t'}(f) \). Then, we have since \( V_{t+1}(u(f)) = V_{t+1}(u(\tilde{f})) \)
\[
u_{t}(\tilde{f}) - V_{t+1}(u(\tilde{f})) = u_t(f) - V_{t+1}(u(f)) - \left( u_t(f) - V_{t+1}(u(f)) \right) = 0.
\]
By the definition of \( V_t \) since (1) is false
\[
V_t(u(\tilde{f})) = V_{t+1}(u(\tilde{f})) = V_{t+1}(u(f)) = V_t(u(f)).
\]
And by (93) \( (a_{0,\ldots,t-1}, f) \sim (a_{0,\ldots,t-1}, \tilde{f}) \) and by (87) for all \( t' \geq t \ u_{t'}(c_\ast) \leq u_{t'}(\tilde{f}) \leq u_{t'}(c_\ast) \). Thus by (86) exists \( \alpha \in (0, 1) \) such that
\[
(a_{0,\ldots,t-1}, f) \sim (a_{0,\ldots,t-1}, \tilde{f}) \sim (a_{0,\ldots,t-1}, c_\ast \alpha c_\ast) \) and \( V_t(u(f)) = V_t(u(\tilde{f})) = V_t(u(c_\ast \alpha c_\ast)) \).
The other case \( u_t(f) < V_{t+1}(u(f)) \) follows symmetrically if (2) is true by using \( p^2 \) instead of \( p^1 \) and if (2) is not true then similar to the case when (1) is not true.

Let us finally show that \( V_t \circ u \) represents \( \succeq_t \). Let \( f, g \in H_t \) be such that

\[
V_t(u(f)) \geq V_t(u(g))
\]

and let \( a \in H_t \). Let us show that \( (a_{\{0, \ldots, t-1\}}, f) \succeq (a_{\{0, \ldots, t-1\}}, g) \). Now exists \( \alpha^* \in (0, 1) \) such that for all \( t' \geq t \)

\[
|u_{t'}(\alpha^* f + (1 - \alpha^*)(c^*/2c_s)) - 1/2(u_{t'}(c^*) + u_{t'}(c_+))| < 1/4(u_{t'}(c^*) - u_{t'}(c_+))
\]

and

\[
|u_{t'}(\alpha^* f + (1 - \alpha^*)(c^*/2c_s)) - 1/2(u_{t'}(c^*) + u_{t'}(c_+))| < 1/4(u_{t'}(c^*) - u_{t'}(c_+)).
\]

Thus by above exists \( \alpha^f, \alpha^g \in [0, 1] \) such that

\[
\begin{align*}
(a\alpha^*(c^*/2c_s)_{\{0, \ldots, t-1\}}, f\alpha^*(c^*/2c_s)_{\{t, \ldots, T\}}) & \sim (a\alpha^*(c^*/2c_s)_{\{0, \ldots, t-1\}}, c^*\alpha^f c_+_{\{t, \ldots, T\}},) \quad \text{(95)} \\
(a\alpha^*(c^*/2c_s)_{\{0, \ldots, t-1\}}, g\alpha^*(c^*/2c_s)_{\{t, \ldots, T\}}) & \sim (a\alpha^*(c^*/2c_s)_{\{0, \ldots, t-1\}}, c^*\alpha^g c_+_{\{t, \ldots, T\}},) \quad \text{(96)}
\end{align*}
\]

\[
V_t(u(f\alpha^*(c^*/2c_s)_{\{t, \ldots, T\}})) = V_t(u(c^*\alpha^f c_+_{\{t, \ldots, T\}})), \quad \text{and} \quad V_t(u(g\alpha^*(c^*/2c_s)_{\{t, \ldots, T\}})) = V_t(u(c^*\alpha^g c_+_{\{t, \ldots, T\}})).
\]

