State Dependent Utility and Ambiguity

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Abstract

Models under uncertainty study choice behavior when outcomes depend on the realized state of the world. The typical assumption is that utilities of outcomes do not depend on the realized state and are state independent. Without this simplifying assumption, it is difficult to separately identify utilities and beliefs. This paper provides novel general foundations for models with state dependent utilities: once we depart from expected utility, it is often possible to uniquely identify utilities and beliefs. Specifically, we show that with general models of non-expected utility under ambiguity we have complete identification of utilities and probabilities under full-dimensional uncertainty. We offer a novel axiomatization for state dependent utility models. Finally, we provide an application to social choice theory. We show the identification of fairness of the society and interpersonal utility comparisons.

1 Introduction

Decision making under uncertainty studies choice behavior when outcomes depend on the realized state of the world. Traditionally, it is assumed that the utilities of outcomes are state independent and do not depend on the realized state. This independence simplifies the identification of utilities and beliefs. However, as observed by Aumann (1971) in many situations outcomes and their utilities might be state dependent. Known examples are when the state of the world is the health of the decision maker, which is likely to affect the utility of outcomes, such as with health insurance (Arrow, 1971). Assuming state independent utilities in these situations may be inaccurate and lead to wrong identification and predictions.

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This paper reconsiders state dependent utilities in the standard framework and shows that, under general conditions, it is possible to separately identify the state dependent utilities and probabilities whenever we have violations of the expected utility hypothesis. Additionally, we provide a novel axiomatic characterization for models with state dependent utilities under ambiguity. We illustrate the usefulness of these results with two applications unrelated to choice under uncertainty.

Before moving on to the results, we highlight the importance of identifying utilities and probabilities with a concrete example. Consider a government that wants to change people’s behavior with a public health campaign, but people find the change difficult or inconvenient e.g. reduce smoking or increase the use of seat belts. Here the choice of an effective campaign depends critically on if the lack of change reflects a taste-based reason, quitting smoking is difficult, seat belts are uncomfortable, or if it reflects a belief-based reason, only heavy smokers get cancer or only reckless drivers get into accidents. In the first case, an effective campaign would make the change of behavior easier by increasing the availability of nicotine replacement products and making smoking socially less acceptable or redesigning seat belts to be more comfortable and convenient. In the second case, an effective campaign would be an information campaign on the effects of behavior changes and risks associated with the current behavior. Here, it is crucial to separate tastes from beliefs in order to choose an effective campaign.

This paper characterizes and identifies general state dependent models under uncertainty. We focus on standard preferences under ambiguity, that is continuous, monotonic, and risk independent preferences. Our first characterization result shows that if these standard preferences have two unambiguous acts, in a sense that will be made precise, then they admit a dual-self expected utility representation (non-convex multiple prior preferences, Chandrasekher et al., 2020) with state dependent utilities.¹

In this context, we show our first main identification result: when the uncertainty about states of the world is full-dimensional, the probabilities and the utilities are fully identified.

¹This result extends Chandrasekher et al.’s (2020) characterization. However, this extension is not entirely straightforward since under state-dependence, constant acts may no longer be unambiguous, so we need to find an alternative way to capture the characterizing properties of the representation. We will infer from the decision maker’s behavior which acts are unambiguous and use this to capture the characterizing properties.
That is, the probabilities are uniquely identified and the state dependent utilities are identified up to a common positive affine transformation. This shows that the impossibility of identification under expected utility is only a knife-edge case, due to the linearity of expected utility: once we depart from such linearity, we regain identification. Instead, with any full-dimensional uncertainty, we can identify the utilities from the violations of the expected utility hypothesis, as we illustrate with an example in the next section. More in general, we show that the identification of utilities and probabilities between two states is characterized by having uncertainty about the relative likelihood between the states. This general identification result has many important special cases. It characterizes the identification of state dependent utilities for models such as maxmin expected utility (multiple prior preferences, Gilboa and Schmeidler, 1989), Choquet expected utility (Schmeidler, 1989), α-maxmin expected utility (Ghirardato et al., 2004), and invariant biseparable preferences (Ghirardato et al., 2004; Amarante, 2009).

Next, we move on to a more general model under uncertainty. We show that if standard ambiguity preferences have two acts that share the same ambiguity and hedge ambiguity similarly, in a sense that will be made precise later on, then they admit a dual-self variational expected utility representation (non-convex variational preferences, Chandrasekher et al., 2020) with state dependent utilities. This generalizes the characterization from Chandrasekher et al. (2020) to state-dependent utilities. In this more general context, we show our second identification result: when the uncertainty about states is full-dimensional, then probabilities and intensities of preferences can be separated. However, in this case, the levels of utilities are not identified. This general result characterizes the identification for state dependent variations of models such as monotone mean-variance preferences (Maccheroni et al., 2009), multiplier preferences (Hansen and Sargent, 2001; Strzalecki, 2011), variational preferences (Maccheroni et al., 2006), monotonic mean-dispersion preferences (Grant and Polak, 2013), and vector expected utility (Siniscalchi, 2009).

This new foundation for state dependent utilities has applications beyond ambiguity. We illustrate this with an application to social choice theory. Here, we interpret states to be

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2Formally, the probabilities are uniquely identified and the changes of state dependent utilities are identified up to a common positive multiplication.
members of a society and study the preferences of the society over distributions of goods. In this context, we show that the fairness of the society and interpersonal utility comparisons can be identified. In a general class of societies, utilitarian societies are only a literal knife-edge case where they are not identified. Instead, with any other society in this general class, the fairness and the interpersonal utilities can be identified.

This paper contributes to the literature studying the identification of state dependent utilities in models of non-expected utilities following Drèze (1987) and Chew and Wang (2020). This provides an alternative to the literature on using additional information (e.g. preferences conditional on signals with Bayesian updating, hypothetical lotteries of state-outcome pairs, or stochastic choice) for the identification of state dependent utilities as in Karni (2011a; 2011b), Karni and Schmeidler (2016), and Lu (2019).

The remainder of the paper proceeds as follows: We begin, in Section 1.1, by showing a simple example of the fundamental forces behind the identification result. Next, Section 2 studies state dependent dual-self expected utility. We axiomatize its existence in Sections 2.3 and 2.4 and characterize its identification in Section 2.5 with behavioral characterizations for the identification. Section 3 extends the existence and the identification results to state dependent dual-self variational expected utility. Section 4 discusses applications to social choice theory. Section 5 clarifies the underlying assumptions for our identification and discusses the limitations of the identification. Finally, Section 6 discusses related literature and Section 7 concludes. Proofs for all the results are in the Appendix.

1.1 An Example of Identification

We begin with a simple example illustrating that with state dependent expected utility the intensities of preferences and the probabilities cannot be separated. However, this is only an unidentified knife-edge case. In the second part of the example, we show that with state dependent maxmin expected utility these can be separated and identified from the violations of the expected utility hypothesis.

To make the problem of separating utilities and probabilities concrete, consider an unemployed person, Bob. Unemployed Bob is offered a choice between $1,000 if he is employed
in six months and $1,000 if he is unemployed in six months. He chooses the unemployment insurance that pays in the case of unemployment. What can we infer from his choice in this situation? Does the choice mean that Bob considers it more likely that he is unemployed in six months than that he would find a job? Or does the choice reflect that without employment Bob could use the additional money to pay for overdue bills and would like to insure against this prospect?

This is the fundamental problem in identifying utilities and probabilities. With state independent utilities we can always infer which event is more likely by comparing the same payoffs in different states as above. However, as soon as we consider the possibility that utilities might be state dependent, the same payoffs in different states are no longer the same in utilities and these two forces become inseparable. Utilities and probabilities are always only observed jointly, and it is not possible to separate them using simple comparisons.

Next, we formalize the above identification problem. Consider Anscombe-Aumann setup with two states of the world, 1 and 2, and outcomes that are lotteries on consequences $\Delta(X)$. Acts consist of consequences for each state, $(f_1, f_2)$. Assume that preferences over acts $(f_1, f_2)$ have a state dependent expected utility representation with an equal probability for both of the states

$$0.5u_1(f_1) + 0.5u_2(f_2)$$

where $u_1, u_2 : \Delta(X) \to \mathbb{R}$ are von Neumann-Morgenstern utilities.

Now these preferences have an alternative expected utility representation with any probability $p \in (0, 1)$ for state 1 since

$$0.5u_1(f_1) + 0.5u_2(f_2) = p\left(\frac{0.5}{p}u_1(f_1)\right) + (1-p)\left(\frac{0.5}{1-p}u_2(f_2)\right) = p\tilde{u}_1(f_1) + (1-p)\tilde{u}_2(f_2),$$

where the terms inside the parentheses define new state dependent utility functions $\tilde{u}_1, \tilde{u}_2$.

In this alternative representation following the above example, we have replaced the probability for employment with an intensity of preference for money. This highlights the impossibility of identifying the state dependent expected utility since the intensities of preferences are inseparable from the probabilities.

However, this lack of identification is only due to the fact that we were using expected utility. Instead, we can identify the utilities across the states from the violations of the
expected utility hypothesis. Going back to the example, assume this time that Bob is uncertain about the probability of employment and is uncertainty averse focusing on the worst-case probability. Assume that we observe many pairwise choices and there is a violation of the expected utility hypothesis at an unemployment insurance of size $x$ in a sense that will be made precise later on. What should we infer from this violation? Since Bob is uncertain about the probability of employment, this violation reflects a change in the worst-case probability. However, with two states of the world, there can be a change in the worst-case probability only if the utility in both states is the same.\(^3\) So we can identify the utilities across states using changes in Bob’s uncertainty attitude.

Next, we formalize this identification from changes in the uncertainty attitude. We move on to a state dependent maxmin expected utility over acts \((f_1, f_2)\) defined by two probabilities \(p_* < p^*\) for state 1 and affine von Neumann-Morgenstern utilities \(u_1, u_2\)

\[
\min_{p \in [p^*, p_*]} pu_1(f_1) + (1 - p)u_2(f_2).
\]

We show that utilities are identifiable across states from non-linearities or violations of Independence axiom\(^4\). For this, let \(f_2\) be a consequence for state 2 and let \(x, y\) be consequences such that \(u_1(0.5x + 0.5y) = 0.5u_1(x) + 0.5u_1(y) = u_2(f_2)\) and \(u_1(x) < u_1(y)\).\(^5\) Next, we will show that there is a violation of Independence axiom at \((\frac{1}{2}x + \frac{1}{2}y, f_2)\).

We focus on the maxmin expected utility for acts \((\alpha x + (1 - \alpha)y, f_2)\) when \(\alpha\) changes from 0 to 1 as in Figure 1. First, between 0 and 0.5, \(\alpha x + (1 - \alpha)y\) gives more weight to \(x\) than to \(y\) and the act \((\alpha x + (1 - \alpha)y, f_2)\) gives lower utility in state 1 than in state 2. So the maxmin expected utility uses probability \(p^*\). Thus the maxmin expected utility for the act increases linearly at the rate \(p^* (u_1(y) - u_1(x))\). Second, between 0.5 and 1, \(\alpha x + (1 - \alpha)y\) has lower utility in state 2 and so the maxmin expected utility for the act \((\alpha x + (1 - \alpha)y, f_2)\) uses probability \(p_*\). This time the expected utility for the act increases linearly at the rate \(p_* (u_1(y) - u_1(x))\). Since the rate of increase changes at \(\alpha = 0.5\), there is a non-linearity at

\(^3\)The worst-case probability always maximizes the probability of the state with a lower utility.

\(^4\)Independence axiom from Anscombe and Aumann (1963) characterizes the linearity of the subjective expected utility. It states that for all acts \(f, g, h\) and \(\alpha \in (0, 1)\),

\[
f \succeq g \iff \alpha f + (1 - \alpha)h \succeq \alpha g + (1 - \alpha)h
\]

when consequences are gambles and mixtures of acts are defined statewise.

\(^5\)The first inequality follows from \(u_1\) being a von Neumann-Morgenstern utility. If \(u_1\) is unbounded, these \(x\) and \(y\) always exist.
that point. This represents a violation of Independence axiom at the act $(0.5x + 0.5y, f_2)$ that has the same utility for both states.

Finally, there can be violations of the independence axiom only if the utilities across states are the same. The only points where there can be non-linearities as in Figure 1 are points where the used probability changes. However, maxmin expected utility with two states always maximizes the probability for the state with lower utility. Thus the only points where there are changes of probabilities are points where the utility order of states changes. Especially, at that point, the utilities across states are exactly the same.

In summary, the acts where the utilities across states are equal are characterized by the violations of Independence axiom and especially they are identifiable. This shows the identification of the utilities across states. Finally, this also shows the identification of probabilities since after identifying the utilities, we can apply the state-independent identification result directly.

This example illustrates that we can behaviorally observe when the utilities across states are equal from the violations of the expected utility hypothesis or Independence axiom. The identification came from the two different probabilities that is from the full-dimensional set of probabilities.
This identification is generalized in our main result, Theorem 2, to finitely many states of the world and to non-convex dual-self expected utility. There we show that if there is full-dimensional uncertainty, then we recover the state independent identification: the set of probabilities is unique and the utilities are unique up to a common positive affine transformation.

2 State Dependent Dual-Self Expected Utility

In order to study identification in the most general setup, we begin by studying a state dependent version of dual-self expected utility (Chandrasekher et al., 2020). The state-independent version is a general model that includes as special cases maxmin expected utility (Gilboa and Schmeidler, 1989), Choquet expected utility (Schmeidler, 1989), and \( \alpha \)-maxmin expected utility (Ghirardato et al., 2004) and that is an alternative representation for invariant biseparable preferences (Ghirardato et al., 2004; Amarante, 2009). Here we characterize the existence and the identification of the state dependent version. Our results thus encompass those for state dependent versions of all the special cases and alternative representations.

2.1 Preliminaries and Notation

Consider the finite Anscombe-Aumann (1963) framework with state dependent consequences. \( S \) is a finite state space, for each \( s \in S \), \( X_s \) is a set of state dependent consequences and \( \Delta(X_s) \) is the set of (simple) lotteries on \( X_s \). Acts are mappings from states to state specific consequences and the set of acts is \( H = \times_{s \in S} \Delta(X_s) \).

Our primitive is a binary relation \( \succsim \) on \( H \). As usual, \( \succ \) and \( \sim \) denote the asymmetric and symmetric parts of \( \succsim \) respectively.

The following notation will be useful. \( \Delta(S) \) is the set of probability measures on \( S \). We endow \( \Delta(S) \) with the Euclidean topology.\(^7\) \( \mathcal{K}(\Delta(S)) \) is the set of all closed, convex, and non-empty subsets of \( \Delta(S) \) endowed with the Hausdorff topology. For \( P \subseteq \Delta(S) \), denote

\(^6\)This includes the standard state independent Anscombe-Aumann setting where \( X_s = X \) for all \( s \).

\(^7\)This is the \(|S|-1\) dimensional Euclidean topology when \( \Delta(S) \) is represented as \( \{p \in \mathbb{R}^{(|S|-1)} \mid \sum_{i=1}^{(|S|-1)} p_i \leq 1 \} \).
the convex closure of $P$ by $\overline{\mathfrak{m}}P$. For $S' \subseteq S$ and $P \subseteq \Delta(S)$, denote the projection of $P$ to $S'$ by $\text{pr}_{S'} P = \{(p_s)_{s \in S'} | p \in P\}$.

For $f \in H, s \in S, x_s \in \Delta(X_s)$, $f_s$ denotes the consequence of the act $f$ in the state $s$ and $(x_s, f_{-s})$ denotes the act where the consequence in the state $s$ is $x_s$ and in the states $s' \in S \setminus \{s\}$, $f_{s'}$. Mixtures of acts are defined statewise: for all $f, g \in H, \alpha \in [0, 1], s \in S$, define $(\alpha f + (1 - \alpha)g)_s = \alpha f_s + (1 - \alpha)g_s$.

If consequences in some state do not affect the preferences, then the utility for these consequences is unobservable. Hence our focus is on proper states:

**Definition** A state $s \in S$ is proper if there exist $x_s, y_s \in \Delta(X_s)$ and $f \in H$ such that $(x_s, f_{-s}) \not\sim (y_s, f_{-s})$.

The collection of proper states is denoted $S^P$.

We infer the preferences on consequences within each state as follows:

**Definition** For each $s \in S$, define $\succeq_s$ on $\Delta(X_s)$ by for all $x_s, y_s \in \Delta(X_s)$,

$$x_s \succeq_s y_s \iff (x_s, f_{-s}) \succeq (y_s, f_{-s}) \text{ for all } f \in H.$$ 

Additionally, $\succ_s$ is the asymmetric part of $\succeq_s$.

### 2.2 State Dependent Dual-Self Expected Utility

The state independent dual-self expected utility was introduced by Chandrasekher et al. (2020) as a general model for preferences under ambiguity. This model generalizes the maxmin expected utility by allowing for multiple sets of beliefs that are aggregated optimistically. Axiomatically this model corresponds to relaxing the uncertainty aversion of the maxmin expected utility. We will use a state dependent variation of it:

**Definition** $(u, \mathbb{P})$ is a state dependent dual-self expected utility for $\succeq$ if $u = (u_s)_{s \in S}$ and for all $s \in S$, $u_s : \Delta(X_s) \to \mathbb{R}$ is affine and $\mathbb{P} \subseteq K(\Delta(S))$ is compact and non-empty such that for each $P \in \mathbb{P}, p \in P, s \notin S^P$, $p_s = 0$, and for all $f, g \in H$,

$$f \succeq g \iff \max_{P \in \mathbb{P}} \min_{p \in P} \sum_{s \in S} p_s u_s(f_s) \geq \max_{P \in \mathbb{P}} \min_{p \in P} \sum_{s \in S} p_s u_s(g_s).$$
This representation is the maxmin expected utility when $P$ is a singleton and the maxmax expected utility when each $P \in \mathcal{P}$ is a singleton. Additionally, since this representation is a non-convex generalization of maxmin expected utility, especially any uncertainty aversion and seeking of the preferences can be represented by the interplay of the sets of beliefs and the collection of sets of beliefs. That is by the interplay of extreme concavity (min) and convexity (max).

As observed in Chandrasekher et al. (2020), the state independent dual-self expected utility is not unique. However, in the state independent case the convex and closed smallest set of probabilities used by the representation is unique which gives us tight dual-self expected utility. We will use this tight representation to study the identification of the state dependent dual-self expected utility.

**Definition** Let $\succsim \subseteq H \times H$. $(u, \mathcal{P})$ is a state dependent tight dual-self expected utility for $\succsim$ if $(u, \mathcal{P})$ is a state dependent dual-self expected utility for $\succsim$ and if $(u, \tilde{\mathcal{P}})$ is another state dependent dual-self expected utility for $\succsim$, then $\overline{\mathcal{C} \bigcup_{P \in \mathcal{P}} P} \supseteq \overline{\mathcal{C} \bigcup_{P \in \tilde{\mathcal{P}}} P}$.

### 2.3 Axioms for Existence

This section introduces six axioms that characterize the existence of a state dependent dual-self expected utility representation. This axiomatization highlights the generality of the representation by showing that essentially the only standard preferences under ambiguity that do not have a dual-self expected utility representation are such that every act is ambiguous or every act contains different ambiguity. A reader interested only in the identification can skip this and the following existence section.

The first four axioms define standard preferences under ambiguity following Cerreia-Vioglio et al. (2011) in the state dependent context. The first two axioms are standard rationality assumptions that the preferences are a nontrivial weak order that satisfy continuity.

**Axiom 1** $\succsim$ is complete, transitive, and non-trivial.

**Axiom 2** For all $f, g, h \in H$, the sets $\{\alpha \in [0, 1] | \alpha f + (1 - \alpha) g \succsim h\}$ and $\{\alpha \in [0, 1] | h \succsim \alpha f + (1 - \alpha) g\}$ are closed in $\mathbb{R}$. 
The next assumption for standard preferences under ambiguity is Monotonicity.\footnote{The state independent monotonicity assumes that if $f, g \in H$ are such that for all $s \in S$, $f_s \succeq g_s$, then $f \succeq g$ where $f_s$ and $g_s$ are acts that give the consequences $f_s$ and $g_s$, respectively, in every state.} This is the main axiom ruling out state dependent utilities by assuming that the consequences within each state of the world are ranked in the same order. We weaken this axiom to only within state monotonicity which allows for fully separate preferences within each state of the world and for state dependent consequences.