Thus by C-additivity and positive homogeneity of \( V_t \) we have

\[
V_t(u(c^*\alpha^f c_+_{\{t, \ldots, T\}})) = V_t(u(f\alpha^*(c^*/2c_s)_{\{t, \ldots, T\}})) = \alpha^*V_t(u(f)) + (1 - \alpha^*)V_t(u(c^*/2c_s)) \geq \alpha^*V_t(u(g)) + (1 - \alpha^*)V_t(u(c^*/2c_s)) = V_t(u(g\alpha^*(c^*/2c_s)_{\{t, \ldots, T\}})) = V_t(u(c^*\alpha^g c_+_{\{t, \ldots, T\}})).
\]

Thus by the C-additivity and positive homogeneity of \( V_t, \alpha^f \geq \alpha^g \). Thus we have by the monotonicity of \( I \) and since \( I \circ u \) represents \( \succeq \)

\[
(a\alpha^*(c^*/2c_s)_{\{0, \ldots, t-1\}}, c^*\alpha^f c_+_{\{t, \ldots, T\}}) \succeq (a\alpha^*(c^*/2c_s)_{\{0, \ldots, t-1\}}, c^*\alpha^g c_+_{\{t, \ldots, T\}}).
\]

Thus by (95,96)

\[
(a\alpha^*(c^*/2c_s)_{\{0, \ldots, t-1\}}, f\alpha^*(c^*/2c_s)_{\{t, \ldots, T\}}) \succeq (a\alpha^*(c^*/2c_s)_{\{0, \ldots, t-1\}}, g\alpha^*(c^*/2c_s)_{\{t, \ldots, T\}}).
\]

Finally by the crispness of \( c^*/2c_s \)

\[
(a_{\{0, \ldots, t-1\}}, f) \succeq (a_{\{0, \ldots, t-1\}}, g).
\]

Thus since \( a \in H_t \) was arbitrary, we have \( f \succeq_t g \). Thus \( V_t \circ u \) is a weak representation for \( \succeq_t \) and \( V_t \) is C-additive, positive homogeneous, and monotonic.
Finally by denoting

$$
\delta_1 := \begin{cases} 
\frac{p_t^2}{\sum_{t'=t}^T p_{t'}}^1, & \text{if } (1), (2), \ \frac{p_t^1}{\sum_{t'=t}^T p_{t'}} \geq \frac{p_t^2}{\sum_{t'=t}^T p_{t'}} \\
\frac{p_t^1}{\sum_{t'=t}^T p_{t'}}^2, & \text{if } (1), (2), \ \frac{p_t^1}{\sum_{t'=t}^T p_{t'}} < \frac{p_t^2}{\sum_{t'=t}^T p_{t'}} \\
1, & \text{if } \neg(2)
\end{cases}
$$

and

$$
\delta_2 := \begin{cases} 
\frac{p_t^1}{\sum_{t'=t}^T p_{t'}}^1, & \text{if } (1), (2), \ \frac{p_t^1}{\sum_{t'=t}^T p_{t'}} \geq \frac{p_t^2}{\sum_{t'=t}^T p_{t'}} \\
\frac{p_t^2}{\sum_{t'=t}^T p_{t'}}^2, & \text{if } (1), (2), \ \frac{p_t^1}{\sum_{t'=t}^T p_{t'}} < \frac{p_t^2}{\sum_{t'=t}^T p_{t'}} \\
1, & \text{if } \neg(1)
\end{cases}
$$

Then by considering cases by the definition of $\delta_1, \delta_2$ for all $h \in H$

$$
V_t(u(h)) = \begin{cases} 
\min_{\delta \in [\delta_1, \delta_2]} (1 - \delta)u_t(h) + \delta V_{t+1}(u(h)), & \text{if } \delta_1 \leq \delta_2 \\
\max_{\delta \in [\delta_1, \delta_2]} (1 - \delta)u_t(h) + \delta V_{t+1}(u(h)), & \text{if } \delta_1 > \delta_2.
\end{cases}
$$