**Axiom 3** For all $s \in S, x_s, y_s \in \Delta(X_s), f, g \in H$,

$$\left(x_s, f_{s^-} \right) \succ \left(y_s, f_{s^-} \right) \implies \left(x_s, g_{s^-} \right) \succeq \left(y_s, g_{s^-} \right).$$

This axiom states that the decision maker’s revealed statewise preferences are consistent with monotonicity. If the decision maker reveals to prefer $x_s$ over $y_s$ in some situation, then by monotonicity the decision maker cannot reveal to strictly prefer $y_s$ over $x_s$ in another situation. This axiom is a state dependent version of the standard monotonicity. It is assuming that the decision maker is a consequentialist that only cares about the outcomes received in each state.\footnote{Formally, this axiom is a weak version of Savage’s (1954) Sure Thing Principle applied only to a single state.} This axiom guarantees that $\succeq_s$ is complete.

The next assumption for standard preferences under ambiguity is Risk Independence assuming Independence on lotteries.\footnote{This is usually assumed implicitly through Certainty Independence or Weak Certainty Independence axioms.} The next axiom assumes a weak version of Independence, ruling out strict preference reversals, for the lotteries within each state of the world. This Weak Independence was introduced in Einy (1989).\footnote{In Appendix Section B, we consider expected utility when Independence is relaxed to this Weak Independence. We show that this gives a latent expected utility representation that is an affine utility $u$ and a continuous and weakly increasing censoring function $H$ such that for all lotteries $P, Q$ on prizes $X$, $x \succeq y \iff H\left(\sum_x P(x)u(x)\right) \geq H\left(\sum_x Q(x)u(x)\right)$. Additionally, $u$ is unique up to a positive affine transformation.}

**Axiom 4** For all $s \in S, x_s, y_s, z_s \in \Delta(X_s), \alpha \in (0, 1)$,

$$x_s \succ y_s \implies \alpha x_s + (1-\alpha)z_s \succeq \alpha y_s + (1-\alpha)z_s.$$

Under standard ambiguity preferences, the dual-self expected utility is characterized by Certainty Independence which assumes that constant acts are unambiguous. In order to
discuss later on preferences that do not satisfy this axiom, we divide this assumption into two parts.

The first part of the Certainty Independence is the existence of a single constant act that is unambiguous and cannot be used for ambiguity hedging. The intuition for this part is that there is no uncertainty about the utility of constant acts since they give the same consequence with the same utility in every state of the world.

However, in the state dependent setting, it is not known which of the acts are unambiguous. Instead, we will infer from the decision maker’s behavior which of the acts are unambiguous. The behavioral interpretation for these acts is that they never hedge ambiguity and Independence axiom is satisfied when mixing with them as defined in Ghirardato et al. (2004).

Definition \( h \in H \) is a crisp act\(^{12} \) if for all \( f, g \in H, \alpha \in (0, 1) \),

\[
\alpha f + (1 - \alpha)h \succeq \alpha g + (1 - \alpha)h.
\]

The Certainty Independence assumes that all constant acts are crisp. The next axiom relaxes this to assuming that there exists some crisp act.

**Axiom 5** There exists a crisp act \( c \in H \).

Here, it is not necessary that the crisp act is unambiguous. Instead, it suffices that the crisp act is a least ambiguous act. That is the ambiguity of the crisp act is contained also in every other act and so it cannot be used for ambiguity hedging.

Our first part of Certainty Independence assumed the existence of only a single crisp act instead of a continuum of crisp acts as Certainty Independence assumes. Thus our second part of Certainty Independence captures this difference by assuming that all the constant acts contain exactly the same ambiguity since they are unambiguous.

In the state dependent setting, we will need to infer from the decision maker’s behavior which acts contain the same ambiguity. Behaviorally this means that the acts always hedge ambiguity similarly and so trading one of the acts for another with the same ambiguity

\(^{12}\)Cerreia-Vioglio et al. (2011) use a different definition for crisp acts that does not extend to state dependent setting.
cannot change the ambiguity hedged or the preferences. This is formalized in the following property from Maccheroni et al.’s (2006) Weak Certainty Independence assumption.

**Definition** \( f, g \in H \) are equally crisp acts if for all \( h, h' \in H, \alpha \in (0, 1), \)

\[
\alpha h + (1 - \alpha)f \succeq \alpha h' + (1 - \alpha)f \iff \alpha h + (1 - \alpha)g \succeq \alpha h' + (1 - \alpha)g.
\]

In this definition, \( f \) and \( g \) are equally crisp if trading \( f \) for \( g \) in any mixtures does not affect the preferences.

Under Certainty Independence, all constant acts are equally crisp. We relax this by assuming that there exist some equally crisp acts that are statewise ordered.

**Axiom 6** There exist \( f^*, g^* \in H \) such that \( f^* \) and \( g^* \) are equally crisp and for all \( s \in S^p, f^*_s \succ_s g^*_s. \)

The intuition for this axiom is that it captures the dispersive or relative nature of ambiguity: Ambiguity of an act comes from it having good outcomes in some states and bad outcomes in some other states and from the uncertainty if the realized state and outcome is good or bad. However, these bad states and outcomes are only bad in the relative sense that the realized outcome could have been better. Now, this axiom is assuming that it is always possible to make all the outcomes better and increase the level of the outcomes without changing the ambiguity of the act by using the two equally crisp acts. Since we are making all the outcomes better, this does not change the relative comparisons of the outcomes. So this axiom is stating that the ambiguity of an act only depends on the relative comparisons of outcomes or the dispersion of outcomes. Finally, this axiom does not restrict how the levels of consequences are changed as long as the changes are strict improvements in all the proper states.

The assumption that the two equally crisp acts are statewise ordered is a weak restriction. Nonordered acts have additional possibilities for different ambiguity from the trade-offs between states. So essentially, the only preferences that do not satisfy this axiom are such that every act contains different ambiguity.

An assumption closer to Certainty Independence than Axioms 5 and 6 would be to assume that there exist two crisp acts that are statewise ordered. However, this is a stronger assumption than Axioms 5 and 6 which is formalized in the next remark.
Remark 1 If there exist crisp acts $f, g$ such that for all $s \in S^P$, $f_s \succ_s g_s$, then $\succeq$ satisfies Axioms 5 and 6.

2.4 Existence

The previous six axioms characterize the existence of a state dependent dual-self expected utility representation for preferences.

Theorem 1 (Existence) The following three conditions are equivalent:

1. $\succeq$ satisfies Axioms 1-6.

2. There exists $(u, \mathcal{P})$ that is a state dependent tight dual-self expected utility for $\succeq$ such that

$$\bigcap_{s \in S} u_s\left(\Delta(X_s)\right) \neq \emptyset.$$

3. There exists $(u, \mathcal{P})$ that is a state dependent tight dual-self expected utility for $\succeq$ and $f \in H$ such that for all $p, q \in \bigcup_{P \in \mathcal{P}} P$, $\sum_{s \in S} p_su_s(f_s) = \sum_{s \in S} q_su_s(f_s)$.

This result shows that the Axioms 1-6 characterize the existence of a state dependent dual-self expected utility with the additional property that there exists an act without uncertainty about its expected utility. A representation with this additional property always has an alternative state dependent tight dual-self expected utility with a utility overlap as shown in the second condition. This additional property follows from Axiom 5.

This result shows the generality of the state dependent dual-self expected utility. Assume that the preferences satisfy Axioms 1-4 that are standard assumptions in the ambiguity literature. Then the preferences have a state dependent dual-self expected utility representation if there exist some well-behaving acts such that one of them is unambiguous and two of them share the same ambiguity and are statewise ordered. This statewise ordering is a weak restriction as discussed above. In other words, the only standard ambiguity preferences that do not have a dual-self representation are such that every act is ambiguous or, essentially, every act contains different ambiguity. We will study in Section 3 a generalization that might have uncertainty about every act.

On the other hand, this axiomatization shows how the state dependent dual-self expected utility generalizes the state independent dual-self expected utility by relaxing Monotonicity...
and Certainty Independence. First, Monotonicity and Risk Independence as a part of Certainty Independence is relaxed to weak statewise Monotonicity and weak Risk Independence on lotteries within each state of the world. These generalizations capture the original intuitions of Monotonicity and Risk Independence that preferences are well-behaving on lotteries and there is no taste uncertainty since the statewise comparisons are done on lotteries within each state.

Second, for the rest of Certainty Independence instead of assuming which acts do not have uncertainty, we infer from the decision maker’s behavior which acts are crisp and equally crisp. Our generalization of Certainty Independence is assuming that there exist a crisp act and two statewise ordered equally crisp acts. These existential generalizations can be interpreted as an assumption on the decision maker’s behavior or on the structure of ambiguity under consideration as discussed in the previous section.

The next remark formalizes the connection of this result to the state independent case by considering what additional assumptions would be required for state independent utility.

**Remark 2** Assume a state independent setting where for all $s \in S$, $X_s = X$. Then there exists a (non-trivial) state independent dual-self expected utility for $\succsim$ if and only if the following conditions hold.

1. $\succsim$ satisfies Axioms 1-6.
2. $\succsim$ satisfies monotonicity with between states comparisons: For all $s, s' \in S, f, g \in H, x, y \in \Delta(X)$,
   \[
   (x_s, f_{-s}) \succ (y_s, f_{-s}) \implies (x_{s'}, g_{-s'}) \succsim (y_{s'}, g_{-s'}),
   \]
3. There exists a constant act $z \in \Delta(X)$ such that $z$ is a crisp act.
4. There exist constant acts $x, y \in \Delta(X)$ such that $x \succ y$ and $x$ and $y$ are equally crisp.

This result shows that the state dependent dual-self expected utility relaxes the monotonicity assumption to statewise monotonicity and does not restrict which acts are crisp or equally crisp. An example of a state dependent dual-self expected utility that satisfies the first two conditions but not necessarily the other ones is if all the state dependent utilities are positive affine transformations of each other.
2.5 Identification

This section provides the main result of the paper. We introduce a novel simple axiom stating that the decision maker has full-dimensional uncertainty. This axiom characterizes the full identification of state dependent utilities and probabilities in the state dependent dual-self expected utility. Additionally, we characterize the identification of probabilities for a single state, the identification of a relative likelihood between two states, and the general partial identification of the representation when there does not exist uncertainty about every state of the world.

2.5.1 Full Identification

To make the underlying intuition for our identifying condition clear, we first derive it informally from the idea that the decision maker’s uncertainty about the proper states is full-dimensional. Full-dimensional uncertainty means that there is uncertainty about the likelihood ratio between any proper states.\(^\text{13}\) Then especially there is uncertainty about every trade-off between proper states. Now consider two acts \(f\) and \(g\) that have trade-offs between proper states that is the act \(f\) is better than \(g\) in some states and \(g\) is better than \(f\) in some other states. Then under full-dimensional uncertainty, there is uncertainty about the trade-offs between \(f\) and \(g\) and uncertainty about their relative value. This uncertainty about their relative value can be observed as a difference in the ambiguity hedging of \(f\) and \(g\) when mixing with some act \(h\). This is formalized in the next identification axiom when \(f\) and \(g\) are indifferent to guarantee observability in the difference for ambiguity hedging.\(^\text{14}\)

**Axiom 7** If \(f, g \in H\) are such that \(f \sim g\) and there exist \(s, s' \in S\) such that \(f_s \succ_s g_s\) and \(g_{s'} \succ_{s'} f_{s'}\), then there exist \(h \in H\) and \(\alpha \in (0, 1)\) such that

\[
\alpha h + (1 - \alpha) f \not\sim \alpha h + (1 - \alpha) g.
\]

\(^{13}\)Formally full-dimensional uncertainty states that the set of probabilities has a non-empty interior. So especially there is uncertainty about the likelihood ratio between any states.

\(^{14}\)This axiom is closely related to the notion of unambiguous preferences from Ghirardato et al. (2004). A decision maker is said to unambiguously prefer \(f\) to \(g\) if for all acts \(h\) and \(\alpha \in [0, 1]\), \(\alpha f + (1 - \alpha) h \succeq \alpha g + (1 - \alpha) h\). Axiom 7 states that if there are trade-offs between two acts across states and they are indifferent, then the acts are not unambiguously indifferent.
The main result of this paper is the following uniqueness result for the state dependent dual-self expected utility. The result states that Axiom 7 characterizes the separation and identification of the probabilities and state dependent utilities.

**Theorem 2 (Full Identification)** Let \((u, \mathbb{P})\) be a state dependent tight dual-self expected utility for \(\succeq\). The following four conditions are equivalent:

1. \(\succeq\) satisfies Axiom 7.
2. \(\text{pr}_{S^\mathbb{P}} \bigcap \cup_{P \in \mathbb{P}} P\) has a non-empty interior in \(\Delta(S^\mathbb{P})\).
3. If \((\tilde{u}, \tilde{\mathbb{P}})\) is a state dependent tight dual-self expected utility for \(\succeq\), then there exist \(\alpha \in \mathbb{R}_+\) and \(\beta \in \mathbb{R}\) such that for all \(s \in S^\mathbb{P}\),
   \[
   \tilde{u}_s = \alpha u_s + \beta.
   \]
4. If \((\tilde{u}, \tilde{\mathbb{P}})\) is a state dependent tight dual-self expected utility for \(\succeq\), then
   \[
   \bigcap_{P \in \tilde{\mathbb{P}}} \bar{P} = \bigcap_{P \in \mathbb{P}} P.
   \]

The second condition shows that Axiom 7’s uncertainty about every trade-off between proper states is the behavioral characterization of full-dimensional uncertainty for the proper states.

The third and fourth conditions show that the full-dimensional uncertainty characterizes the identification of the state dependent utilities for the proper states up to a common positive affine transformation and the identification of the convex closure of the probabilities. That is we have recovered the state independent identification. The equivalency between identifying the utilities and the probabilities shows they can be identified separately but identifying one of them means that the other one is also identified. This theorem with the existence theorem, Theorem 1, provides the foundations for state dependent utility under uncertainty. Next, we discuss this result.

First, this identification result has important special cases. This result shows the identification for state dependent maxmin expected utility (Gilboa and Schmeidler, 1989), which will be discussed more in Section 4, for state dependent \(\alpha\)-maxmin expected utility (Ghirardato et al., 2004), and for state dependent Choquet expected utility (Schmeidler, 1989).
Second, the full-dimensionality of the set of probabilities for the proper states is a generic property for a given utility function.\textsuperscript{15} Thus it is not a restrictive condition but rules out important special cases: If $\succeq$ satisfies Independence axiom, then Axiom 7 will not be satisfied. This is the observation from Section 1.1 that state dependent expected utility is not identified. More generally, if there are unambiguous events, that is each of the probabilities agrees on the probability for an event, then Axiom 7 will not be satisfied for these unambiguous events and the utilities and probabilities are not identified across the unambiguous event.

Third, under Axiom 7 the state dependent utilities are identified for all the non-null states and the tight set of probabilities is identified. However, the two-stage structure of probability set collections is not identified.\textsuperscript{16} This is symmetrical to the multiplicity of state independent dual-self expected utility where only the largest representing collection of probability sets is identified (Chandrasekher et al., 2020, Proposition S.1.1). Additionally, the utilities for null states are not identifiable since they have a zero probability and do not affect preferences.

Fourth, Section 1.1 sketched the proof for this result. The intuition underlying that example generalizes to the state dependent dual-self expected utility and to any finite number of states.

\subsection*{2.5.2 Partial Identification}

The previous result showed a characterization for the full identification. However, some states of the world might not be uncertain and in this case, the full identification is not possible. The next two results study the identification of probabilities for a single state and the identification of a relative likelihood between two states.

The next result shows the identification of probabilities for a single state and how full-dimensionality can be extended to a single state by applying Axiom 7 only to a single state. Additionally, it provides a formalization for a notion of full-dimensionality of probabilities for a single state.

\textsuperscript{15}Generic property in the topological sense that the closure of all the probability set collections not satisfying this property has an empty interior in the space of compact sets of $\mathcal{K}(\Delta(S))$ that is they are nowhere dense.

\textsuperscript{16}This can be easily observed by the following: If $P \in \mathbb{P}$, $P' \supseteq P$, and $P'$ is convex and closed, then $\mathbb{P} \cup \{P'\}$ also gives a dual-self representation since $P'$ is dominated and never used.
Proposition 3 (Probability Identification)  Let \((u, \mathbb{P})\) be a state dependent tight dual-self expected utility for \(\succsim\) and \(s \in S^P\). The following three conditions are equivalent:

1. If \(f, g \in H\) are such that \(f \sim g\), \(f_s \succ g_s\), and there exists \(s' \in S\) such that \(g_{s'} \succ s' f_{s'}\), then there exist \(h \in H\) and \(\alpha \in (0, 1)\) such that
   \[
   \alpha h + (1 - \alpha) f \not\sim \alpha h + (1 - \alpha) g.
   \]

2. \(S^P = \{s\}\) or there exist \(p, q \in \overline{\mathbb{P}} \bigcup_{P \in \mathbb{P}} P\) such that \(p_s \neq q_s\) and for all \(\tilde{s} \in S \setminus \{s\}\),
   \[
   \frac{p_{\tilde{s}}}{1 - p_s} = \frac{q_{\tilde{s}}}{1 - q_s}.
   \]

3. If \((\tilde{u}, \overline{\mathbb{P}})\) is a state dependent tight dual-self expected utility for \(\succsim\), then
   \[
   \left\{\tilde{p}_s \mid \tilde{p} \in \overline{\mathbb{P}} \bigcup_{P \in \mathbb{P}} P\right\} = \left\{p_s \mid P \in \overline{\mathbb{P}} \bigcup_{P \in \mathbb{P}} P\right\}.
   \]

full-dimensional uncertainty about a state \(s\) means behaviorally that there is uncertainty about any trade-offs involving state \(s\) as stated by the first condition. Our first result shows that a full-dimensional uncertainty about a state \(s\) is formalized by the second condition: uncertainty about the probability of the state \(s\) while keeping all the probabilities conditional on the event \(S \setminus \{s\}\) as constant. In other words, there exist independent uncertainty about the state \(s\).

Our second result shows that full-dimensional uncertainty about a state characterizes the identification of the probabilities for that state. However, this time the identification of utilities for a single state is not equivalent to identifying the probabilities for that state since utilities are always identified only up to a common positive affine transformation.

This result has an important corollary: if there exists an event \(E\) such that Axiom 7 is satisfied for the event \(E\): For all \(f, g \in H\) and \(s^* \in E\), \(s \in S\) if \(f_{s^*} \succ s^* g_{s^*}\) and \(g_s \succ s f_s\), then there exist \(h \in H\) and \(\alpha \in (0, 1)\) such that
   \[
   \alpha h + (1 - \alpha) f \not\sim \alpha h + (1 - \alpha) g.
   \]

Then this result shows that the tight probabilities for the states in \(E\) are identified and the utilities in \(E\) are identified up to a common positive affine transformation.\(^{17}\)

Next, we move on to the identification of relative likelihood of two states. When considering a gamble between two states the identification of the probabilities is not important since

\(^{17}\)The identification of utilities in \(E\) is not a direct corollary but follows symmetrically.
The gamble depends only on the relative likelihood of the states. The next result provides a behavioral condition for full-dimensional uncertainty about a relative likelihood and shows how that characterizes the identification of utilities and the relative likelihoods between two states. Additionally, it provides a formalization for a notion of full-dimensionality of relative likelihood for two states.

**Proposition 4 (Relative Likelihood Identification)** Let \((u, \mathbb{P})\) be a state dependent tight dual-self expected utility for \(\succeq\) and \(s, s' \in S^\mathbb{P}, s \neq s'\). The following four conditions are equivalent:

1. If \(f, g \in H\) are such that and \(f_s \succ g_s\) and \(g_{s'} \succ f_{s'}\), then there exist \(h, h' \in H\) and \(\alpha \in (0, 1)\) such that
   \[
   \alpha h + (1 - \alpha) f \succ \alpha h' + (1 - \alpha) f \quad \text{and} \quad \alpha h + (1 - \alpha) g \prec \alpha h' + (1 - \alpha) g.
   \]

2. There exist \(p, q \in \overline{\bigcup_{P \in \mathbb{P}} P}\) such that \(\frac{p_s}{p_{s'}} \neq \frac{q_s}{q_{s'}}\) and for all \(\tilde{s} \in S \setminus \{s, s'\}\), \(p_{\tilde{s}} = q_{\tilde{s}}\).

3. If \((\tilde{u}, \tilde{\mathbb{P}})\) is a state dependent tight dual-self expected utility for \(\succeq\), then there exist \(\alpha \in \mathbb{R}_+\) and \(\beta \in \mathbb{R}\) such that
   \[
   \tilde{u}_s = \alpha u_s + \beta \quad \text{and} \quad \tilde{u}_{s'} = \alpha u_{s'} + \beta.
   \]

4. If \((\tilde{u}, \tilde{\mathbb{P}})\) is a state dependent tight dual-self expected utility for \(\succeq\), then
   \[
   \left\{ \frac{P_s}{P_{s'}} \mid \tilde{P} \in \overline{\bigcup_{P \in \mathbb{P}} P} \right\} = \left\{ \frac{P_s}{P_{s'}} \mid \tilde{p} \in \overline{\bigcup_{P \in \mathbb{P}} P} \right\}.
   \]

The first condition is a generalization of Axiom 7 when applied to only two states. In Theorem 2, Axiom 7 is equivalent to the first condition when applied for any states \(s, s'\) but there we could use the simpler formulation to capture all relevant trade-offs between states. However, when considering only two states \(s\) and \(s'\), this is not possible anymore since there can be trade-offs where the two acts are not indifferent. Thus the interpretation for this condition is the same as before that there is uncertainty about any trade-offs between \(s\) and \(s'\). However, this time \(f\) and \(g\) are not indifferent so we need the stronger condition with two convex combinations to observe the uncertainty about the relative value of \(f\) and \(g\) as a difference in hedging.

Our first result provides a formalization for a notion of full-dimensional uncertainty about the relative likelihood of two states by the second condition: Uncertainty about the relative
likelihood while keeping the probabilities for all the other states constant. That is there exist
independent uncertainty about the relative likelihood of these states.

The second result shows that this full-dimensional uncertainty about states $s$ and $s'$
characterizes the identification of utilities between these states in the sense that they are
identified up to a common positive affine transformation and the identification of the rel-
ative likelihoods between these states. As in Theorem 2 the identification of utilities and
probabilities is equivalent.

Finally, we move on to the general partial identification of the state dependent dual-self
expected utility. The next result shows that the value of the representation is unique up to
a positive affine transformation, but there can be additional transformations for the utilities
and probabilities as long as they do not affect the value of the representation: First, for the
probabilities all statewise multiplicative transformations for which the probabilities remain
as probabilities are possible when we do the statewise reciprocal transformations for the
utilities. Second, all statewise additive transformations for the utilities that do not affect
the expected utility for any probabilities are possible. Finally, all common positive affine
transformations are possible for the utilities.

Before stating the next result, we introduce some notation. For all probabilities $p \in \Delta(S)$
and numbers for each state $x \in \mathbb{R}^S_+$, we define multiplications statewise, $xp = (x_sp_s)_{s \in S}$.
This induces multiplications for sets of probabilities and collections of probability sets with
$x \in \mathbb{R}^S_+$. Especially for $P \in \mathcal{K}(\Delta(S))$, $xP = \{(xp|p \in P)|P \in P \}$.  

**Theorem 5 (Partial Identification)** Let $(u, P)$ be a state dependent tight dual-self
expected utility for $\succsim$, and $\bar{u} = (\bar{u}_s)_{s \in S}$ be such that for all $s \in S$, $\bar{u}_s : \Delta(X_s) \to \mathbb{R}$ is affine
and $\tilde{P} \subseteq \mathcal{K}(\Delta(S))$ be compact and non-empty. Then $(\bar{u}, \tilde{P})$ is a state dependent dual-self
expected utility for $\succsim$ if and only if there exist $x \in \mathbb{R}^S_+, y \in \mathbb{R}^S$ such that for all $p, q \in \bigcup_{P \in \tilde{P}} P$,

$$\sum_{s \in S} x_sp_s = \sum_{s \in S} x_sq_s =: \alpha \text{ and } \sum_{s \in S} y_sp_s = \sum_{s \in S} y_sq_s,$$

for all $s \in S^P$,

$$\bar{u}_s = \frac{1}{x_s} (u_s + y_s),$$

and for all $f \in H$,

$$\max_{P \in \tilde{P}} \min_{\tilde{p} \in P} \sum_{s \in S} \tilde{p}_s \bar{u}_s(f_s) = \max_{P \in \mathbb{P}} \min_{\tilde{p} \in P} \sum_{s \in S} p_s \bar{u}_s(f_s).$$
Especially, \(( \tilde{u}, x \tilde{P} )\) is a state dependent tight dual-self expected utility for \(\succsim\) and if \(( \tilde{u}, \tilde{P} )\) is a tight representation, then

\[
\mathcal{V} \bigcup_{\tilde{P} \in \tilde{P}} \tilde{P} = \mathcal{V} \bigcup_{\tilde{P} \in \tilde{P}} \frac{x}{\alpha} \tilde{P}.
\]

This theorem characterizes the general partial identification. For any other state dependent dual-self representation the values of the representation are identified up to a positive affine transformation and the utilities and the tight set of probabilities are partially identified. In the above formulation, \(\frac{1}{\alpha}\) is the common positive multiplicative transformations for utilities and \(x\) is the multiplicative transformation for the probabilities that keeps them as probabilities. These two multiplicative transformations are combined into a single term \(x\). Similarly, the term \(y\) combines all the additive transformations for the utilities.\(^{18}\)

This theorem shows how the size of the set of probabilities restricts the possible transformations. If there are more probabilities, then there are fewer transformations \(x \in \mathbb{R}^S\) such that for all probabilities \(p\) and \(q\), \(\sum_{s \in S} x_s p_s = \sum_{s \in S} x_s q_s\). Especially these transformations are restricted by how many linearly independent probabilities there are in \(\bigcup_{\tilde{P} \in \tilde{P}} P\).\(^{19}\)

The rest of this section uses the definitions from Section 2.3 to provide an insight into the identification through the connection to the underlying preferences. A reader interested only in the identification can skip the rest of this section.

The next result shows the connection of possible transformations to crisp acts.

**Corollary 6** Let \((u, P)\) be a state dependent dual-self expected utility for \(\succsim\) and \(x \in \mathbb{R}^S_{++}, y \in \mathbb{R}^S\). The following two conditions are equivalent:

1. \((\tilde{u}, \tilde{P})\) is a state dependent dual-self expected utility for \(\succsim\) such that
   \[
   \tilde{u}_s = \frac{1}{x_s} (u_s + y_s) \quad \text{for all } s \in S^P \text{ and } 0, 1 \in \bigcap_{s \in S^P} \tilde{u}_s \left( \Delta(X_s) \right).
   \]

2. There exist crisp acts \(f^*, g^*\) such that \(u_s(g^*_s) = -y_s\) and \(u_s(f^*_s) = -y_s + x_s\).

\(^{18}\)Denoting \(p \in \bigcup_{\tilde{P} \in \tilde{P}} P\), the additive transformations that do not affect the expected utility are \(y - \sum_{s \in S} y_s p_s\), where the subtraction is statewise, and the common additive transformations are \(\sum_{s \in S} y_s p_s\).

\(^{19}\)This follows from the following observation. If \(x \in \mathbb{R}^S\) and \(\alpha \in \mathbb{R}\) are such that for all probabilities \(p\), \(\sum_{s \in S} x_s p_s = \alpha\), then we can write \(x = \alpha + (x - \alpha)\) where for all probabilities \(p\), \(\sum_{s \in S} (x_s - \alpha) p_s = 0\). Thus we can decompose any transformation to a sum of a constant and a vector orthogonal (perpendicular) to the set of probabilities. Since \(\mathbb{R}^S\) can be decomposed to the sum of the linear span of \(\bigcup_{\tilde{P} \in \tilde{P}} P\) and its orthogonal complement, this shows the close connection between the size of the set of probabilities and the set of possible transformations.
This corollary with Theorem 5 shows the idea of the partial identification. The multiplicity of representations comes from the freedom of choosing any two crisp acts that are statewise ordered to provide the constant utility of zero and one. However, Theorem 5 extends the partial identification to situations without utility overlap or crisp acts.

This corollary provides an insight into our main identification result Theorem 2. The full identification is only possible if all crisp acts have a constant utility. This is what Axiom 7 guarantees by assuming that for all pairs of crisp acts, one of the crisp acts (weakly) dominates the other one statewise.

3 State Dependent Dual-Self Variational Expected Utility

This section studies a more general setup with a state dependent version of dual-self variational expected utility (Chandrasekher et al., 2020). We characterize its existence and identification. The state-independent version is a general model that includes as special cases monotone mean-variance preferences (Maccheroni et al., 2009), multiplier preferences (Hansen and Sargent, 2001; Strzalecki, 2011), variational preferences (Maccheroni et al., 2006), monotonic mean-dispersion preferences (Grant and Polak, 2013), and vector expected utility (Siniscalchi, 2009). None of these representations is a special case of the previous dual-self expected utility. Our identification results in this section encompass those for state dependent versions of all the above special cases.

Dual-self variational expected utility generalizes variational expected utility. In a variational expected utility, the decision maker has an index of ambiguity attitude $c: \Delta(S) \to \mathbb{R} \cup \{\infty\}$ for each probability and the preferences are represented with utility $u$ as

$$f \succsim g \iff \min_{p \in \Delta(S)} \sum_{s \in S} p_s u(f_s) + c(p) \geq \min_{p \in \Delta(S)} \sum_{s \in S} p_s u(g_s) + c(p).$$

This corresponds to the maxmin expected utility if $c$ is always 0 or $\infty$. Here the effective domain of $c$ denoted by $\text{dom } c = \{p \in \Delta(S) | c(p) \in \mathbb{R}\}$ captures the set of subjective probabilities that the decision maker uses.

The dual-self variational expected utility generalizes this representation by allowing for multiple cost functions that are aggregated optimistically. Axiomatically, this generalizes
variational preferences by relaxing Uncertainty Aversion. We will use a state dependent variation of it.

**Definition** Let \( \succsim \subseteq H \times H \). \((u, C)\) is a state dependent dual-self variational expected utility for \( \succsim \) if \( u = (u_s)_{s \in S} \) and for all \( s \in S \), \( u_s : \Delta(X_s) \to \mathbb{R} \) is affine, \( C \subseteq \{ c : \Delta(S) \to \mathbb{R} \cup \{\infty\} \mid c \text{ is convex} \} \) is such that \( \max_{c \in C} \min_{p \in \Delta(S)} c(p) = 0 \) and for all \( c \in C \) and \( p \in \text{dom} c, s \notin S^p \), \( p_s = 0 \), and for all \( f, g \in H \),

\[
f \succsim g \iff \max_{c \in C} \min_{p \in \Delta(S)} \sum_{s \in S} p_s u_s(f_s) + c(p) \geq \max_{c \in C} \min_{p \in \Delta(S)} \sum_{s \in S} p_s u_s(g_s) + c(p).
\]

This representation is the variational expected utility when \( C \) is a singleton. Additionally, since this representation is a non-convex generalization of variational preferences, especially any uncertainty aversion and seeking of the preferences can be represented by the interplay of a cost function (min) and the set of cost functions (max).

Similarly to the dual-self expected utility, the dual-self variational expected utility is not unique. However, the smallest convex and closed effective domain of the cost functions is unique which gives us tight dual-self variational expected utility.

**Definition** Let \( \succsim \subseteq H \times H \). \((u, C)\) is a state dependent tight dual-self variational expected utility for \( \succsim \), if \((u, C)\) is a state dependent dual-self variational expected utility for \( \succsim \) and if \((\tilde{u}, \tilde{C})\) is another state dependent dual-self variational expected utility for \( \succsim \), then \( \overline{\mathbb{W} \cup_{c \in C} \text{dom} \tilde{c}} \supseteq \overline{\mathbb{W} \cup_{c \in C} \text{dom} c} \).

### 3.1 Existence, Variational

We start off with characterizing the existence of the state dependent dual-self variational expected utility. This characterization highlights the generality of the state dependent dual-self variational expected utility by showing that the only standard ambiguity preferences that do not have this representation are such that, essentially, every act contains different ambiguity. This section uses the axioms from Section 2.3. A reader only interested in the identification can skip this section.

The next result shows that the existence of state dependent dual-self variational expected utility is characterized by the same axioms as state dependent dual-self expected utility with the exception that there might not be crisp acts and Axiom 5 might not be satisfied.
Theorem 7 (Existence, Variational) The following two conditions are equivalent:

(1) $\succeq$ satisfies Axioms 1-4, and 6.

(2) There exists $(u, \mathcal{C})$ that is a state dependent tight dual-self variational expected utility for $\succeq$.

This result shows that a state dependent dual-self variational expected utility is characterized by relatively weak axioms capturing the dispersive nature of ambiguity without imposing any additional restrictions on it. This representation has multiple special cases as discussed above and this theorem shows the state dependent foundations for these models.

This result highlights the generality of the dual-self variational expected utility. Under the standard assumptions under ambiguity, Axioms 1-4, the only requirement for the state dependent dual-self variational expected utility is the existence of two acts that share the same ambiguity and that they are ordered statewise. As discussed in Section 2.3, the statewise ordering is a weak restriction. In other words, essentially the only standard preferences that do not have a state dependent dual-self variational representation are such that all the acts contain different ambiguity.

3.2 Identification, Variational

Next, we move on to the identification of the state dependent dual-self variational expected utility. The following identification result shows the main result of this section: Under full-dimensional uncertainty, the intensities of preferences and the probabilities can be separated.

Theorem 8 (Full Identification, Variational) Let $(u, \mathcal{C})$ be a state dependent tight dual-self variational expected utility for $\succeq$. The following four conditions are equivalent:

(1) $\succeq$ satisfies Axiom 7.

(2) $\Pr_{s^p} \mathcal{S} \cup_{c \in \mathcal{C}} \text{dom } c$ has a non-empty interior in $\Delta(S^p)$.

(3) If $(\tilde{u}, \tilde{\mathcal{C}})$ is a state dependent tight dual-self variational expected utility for $\succeq$, then there exist $\alpha \in \mathbb{R}_+$ and $B \in \mathbb{R}^S$ such that for all $s \in S^p$,

$$\tilde{u}_s = \alpha u_s + B_s.$$
(4) If \((\bar{u}, \bar{C})\) is a state dependent tight dual-self variational expected utility for \(\succeq\), then

\[
\bigcup_{\bar{c} \in \bar{C}} \text{dom } \bar{c} = \bigcup_{c \in C} \text{dom } c.
\]

This result states that under Axiom 7 the changes of state dependent utilities are identified. However, the levels are not identified and instead any additive transformations are possible. This follows from the interchangeability of cost functions and additive constants as shown in the next result. Additionally, the convex closure of the tight set of probabilities is uniquely identified as with the state dependent dual-self expected utility. The symmetric result for the state dependent dual-self expected utility Theorem 2 was discussed extensively in Section 2.5 and the same discussion applies here. Especially the non-empty interior condition is a weak restriction for state dependent dual-self variational expected utility and Axiom 7 is not demanding but rules out important special cases.

This theorem shows that under the more general state dependent tight dual-self variational expected utility and Axiom 7, the utility differences between acts are still meaningful. Especially, it is possible to study the utility gains and losses from trading one act for another one. Additionally, states of the world have meaningful probabilities which is crucial for modeling decision making under uncertainty.

Next, we move on to the general partial identification. Before stating the partial identification result, we introduce some notation. For all cost functions \(c : \Delta(S) \to \mathbb{R} \cup \{\infty\}\) and numbers for each state \(x \in \mathbb{R}^+_S\), we define the product \(c \circ x : \Delta(S) \to \mathbb{R} \cup \{\infty\}\) as for all \(p \in \Delta(S)\), \(c \circ x(p) = c(xp)\) where \(c(xp) = \infty\) if \(xp \notin \Delta(S)\) and the product \(xp\) is the elementwise product as in Section 2.5. This induces products of cost sets with \(x \in \mathbb{R}^+_S\) as elementwise products. Second, for \(B \in \mathbb{R}^S\) define \(dB\), the linear cost function (or a measure) associated with \(B\), by for all \(p \in \Delta(S)\), \(dB(p) = \sum_{s \in S} B_sp_s\).

**Theorem 9 (Partial Identification, Variational)** Let \((u, C)\) be a state dependent tight dual-self variational expected utility for \(\succeq\), and \(\bar{u} = (\bar{u}_s)_{s \in S}\) be such that for all \(s \in S\), \(\bar{u}_s : \Delta(X_s) \to \mathbb{R}\) is affine and \(\bar{C} \subseteq \{c : \Delta(S) \to \mathbb{R} \cup \{\infty\} | c \text{ is convex}\}\) be such that \(\max_{c \in \bar{C}} \min_{p \in \Delta(S)} \bar{c}(p) = 0\). Then \((\bar{u}, \bar{C})\) is a state dependent dual-self variational expected utility for \(\succeq\) if and only if there exist \(x \in \mathbb{R}^+_S\), \(B \in \mathbb{R}^S\) such that for all \(p, q \in \cup_{c \in C} \text{dom } c\),

\[
\sum_{s \in S} x_sp_s = \sum_{s \in S} x_sq_s =: \alpha,
\]

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for all $s \in S^p$,
\[
\tilde{u}_s = \frac{1}{x_s} u_s + B_s,
\]
and by denoting $\beta = -\max_{c \in \tilde{C}} \min_{p \in \Delta(S)} \alpha c(x/\alpha p) - \sum_{s \in S} p_s B_s$ for all $f \in H$,
\[
\max_{c \in C} \min_{p \in \Delta(S)} \sum_{s \in S} p_s \tilde{u}_s(f_s) + \tilde{c}(p) = \max_{c \in \tilde{C}} \min_{c \in \tilde{C}} \sum_{s \in S} p_s \tilde{u}_s(f_s) + \alpha c(x/\alpha p) - \sum_{s \in S} B_s p_s + \beta.
\]
Especially $(\tilde{u}, \alpha(C \circ x/\alpha - dB + \beta))$, where multiplication and subtractions are done elementwise for each cost function, is a state dependent tight dual-self variational expected utility for $\succsim$ and if $(\tilde{u}, \tilde{C})$ is a tight representation, then
\[
\bigcup_{\tilde{c} \in \tilde{C}} \text{dom} \tilde{c} = \bigcup_{c \in C} \frac{x}{\alpha} \text{dom} c.
\]

This theorem shows that for any other state dependent dual-self variational representation the values of the representation are identified up to a positive affine transformation. All the additional transformations are the ones that do not affect the representation: For the tight set of probabilities, all common statewise multiplicative transformations for which the probabilities still sum to a constant number are allowed when we do the statewise reciprocal transformation for the utilities and scale the probabilities to sum to 1. Additionally, additive transformations for the utilities are allowed when we do the negative of this transformation for all the cost functions and normalize the set of cost functions with $\beta$.

Symmetrical results to Propositions 3 and 4 for the identification of probabilities for a single state or for the identification of relative likelihoods between two states hold under the state dependent dual-self variational expected utility. The only difference is that the additive utility transformations are not identified anymore.

Finally, we connect the state dependent dual-self and dual-self variational expected utilities to each other by linear cost functions.

**Proposition 10 (Connecting Dual-Self and Dual-Self Variational)** The following two conditions are equivalent:

1. There exists $(u, C)$ that is a state dependent dual-self variational expected utility for $\succsim$ and there exists $B \in \mathbb{R}^S$ such that for all $c \in C, p \in \text{dom}(c)$, $c(p) = \sum_{s \in S} B_s p_s$.
2. There exists $(\tilde{u}, \tilde{P})$ that is a state dependent dual self expected utility for $\succsim$. 

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This proposition shows that state dependent dual-self variational expected utility with a linear cost function corresponds to state dependent dual-self expected utility. This gives a new insight into the state dependent dual-self expected utility and into unambiguous acts since these linear cost functions are underlying the intuition for unambiguous acts.\textsuperscript{20} It is important to contrast this result to the state independent case where only state independent dual-self variational expected utility with a cost function that is constant at 0 correspond to state independent dual-self expected utility.

The rest of this section uses the definitions from Section 2.3 to provide an insight into the identification through the connection to the underlying preferences. A reader interested only in the identification can skip the rest of this section.

The next result connects the partial identification from Theorem 9 to equally crisp acts.