Lemma 100 Let $\succsim$ satisfies Axioms 1-4,5',6, and 10. There exists $(u, (\delta_t^1, \delta_t^2)_{t \in T})$ that is a state dependent recursive dual-self representation for $\succsim$ such that

$$
\text{int } \bigcap_{t \in T} u_t(\Delta(X_t)) \neq \emptyset.
$$

Proof. By Propositions 39 and 42, exists $(u, I)$ such that $I : u(H) \to \mathbb{R}$ is C-additive on non-null states, positive homogeneous and monotonic and $I \circ u$ represents $\succsim$ and for all $s \in S$ for the crisp act from Axiom 5' $u_s(c) = 0$. Now by Axiom 5' exists an act $f \in H$ such that for all $s \in S^P c_s \succ_s f_s$ or for all $s \in S^P c_s \succ_s f_s$. Thus by the monotonicity of $I$ and the definition of $\succsim$ for all $s \in S^P u_s(c_s) > u_s(f_s)$ or for all $s \in S^P u_s(f_s) > u_s(c_s)$. Thus by the convexity of $u(H)$ and since $S$ is finite exists $c^*, c_s \in H$ such that for all $s, s' \in S^P u_s(c_s) = u_{s'}(c_{s'})$ and $u_s(c_s) = u_{s'}(c_{s'})$ and $u_s(c_s) > u_s(c_{s'}).$ Since for all $s \in S$ $|X_s| \geq 2$, by redefining utilities for null states which does not affect the values of the representation by the definition of state dependent dual-self representation and by redefining $c_s$ and $c^*$ for null states, we can assume that for all $s, s' \in S$ $u_s(c_s^*) = u_{s'}(c_{s'})$ and $u_s(c_{s'}) = u_{s'}(c_{s''})$. Now $c^*$ and $c_s$ are crisp acts as constant utility acts by the representation.

First, by the monotonicity of $I$, $u_T$ is a weak representation for $\succsim_T$. Thus define $V_T : u(H) \to \mathbb{R}$ by for all $f \in H V_T(u(h)) = u_T(h)$. Then $V_t$ is C-additive, positively homogeneous, and monotonic.
and \( V_T \circ u \) represents \( \succeq^T=\succeq_T \). Now by recursively applying Lemma 99 for all \( 0 \leq t \leq T-1 \) exists \( V_t : u(H) \to \mathbb{R} \) that is C-additive, positively homogeneous, and monotonic, and \( V_t \circ u \) weakly represents \( \succeq^t \) and exists \( \delta_1, \delta_2 \in [0,1] \) such that for all \( h \in H \)

\[
V_t(u(h)) = \begin{cases} 
\min_{\delta \in [\delta_1, \delta_2]} (1-\delta)u_t(h) + \delta V_{t+1}(u(h)), & \text{if } \delta_1 \leq \delta_2 \\
\max_{\delta \in [\delta_1, \delta_2]} (1-\delta)u_t(h) + \delta V_{t+1}(u(h)), & \text{if } \delta_1 > \delta_2.
\end{cases}
\]

Let us finally show that \( V_0 \) represents \( \succeq \). Since \( \succeq^0=\succeq \) \( V_0 \) weakly represents \( \succeq \). So it only remains to show that for all \( f,g \in H \) if \( V_0(f) \geq V_0(g) \) then \( f \succ g \). Let \( f,g \in H \) be such that \( u(f), u(g) \in \text{int}_{\mathcal{S}^V} u(H) \) and \( V_0(u(f)) > V_0(u(g)) \). Assume, per contra, \( g \succeq f \). Let \( \varepsilon = \left( 0, V_0(u(f)) - V_0(u(g)) \right) \) be such that \( \text{pr}_{\mathcal{S}^V} B_\varepsilon(u(g), \varepsilon) \subseteq \text{pr}_{\mathcal{S}^V} u(H) \). Thus exists \( \bar{g} \in H \) such that \( u(\bar{g}) = u(g) + \varepsilon \mathbf{1}_{\mathcal{S}^V} \).