**Corollary 11** Let \((u, C)\) be a state dependent dual-self variational expected utility for \(\succeq\) and \(x \in \mathbb{R}^S_{++}, B \in \mathbb{R}^S\). The following two conditions are equivalent:

1. \((\tilde{u}, \tilde{C})\) is a state dependent dual-self variational expected utility for \(\succeq\) such that
   \[
   \tilde{u}_s = \frac{1}{x_s}(u_s + B_s) \text{ for all } s \in S^P \text{ and } 0, 1 \in \bigcap_{s \in S^P} \tilde{u}_s(\Delta(X_s)).
   \]
2. There exist equally crisp acts \(f^*, g^*\) such that \(u_s(g^*_s) = -B_s\) and \(u_s(f^*_s) = -B_s + x_s\).

This corollary with Theorem 9 shows the idea of the partial identification. The multiplicity of representations comes from the freedom of choosing any act \(g^*\) to provide the constant utility of zero and choosing \(f^*\) such that \(g^*\) and \(f^*\) are equally crisp and statewise ordered to provide the constant utility of one.

This highlights the difference to the identification of state dependent dual-self expected utility. Corollary 6 showed that the partial identification of dual-self expected utility comes from crisp acts. However, if an act is equally crisp with a crisp act, then both of the acts are crisp acts. So the only difference between these two identifications is that with dual-self expected utility one of the acts has to be crisp, whereas with dual-self variational expected utility there might not exist crisp acts and we are free to choose one of the acts.

\textsuperscript{20}This highlights how mixing with unambiguous or crisp acts changes ambiguity linearly that is the underlying property in the definition of crisp acts in Section 2.3.
Symmetrically to the dual-self expected utility, this corollary provides an insight into the full identification of dual-self variational expected utility, Theorem 8. This identification is only possible if the utility differences between all equally crisp acts are constant. This is what Axiom 7 guarantees by assuming that in every pair of equally crisp acts, one of the acts (weakly) dominates the other one statewise.

4 Application to Social Choice Theory

This section presents an application to social choice theory for state dependent utility models beyond uncertainty by changing the interpretations of the primitives of the models. In this application, we study the preferences of the society over distributions of goods to members of the society. We show the identification of fairness of the society and interpersonal utility comparisons using state dependent maxmin expected utility.

In this context, we will interpret states of the world to be the members of the society, acts to be distributions of goods to each member, and \( \succsim \) to represent the preferences of the society. Here, ambiguity reflects the fairness of the distribution. We focus on a state dependent maxmin expected utility \((u, P)\) with a symmetric set of probabilities that is for all permutations of states\(^{21}\) \(\pi\), if \((p_s)_{s \in S} \in P\), then \((p_{\pi(s)})_{s \in S} \in P\). This symmetry property captures anonymity of the societal preferences.

In the current context, this symmetric state dependent maxmin expected utility is a generalized Rawlsian social welfare function. Here, the set of probabilities captures the fairness of the society. The least unfairness-averse generalized Rawlsian function is the utilitarian social welfare function which is the average utility. This corresponds to a state dependent expected utility with a uniform probability. The most unfairness-averse generalized Rawlsian function is the Rawlsian social welfare function which is the minimum utility. This corresponds to state dependent maxmin expected utility with a full set of probabilities \(\Delta(S)\).

The main result of this section characterizes the existence of a generalized Rawlsian social welfare function and its identification. Especially we show that the utilitarian social welfare function is an unidentified knife-edge case of a generalized Rawlsian social welfare function:

\(^{21}\pi : S \rightarrow S\) is a permutation if it is a one-to-one (i.e. bijective) function.
All representations except this knife-edge case are fully identified. First, we interpret shortly the axioms from Section 2.3 in the current context and introduce a new anonymity axiom.

The society has continuous preferences (Axioms 1 and 2) and is benevolent (Axiom 3) caring about the well-being of each person that is the society’s preferences respect individual Pareto improvements. Additionally, the society consists of rational expected utility maximizers (Axiom 4) and cares about the fairness of the distribution (Axiom 8).\textsuperscript{22}

Axiom 5 assumes that there exists a fair distribution that does not hedge the fairness of any other distribution. Axiom 6 assumes that the unfairness is over the variation of well-being across the society. That is fairness depends only on what each person receives relative to others.

The next axiom is Uncertainty Aversion from Gilboa and Schmeidler (1989). This captures that the society is inequality averse.

**Axiom 8** For all $f, g \in H$, $\alpha \in [0, 1]$, if $f \succsim g$, then $\alpha f + (1 - \alpha) g \succsim g$.

Finally, we have a new anonymity axiom. This axiom states that there exist a better and a worse fair distribution such that the society is indifferent on the identity of the members who receive the better or the worse fair distribution.

**Axiom 9** There exist crisp acts $f^*, g^*$ such that $f^* \succ g^*$ and for all $\gamma \in \Delta(S)$ and permutations $\pi : S \to S$,

$$
(\gamma_s f^*_s + (1 - \gamma_s) g^*_s)_{s \in S} \sim (\gamma_{\pi(s)} f^*_s + (1 - \gamma_{\pi(s)}) g^*_s)_{s \in S}.
$$

In this axiom, $\gamma$ is a lottery across the members of the society and one of the members is awarded the better fair consequence $f^*_s$ and everybody else receives the worse fair consequence $g^*_s$. That is $\gamma$ captures the distribution of the better and worse consequences in the society. Then the identity of the winning member does not matter for societal preferences but only the distribution of the consequences matters.

These axioms characterize the generalized Rawlsian social welfare function. That is the symmetric state dependent maxmin expected utility. As a notation, a state depen-\textsuperscript{22}This assumption can be also interpreted directly as an aversion to ambiguity under a Rawlsian interpretation: The preferences of the society are determined by a person behind the veil of ignorance. However, the person deciding the preferences is uncertain about their identity in the society. This uncertainty about the identity gives a rational for the uncertainty aversion.
dent maxmin expected utility is a state dependent dual-self expected utility with a single probability set \((u, \{P\})\) and we omit the inner brackets in the following.

**Theorem 12 (Symmetric State Dependent MaxMin)** \(\succeq_s\) satisfies Axioms 1-6, 8, and 9 if and only if (1) or (2) of the following conditions holds:

1. For all \(p \in \text{int } \Delta(S)\), there exists \(u = (u_s)_{s \in S}\) such that for all \(s \in S\), \(u_s : \Delta(X_s) \to \mathbb{R}\) is affine and \(\text{int } u_s(\Delta(X_s)) \neq \emptyset\) and for all \(f, g \in H\),
   \[
   f \succeq g \iff \sum_{s \in S} p_s u_s(f_s) \geq \sum_{s \in S} p_s u_s(g_s).
   \]

2. There exists \((u, P)\) that is a state dependent maxmin expected utility for \(\succeq_s\) such that \(\text{int } P \neq \emptyset\), for all \(p \in P\), permutations \(\pi : S \to S\), \((p_{\pi(s)})_{s \in S} \in P\), and
   \[
   \text{int } \bigcap_{s \in S} u_s(\Delta(X_s)) \neq \emptyset.
   \]

Additionally, if (2) holds and \((\tilde{u}, \tilde{P})\) is another state dependent maxmin expected utility for \(\succeq_s\), then there exist \(\alpha > 0, \beta \in \mathbb{R}\) such that
   \[
   \tilde{P} = P \quad \text{and} \quad \tilde{u}_s = \alpha u_s + \beta \quad \text{for all } s \in S.
   \]

First, this result highlights that utilitarianism is a knife-edge case of generalized Rawlsian social welfare functions where identification is not possible. There are only two possibilities: Either the preferences have a fully non-identified utilitarian representation or a fully identified generalized Rawlsian social welfare function with a non-singleton set of probabilities. In the ambiguity context, this result also highlights how state dependent expected utility is a non-identified knife-edge case of state dependent maxmin expected utility.

Second, this result provides the identification for the fairness of the society and interpersonal utility comparisons when the societal preferences are not utilitarian. The identification of fairness allows us to compare the fairness of different societies to each other independently of the members that form the society.\(^{23}\) Additionally, the identification of interpersonal utilities allows us to do comparisons within the society and compare the welfare of its members. As an important special case, this result shows that the Rawlsian social welfare function is well identified if there is utility overlap.

\(^{23}\)Under the Rawlsian interpretation, the weights that society gives to its members are the probability assessments of the person behind the veil of ignorance for their identity.
Third, Theorem 2 generalizes this identification to non-symmetric generalized Rawlsian social welfare functions. In this context, the full-dimensionality axiom, Axiom 7, assumes that any redistribution that involves redistributing from one person to another makes the distribution of well-being less fair in some situations. Essentially this assumes that starting from a fair (crisp) distribution, redistribution decreases the fairness of the distribution. Under this full-dimensionality condition, we achieve the same identification of the fairness of the society and interpersonal utilities as above in the more general case without anonymity of the social welfare function. Especially, we can identify if the social welfare function is anonymous based on Axiom 9.

**Remark** Applying state dependent dual-self variational expected utility in this context would give the identification of fairness of the society independently of the members and the identification of interpersonal utility gain comparisons. This is a useful identification for example in the context of redistribution. In studying redistribution, the focus is on people who gain and who lose from the redistribution but not necessarily on the absolute levels of utilities that are not identified here.

5 Discussion on Identification

The example in Section 1.1 showed how the identification in our models comes from attributing the changes in uncertainty attitude to changes in the probabilities. Next, we formalize this underlying assumption and consider state dependent Bewley preferences that highlight the differences between state dependent and state independent identifications.

First, we consider the state dependent dual-self expected utility. Here, the identification comes from the assumption that unambiguous acts do not have uncertainty in the representation. One of the characterizing axioms for this representation, Axiom 5, was that there exists an act that does not hedge ambiguity. That is, it is behaviorally revealed to be unambiguous as a crisp act. In the representation, these behaviorally unambiguous acts do not have uncertainty and they are the only acts that can have a constant utility.\(^{24}\) Especially, under

\(^{24}\)Formally, this is shown in Corollary 6.
full-dimensional uncertainty, every behaviorally unambiguous act has a constant utility. Our identification results follow from this observation.

The underlying assumption here is that the decision maker considers some acts as unambiguous: The condition that an act is behaviorally revealed to be unambiguous as a crisp act is only a necessary condition for the decision maker to consider it as unambiguous. As illustrated in Proposition 10, there exist alternative dual-self variational representations for these preferences with linear costs functions where these behaviorally unambiguous acts are not unambiguous in the representation.

Next, we move on to the state dependent dual-self variational expected utility that has a similar underlying assumption. Here, the identification uses a pair of acts that hedge ambiguity always similarly. That is they are behaviorally revealed to share the same ambiguity as equally crisp acts. In the representation, there is no uncertainty about the utility difference between these equally crisp acts and especially under full-dimensionality, these acts have a constant utility difference. In this case, the underlying assumption is that the decision maker considers some acts as sharing the same ambiguity. Without this underlying assumption, these preferences have more general representations where the acts that behaviorally share the same ambiguity do not share the same ambiguity in the representation. Especially, the probabilities and the intensities of utilities would not be separated and identified.

These underlying assumptions have a counterpart in the state independent representations. In the state independent case, the underlying assumption is that constant acts, giving the same consequence in every state, are unambiguous for every decision maker (Ghirardato and Marinacci, 2002). This gives the identification since the ambiguity of other acts can be measured relative to these known unambiguous constant acts. Our underlying assumptions relax this.

Finally, consider the possibility that the decision maker has also uncertainty about tastes. Then the above example’s change in uncertainty attitude could also be attributed to changes in tastes weakening the identification. In this case, acts that do not hedge ambiguity would not necessarily be unambiguous since the lack of hedging could also reflect certainty about

\[25\text{The existence of this pair of equally crisp is one of the characterizing properties for this representation and formalized in Axiom 6.}\]

\[26\text{Formally, this is shown in Corollary 11.}\]
tastes or utility for these acts. Karni (2020) has recently illustrated this lack of identification with rank dependent probabilities when state dependent utilities are also rank dependent.

5.1 State Dependent Bewley Expected Utility

Next, we move on to state dependent Bewley expected utility that highlights the difference between state independent and state dependent identifications. With state independent utility, the decision maker’s uncertainty about states can be inferred from the Independence satisfying core of the preferences. These are incomplete preferences that have a multi-prior Bewley representation (Bewley, 2002) as defined below. However, this identification of uncertainty does not extend to state dependent setting as we show next with a state dependent Bewley representation.

**Definition** \((u, P)\) is a state dependent Bewley expected utility for \(\succsim\) if \(u = (u_s)_{s \in S}\) and for all \(s \in S, u_s : \Delta(X_s) \rightarrow \mathbb{R}\) is affine and \(P \subseteq \Delta(S)\) is closed, convex, and non-empty such that for each \(p \in P, s \notin S^P, p_s = 0\) and for all \(f, g \in H\),

\[
f \succsim g \iff \sum_{s \in S} p_s u_s(f_s) \geq \sum_{s \in S} p_s u_s(g_s) \text{ for all } p \in P.
\]

In this representation, the set of probabilities \(P\) captures the decision maker’s uncertainty. However, here, the decision maker does not aggregate the uncertainty for complete preferences. For example, in the state independent case, maxmin expected utility corresponds to cautious aggregation of the uncertainty using the minimum probability (Gilboa et al., 2010).

The next result shows the lack of identification for the state dependent Bewley representation: the set of probabilities and the state dependent utilities cannot be separated and identified. This is in contrast to the state independent Bewley representation where the set of probabilities is uniquely identified.

**Theorem 13 (Identification, Bewley)** Let \((u, P)\) be a state dependent Bewley representation for \(\succsim\), \(\tilde{u} = (\tilde{u}_s)_{s \in S}\) be such that for all \(s \in S, \tilde{u}_s : \Delta(X_s) \rightarrow \mathbb{R}\) is affine, and \(\tilde{P} \subseteq \Delta(S)\)

\[
27\text{Formally, this is the unambiguous preference } \succsim^* \text{ that is defined as for all acts } f, g \in H, \text{ and for all } h \in H, \alpha \in (0, 1].
\]

Ghirardato et al. (2004) show that \(\succsim^*\) is the largest Independence satisfying restriction of \(\succsim\) and captures the beliefs and tastes of the decision maker.
be closed, convex and nonempty. Then \((\tilde{u}, \tilde{P})\) is a state dependent Bewley representation for \(\succeq\) if and only if there exist \(a \in \mathbb{R}^S_{++}, b \in \mathbb{R}^S\) such that
\[
\tilde{P} = \left\{ \left( \frac{a^{-1}s p}{\sum_{s' \in S} a^{-1}s' p_{s'}} \right)_{s \in S} \mid p \in P \right\}
\]
and for all \(s \in S^P, \tilde{u}_s = a_s u_s + b_s.

This result highlights the difference in the identifications between state dependent and state independent representations. With state dependent utilities, the largest Independence satisfying core of the preferences does not allow us to separate the probabilities and the utilities. Instead, the intensities of preferences and the probabilities are interchangeable. This follows from the observation that the incomplete preferences do not reveal which acts are unambiguous. Instead, any act could be unambiguous.

In contrast, in the state independent case, the Independence satisfying core allows us to separate beliefs and tastes. This follows from the underlying assumption that constant acts are unambiguous. This highlights how the state dependent identification is more general since we do not make any assumptions on which acts are unambiguous. Instead, we infer it from the decision maker’s behavior using the violations of the Independence axiom. These violations are not revealed in the Bewley preferences that satisfy Independence axiom and linearity.

In Appendix Section F, we also axiomatically characterize the existence of state dependent Bewley expected utility. Especially, with state independent prizes,\(^{28}\) we show that the only axiom we relax is monotonicity with between states comparisons as stated in Remark 2 in Section 2.4.

6 Related Literature

State independent dual-self and dual-self variational expected utilities were introduced in Chandrasekher et al. (2020) building on Ghirardato et al.’s (2004) approach of using Clarke derivatives to capture the beliefs of the decision maker. These models are non-convex generalizations of multiple prior preferences (Gilboa and Schmeidler, 1989) and variational preferences (Maccheroni et al., 2006).

\(^{28}\)Here, for all \(s \in S, X_s = X.\)
The previous literature on state dependent utility has mainly focused on axiomatizing the state dependent expected utility using some additional information: Karni (2007) assumes preferences on conditional acts which are acts conditional on a given event happening. Karni (2011a; 2011b) assumes preferences conditional on signals and uses updating of probabilities for identification. Karni et al. (1983) and Karni and Schmeidler (2016) assume preferences on hypothetical lotteries that are lotteries on state-consequence pairs. Lu (2019) establishes uniqueness for the utilities up to a common positive multiplication and addition of any state specific constants by using two different random choice data based on updated random beliefs.

Chew and Wang (2020) exemplifies that state and rank dependent expected utility can be identified under two states of the world. State and rank dependent expected utility is a special case of state dependent dual-self expected utility. Karni (2020) shows that in state and rank dependent expected utility with rank-dependent probabilities the utilities and probabilities are not identified.

Drèze (1958; 1961; 1987; 2004) studies state dependent maxmax expected utility when the acts are lotteries of Anscombe-Aumann acts in the context of moral hazard. He characterizes the existence and the uniqueness of the state dependent maxmax expected utility when the intersection of utilities has a non-empty interior. In contrast, we characterize the uniqueness of any state dependent dual-self expected utility in the standard state dependent Anscombe-Aumann setting and characterize behaviorally when the utilities and the set of probabilities are fully or partially identified. Baccelli (2019) discusses Drèze’s contribution extensively.

Hill (2019) studies state dependent maxmin expected utility such that the best and the worst acts have a constant utility in the state independent Anscombe-Aumann setting without Risk Independence. He assumes that the best and the worst acts are crisp or unambiguous acts. In contrast under Risk Independence, we show the full generality of the dual-self and the maxmin expected utilities as assuming that there exist some unambiguous act and two acts that share the same ambiguity. Hill (2019) shows the identification of the representation when the best and the worst acts have a constant utility and the representation when the best and the worst acts have a constant utility and the repre-

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29This additional property is used in the proof but not stated in the theorem.
sentation is linear between these best and worst acts. Under these restrictions, we have the standard state independent identification. In contrast under Risk Independence, we show the full identification of the representation among any state dependent dual-self representations which were ruled out by the above restriction.

Our application to social choice theory with symmetric state dependent maxmin contributes to the literature on preference aggregation. Harsanyi’s (1953; 1955; 1977) seminal contribution axiomatized utilitarian aggregation of preferences. We generalize this aggregation to generalized Rawlsian social welfare functions. Grant et al. (2010) considers this axiomatization and generalizes it to generalized utilitarianism where the society has different risk attitudes over individual’s utilities. Second, we contribute to the literature discussing the utility comparability requirements for Harsanyi’s axiomatization. This is discussed in Sen (1986) and Weymark (1991). Third, our application contributes to the literature on generalizing Rawlsian social welfare function for a common class of social welfare functions with the utilitarian social welfare function. Symmetric state dependent maxmin generalizes mixed utilitarian–maximin welfare function that is a convex combination of Rawlsian and utilitarian social welfare functions as suggested in Roberts (1980). This was axiomatized in Bossert and Kamaga (2020) assuming that utilities are observable. We additionally show the identification of this representation and axiomatize its generalization when utilities are not observable.

7 Conclusion

The assumption of state independent utilities has been the simplifying, but non-ideal, assumption to separate the subjective probabilities from the utilities. This paper provided a novel foundation for state dependent utility by studying models of non-expected utilities. We showed that with a state dependent version of general dual-self expected utility (non-convex multiple priors preferences, Chandrasekher et al., 2020) state dependent utilities and probabilities can be identified. Additionally, with a state dependent version of more general dual-self variational expected utility (non-convex variational preferences, Chandrasekher et al., 2020) the intensities of preferences and probabilities can be separated; however, the levels
of utilities cannot be identified. These identifications are characterized by full-dimensional set of probabilities.

These identifications encompass those for state dependent versions of all the special cases and alternative representations of dual-self and dual-self variational expected utilities: maxmin expected utility (multiple prior preferences, Gilboa and Schmeidler, 1989), Choquet expected utility (Schmeidler, 1989), α-maxmin expected utility (Ghirardato et al., 2004), invariant biseparable preferences (Ghirardato et al., 2004; Amarante, 2009), monotone mean-variance preferences (Maccheroni et al., 2009), multiplier preferences (Hansen and Sargent, 2001; Strzalecki, 2011), variational preferences (Maccheroni et al., 2006), monotonic mean-dispersion preferences (Grant and Polak, 2013), and vector expected utility (Siniscalchi, 2009).