Thus by C-additivity and monotonicity for \( V_0 \)

\[
V_0(u(\bar{g})) \leq V_0(u(g)) + \varepsilon < V_0(u(g)) + V_0(u(f)) - V_0(u(g)) = V_0(u(f)).
\]

But by C-additivity for non-null states of \( I \)

\[
I(u(\bar{g})) = I(u(g)) + \varepsilon > I(u(g)).
\]

Thus \( \bar{g} \succ g \succeq f \). Thus since \( V_0 \circ u \) is a weak representation for \( \succeq \)

\[
V_0(u(\bar{g})) > V_0(u(f))
\]

which contradicts (97). Thus \( f \succ g \). Finally let \( f,g \in H \) be such that \( u(f), u(g) \in \text{int}_{\mathcal{S}^V} u(H) \) and \( V_0(u(f)) > V_0(u(g)) \). Now by C-additivity and positive homogeneity of \( V_0 \), we have \( V_0\left( u(f^{1/2}(c^*1/2c_*)) \right) > V_0\left( u(g^{1/2}(c^*1/2c_*)) \right) \).

Thus by above since \( u(c^*1/2c_*) \in \text{int} u(H) \), \( f^{1/2}(c^*1/2c_*) \succ g^{1/2}(c^*1/2c_*) \) and since \( c^*1/2c_* \) is a crisp act \( f \succ g \), which shows the claim. \( \square \)

**Lemma 101** Let \( \succeq \subseteq H \times H \) and \( \left( u, (\delta^1_t, \delta^2_t)_{t \in T} \right) \) be a state dependent recursive dual-self representation for \( \succeq \) such that

\[
\text{int} \bigcap_{t \in T} u_t \left( \Delta(X_t) \right) \neq \emptyset.
\]

If there are no null states and \( \succeq \) satisfies Axiom 7 and \( \left( \tilde{u}, (\tilde{\delta}^1_t, \tilde{\delta}^2_t)_{t \in T} \right) \) is another state dependent recursive dual-self representation for \( \succeq \), then exists \( \alpha \in \mathbb{R}_{++}, \beta \in \mathbb{R} \) such that for all \( t \in T \)

\[
\tilde{u}_t = \alpha u_t + \beta
\]

and for all \( 0 \leq t \leq T-1 \)

\[
\delta^1_t = \bar{\delta}^1_t \text{ and } \delta^2_t = \bar{\delta}^2_t.
\]
Proof. Let $V_0$ and $\tilde{V}_0$ be the two recursive solutions. Then define $I : u(H) \to \mathbb{R}$ and $I : \tilde{u}(H) \to \mathbb{R}$ by for all $f \in H$

$$I(u(f)) = V_0(f) \text{ and } \tilde{I}(\tilde{u}(f)) = \tilde{V}_0(f)$$

which are well-defined by the definition of $V_0, \tilde{V}_0$ and $I, \tilde{I}$ are $C$-additive, positive homogeneous and monotonic and $I \circ u$ and $\tilde{I} \circ \tilde{u}$ represents $\preceq$. Thus by Lemmas 78 and 83 exists $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$ such that for all $t \in T$

$$\tilde{u}_t = \alpha u_t + \beta$$ (98)

and for all $f \in H$ by Lemma 77

$$\alpha I(u(f)) + \beta = \tilde{I}(\tilde{u}(f)).$$ (99)

Let $0 \leq t \leq T - 1$ and let us show that $\delta_t^1 = \tilde{\delta}_t^1$ and $\delta_t^2 = \tilde{\delta}_t^2$. Since

$$\text{int} \bigcap_{t \in T} u_t(\Delta(X_t)) \neq \emptyset$$

exists $c^*, c_*$ such that for all $t, t' \in T$

$$u_t(c^*) = u_{t'}(c^*) > u_t(c_*) = u_{t'}(c_*)$$

and by (98)

$$\tilde{u}_t(c^*) = \tilde{u}_{t'}(c^*) > \tilde{u}_t(c_*) = \tilde{u}_{t'}(c_*).$$