We provided new characterizations for state dependent dual-self and dual-self variational expected utilities. First, we generalized standard preferences under ambiguity to state dependent setting. Under these preferences, state dependent dual-self and dual-self variational expected utilities are characterized by assumptions on the structure of ambiguity: dual-self variational was characterized by an axiom stating that each act does not contain a different source of ambiguity but instead there exist two acts that are statewise ranked and share exactly the same sources of ambiguity. This was operationalized as the two acts hedging ambiguity similarly. Dual-self expected utility was characterized by an additional assumption stating that there exists a least ambiguous act that shares the sources of ambiguity with all the other acts. This was operationalized as the least ambiguous act not hedging ambiguity.

Finally, we showed that these identifications have applications beyond ambiguity. We applied it to social choice theory. There we provided the identification of fairness of the society and for interpersonal utility comparisons and showed that a utilitarian social welfare function is a non-identified knife-edge case of Rawlsian type social welfare functions.
Online Appendix to “State Dependent Utility and Ambiguity”

Not intended for publication

This appendix is organized as follows. Section A shows the general identification result for the special cases and alternative representations of state dependent dual-self expected utility and state dependent dual-self variational expected utility. We do this by considering a representation consisting only of utility $u$ and aggregator $I : u(H) \rightarrow \mathbb{R}$.

Section B considers expected utility when we relax Independence to Weak Independence. We show that this gives expected utility with censoring for some utility intervals. That is there exist an affine utility $u$ and a weakly increasing censoring function $H$ such that for all $x, y \in \Delta(X)$,

$$x \succsim y \iff H(u(x)) \geq H(u(y)).$$

Section C proves the existence of state dependent dual-self expected utility and state dependent dual-self variational expected utilities, Theorems 1 and 7. Section C.5 proves Remark 2.

Section D proves the identification results for state dependent dual-self expected utility and state dependent dual-self variational expected utilities. First, Section D.1 shows the dual-self variational representation is additive between any equally crisp acts. Second, Section D.2 shows how we can capture beliefs or derivatives of the representation behaviorally. Third, Section D.3 shows the general partial identification results, Theorems 5 and 9. Next, Section D.4 shows the full-identification results under Axiom 7, Theorems 2 and 8. Next, Section D.5 shows how the identification relates to underlying preferences and crisp and equally crisp acts with Corollaries 6 and 11. Finally, Section D.6 shows the relative likelihood and probability characterizations by showing how the general partial identification result simplifies in some cases with Propositions 3 and 4.

Section E proves our application to social choice theory and characterizes symmetric state dependent maxmin expected utility with Theorem 12.

Finally, Section F considers state dependent Bewley expected utility. We axiomatize its existence and characterize its uniqueness in Sections F.1 and F.2 respectively. Section F.3
axiomatizes state independent Bewley expected utility and shows how the state dependent version only relaxes monotonicity with between states comparisons.

A General Identification

This section shows the general identification results for any aggregator function $I : u(H) \to \mathbb{R}$. This shows the identification for all the special cases and alternative representations of state dependent dual-self expected utility and dual-self variational expected utility.

For this section, we assume that $S = S^P$. The results extend to situations where not all states are proper and utilities are not identified on these states.

For $\psi, \varphi \in \mathbb{R}^S$, denote $\varphi \cdot \psi = \sum_{s \in S} \varphi_s \psi_s$. Let $A \subseteq \mathbb{R}^S$ be a convex set and $I : A \to \mathbb{R}$ be a function. We denote $\bar{I} \in \mathbb{R}^S$ such that for all $s \in S$, $\bar{I}_s = 1$. We say that

- For state dependent utility $u$, $(u, I)$ is a state dependent representation for $\succeq$ if for all $f, g \in H$, $f \succeq g \iff I(u(f)) \geq I(u(g))$.
- $I$ is C-additive if for all $\varphi \in A, \alpha \geq 0$ such that $\varphi + \alpha \bar{I} \in A$, $I(\varphi + \alpha \bar{I}) = I(\varphi) + \alpha$.
- $I$ is positive homogeneous if for all $\varphi \in A, \alpha > 0$ such that $\alpha \varphi \in A$, $I(\alpha \varphi) = \alpha I(\varphi)$.
- $I$ is monotonic if for all $\varphi, \psi \in A$ such that for all $s \in S$, $\varphi_s \geq \psi_s$, $I(\varphi) \geq I(\psi)$.

Dual-self variational aggregators correspond to C-additive and monotonic aggregators. Dual-self aggregators correspond to C-additive, positive homogeneous, and monotonic aggregators.

Let $A \subseteq \mathbb{R}^S$ be a convex set. For every $\varphi \in \text{int} A, \xi \in \mathbb{R}^S$, the Clarke upper derivative of $I$ at $\varphi$ in the direction $\xi$ is

$$I^o(\varphi; \xi) = \limsup_{\psi \to \varphi, t \searrow 0} \frac{I(\psi + t\xi) - I(\psi)}{t}.$$ 

The Clarke subdifferential of $I$ at $\varphi$ is the set

$$\partial I(\varphi) = \{ \chi \in \mathbb{R}^S | \forall \xi \in \mathbb{R}^S, \chi \cdot \xi \leq I^o(\varphi; \xi) \}.$$ 

Ghirardato et al. (2004) has shown that the union of all Clarke subdifferentials, $\bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi)$, captures the decision maker’s beliefs.
Our first result generalizes the partial identification for state dependent dual-self variational expected utility, Theorem 7. Here the identification is the same but for the general aggregator $I$.

**Proposition 14** Let $(u, I)$ and $(\tilde{u}, \tilde{I})$ be state dependent monotonic and C-additive representations for $\succsim$. Then there exist $x \in \left( \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \right)^\perp$, $\alpha > 0$, $B \in \mathbb{R}^S$, $\beta \in \mathbb{R}$ such that for all $f \in H$ such that $u(f) \in \text{int} u(H)$,

$$\partial \tilde{I}(u(f)) = (1 + x)\partial I(\tilde{u}(f)),$$

especially

$$\bigcup_{\varphi \in \text{int} \tilde{u}(H)} \partial \tilde{I}(\varphi) = (1 + x) \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi),$$

for all $s \in S$,

$$\tilde{u}_s = \frac{\alpha}{1 + x_s} u_s + B_s,$$

and for all $f \in H$,

$$\tilde{I}(\tilde{u}(f)) = \alpha I(u(f)) + \beta.$$

Our second result generalizes the partial identification for state dependent dual-self expected utility, Theorem 5.

**Proposition 15** Let $(u, I)$ and $(\tilde{u}, \tilde{I})$ be state dependent monotonic, C-additive, and positive homogeneous representations for $\succsim$. Then there exist $x, y \in \left( \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \right)^\perp$, $\alpha > 0$, $\beta \in \mathbb{R}$ such that for all $f \in H$ such that $u(f) \in \text{int} u(H)$,

$$\partial \tilde{I}(u(f)) = (1 + x)\partial I(\tilde{u}(f)),$$

especially

$$\bigcup_{\varphi \in \text{int} \tilde{u}(H)} \partial \tilde{I}(\varphi) = (1 + x) \bigcup_{\varphi \in \text{int} \tilde{u}(H)} \partial \tilde{I}(\varphi),$$

for all $s \in S$,

$$\tilde{u}_s = \frac{\alpha}{1 + x_s} (u_s + y_s) + \beta,$$

and for all $f \in H$,

$$\tilde{I}(\tilde{u}(f)) = \alpha I(u(f)) + \beta.$$
These two partial identifications give all the following full identifications and identifications for probabilities for a single state and relative likelihoods between two states. The next result shows the full identification and gives Theorems 2 and 8.

**Proposition 16** Let \((u, I)\) be a state dependent monotonic, C-additive representation for \(\succsim\). The following conditions are equivalent:

1. \(\succsim\) satisfies Axiom 7.
2. \((-\bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi)) = \emptyset\).
3. \(\overline{\text{co}} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi)\) has a non-empty interior in \(\Delta(S)\).

The next result shows the identification for the probabilities for a single state. This generalizes Proposition 3 to variational representation.

**Proposition 17** Let \((u, I)\) be a state dependent monotonic, C-additive representation for \(\succsim\) and \(s \in S^P\). The following conditions are equivalent:

1. If \(f, g \in H\) are such that \(f \sim g\), \(f_s \succ g_s\), and there exists \(s' \in S\) such that \(g_{s'} \succ f_{s'}\), then there exist \(h \in H\) and \(\alpha \in (0, 1)\) such that
   \[\alpha h + (1 - \alpha)f \not\succ \alpha h + (1 - \alpha)g\.
\]
2. \(\text{pr}_s\left(\bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi)\right) = \emptyset\).
3. \(S^P = \{s\}\) or there exist \(p, q \in \overline{\text{co}} \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi)\) such that \(p \neq q\) and for all \(\tilde{s} \in S \setminus \{s\}\),
   \[\frac{p_s}{1-p_s} = \frac{q_{\tilde{s}}}{1-q_{\tilde{s}}}.
\]

The last result shows the identification for the relative likelihoods for a single state. This generalizes Proposition 4 to variational representation.

**Proposition 18** Let \((u, I)\) be a state dependent monotonic, C-additive representation for \(\succsim\) and \(s, s' \in S^P\), \(s \neq s'\). The following conditions are equivalent:

1. If \(f, g \in H\) are such that \(f \sim g\), \(f_s \succ g_s\), and \(g_{s'} \succ f_{s'}\), then there exist \(h, h' \in H\) and \(\alpha \in (0, 1)\) such that
   \[\alpha h + (1 - \alpha)f \succ \alpha h' + (1 - \alpha)f\] and \(\alpha h + (1 - \alpha)g < \alpha h' + (1 - \alpha)g\).
2. Either
   \[\text{pr}_{s, s'}\left(\bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi)\right) = \{0, 0\}\] or \(\text{pr}_{s, s'}\left(\bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi)\right) = \{(a, a) | a \in \mathbb{R}\}.

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(3) There exist \( p, q \in \mathbb{C} \cup \varphi \in \text{int}(u(H)) \) \( \partial I(\varphi) \) such that \( \frac{p}{p_{s'}} \neq \frac{q}{q_{s'}} \) and for all \( s \in S \setminus \{s, s'\} \), \( p_s = q_s \).

**B Latent Expected Utility Representation**

For this section, we assume that \( X \) is a nonempty set, \( \Delta(X) \) is the set of all (simple) lotteries on \( X \), \( H = \Delta(X) \), and \( \succcurlyeq \subseteq H \times H \). This section studies the following weak independence assumption.

**Axiom 4’** For all \( f, g, h \in H, \alpha \in (0, 1) \),

\[
f \succ g \implies \alpha f + (1 - \alpha)h \succcurlyeq \alpha g + (1 - \alpha)h.
\]

We will show that under continuity and weak order assumptions, Weak Independence characterizes censored expected utility where some utility differences are censored with the function \( H \) below. This gives a latent expected utility model.

**Theorem 19 (Latent Expected Utility)** \( \succcurlyeq \) satisfies Axioms 1, 2, 4’ iff there exist affine \( u : \Delta(X) \to \mathbb{R} \) and continuous and weakly increasing \( H : u(\Delta(X)) \to \mathbb{R} \) such that for all \( P, Q \in \Delta(X) \),

\[
P \succcurlyeq Q \iff H\left( \sum_x P(x)u(x) \right) \geq H\left( \sum_x Q(x)u(x) \right).
\]

Additionally, if \((\tilde{u}, \tilde{H})\) is another censored expected utility for \( \succcurlyeq \), then there exist \( a > 0, b \in \mathbb{R} \) such that \( \tilde{u} = au + b \) and a strictly increasing \( \varphi : H(u(\Delta(X))) \to \mathbb{R} \) such that for all \( v \in u(\Delta(X)) \),

\[
\tilde{H}(av + b) = \varphi(H(v)).
\]

The uniqueness part of this theorem shows that the affine utility is unique up to a positive affine transformation and we can determine for every lottery \( P \in \Delta(X) \) if it is censored. That is the intervals on which the utility is censored are unique given the utility. However, the values of the censoring function are not unique and any ordinal transformation are allowed.

Secondly, in this representation, \( u \) gives a weak affine representation for \( \succcurlyeq \).

**Definition** \( u \) is a weak affine representation for \( \succcurlyeq \) if \( u : \Delta(X) \to \mathbb{R} \) is affine and for all \( x, y \in \Delta(X) \),

\[
P \succ Q \implies \sum_x P(x)u(x) > \sum_x Q(x)u(x).
\]
B.1 Proof for Latent Expected Utility

We denote the elements of $\Delta(X)$ by $x, y$. We denote for $\alpha \in [0, 1]$, $x\alpha y = \alpha x + (1 - \alpha)y$. We start with simple lemmas. The first shows that there are strictly middle points of preferences in convex combinations.

**Lemma 20** Assume that $\succsim$ satisfies Axioms 1, 2. If $x \succ y$, then there exist $\alpha^1, \alpha^2 \in (0, 1)$ such that

$$x \succ x\alpha^1 y, x\alpha^2 y \succ y$$

and for all $\alpha \in (\alpha^1, 1], \alpha' \in [0, \alpha^2)$,

$$x\alpha y \succ x\alpha^1 y \text{ and } x\alpha^2 y \succ x\alpha' y.$$  

**Proof.** Let $x \succ y$. By Axiom 2, the set

$$\{\alpha \in [0, 1] | x \succ \alpha x + (1 - \alpha)y\}$$

is open and $$\{\alpha \in [0, 1] | y \succsim \alpha x + (1 - \alpha)y\}$$ is closed.

Thus there exists $\alpha^0 \in (0, 1)$ such that $\alpha^0 \in \{\alpha \in [0, 1] | x \succ \alpha x + (1 - \alpha)y\}$ and $\alpha^0 \notin \{\alpha \in [0, 1] | y \succsim \alpha x + (1 - \alpha)y\}$. Thus $x \succ x\alpha^0 y \succ y$. Next denote

$$\alpha^1 = \max\{\alpha \in [0, 1] | x\alpha^0 y \succsim \alpha x + (1 - \alpha)(x\alpha^0 y)\},$$

$$\alpha^2 = \max\{\alpha \in [0, 1] | y \succsim \alpha x + (1 - \alpha)(x\alpha^0 y)\}.$$

Now define $\alpha^3 = \alpha^1 + (1 - \alpha^1)\alpha^0$ and $\alpha^4 = (1 - \alpha^2)\alpha^0$. Then we have

$$x\alpha^0 y \sim x\alpha^3 y \sim x\alpha^4 y$$

and for all $\alpha \in (\alpha^3, 1], \alpha' \in (0, \alpha^4)$,

$$x\alpha y \succ x\alpha^3 y \text{ and } x\alpha^4 y \succ x\alpha' y.$$  

\qed

The next two lemmas show that indifference sets are convex and strict upper and lower contour sets are convex.

**Lemma 21** Assume that $\succsim$ satisfies Axioms 1, 2, 4. If $x \sim y$, then for all $\alpha \in (0, 1)$, $x \sim \alpha x + (1 - \alpha)y$. 

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Proof. Assume, per contra, there exist $x, y \in \Delta(X), \alpha^0 \in (0, 1)$ such that $x \sim y$ and $x \not\sim \alpha^0 x + (1 - \alpha^0) y$. Assume w.l.o.g. $\alpha^0 x + (1 - \alpha^0) y \succ x$. By Lemma 20, there exists $\alpha^1 \in (\alpha^0, 1)$ such that for all $\alpha \in (\alpha^1, 1)$,
\[ x^0 \alpha y \succ x^0 y. \] (1)
Now $x^0 y \succ y$ and thus by Weak Independence,
\[ x^1 y \preceq x^1 (x^0 y) \equiv x (\alpha^1 + (1 - \alpha^1) \alpha^0) y \lessdot x^1 y, \]
which contradicts transitivity.

Lemma 22 Assume that $\succsim$ satisfies Axioms 1, 2, 4'. If $x \succ z$ and $y \succ z$, then for all $\alpha \in (0, 1)$, $ax + (1 - \alpha)y \succ z$. If $z \succ x$ and $z \succ y$, then for all $\alpha \in (0, 1)$, $z \succ ax + (1 - \alpha)y$.

Proof. Assume, per contra, there exist $x, y, z \in \Delta(X), \alpha^0 \in (0, 1)$ such that $x \not\succ z, y \succ z$ and $\alpha^0 x + (1 - \alpha^0) y \sim z$. By Lemma 20, there exists $\alpha^1 \in (\alpha^0, 1)$ such that for all $\alpha \in (\alpha^1, 1)$, 
\[ x^0 \alpha y \succ x^0 y. \] (2)
Now $y \succ x^0 y$ and thus by Weak Independence,
\[ x^1 y \preceq x^1 (x^0 y) \equiv x (\alpha^1 + (1 - \alpha^1) \alpha^0) y \lessdot x^1 y, \]
which contradicts transitivity. The other case follows symmetrically.

Next, we show the existence of an affine utility that gives a weak representation for the preferences. This follows similarly to vNM theorem but in here we define utility at a point where preferences are strictly increasing.

Proposition 23 Assume that $\succsim$ satisfies Axioms 1, 2, 4'. Then there exists $u : \Delta(X) \rightarrow \mathbb{R}$ that is a weak affine representation for $\succsim$.

Proof. By nontriviality, there exist $x^*, x_\ast \in \Delta(X)$ such that $x^* \succ x_\ast$. By applying Lemma 20 twice, there exist $\alpha^*, \alpha_\ast \in (0, 1)$ such that $\alpha^* > \alpha_\ast$ and for all $\alpha \in (\alpha^*, 1), \alpha' \in [0, \alpha_\ast)$,
\[ x^* \alpha x_\ast \succ x^* \alpha^* x_\ast \succ x^* \alpha_\ast x_\ast \succ x^* \alpha' x_\ast. \]
Next, we define the representation by considering cases. If $x \succ x^* \alpha^* x_\ast$,
\[ \alpha^x = \min\{\alpha \in [0, 1]| \alpha x + (1 - \alpha)(x^* \alpha^* x_\ast) \succ x^* \alpha^* x_\ast\} \]
and define

\[ u(x) := \frac{1}{\alpha^x}. \]

Now especially \( u(x) > 1 \). If \( x^* \alpha^*_x \succ x \),

\[ \alpha^x = \max\{\alpha \in [0, 1]|x^* \alpha^*_x \succ \alpha(x^* \alpha^*_x) + (1 - \alpha)x\} \]

and define

\[ u(x) := \frac{-\alpha^x}{1 - \alpha^x}. \]

Now especially \( u(x) < -1 \). If \( x^* \alpha^*_x \succsim x \succsim x^* \alpha^*_x \),

\[ \alpha^x = \max\{\alpha \in [0, 1]|x^* \alpha^*_x \succsim \alpha x^* + (1 - \alpha)x\} \]

and define

\[ u(x) := \frac{1 - u(x^*)\alpha^x}{1 - \alpha^x}. \]

Now especially \( 1 \leq u(x) \leq -1 \). We show that \( u \) is a weak representation for \( \succsim \). Let \( x, y \in \Delta(X) \) be such that \( u(x) \geq u(y) \). Assume, per contra, \( y \succ x \). If \( y \succ x^* \alpha^*_x \succ x \) or \( y \succsim x^* \alpha^*_x \succ x \), then \( u(x) > u(y) \). So we have three cases to consider. 1) \( y \succ x \succ x^* \alpha^*_x \). Now there exists \( \alpha^0 \in (0, 1) \) such that \( y^{\alpha^0}(x^* \alpha^*_x) \succ x \). Then by Weak Risk Independence and Lemma 21, for all \( \alpha \in (0, \alpha^x) \),

\[ \alpha(x^* \alpha^*_x) + (1 - \alpha)y^{\alpha^0}(x^* \alpha^*_x) \succsim \alpha(x^* \alpha^*_x) + (1 - \alpha)x \succsim x^* \alpha^*_x. \]

Thus \( \alpha^y \geq \alpha^x + (1 - \alpha^x)(1 - \alpha^0) \) which contradicts \( u(x) \geq u(y) \).

2) \( x^* \alpha^*_x \succ y \succ x \) follows symmetrically since there exists \( \alpha^0 \in (0, 1) \) such that \( y \succ x^{\alpha^0}(x^* \alpha^*_x) \). 3) \( x^* \alpha^*_x \succsim y \succsim x^* \alpha^*_x \) follows symmetrically since there exists \( \alpha^0 \in (0, 1) \) such that \( y \succ x^{\alpha^0}x^* \).

We show the linearity of \( u \) within each of the three above areas. 1) Let \( y, x \succ x^* \alpha^*_x, \alpha \in (0, 1) \) and \( z := \alpha x + (1 - \alpha)y \). Next, we show that \( \alpha^z = \frac{\alpha^x \alpha^y}{\alpha^z(1 - \alpha) + \alpha^y \alpha} \). First, by Lemma 21,

\[
\frac{\alpha \alpha^y}{\alpha^z(1 - \alpha) + \alpha^y \alpha} \left( \alpha^x z + (1 - \alpha^x)(x^* \alpha^*_x) \right) + \frac{\alpha^x(1 - \alpha)}{\alpha^z(1 - \alpha) + \alpha^y \alpha} \left( \alpha^y y + (1 - \alpha^y)(x^* \alpha^*_x) \right) \\
\equiv \frac{\alpha \alpha^y}{\alpha^z(1 - \alpha) + \alpha^y \alpha} z + \left( 1 - \frac{\alpha^x \alpha^y}{\alpha^z(1 - \alpha) + \alpha^y \alpha} \right)(x^* \alpha^*_x) \sim x^* \alpha^*_x.
\]

Additionally, for all \( \alpha \succ \frac{\alpha^x \alpha^y}{\alpha^z(1 - \alpha) + \alpha^y \alpha} \), by Lemma 22,

\[ \alpha z + (1 - \alpha)(x^* \alpha^*_x) \succ x^* \alpha^*_x. \]
Thus the linearity follows from the definition of $u$. The cases $x^*\alpha^* x_\ast \succeq x, y \succ x^*\alpha_\ast x_\ast$ and $x^*\alpha^* x_\ast \succ x, y$ follow similarly.

Next we show that $u$ is linear between $x^*$ and $x_\ast$. By the choice of $\alpha^*$ and $\alpha_\ast$, we have for all $\alpha \in (\alpha^*, 1]$, $x^*\alpha_\ast x_\ast \succeq x^*\alpha^* x_\ast$ and

$$\alpha^*\alpha_\ast = \frac{\alpha - \alpha_\ast}{\alpha - \alpha^*}.$$

Thus

$$u(x^*\alpha_\ast x_\ast) = \frac{\alpha - \alpha_\ast}{\alpha^* - \alpha}.$$

We have for all $\alpha \in [\alpha_\ast, \alpha^*]$, $x^*\alpha_\ast x_\ast \succeq x^*\alpha^* x_\ast$ and

$$\alpha^*\alpha_\ast = \frac{\alpha^* - \alpha}{1 - \alpha^*}.$$

Thus

$$u(x^*\alpha_\ast x_\ast) = \frac{1 - \frac{\alpha - \alpha_\ast}{\alpha^* - \alpha}}{1 - \frac{\alpha^* - \alpha}{1 - \alpha^*}} = \frac{\alpha - \alpha_\ast}{\alpha^* - \alpha}.$$

We have for all $\alpha \in [0, \alpha_\ast)$, $x^*\alpha_\ast x_\ast \succeq x^*\alpha^* x_\ast$ and

$$\alpha^*\alpha_\ast = \frac{\alpha_\ast - \alpha}{\alpha^* - \alpha}.$$

Thus

$$u(x^*\alpha_\ast x_\ast) = \frac{-\frac{\alpha_\ast - \alpha}{\alpha^* - \alpha}}{1 - \frac{\alpha^* - \alpha}{1 - \alpha^*}} = \frac{\alpha - \alpha_\ast}{\alpha^* - \alpha}.$$

Thus the utility is linear between $x^*$ and $x_\ast$. Now let $x, y \in \Delta(X)$ be such that $x \succ x^*\alpha^* x_\ast \succeq y \succeq x^*\alpha_\ast x_\ast$ and we show that $u$ is affine between $x$ and $y$. Let $\alpha^1 \in (\alpha^*, 1]$ be such that $x \succ x^*\alpha^1 x_\ast$. Then there exist $\alpha^2 \in (0, 1)$ such that $u(x\alpha^2 y) = u(x^*\alpha^1 x_\ast)$ and $\alpha^3 \in [\alpha_\ast, \alpha^*]$ such that $u(y) = u(x^*\alpha^3 x_\ast)$. Now since $u$ is a weak representation, we have by the linearity, for all $z, z' \in \Delta(X)$ such that $z, z' \succ x^*\alpha^* x_\ast$ and for all $\alpha \in (\frac{\alpha^* - \alpha^3}{\alpha^* - \alpha}, 1]$,

$$u(x\alpha^2 y) = u\left(x^*\left(\alpha^1\alpha + \alpha^3(1 - \alpha)\right)x_\ast\right).$$

And by the linearity, for all $z, z' \in \Delta(X)$ such that $x^*\alpha^* x_\ast \succeq z, z' \succeq x^*\alpha^* x_\ast$ and for all $\alpha \in [0, \frac{\alpha^* - \alpha^3}{\alpha^* - \alpha}]$,

$$u(x\alpha^2 y) = u\left(x^*\left(\alpha^1\alpha + \alpha^3(1 - \alpha)\right)x_\ast\right).$$

Thus by the linearity of $u$ between $x^*$ and $x_\ast$, $u$ is linear between $x\alpha^2 y$ and $y$. Additionally, by the linearity, for all $z, z' \in \Delta(X)$ such that $z, z' \succ x^*\alpha^* x_\ast$, $u$ is linear between $x$ and $x\left(\alpha^2\left(\frac{\alpha^1 + \alpha^*}{\alpha^* - \alpha^3} - \alpha^3\right)\right)y$. Thus $u$ is linear between $x$ and $y$. 

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Symmetrically if \( x, y \in \Delta(X) \) are such that \( x^* \alpha^* x_\ast \succeq x \succeq x^* \alpha_\ast x_\ast \succ y \), then \( u \) is affine between \( x \) and \( y \). Thus finally for all \( x, y \in \Delta(X) \), \( u \) is affine between \( x \) and \( y \). This shows the claim. \( \square \)

The next result shows the necessity of Weak Independence for an affine weak representation.

**Lemma 24** Assume that \( \succeq \) satisfies Axioms 1, 2. If there exists \( u: \Delta(X) \to \mathbb{R} \) that is a weak affine representation for \( \succeq \), then \( \succeq \) satisfies Axiom \( 4' \).

**Proof.** Assume, per contra, there exist \( x, y, z \in \Delta(X), \alpha \in (0,1) \) such that \( x \succ y \) and \( \alpha y + (1-\alpha)z \succ \alpha x + (1-\alpha)z \). Then since \( u \) is a weak representation for \( \succeq \) and affine, we have

\[
u(x) > u(y) \quad \text{and} \quad \alpha u(y) + (1-\alpha)u(z) = u(\alpha y + (1-\alpha)z) > u(\alpha x + (1-\alpha)z) = \alpha u(x) + (1-\alpha)u(z),\]

which is a contradiction. \( \square \)

The next result shows the uniqueness of the affine weak representation.

**Proposition 25** Assume that \( \succeq \) satisfies Axioms 1, 2, \( 4' \). If \( u: \Delta(X) \to \mathbb{R} \) and \( \tilde{u}: \Delta(X) \to \mathbb{R} \) are weak affine representations for \( \succeq \), then there exist \( \alpha > 0, \beta \in \mathbb{R} \) such that

\[
\tilde{u} = \alpha u + \beta.
\]

**Proof.** First, we show that for all \( x, y \in \Delta(X) \),

\[
u(x) \geq u(y) \iff \tilde{u}(x) \geq \tilde{u}(y).
\]

Assume, per contra, there exist \( \tilde{u}(x) > \tilde{u}(y) \) and \( u(x) \leq u(y) \). By Axiom 1, there exist \( x^*, x_\ast \in \Delta(X) \) such that \( x^* \succ x_\ast \). By applying Lemma 20 twice, there exist \( \alpha^*, \alpha_\ast \in (0,1) \) such that \( \alpha^* > \alpha_\ast \) and for all \( \alpha \in (\alpha^*, 1], \alpha' \in [0, \alpha_\ast) \),

\[
x^* \alpha x_\ast \succ x^* \alpha^* x_\ast \succ x^* \alpha_\ast x_\ast \succ x^* \alpha' x_\ast.
\] (3)

Now since \( u \) is a weak representation and so

\[
u(x^*) > u(x_\ast),
\] (4)
and either $u(x^* \alpha x) \geq u(x)$ or $u(x^* \alpha x) \leq u(y)$. Assume w.l.o.g. $u(x^* \alpha x) \geq u(x)$. By the affinity of $u$ and $\tilde{u}$, there exist $\alpha^x, \alpha^y$ such that

$$\tilde{u}(x \alpha^x x) > \tilde{u}(y \alpha^y x)$$

and thus $u(x \alpha^x x) < u(y \alpha^y x)$. Denote $\tilde{x} := x \alpha^x x$ and $\tilde{y} := y \alpha^y x$. By (4), there exists $\alpha^0 \in (0, 1]$ such that $u(x^* \alpha^x x) = u(\tilde{x} \alpha^0 x)$. Thus especially by the affinity of $u$, we have $u(\tilde{y} \alpha^0 x) > u(x^* \alpha^x x)$. By affinity of $u$, there exists $\alpha^1 \in (\alpha^*, 1]$ such that $u(\tilde{y} \alpha^0 x) > u(x^* \alpha^1 x)$. Thus since $u$ is a weak representation for $\succ$, we have

$$\tilde{y} \alpha^0 x \succ x^* \alpha^1 x \sim \tilde{x} \alpha^0 x.$$ 

Thus especially since $\tilde{u}$ is a weak representation for $\succ$

$$\alpha^0 \tilde{u}(\tilde{y}) + (1 - \alpha^0)u(x^*) = \tilde{u}(\tilde{y} \alpha^0 x) > \tilde{u}(\tilde{x} \alpha^0 x) = \alpha^0 \tilde{u}(\tilde{x}) + (1 - \alpha^0)u(x^*).$$ 

However, this contradicts (5).

Thus for all $x, y \in \Delta(X)$,

$$u(x) \geq u(y) \iff \tilde{u}(x) \geq \tilde{u}(y).$$

Define $\succeq \subseteq \Delta(X) \times \Delta(X)$ by

$$x \succeq y \iff u(x) \geq u(y).$$

Now $u$ and $\tilde{u}$ are affine representations for $\succeq$ and hence by Herstein and Milnor (1953) and Fishburn (1970) there exist $\alpha > 0, \beta \in \mathbb{R}$ such that $\tilde{u} = \alpha u + \beta$.

Finally, the weak affine representation gives us a full representation with a censoring function.

**Theorem 19 (Latent Expected Utility)** $\succsim$ satisfies Axioms 1, 2, 4’ iff there exist affine $u : \Delta(X) \to \mathbb{R}$ and continuous and weakly increasing $H : u(\Delta(X)) \to \mathbb{R}$ such that for all $P, Q \in \Delta(X)$,

$$P \succsim Q \iff H\left(\sum_x P(x)u(x)\right) \geq H\left(\sum_x Q(x)u(x)\right).$$

Additionally, if $(\tilde{u}, \tilde{H})$ is another censored expected utility for $\succsim$, then there exist $a > 0, b \in \mathbb{R}$ such that $\tilde{u} = au + b$ and a strictly increasing $\varphi : H(\Delta(X)) \to \mathbb{R}$ such that for all $v \in u(\Delta(X))$,

$$\tilde{H}(av + b) = \varphi(H(v)).$$
Proof. We first show the sufficiency of the axioms. By Proposition 23, there exists \( u : \Delta(X) \to \mathbb{R} \) that is affine and a weak representation for \( \succeq \). Define \( \succeq \subseteq u(\Delta(X)) \times u(\Delta(X)) \) by for all \( x, y \in \Delta(X) \),

\[
u(x) \succeq u(y) \iff x \succeq y.
\]

First, \( u(\Delta(X)) \) is an interval since \( u \) is affine. Second, \( \succeq \) is well-defined that is there does not exist \( x, y \in \Delta(X) \) such that \( u(x) = u(y) \) and \( x \not\succ y \). If \( x \not\succ y \), then since \( u \) is a weak representation for \( \succeq \) and \( \succeq \) is complete, we have \( x \succ y \) or \( y \succ x \) and so \( u(x) > u(y) \) or \( u(y) > u(x) \) respectively.

By Axiom 1, \( \succeq \) is complete and transitive.

We show that \( \succeq \) is continuous. We show that upper contour sets of preferences are closed. The lower contour sets follow symmetrically. Let \( \{x_i\}_{i=1}^\infty \subseteq \Delta(X), x, y \in \Delta(X) \) be such that \( u(x_i) \to u(x) \) as \( i \to \infty \). Assume that for all \( i \in \mathbb{N} \), \( u(x_i) \succeq u(y) \). We show that \( u(x) \succeq u(y) \).

First, we define \( x^*, x_\ast \in \Delta(X) \) such that for all \( i \in \mathbb{N} \), \( u(x^*) \geq u(x_i) \geq u(x_\ast) \). We only define \( x^* \) since \( x_\ast \) follows symmetrically. If \( \sup u(\Delta(X)) \in u(\Delta(X)) \), then define \( x^* = \sup u(\Delta(X)) \). So assume that \( \sup u(\Delta(X)) \not\in u(\Delta(X)) \). Since \( u(\Delta(X)) \) is an interval, there exists \( \varepsilon > 0 \) such that \( u(x) + \varepsilon \in u(\Delta(X)) \). Since \( u(x^i) \to u(x) \) as \( i \to \infty \), there exists \( i^\varepsilon \in \mathbb{N} \) such that for all \( i > i^\varepsilon \), \( u(x) + \varepsilon > u(x_i) \). Let

\[
u^* = \max\{u(x^i) | 1 \leq i \leq i^\varepsilon\} \cup \{u(x) + \varepsilon\}.
\]

Since \( u^* \) is a maximum over a finite set, it exists and so there exists \( x^* \in \Delta(X) \) such that \( u^* = u(x^*) \).

So there exist \( x^*, x_\ast \in \Delta(X) \) such that for all \( i \in \mathbb{N} \), \( u(x^*) \geq u(x_i) \geq u(x_\ast) \) and since \( u(\Delta(X)) \) is not a singleton, \( u(x^*) > u(x_\ast) \). Thus for each \( i \in \mathbb{N} \), there exists a unique \( \alpha^i \in [0, 1] \) such that \( \alpha^iu(x^*) + (1 - \alpha^i)u(x_\ast) = u(x_i) \). Now since \( u(x_i) \to u(x) \) as \( i \to \infty \) there exists \( \alpha \in [0, 1] \) such that \( \alpha^i \to \alpha \) as \( i \to \infty \) and \( \alpha u(x^*) + (1 - \alpha)u(x_\ast) = u(x) \). Now we have for all \( i \in \mathbb{N} \), \( u(x_i) \succeq u(y) \) and so \( x_i \succeq y \). By above, since \( \succeq \) is well-defined and \( u \) is affine, we have for all \( i \in \mathbb{N} \), \( x_i \sim \alpha^ix^* + (1 - \alpha^i)x_\ast \). Thus by Axiom 2, \( \alpha x^* + (1 - \alpha)x_\ast \succeq y \). And so \( \alpha u(x^*) + (1 - \alpha)u(x_\ast) = u(x) \geq u(y) \). This shows the claim.
Now \( u(\Delta(X)) \) is connected and separable. By Debreu (1954), there exists a continuous function \( H : u(\Delta(X)) \to \mathbb{R} \) such that for all \( x, y \in \Delta(X) \),

\[
H(u(x)) \geq H(u(y)) \iff u(x) \succeq u(y) \iff x \succeq y.
\]

Since \( u \) is affine, we have for all (simple) lotteries \( P \in \Delta(X) \),

\[
H(u(P)) = H\left( \sum_{x \in X} P(x)u(x) \right).
\]

Finally, we show that \( H \) is weakly increasing. We have for all \( x, y \in \Delta(X) \), if \( H(u(x)) > H(u(y)) \), then \( x \succ y \) and since \( u \) is a weak representation \( u(x) > u(y) \). This shows that \( H \) is weakly increasing.

Next, we show the necessity of the axioms. Assume that there exist affine \( u : \Delta(X) \to \mathbb{R} \) and continuous and weakly increasing \( H : u(\Delta(X)) \to \mathbb{R} \) such that for all \( x, y \in \Delta(X) \),

\[
x \succeq y \iff H(u(x)) \geq H(u(y)).
\]

First, since \( \succeq \) has a representation it is complete and transitive. Second, since \( u \) is affine and \( H \) is continuous, \( \succeq \) satisfies mixture continuity, Axiom 2.

Third, we show Weak Independence. Let \( x, y, z \in \Delta(X) \), \( \alpha \in (0, 1) \) and \( H(u(x)) > H(u(y)) \). Since \( H \) is weakly increasing, \( u(x) > u(y) \). Since \( u \) is affine, \( u(\alpha x + (1 - \alpha)z) > u(\alpha y + (1 - \alpha)z) \).

Since \( H \) is weakly increasing,

\[
H(u(\alpha x + (1 - \alpha)z)) \geq H(u(\alpha y + (1 - \alpha)z))
\]

that shows the claim.

Next we move on to the uniqueness. By Proposition 25, there exist \( a > 0, b \in \mathbb{R} \) such that \( \tilde{u} = au + b \). Now \( H \circ u \) and \( H \circ \tilde{u} \) are representations for \( \succeq \). By the ordinal uniqueness of the representation, there exists a strictly increasing function \( \varphi : H(u(\Delta(X))) \to \mathbb{R} \) such that for all \( x \in \Delta(X) \),

\[
\varphi(H(u(x))) = \tilde{H}(\tilde{u}(x)).
\]

Thus for all \( v \in u(\Delta(X)) \), we have \( av + b \in \tilde{u}(\Delta(X)) \) and by the uniqueness of \( u \),

\[
\tilde{H}(av + b) = \varphi(H(v))
\]

\[\Box\]
C Existence of State Dependent Dual-Self Expected Utility and Dual-Self Variational Expected Utility

For acts \( f, g \in H \) and \( \alpha \in [0,1] \), denote \( f\alpha g := \alpha f + (1 - \alpha)g \).

For \( x \in \mathbb{R}, \bar{x}_S \in \mathbb{R}^S \) denotes the constant vector of \( x \). Denote \( h^* = \frac{1}{2}f^* + \frac{1}{2}g^* \) for equally crisp acts \( f^*, g^* \) from Axiom 5.

We start off with a simple observation.

Lemma 26 Let \( \succsim \) satisfy transitivity. If \( f, g \in H \) s.t. for all \( s \in S \), \( f_s \succsim_s g_s \), then \( f \succsim g \).

Proof. Since \( S \) is finite we can enumerate \( S = \{s_1, \ldots, s_{|S|}\} \). Now we have by transitivity of \( \succsim \) and the definition of \( \succsim_s \),

\[
\begin{align*}
f = (f_{s_1}, f_{s_2}, \ldots, f_{s_{|S|}}) \succsim (g_{s_1}, f_{s_2}, \ldots, f_{s_{|S|}}) \succsim \cdots \succsim (g_{s_1}, g_{s_2}, \ldots, g_{s_{|S|-1}}, f_{s_{|S|}}) \succsim (g_{s_1}, g_{s_2}, \ldots, g_{s_{|S|}}) = g.
\end{align*}
\]

Thus by transitivity, \( f \succsim g \). \qed

C.1 Statewise Weak Affine Representations

Next we show that for each \( s \in S \), \( \succsim_s \) has a weak affine representation as defined in Section B. We show that \( \succsim_s \) is weak order that is mixture continuous and satisfy Weak Independence by assumption. First, we show weak order part.

Lemma 27 Let \( s \in S \). If \( \succsim \) satisfies Axioms 1-3, then \( \succsim_s \) satisfies

1. completeness, i.e., for all \( x_s, y_s \in \Delta(X_s) \), \( x_s \succsim_s y_s \) or \( y_s \succsim_s x_s \),

2. transitivity, i.e., for all \( x_s, y_s, z_s \in \Delta(X_s) \), if \( x_s \succsim_s y_s \) and \( y_s \succsim_s z_s \), then \( x_s \succsim_s z_s \), and

Proof. By the definition of \( \succsim_s \), for all \( x_s, y_s \in \Delta(X_s) \),

\[
x_s \succsim_s y_s \iff \forall f \in H, (x_s, f_{-s}) \succsim (y_s, f_{-s}) \tag{6}
\]

and by Axiom 3,

\[
x_s \succsim_s y_s \iff \exists f \in H, (x_s, f_{-s}) \succ (y_s, f_{-s}). \tag{7}
\]

Complete: Let \( x_s, y_s \in \Delta(X_s) \). If there exists \( f \in H \) such that \( (x_s, f_{-s}) \succ (y_s, f_{-s}) \), then by (7), \( x_s \succsim_s y_s \) and so especially by the definition of \( \succsim_s \), \( x_s \succsim_s y_s \). If the above \( f \in H \) does
not exist, then by the completeness of $\succeq_s$, for all $f \in H$, $(y_s, f_{-s}) \succeq (x_s, f_{-s})$ and so by (6) $y_s \succeq_s x_s$.