Let us consider the mappings

$$\alpha \in (0, 1) \mapsto I\left(u(c^{*1/2}_t,c_*)\alpha_t{t_1},c^*_t...t_3T\right) \text{ and } \alpha \in (0, 1) \mapsto \tilde{I}\left(\tilde{u}(c^{*1/2}_t,c_*)\alpha_t{t_1},c^*_t...t_3T\right).$$

By the definition of $I, \tilde{I}$

$$\nabla J(3/4) = (u_t(c^*) - u_t(c_*))\delta_1 \text{ and } \nabla \tilde{J}(3/4) = \alpha(u_t(c^*) - u_t(c_*))\tilde{\delta}_1$$

and

$$\nabla J(1/4) = (u_t(c^*) - u_t(c_*))\delta_2 \text{ and } \nabla \tilde{J}(1/4) = \alpha(u_t(c^*) - u_t(c_*))\tilde{\delta}_2.$$ 

Thus by (99)

$$\delta_1 = \tilde{\delta}_1 \text{ and } \delta_2 = \tilde{\delta}_2.$$
Lemma 102. Let $\succeq \subseteq H \times H$, there are no null states, and $(u, (\delta^1_t, \delta^2_t)_{t \in T})$ be a state dependent recursive dual-self representation for $\succeq$ such that

$$\int \bigcap_{t \in T} u_t \left( \Delta(X_t) \right) \neq \emptyset.$$ 

Assume that if $(\tilde{u}, (\tilde{\delta}^1_t, \tilde{\delta}^2_t)_{t \in T})$ is another state dependent recursive dual-self representation for $\succeq$, then exists $\alpha \in \mathbb{R}_{++}, \beta \in \mathbb{R}$ such that for all $t \in T$

$$\tilde{u}_t = \alpha u_t + \beta$$

and for all $0 \leq t \leq T - 1$

$$\delta^1_t = \tilde{\delta}^1_t \text{ and } \delta^2_t = \tilde{\delta}^2_t.$$ 

Then for all $0 \leq t \leq T - 1 \delta^1_t \neq \delta^2_t$.

\textbf{Proof}. Assume per contra that exists $0 \leq t \leq T - 1$ such that $\delta^1_t = \delta^2_t$. By the representation if $\delta^1_t = 1$, then $\succeq_t$ is null and if $\delta^1_t = 0$, then $\succeq_{t+1}$ is null. Thus $\delta^1_t \in (0, 1)$. Let us define for all $t' \in T$

$$\tilde{u}_{t'} = \begin{cases} u_{t'}, & \text{if } t' < t \\ u_{t'} + \frac{1}{1 - \delta^1_t}, & \text{if } t' = t \\ u_{t'} - \frac{1}{\delta^1_t}, & \text{if } t' > t. \end{cases}$$

Let us show that $(\tilde{u}, (\tilde{\delta}^1_t, \tilde{\delta}^2_t)_{t \in T})$ is a state dependent recursive dual-self representation for $\succeq$. For $t \in T$ $\tilde{V}_t$ be the corresponding recursive equations. Let $f \in H$. Let us show that $\tilde{V}_0(f) = V_0(f)$ by induction. First, let us show that for all $t + 1 \leq t' \leq T \tilde{V}_{t'}(f) = V_{t'}(f) - \frac{1}{\delta^1_{t'}}$. First, we have $V_T(f) = u_T(f)$ and $\tilde{V}_T(f) = u_T(f) - \frac{1}{\delta^1_T}$. Let $t + 1 \leq t' \leq T$ and assume that $\tilde{V}_t(f) = V_t(f) - \frac{1}{\delta^1_t}$. Assume w.l.o.g. $\delta^1_t \leq \delta^2_t$. Then we have by the definition of $\tilde{u}$ and the recursive representation

$$\tilde{V}_{t'}(f) = \min_{\delta \in [\delta^1_t, \delta^2_t]} (1 - \delta) u_{t'}(f) + \delta V_{t'}(f)$$

$$= \min_{\delta \in [\delta^1_t, \delta^2_t]} (1 - \delta) u_{t'}(f) + \delta V_{t'}(f) - (1 - \delta) \frac{1}{\delta^1_t} - \delta \frac{1}{\delta^1_t} = V_{t'}(f) = V_t(f).$$

Thus by induction, we have $\tilde{V}_{t+1}(f) = V_{t+1}(f) - \frac{1}{\delta^1_{t+1}}$. Finally, let us show by induction that for all $0 \leq t' \leq t \tilde{V}_{t'}(f) = V_t(f)$. First, let us show that $\tilde{V}_t(f) = V_t(f) - \frac{1}{\delta^1_t}$.

Now we have by the definition of $\tilde{u}$ and the recursive representation since $\delta^1_t = \delta^2_t$

$$\tilde{V}_t(f) = (1 - \delta^1_t) \tilde{u}_t(f) + \delta^1_t \tilde{V}_{t+1}(f)$$

$$=(1 - \delta^1_t) u_t(f) + \delta^1_t V_{t+1}(f) + (1 - \delta^1_t) \frac{1}{1 - \delta^1_t} - \delta^1_t \frac{1}{\delta^1_t} = V_t(f).$$
Next, let $0 \leq t' \leq t - 1$ and assume that $V_{t+1}(f) = V_{t+1}(f)$. Assume w.l.o.g. $\delta^1_t \leq \delta^2_t$. Then by the definition of $\tilde{u}$ and the recursive representation

$$
\tilde{V}_t(f) = \min_{\delta \in [\delta^1_t, \delta^2_t]} (1 - \delta) \tilde{u}_t'(f) + \delta \tilde{V}_t'(f)
$$

Thus by induction $\tilde{V}_0(f) = V_0(f)$. Hence since $V_0$ represents $\succeq$ especially $\tilde{V}_0$ represents $\succeq$. Thus $(\tilde{u}, (\delta^1_t, \delta^2_t)_{t \in \mathcal{T}})$ is a state dependent recursive dual-self representation for $\succeq$ which contradicts the assumed uniqueness. \hfill \Box

**Lemma 103** Let $\succeq \subseteq H \times H$, there are no null states, and $(\tilde{u}, (\delta^1_t, \delta^2_t)_{t \in \mathcal{T}})$ be a state dependent recursive dual-self representation for $\succeq$ such that

$$
\text{int } \bigcap_{t \in \mathcal{T}} u_t\left(\Delta(X_t)\right) \neq \emptyset.
$$

If for all $0 \leq t \leq T - 1$ $\delta^1_t \neq \delta^2_t$, then $\succeq$ satisfies Axiom 7.

**Proof.** Let us denote $\mathcal{T}_{-1} = \{0, \ldots, T - 1\}$. Let $V_0$ be the corresponding recursive solution and define for all $t \in \mathcal{T}$ $I_t : u(H) \to \mathbb{R}$ by for all $f \in H$

$$
I_t\left(u(f)\right) = V_0(f)
$$

and denote $I := I_0$, are well-defined by the definition of $V_t$ and $I_t$ depend only on utility for periods $\{t, \ldots, T\}$ are C-additive on $\{t, \ldots, T\}$, positive homogeneous and monotonic and $I \circ u$ represents $\succeq$. Let $c^*, c_\ast \in H$ be such that for all $t, t' \in \mathcal{T}$

$$
u_t(c^*) = u_{t'}(c^*) > u_t(c_\ast) = u_{t'}(c_\ast).
$$

Let $C := \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi)$ Let us show that for all $\theta \in \{1, 2\}^{T-1}$ exists, $p^\theta \in C$ where for each $t \in \mathcal{T}$

$$
p^\theta_t := \begin{cases} 
\prod_{t' = 0}^{t-1} \delta^\theta_{t'} (1 - \delta^\theta_{t'}), & \text{if } t < T \\
\prod_{t' = 0}^{t-1} \delta^\theta_{t'}, & \text{if } t = T
\end{cases}
$$