Transitive: Let $x_s, y_s, z_s \in \Delta(X_s)$ such that $x_s \succeq_s y_s \succeq_s z_s$, then $\forall f \in H$, $(x_s, f_{-s}) \succeq_s (y_s, f_{-s}) \succeq_s (z_s, f_{-s})$ and so by Axiom 1, $\forall f \in H$, $(x_s, f_{-s}) \succeq (z_s, f_{-s})$. Thus by (6) $x_s \succeq_s z_s$. □

Next, we show mixture continuity for $\succeq_s$.

**Lemma 28** Let $s \in S$. If $\succeq_s$ satisfies Axioms 1-3, then for all $x_s, y_s, z_s \in \Delta(X_s)$, the sets

\[
\{ \alpha \in [0,1] | \alpha x_s + (1 - \alpha)y_s \succeq_s z_s \}
\]

and

\[
\{ \alpha \in [0,1] | z_s \succ_s \alpha x_s + (1 - \alpha)y_s \}
\]

are closed in $\mathbb{R}$.

**Proof.** Let $s \in S$ and $x_s, y_s, z_s \in \Delta(X_s)$. We show that the sets $\{ \alpha \in [0,1] | \alpha x_s + (1 - \alpha)y_s \succeq_s z_s \}$ and $\{ \alpha \in [0,1] | z_s \succ_s \alpha x_s + (1 - \alpha)y_s \}$ are open in $\mathbb{R}$ which shows the claim since by Lemma 27, $\succeq_s$ is complete. We show that $\{ \alpha \in [0,1] | \alpha x_s + (1 - \alpha)y_s \succeq_s z_s \}$ is open since $\{ \alpha \in [0,1] | z_s \succ_s \alpha x_s + (1 - \alpha)y_s \}$ is open by a symmetric argument. Let $\alpha^0 \in \{ \alpha \in [0,1] | \alpha x_s + (1 - \alpha)y_s \succeq_s z_s \}$. Thus $\alpha^0 x_s + (1 - \alpha^0)y_s \succeq_s z_s$. By the definition of $\succ_s$, there exists $f \in H$ such that $(\alpha^0 x_s + (1 - \alpha^0)y_s, f_{-s}) \succ (z_s, f_{-s})$. Thus $\alpha^0 \in \{ \alpha \in [0,1] | (\alpha^0 x_s + (1 - \alpha^0)y_s, f_{-s}) \succ (z_s, f_{-s}) \}$. By Axiom 2, there exists a neighborhood $V$ of $\alpha^0$ such that for all $\alpha \in V \cap [0,1]$, $(\alpha x_s + (1 - \alpha)y_s, f_{-s}) \succ (z_s, f_{-s})$. Thus for all $\alpha \in V \cap [0,1]$, $\alpha x_s + (1 - \alpha)y_s \succ z_s$. Hence $V \cap [0,1] \subseteq \{ \alpha \in [0,1] | z_s \succ \alpha x_s + (1 - \alpha)y_s \}$ and since $V$ is a neighborhood of $\alpha^0$ this shows the claim. □

Next, we put the above together for weak affine representation.

**Lemma 29** Let $s \in S$. If $\succeq_s$ satisfies Axioms 1-4, then there exists an affine $u_s : \Delta(X_s) \to \mathbb{R}$ such that $u_s$ is a weak affine representation for $\succeq_s$.

**Proof.** By Lemmas 27 and 28, $\succeq_s$ is a nontrivial weak order and satisfies continuity. By Axiom 4, $\succeq_s$ satisfies Weak Independence. By Proposition 23, there exists an affine $u_s : \Delta(X_s) \to \mathbb{R}$ such that for all $x_s, y_s \in \Delta(X_s),

\[ x_s \succ_s y_s \Rightarrow u_s(x_s) > u_s(y_s). \] □
The next result shows statewise monotonicity results with the weak affine representation.

**Lemma 30** Assume that for each \( s \in S \), \( \succsim_s \) has a weak affine representation with \( u_s \). Let \( s \in S \). Then

1. For all \( x_s, y_s \in \Delta(X_s) \), \( u_s(x_s) \geq u_s(y_s) \implies x_s \succsim_s y_s \).
2. For all \( f, g \in H \), if for all \( s \in S^P \), \( u_s(f_s) \geq u_s(g_s) \), then \( f \succsim g \).

**Proof.** The first one follows from the negation of the definition of weak representation. The second one follows from the first one and Lemma 26. \( \square \)

### C.2 Preferences Monotonic Between Equally Crisp Acts

Next we show that \( \succsim \) is monotonic when moving from one crisp act to another one.

**Lemma 31** Let \( \succsim \) satisfy Axioms 1-5. Let for all \( s \in S^P \), \( u_s : \Delta(X_s) \to \mathbb{R} \) be an affine weak representation for \( \succsim_s \). Let \( f^*, g^* \) be equally crisp acts such that for all \( s \in S^P \),

\[
u_s(f^*_s) > u_s(g^*_s).
\]

(8)

Then for all \( \alpha, \alpha' \in [0,1] \),

\[
\alpha \geq \alpha' \iff f^* \alpha g^* \succsim f^* \alpha' g^*.
\]

**Proof.** Now \( u(f^*/2g^*) \in \text{int } u(H) \). Assume, per contra, that there exist \( 1 > \alpha^* > \alpha_s > 0 \) such that

\[
f^* \alpha g^* \nsim f^* \alpha' g^*.
\]

(9)

By the nontriviality, there exists \( f^0 \nsim f^* \alpha g^* \). W.l.o.g. assume that \( f^* \alpha g^* \nsim f^0 \). By continuity, there exists \( \alpha^1 \in (0,1) \) such that \( f^1 = \alpha^1 f^0 + (1 - \alpha^1)(f^*/2g^*) \) and \( f^* \alpha g^* \nsim f^1 \). Now \( u(f^1) \in \text{int } u(H) \). Let

\[
\alpha^f = \inf \{ \alpha \in [0,1] \mid \alpha f^1 + (1 - \alpha) f^* \alpha g^* \succsim f^* \alpha g^* \}.
\]

(10)

By the continuity and the assumption, \( \alpha^* \in [0,1] \) and \( \alpha^f f^0 + (1 - \alpha^f)(f^* \alpha g^*) \sim f^* \alpha g^* \). Now by Corollary 55, \( f^* \alpha g^* \) and \( f^* \alpha^* g^* \) are equally crisp. Thus

\[
\alpha^f f^0 + (1 - \alpha^f)(f^* \alpha g^*) \sim \alpha^f f^* \alpha g^* + (1 - \alpha^f)(f^* \alpha g^*).
\]

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and so
\[ \alpha f^0 + (1 - \alpha f^0) f^* \alpha_s g^* \sim \alpha f^* \alpha_s g^* + (1 - \alpha f^0) f^* \alpha_s g^* = f^* (\alpha f^* \alpha_s g^* + (1 - \alpha f^0) \alpha_s g^*). \quad (11) \]

Let \( 0 < \varepsilon < 1 - \alpha f^0 \) and
\[ 0 < \varepsilon < \frac{(1 - \alpha f^0)(\alpha^* - \alpha_s) \min_{s \in S^P} |u_s(f^0_s) - u(g^*_s)|}{\|u(f^0) - u(f^* \alpha^* g^*)\|_\infty}. \quad (12) \]

Then we have for all \( s \in S^P \),
\[ u_s((\alpha f^0 + \varepsilon) f^0 + (1 - \alpha f^0 - \varepsilon) f^* \alpha_s g^*) \]
\[ = \alpha f^0 u_s(f^0_s) + (1 - \alpha f^0) (u_s(g^*_s) + \alpha^* (u_s(f^0_s) - u_s(g^*_s))) + \varepsilon (u_s(f^0_s) - u_s(f^* \alpha_s g^*_s)) \]
\[ \geq \alpha f^0 u_s(f^0_s) + (1 - \alpha f^0) u_s(g^*_s) + (1 - \alpha f^0) \alpha^* (u_s(f^0_s) - u_s(g^*_s)) - \varepsilon \|u(f^0) - u(f^* \alpha^* g^*)\| \]
\[ \geq \alpha f^0 u_s(f^0_s) + (1 - \alpha f^0) u_s(g^*_s) + (1 - \alpha f^0) \alpha^* (u_s(f^0_s) - u_s(g^*_s)) \]
\[ \geq (\alpha f^0 + \varepsilon) f^0 + (1 - \alpha f^0 - \varepsilon) f^* \alpha_s g^* \]
\[ \geq u_s(\alpha f^0 f^0 + (1 - \alpha f^0) f^* \alpha_s g^*). \quad (13) \]

Thus by Lemma 30,
\[ (\alpha f^0 + \varepsilon) f^0 + (1 - \alpha f^0 - \varepsilon) f^* \alpha_s g^* \sim \alpha f^0 + (1 - \alpha f^0) f^* \alpha_s g^* \]
\[ \sim f^*(\alpha f^* \alpha + (1 - \alpha f^0) \alpha_s) g^* \sim f^* \alpha_s g^* \sim f^* \alpha^* g^*. \quad (8) \]

Thus for all \( 0 < \varepsilon < 1 - \alpha f^0 \) that satisfy (12), \( (\alpha f^0 + \varepsilon) f^0 + (1 - \alpha f^0 - \varepsilon) f^* \alpha_s g^* \sim f^* \alpha^* g^* \). However, this contradicts the definition of \( \alpha f^0 \) in (10). Thus if \( 1 > \alpha^* > \alpha_s > 0 \), then by (8) and Lemma 30 \( f^* \alpha^* g^* \sim f^* \alpha_s g^* \). Finally, if \( 1 \geq \alpha^* > \alpha_s \geq 0 \), then by above, (8) and Lemma 30,
\[ f^* \alpha^* g^* \sim f^*(3/4 \alpha^* + 1/4 \alpha^*) g^* \sim f^*(1/4 \alpha^* + 3/4 \alpha^*) g^* \sim f^* \alpha_s g^* \]
\[ \square \]

The next result generalizes the previous monotonicity when the equally crisp acts are mixed with another act.
Lemma 32  Let $\succeq$ satisfy Axioms 1-5 and let for all $s \in S^p$, $u_s : \Delta(X_s) \to \mathbb{R}$ be a weak affine representation for $\succeq_s$. Let $f^*, g^*$ be equally crisp acts such that for all $s \in S^p$, $u_s(f^*_s) > u_s(g^*_s)$. Then for all $f \in H, \beta \in [0, 1),$

$$\alpha \geq \alpha' \iff \beta f + (1 - \beta)(f^* \alpha g^*) \succeq \beta f + (1 - \beta)(f^* \alpha' g^*).$$

Proof. Let $f \in H$ and $\beta \in (0, 1)$. By Corollary 56, $\beta f + (1 - \beta)f^*, \beta f + (1 - \beta)g^*$ are equally crisp acts and for all $s \in S^p$, by the affinity of $u_s$, $u_s(\beta f + (1 - \beta)f^*) > u_s(\beta f + (1 - \beta)g^*)$. Thus the claim follows from Lemma 31.

C.3 Transfinite Induction

The next steps show the existence of C-additive and monotonic representation. First, we define a representation between the two equally crisp acts. Second, we extend this representation using the equally crisp acts in a manner that maintains C-additivity. This is done using transfinite induction in multiple steps.

The next lemma shows, that extending the starting representation using equally crisp acts and C-additivity gives a unique way to extend the representation.

Lemma 33  Assume that $S = S^p$. Let $\succeq$ satisfy Axioms 1-4,6, and for all $s \in S^p$, let $u_s : \Delta(X_s) \to \mathbb{R}$ be a weak affine representation for $\succeq_s$ such that

$$u_s(f^*_s) - u_s(g^*_s) = 1. \quad (14)$$

Let $A \subseteq \{(a_1, a_2) \subseteq \mathbb{R}|a_1, a_2 \in \mathbb{R} \cup \{-\infty, \infty\}, a_1 \leq a_2\}$ be such that $A \in \mathcal{A}$ if and only if

1. There exists $\bar{I}$ such that $\text{dom} \bar{I} \subseteq \text{int} u(H)$ and $\text{Im} \bar{I} = A$.
2. For all $f \in H$ such that $u(f) \in \text{int} u(H)$, if there exist $f', g' \in H$ such that $u(f')$, $u(g') \in \text{dom} \bar{I}$ and $f' \succeq f \succeq g'$, then $u(f) \in \text{dom} \bar{I}$.
3. For all $\alpha \in (0, 1)$, $u(\alpha f^* + (1 - \alpha)g^*) \in \text{dom} \bar{I}$ and

$$\bar{I} \circ u(\alpha f^* + (1 - \alpha)g^*) = \alpha.$$

4. If $\varphi \in \text{dom} \bar{I}$ and $\alpha^1 > 0$ are such that $\varphi + \alpha^1 \bar{I} \in \text{dom} \bar{I}$, then $\bar{I}(\varphi + \alpha^1 \bar{I}) = \bar{I}(\varphi) + \alpha^1$.
5. For all $f, g \in H$ with $u(f)$, $u(g) \in \text{dom} \bar{I}$, $f \succeq g$ iff $\bar{I}(u(f)) \geq \bar{I}(u(g))$.

Let $A_1, A_2 \in \mathcal{A}$, $A_1 \subseteq A_2$ and $I_1, I_2$ be the associated functions. Then $\text{dom} I_1 \subseteq \text{dom} I_2$ and for all $P \in \text{dom} I_1$, $I_1(P) = I_2(P)$.  

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Proof. Denote $f^*\frac{1}{2}g^* = \frac{1}{2}f^* + \frac{1}{2}g^*$. We will show the claim for all $\varphi \in \text{dom } I_1$ such that $I_1(\varphi) \geq \frac{1}{2}$. The other case follows symmetrically. Denote

$$m := \inf \left\{ a \in [1/2, \infty) \middle| \exists \psi \in \text{dom } I_1, I_1(\psi) = a, [\psi \not\in \text{dom } I_2 \text{ or } I_1(\psi) \neq I_2(\psi)] \right\}. \quad (15)$$

We will show that $m = \infty$. We will first show that $m > 0$. Let $\varphi \in \text{dom } I_1$, be such that $\frac{1}{2} \leq I_1(\varphi) < 1$. Let $f \in H$ be such that $u(f) = \varphi$ and $\alpha^\varphi(I_1(\varphi), 1)$. Now we have by Condition 3,

$$u\left(f^*\frac{1}{2}g^*\right), u\left(\alpha^\varphi f^* + (1 - \alpha^\varphi)g^*\right), u\left(I_1(\varphi)f^* + (1 - I_1(\varphi))g^*\right) \in \text{dom } I_1, \text{dom } I_2.$$

By Conditions 3 and 5 for $I_1$,

$$\alpha^\varphi f^* + (1 - \alpha^\varphi)g^* \gtrsim f \sim I_1(\varphi)f^* + (1 - I_1(\varphi))g^* \gtrsim f^*\frac{1}{2}g^*.$$

By Condition 2 for $I_2$, $u(f) = \varphi \in \text{dom } I_2$. And by Conditions 3 and 5 for $I_2$,

$$I_2(\varphi) = I_2(u(f)) = I_2 \circ u\left(I_1(\varphi)f^* + (1 - I_1(\varphi))g^*\right) = I_1(\varphi).$$

Thus $m > \frac{1}{2}$.

Assume, per contra, $m < \infty$. By Condition 1, there exists $\varphi \in \text{dom } I_1$ and $I_1 \circ u(\varphi) = m$. Since $A_1$ is open, there exists $\bar{f}^1 \in H$ such that $u(\bar{f}^1) \in \text{dom } I_1$ and $I_1 \circ u(f^m) < I_1 \circ u(\bar{f}^1)$.

Since $A_2 \supseteq A_1$, there exists $\bar{f}^2 \in H$ such that $u(\bar{f}^2) \in \text{dom } I_2$ and $I_2 \circ u(\bar{f}^2) = I_1 \circ u(\bar{f}^1)$.

Since $\varphi \in \text{int } u(H)$, there exists $\alpha^* \in (0, 1)$ such that

$$\frac{1}{\alpha^*}\varphi - \frac{1 - \alpha^*}{\alpha^*}u(f^*\frac{1}{2}g^*) \in \text{int } u(H).$$

Thus there exists $f^m \in H$ such that

$$u(f^m) = \frac{1}{\alpha^*}\varphi - \frac{1 - \alpha^*}{\alpha^*}u(f^*\frac{1}{2}g^*).$$

Now

$$\alpha^*u(f^m) + (1 - \alpha^*)u(f^*\frac{1}{2}g^*) = \varphi.$$

By Condition 5 for $I_1$,

$$\bar{f}^1 \succ \alpha^*u(f^m) + (1 - \alpha^*)u(f^*\frac{1}{2}g^*) \succ f^*\frac{1}{2}g^*.$$

By Axiom 2, there exists $\alpha^1 \in (0, 1/2)$ such that for all $\alpha \in (1/2 - \alpha^1, 1/2 + \alpha^1)$,

$$\bar{f}^1 \succ \alpha^*f^m + (1 - \alpha^*)f^*\alpha g^* \succ f^*\frac{1}{2}g^*.$$
By Condition 2, for all \( \alpha \in (\frac{1}{2} - \alpha^\dagger, \frac{1}{2} + \alpha^\dagger) \),
\[
\alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \in \text{dom } I_1.
\]

By (14) and Condition 4, for all \( \alpha \in (\frac{1}{2} - \alpha^\dagger, \frac{1}{2} + \alpha^\dagger) \),
\[
I_1 \left( \alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \right) = I_1 \left( \alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \right) + (1 - \alpha^*) (\alpha - \frac{1}{2}).
\]

Thus by (15) for all \( \alpha \in (\frac{1}{2} - \alpha^\dagger, \frac{1}{2}) \),
\[
\alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \in \text{dom } I_2
\]

and
\[
I_2 \left( \alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \right) = I_1 \left( \alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \right) < I_1 (\varphi) < I_2 \circ u(f^2).
\]

By Condition 5 for \( I_2 \), for all \( \alpha \in (\frac{1}{2} - \alpha^\dagger, \frac{1}{2}) \),
\[
\bar{f}^2 \succ \alpha^* f^m + (1 - \alpha^*) (f^* g^*)
\]

and so by Axiom 2,
\[
\bar{f}^2 \succeq \alpha^* f^m + (1 - \alpha^*) (f^* g^*).
\]

By Condition 2,
\[
\alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \in \text{dom } I_2
\]

and by Condition 4,
\[
I_2 \left( \alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \right) = I_1 \left( \alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \right).
\]

By Condition 5, \( \bar{f}^2 \succ \alpha^* f^m + (1 - \alpha^*) (f^* g^*) \). By Axiom 2, there exists \( \alpha^\dagger \in (0, \alpha^\dagger) \) such that for all \( \alpha \in [\frac{1}{2}, \frac{1}{2} + \alpha^\dagger] \),
\[
\bar{f}^2 \succ \alpha^* f^m + (1 - \alpha^*) f^* g^*.
\]

By Condition 2, for all \( \alpha \in [\frac{1}{2}, \frac{1}{2} + \alpha^\dagger] \),
\[
\alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \in \text{dom } I_1 \cap \text{dom } I_2.
\]

By Condition 4 and (16), for all \( \alpha \in [\frac{1}{2}, \frac{1}{2} + \alpha^\dagger] \),
\[
I_1 \left( \alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \right) = I_2 \left( \alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \right).
\]

Next, let \( h \in H \) and \( u(h) \in \text{dom } I_1 \) be such that
\[
I_1 \left( \alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \right) \geq I_1 \circ u(h) \geq I_1 \left( \alpha^* u(f^m) + (1 - \alpha^*) u(f^* g^*) \right).
\]
Let We show that \( u(h) \in \text{dom}\, I_2 \) and \( I_2 \circ u(h) = I_1 \circ u(h) \). By Condition 5 for \( I_1 \),
\[
\alpha^* f^m + (1 - \alpha^*)(f^* \alpha^1 g^*) \succeq h \succeq \alpha^* f^m + (1 - \alpha^*)(f^*1/2g^*).
\]
By Condition 2 and (17), \( u(h) \in \text{dom}\, I_2 \). By Axiom 2, there exists \( \alpha^h \in [1/2, \alpha^1] \) such that
\[
h \sim \alpha^* f^m + (1 - \alpha^*)(f^* \alpha^h g^*).
\]
By Condition 5 and (17,18),
\[
I_1(h) = I_1(\alpha^* u(f^m) + (1 - \alpha^*)u(f^* \alpha^1 g^*)) = I_2(\alpha^* u(f^m) + (1 - \alpha^*)u(f^* \alpha^h g^*)) = I_2(h).
\]
Thus for all \( \psi \in \text{dom}\, I_1 \) such that \( I_1(\alpha^* u(f^m) + (1 - \alpha^*)u(f^* \alpha^1 g^*)) \geq I_1(\psi) \geq I_1(\varphi) \), we have \( \psi \in \text{dom}\, I_2 \) and \( I_2(\psi) = I_1(\psi) \). This contradicts the choice of \( \varphi \) in (15). Thus \( m = \infty \). □

The next lemma shows that if we measure the utility of other acts by the two equally crisp acts, we maintain C-additivity.