(100)

Let $\theta \in \{1, 2\}^{T-1}$. Define $\varphi^\theta$ as follows for each $t \in \mathcal{T}$.

$$
\varphi^\theta_t := \begin{cases} 
u_t\left(\frac{c^* (t + 1)}{2T + 2} c_\ast\right), & \text{if } \theta_t = 1 \\
\nu_t\left(\frac{c^* (1 - t + 1)}{2T + 2} c_\ast\right), & \text{if } \theta_t = 2
\end{cases}
$$

(101)
Now let \( \psi \in B_\infty(\varphi^0,1/|t+2|)(u_t(c^*) - u_t(c_*)) \). Now since \( I_t \) depend only on \( \{t, \ldots, T\} \) and are \( C \)-additive and monotonic, we have for all \( 0 \leq t \leq T \), \( t \neq t^0 \)

\[
|I_{t+1}(\psi) - I_{t+1}(\varphi^0)| \leq \frac{1}{4T + 2} (u_t(c^*) - u_t(c_*)).
\]  

(102)

Let \( 0 \leq t \leq T - 1 \). Let us show first that if \( \theta_t = 1 \), then \( \psi_t < I_{t+1}(\psi) \). Now by (101) for all \( t < t' \leq T \varphi^0 \geq u_t(c^*(1/2T+2)c_*) \) and thus

\[
I_{t+1}(\varphi^0) \geq I_{t+1}\left(u_t\left(c^*(\frac{1}{2}T + 2)c_*\right)\right) = u_t\left(c^*(\frac{1}{2}T + 2)c_*\right).
\]  

(103)

Thus we have

\[
I_{t+1}(\psi) - \psi_t \geq u_t\left(c^*(\frac{1}{2}T + 2)c_*\right) - \frac{1}{4T + 2} (u_t(c^*) - u_t(c_*)) - \psi_t
\]

\[
= u_t\left(c^*(\frac{1}{2}T + 2)c_*\right) + \frac{1}{4T + 2} (u_t(c^*) - u_t(c_*)) - \psi_t
\]

\[
= \varphi_t^0 - \psi_t + \frac{1}{4T + 2} (u_t(c^*) - u_t(c_*)) > 0.
\]

Second, let us show that if \( \theta_t = 2 \), then \( \psi_t > I_{t+1}(\psi) \). Now by (101) for all \( t < t' \leq T \varphi^0 \leq u_t(c^*(1/2T+2)c_*) \) and thus

\[
I_{t+1}(\varphi^0) \leq I_{t+1}\left(u_t\left(c^*(1 - \frac{1}{2}T + 2)c_*\right)\right) = u_t\left(c^*(1 - \frac{1}{2}T + 2)c_*\right).
\]  

(104)

Thus we have

\[
I_{t+1}(\psi) - \psi_t \leq u_t\left(c^*(1 - \frac{1}{2}T + 2)c_*\right) + \frac{1}{4T + 2} (u_t(c^*) - u_t(c_*)) - \psi_t
\]

\[
= u_t\left(c^*(1 - \frac{1}{2}T + 2)c_*\right) - \frac{1}{4T + 2} (u_t(c^*) - u_t(c_*)) - \psi_t
\]

\[
= \varphi_t^0 - \psi_t - \frac{1}{4T + 2} (u_t(c^*) - u_t(c_*)) < 0.
\]

Thus by the recursive dual-self representation we have \( p^0 \cdot \psi = I(\psi) \). Thus especially \( \varphi^0 \) is a differentiability point of \( I \) and \( p^0 \in C \). Let us show that linear span of the set \( \{p^0\}_{\theta \in \{1,2\}^{T-1}} \) is \( \mathbb{R}^S \) by showing inductively that the linear span contains indicators for each coordinate. First, since for all \( t \in T \) \( \delta^1_t \neq \delta^2_t \), exists \( \theta^* \in \{1,2\}^{T-1} \) such that for all \( t \in T \) \( \delta^\theta_t \neq 0 \).

Let us show that for all \( t \in T \) an element \( p \) is in the linear span of \( \{p^\theta\}_{\theta \in \{1,2\}^{T-1}} \) such that \( p_t \neq 0 \) and for all \( t < t' \leq T \) \( p_t = 0. \) Let \( t \in T \). If \( t = T \), then \( p^\theta \) gives the desired element. So assume \( 0 \leq t \leq T - 1 \). Let \( \theta^t \in \{1,2\}^{T-1} \) be such that for all \( 1 \leq t \leq T - 1 \) \( \theta^t_0 = \theta^t_t \) and \( \theta^0 \neq \theta^0_t \). First if \( \delta^1_t = 0 \) or \( \delta^2_t = 0 \), then by the choice of \( \theta^* \) \( \delta^\theta_t \neq 0 \) and \( \delta^\theta_t = 0 \). Thus \( p^\theta \) is such that \( p^\theta_t \neq 0 \) and for all \( t < t' \leq T \) \( p^\theta_{t'} = 0. \)
Next assume $\delta^1_t \neq 0$ and $\delta^2_t \neq 0$. Now we have by (100)

\[
p_t^\theta - \frac{\delta^\theta_t}{\delta^\theta_t} p_t^\theta = \begin{cases} 
\prod_{t''=0}^{t'-1} \delta^\theta_{t''} (1 - \delta^\theta_t) - \frac{\delta^\theta_t}{\delta^\theta_t} \prod_{t''=0}^{t'-1} \delta^\theta_{t''} (1 - \delta^\theta_t), & \text{if } t' = t \\
\prod_{t''=0}^{t'-1} \delta^\theta_{t''} (1 - \delta^\theta_t) - \frac{\delta^\theta_t}{\delta^\theta_t} \prod_{t''=0, t'' \neq t}^{t'-1} \delta^\theta_{t''} \delta^\theta_t (1 - \delta^\theta_t) = 0, & \text{if } t < t' < T \\
\prod_{t''=0}^{t'-1} \delta^\theta_{t''} - \frac{\delta^\theta_t}{\delta^\theta_t} \prod_{t''=0, t'' \neq t}^{t'-1} \delta^\theta_{t''} \delta^\theta_t = 0, & \text{if } t < t' = T.
\end{cases}
\]

Additionally we have since $\delta^1_t \neq 0$ and $\delta^2_t \neq 0$

\[
(1 - \delta^\theta_t) \neq \frac{\delta^\theta_t}{\delta^\theta_t} (1 - \delta^\theta_t) \iff \frac{1 - \delta^\theta_t}{\delta^\theta_t} \neq \frac{1 - \delta^\theta_t}{\delta^\theta_t}
\]

and this holds since the function $\frac{1-x}{x}$ is strictly decreasing in the interval $(0,1]$ and since $\delta^\theta_t \neq \delta^\theta_t$. Thus $p_t^\theta - \frac{\delta^\theta_t}{\delta^\theta_t} p_t^\theta \neq 0$ and for all $t < t' \leq T$, $p_t^\theta - \frac{\delta^\theta_t}{\delta^\theta_t} p_t^\theta = 0$.

Thus the linear span of $\{p^\theta\}_{\theta \in \{1,2\}^{T-1}}$ is $\mathbb{R}^S$ and hence $\overline{\mathbb{O}} \{p^\theta\} \subseteq C$ has non-empty interior (Boyd and Vandenberghe, 2004, Section 2.5.2). Thus by Lemma 83, $\succsim$ satisfies Axiom 7.

\[\Box\]
References


Chandrasekher, Madhav; Frick, Mira; Iijima, Ryota, and Le Yaouanc, Yves (2020). Dual-self representations of ambiguity preferences.