**Lemma 34** Assume that \( S = S^P \). Let \( \succeq \) satisfy Axioms 1-4,6, and for all \( s \in S^P \), let \( u_s : \Delta(X_s) \to \mathbb{R} \) be a weak affine representation for \( \succeq_s \) such that
\[
u_s(f_s^*) - u_s(g_s^*) = 1.
\]
Let \( \beta^* \in [0, 1], f^\dagger \in H, c \in \mathbb{R} \) and denote
\[
B = \{ \varphi \in \text{int}\, u(H) \mid \exists \, f \in H, \alpha \in (0, 1), u(f) = \varphi, f \sim \beta^* f^\dagger + (1 - \beta^*)(f^* \alpha g^*) \}.
\]
Define \( I : B \to \mathbb{R} \) by the following. For each \( \varphi \in B \), let \( f^\varphi, \alpha^\varphi \in (0, 1) \in H \) be such that
\[
u(f) = \varphi \text{ and } f \sim \beta^* f^\dagger + (1 - \beta^*)(f^* \alpha g^*),
\]
then define
\[
I(\varphi) = (1 - \beta^* )\alpha + c. \tag{19}
\]
If \( \varphi \in B \) and \( \alpha^1 > 0 \) are such that \( \varphi + \alpha^1 \bar{1} \in \text{dom}\, B \), then \( I(\varphi + \alpha^1 \bar{1}) = \bar{I}(\varphi) + \alpha^1 \).

**Proof.** First, for all \( f, g \in H \) such that \( u(f), u(g) \in B \) and \( f \sim g \), we have by Lemma 32,
\[
I\left(u(f)\right) = I\left(u(g)\right). \tag{20}
\]

Let \( \varphi \in B \) and \( \alpha^0 > 0 \) such that \( \varphi + \alpha^0 \bar{1} \in B \). Let \( \varphi^{-1}, f^0 \in H \) be such that \( \varphi = u(\varphi^{-1}), \varphi + \alpha^0 \bar{1} = u(f^0) \). Consider the mapping \( J : [0, \alpha^0] \to \mathbb{R} \) defined by for all \( \alpha \in [0, \alpha^0] \),
\[
J(\alpha) = I(\varphi + \alpha \bar{1}).
\]
We show that this is an affine function with derivative 1. By Lemma 32 and the definition of \( B \),
\[
\beta^* f^\dagger + (1 - \beta^*) f^* \succ f^0, \varphi^{-1} \succ \beta^* f^\dagger + (1 - \beta^*) g^*. \tag{21}
\]

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Let $\alpha^1 \in [0, \alpha^0]$. Since $\phi, \varphi + \alpha^1 \bar{1} \in \text{int} u(H)$, $\varphi + \alpha^1 \bar{1} \in \text{int} u(H)$. Let $f^1 \in H$ be such that $u(f^1) = \varphi + \alpha^1 \bar{1}$. By Lemma 30,

$$
\beta^* f^1 + (1 - \beta^*) f^* \succeq 0 \succeq f^1 \succ \varphi^{-1} \succ \beta^* f^1 + (1 - \beta^*) g^*.
$$

Thus there exists $\alpha^2 \in (0, 1)$ such that $\beta^* f^1 + (1 - \beta^*) (f^* \alpha^2 g^*) \sim f^1$. Since $u(\varphi + \alpha^1 \bar{1}) \in \text{int} u(H)$, there exists $\varepsilon^\varphi > 0$ such that $B_{\infty}(\varphi + \alpha^1 \bar{1}, \varepsilon^\varphi) \subseteq u(H)$.

Let $\beta^\varphi \in (\beta^*, 1)$, which exists since $\beta^* < 1$, be such that

$$
\frac{1 - \beta^\varphi}{\beta^\varphi} (\|\varphi + \alpha^1 \bar{1}\|_{\infty} + \|u(f^* \alpha^2 g^*)\|_{\infty}) < \varepsilon^\varphi.
$$

Thus by the choice of $\beta^\varphi$, there exists $f^\varphi \in H$ such that

$$
u(f^\varphi) = \varphi + \alpha^1 \bar{1} + \frac{1 - \beta^\varphi}{\beta^\varphi} (\varphi + \alpha^1 \bar{1}) - \frac{1 - \beta^\varphi}{\beta^\varphi} u(f^* \alpha^2 g^*).
$$

Especially

$$\beta^\varphi u(f^\varphi) + (1 - \beta^\varphi) u(f^* \alpha^2 g^*) = \varphi + \alpha^1 \bar{1} = u(f^1).
$$

Now by Lemma 30 and since $\beta^\varphi > \beta^*$,

$$
\beta^\varphi f^\varphi + (1 - \beta^\varphi)(f^* \alpha^2 g^*) \sim f^1 \sim \beta^* f^1 + (1 - \beta^*) (f^* \alpha^2 g^*)
\equiv \beta^\varphi \left( \frac{\beta^\varphi}{\beta^\varphi} f^1 + \frac{\beta^\varphi - \beta^*}{\beta^\varphi} (f^* \alpha^2 g^*) \right) + (1 - \beta^\varphi) (f^* \alpha^2 g^*).
$$

Since $f^*, g^*$ are equally crisp, we have for all $\gamma \in [0, 1]$, by Corollary 54,

$$\beta^\varphi f^\varphi + (1 - \beta^\varphi)(f^* \gamma g^*) \sim \beta^\varphi \left( \frac{\beta^\varphi}{\beta^\varphi} f^1 + \frac{\beta^\varphi - \beta^*}{\beta^\varphi} (f^* \alpha^2 g^*) \right) + (1 - \beta^\varphi) (f^* \gamma g^*). \quad (22)
$$

By the linearity of $u$, we have

$$
u(\beta^\varphi f^\varphi + (1 - \beta^\varphi)(f^* \gamma g^*))
\equiv u(f^1) + (1 - \beta^\varphi) \left( u(f^* \gamma g^*) - u(f^* \alpha^2 g^*) \right) = \varphi + \left( \alpha^1 + (1 - \beta^\varphi)(\gamma - \alpha^2) \right) \bar{1}
$$

and since $1 - \beta^* > 1 - \beta^\varphi$

$$\beta^\varphi u\left( \frac{\beta^\varphi}{\beta^\varphi} f^1 + \frac{\beta^\varphi - \beta^*}{\beta^\varphi} (f^* \alpha^2 g^*) \right) + (1 - \beta^\varphi) u(f^* \gamma g^*)
\equiv \beta^* u(f^1) + \left( (\beta^\varphi - \beta^*)(\alpha^2 + (1 - \beta^\varphi) \gamma) u(f^*) + (\beta^\varphi - \beta^*)(1 - \alpha^2)(1 - \gamma) u(g^*) \right)
= \beta^* u(f^1) + (1 - \beta^*) u\left( f^* \left( \alpha^2 + \frac{1 - \beta^\varphi}{1 - \beta^*}\gamma \bar{1} \right) g^* \right)
$$

We have for all $\gamma \in [0, 1]$,

$$J\left( \alpha^1 + (1 - \beta^\varphi)(\gamma - \alpha^2) \right)$$
Hence
\[ J(x) = J\left(\alpha^1 + (1 - \beta^x)(\gamma - \alpha^2)\right) \]
\[ = (1 - \beta^x)\alpha^2 + (1 - \beta^x)(\frac{x - \alpha^1}{1 - \beta^x} + \alpha^2 - \alpha^2) + c = (1 - \beta^x)\alpha^2 + (1 - \beta^x)(\gamma - \alpha^2) + c. \]

Thus for all \( x \in (\alpha^1 - (1 - \beta^x)\alpha^2, \alpha^1 + (1 - \beta^x)(1 - \alpha^2)) \cap [0, \alpha^1] \), first \( \frac{x - \alpha^1}{1 - \beta^x} + \alpha^2 \in (0, 1) \) and so by above,
\[ J(x) = J\left(\alpha^1 + (1 - \beta^x)\left(\frac{x - \alpha^1}{1 - \beta^x} + \alpha^2 - \alpha^2\right)\right) \]
\[ = (1 - \beta^x)\alpha^2 + (1 - \beta^x)(\frac{x - \alpha^1}{1 - \beta^x} + \alpha^2 - \alpha^2) + c = (1 - \beta^x)\alpha^2 + x + c. \]

So \( J \) is affine in the neighborhood \( (\alpha^1 - (1 - \beta^x)\alpha^2, \alpha^1 + (1 - \beta^x)(1 - \alpha^2)) \cap [0, \alpha^1] \) of \( \alpha^1 \) and the derivative is 1. Since the point \( \alpha^1 \in [0, \alpha^0] \) was arbitrary, the derivative of \( J \) is locally constant at 1 and \( J \) is affine locally at 0 and \( \alpha^0 \). Since the interval \( [0, \alpha^0] \) is connected, by (Viro et al., 2008, Problem 12.2x) the derivative of \( J \) is constant at 1 on the set \( (0, \alpha^0) \).

Thus \( J \) is affine function on the set \( [0, \alpha^0] \). Thus by the Fundamental theorem of calculus and since \( J \) is locally affine at 0 and \( \alpha^0 \),
\[ J(\alpha^0) - J(0) = J(\alpha^0) - \lim_{\alpha \to \alpha^0} J(\alpha) - J(0) + \lim_{\alpha \to \alpha^0} J(\alpha) + \int_0^{\alpha^0} \nabla J(\alpha)d\alpha = \int_0^{\alpha^0} 1d\alpha = \alpha^0. \]

Hence
\[ I(\varphi + \alpha^0) = J(\alpha^0) = J(0) + \alpha^0 = I(\varphi) + \alpha^0. \]

The next lemma is the first step to using Zorn’s lemma for the extension and shows that the set of extensions is non-empty.

**Lemma 35** Assume that \( S = S^p \). Let \( \succcurlyeq \) satisfy Axioms 1-4,6, and for all \( s \in S^p \), let \( u_s : \Delta(X_s) \to \mathbb{R} \) be a weak affine representation for \( \succcurlyeq_s \) such that
\[ u_s(f_s^*) - u_s(g_s^*) = 1. \]

Let \( \mathcal{A} \subseteq \{(a_1, a_2) \subseteq \mathbb{R} | a_1, a_2 \in \mathbb{R} \cup \{-\infty, \infty\}, a_1 \leq a_2\} \) be such that \( A \in \mathcal{A} \) if and only if
(1) There exists \( I \) such that \( \text{dom } I \subseteq \text{int } u(H) \) and \( \text{Im } I = A \).
(2) For all $f \in H$ such that $u(f) \in \text{int} u(H)$, if there exist $f', g' \in H$ such that $u(f'), u(g') \in \text{dom} \bar{I}$ and $f' \succ f \succ g'$, then $u(f) \in \text{dom} \bar{I}$.

(3) For all $\alpha \in (0, 1)$, $u(\alpha f^* + (1 - \alpha)g^*) \in \text{dom} \bar{I}$ and

$$\bar{I} \circ u(\alpha f^* + (1 - \alpha)g^*) = \alpha.$$

(4) If $\varphi \in \text{dom} \bar{I}$ and $\alpha^1 > 0$ are such that $\varphi + \alpha^1 \bar{1} \in \text{dom} \bar{I}$, then $\bar{I}(\varphi + \alpha^1 \bar{1}) = \bar{I}(\varphi) + \alpha^1$.

(5) For all $f, g \in H$ with $u(f), u(g) \in \text{dom} \bar{I}$, $f \succ g$ iff $\bar{I}(u(f)) \geq \bar{I}(u(g))$.

Then $A \neq \emptyset$.

**Proof.** We show that $(0, 1) \in A$. Denote

$$B = \{ \varphi \in \text{int} u(H) | \exists f \in H, u(f) = \varphi, f^* \succ f \succ g^* \}.$$

Define $I^0 : B \to \mathbb{R}$ by the following. Let $\varphi \in B$ and $f \in H$ such that $u(f) = \varphi, f^* \succ f \succ g^*$. By Axiom 2, there exists $\alpha^f \in (0, 1)$ such that $f^* \alpha^f + (1 - \alpha^f)g^* \sim f$. Define $I^0(\varphi) = \alpha^f$. By Lemmas 30 and 31, this is well-defined function. We show that $(0, 1)$ and with the function $I^0$ satisfies Conditions 1-5.

- **Conditions 1-3** follow from the definition of $I^0$.
- **Condition 4:** Follows from Lemma 34.
- **Condition 5:** Since for all $h \in \text{dom} I^0$, $f^* \succ h \succ g^*$, there exist $\alpha^f, \alpha^g \in (0, 1)$ such that $f \sim \alpha^f f^* + (1 - \alpha^f)g^*$ and $g \sim \alpha^g f^* + (1 - \alpha^g)g^*$. By the definition of $I^0$ and Lemma 32, we have

$$I^0(u(f)) \geq I^0(u(g)) \iff \alpha^f \geq \alpha^g \iff f^* \alpha^f g^* \succ f^* \alpha^g g^* \iff f \succ g.$$

Thus $(0, 1) \in A$. \qed

The next lemma is the second step to using Zorn’s lemma and shows that any chain of extensions has an upper bound.

**Lemma 36** Assume that $S = S^P$. Let $\succsim$ satisfy Axioms 1-4,6, and for all $s \in S^P$, let $u_s : \Delta(X_s) \to \mathbb{R}$ be a weak affine representation for $\succsim_s$ such that

$$u_s(f_s^*) - u_s(g_s^*) = 1.$$

Let $A \subseteq \{(a_1, a_2) \subseteq \mathbb{R} | a_1, a_2 \in \mathbb{R} \cup \{-\infty, \infty\}, a_1 \leq a_2 \}$ be such that $A \in A$ if and only if

(1) There exists $\bar{I}$ such that $\text{dom} \bar{I} \subseteq \text{int} u(H)$ and $\text{Im} \bar{I} = A$. 

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(2) For all \( f \in H \) such that \( u(f) \in \text{int} u(H) \), if there exist \( f', g' \in H \) such that \( u(f'), u(g') \in \text{dom} \tilde{I} \) and \( f' \succ f \succ g' \), then \( u(f) \in \text{dom} \tilde{I} \).

(3) For all \( \alpha \in (0, 1) \), \( u(\alpha f^* + (1 - \alpha)g^*) \in \text{dom} \tilde{I} \) and 
\[
\tilde{I} \circ u(\alpha f^* + (1 - \alpha)g^*) = \alpha.
\]

(4) If \( \varphi \in \text{dom} \tilde{I} \) and \( \alpha^1 > 0 \) are such that \( \varphi + \alpha^1 \tilde{I} \in \text{dom} \tilde{I}, \) then \( \tilde{I}(\varphi + \alpha^1 \tilde{I}) = \tilde{I}(\varphi) + \alpha^1. \)

(5) For all \( f, g \in H \) with \( u(f), u(g) \in \text{dom} \tilde{I}, f \succ g \iff \tilde{I}(u(f)) \geq \tilde{I}(u(g)). \)

Let \( B \subseteq A \) be such that for each \( B_1, B_2 \in B, B_1 \subseteq B_2 \) or \( B_2 \subseteq B_1 \). Then \( \bigcup_{B \in B} B \subseteq A. \)

Proof. Denote \( B^* := \bigcup_{B \in B} B. \) For each \( B \in B \) denote by \( I^B \) the associated function. Since each \( B \) consists of nested open intervals, \( B^* \) is an open interval. Next, define the associated extension \( I^* \). For all \( \varphi \in \text{int} u(H) \), if there exists \( B \in B \) such that \( \varphi \in \text{dom} I^B \), define \( I^*(\varphi) = I^B(\varphi). \) Since \( B \) is a chain by Lemma 33, \( I^* \) is well-defined. We will show that \( I^* \) and \( B^* \) satisfies Conditions 1-5.

Condition 1 follows from the definition of \( I^* \).

Condition 2: Let \( f \in H \) be such that \( u(f) \in \text{int} u(H) \) and \( f', g' \in H \) such that \( u(f'), u(g') \in \text{dom} I^* \) and \( f' \succ f \succ g' \). Now there exist \( B_1, B_2 \) such that \( f' \in \text{dom} I^{B_1}, g' \in \text{dom} I^{B_2} \) and w.l.o.g. \( B_1 \subseteq B_2 \). By Lemma 33, \( f' \in \text{dom} I^{B_2} \). By Condition 2 for \( B_2, u(f) \in \text{dom} I^{B_2} \).

Condition 3 follows from the definition of \( I^* \).

Conditions 4 and 5 follow similarly as 2 since \( B \) is a chain.

The next lemma is the last step to using Zorn’s lemma for the extension and shows that any extension can be extended further unless they are the desired representation defined on the interior of \( u(H) \).

Lemma 37 Assume that \( S = S^P \). Let \( \succcurlyeq \) satisfy Axioms 1-4,6, and for all \( s \in S^P \), let \( u_s : \Delta(X_s) \to \mathbb{R} \) be a weak affine representation for \( \succcurlyeq_s \) such that 
\[
u_s(f_s^*) - u_s(g_s^*) = 1.
\]

Let \( \mathcal{A} \subseteq \{(a_1, a_2) \subseteq \mathbb{R} | a_1, a_2 \in \mathbb{R} \cup \{-\infty, \infty\}, a_1 \leq a_2 \} \) be such that \( A \in \mathcal{A} \) if and only if

(1) There exists \( \tilde{I} \) such that \( \text{dom} \tilde{I} \subseteq \text{int} u(H) \) and \( \text{Im} \tilde{I} = A. \)
(2) For all \( f \in H \) such that \( u(f) \in \text{int} u(H) \), if there exist \( f', g' \in H \) such that \( u(f'), u(g') \in \text{dom } I \) and \( f' \succ f \succ g' \), then \( u(f) \in \text{dom } I \).

(3) For all \( \alpha \in (0, 1) \), \( u(\alpha f^* + (1 - \alpha)g^*) \in \text{dom } I \) and

\[
I \circ u(\alpha f^* + (1 - \alpha)g^*) = \alpha.
\]

(4) If \( \varphi \in \text{dom } I \) and \( \alpha^1 > 0 \) are such that \( \varphi + \alpha^11 \in \text{dom } I \), then \( I(\varphi + \alpha^11) = I(\varphi) + \alpha^1 \).

(5) For all \( f, g \in H \) with \( u(f), u(g) \in \text{dom } I \), \( f \succ g \iff I(u(f)) \geq I(u(g)) \).

If \( A \in A \) with an associated function \( I^A \) and there exists \( \varphi \in \text{int} u(H) \) such that \( \varphi \notin \text{dom } I^A \), then there exists \( A^* \supseteq A \) such that \( A^* \in A \).

**Proof.** Let \( a_1, a_2 \in \mathbb{R} \cup \{\infty, -\infty\} \) be such that \( A = (a_1, a_2) \). Assume that there exists \( f^1 \in H \) such that \( u(f^1) \in \text{int} u(H) \) and \( u(f^1) \notin \text{dom } I^A \). Assume w.l.o.g. \( f^1 \succ h^* \). Let

\[
\alpha^2 \coloneqq \sup\{\alpha \in [0, 1] | u(\alpha f^1 + (1 - \alpha)h^*) \notin \text{dom } I^A\}.
\]

Since \( u(f^1) \notin \text{dom } I^A \), \( \alpha^2 \in [0, 1] \). Denote

\[
f^2 \coloneqq \alpha^2 f^1 + (1 - \alpha^2)h^*.
\]

Now \( u(f^2) \in \text{int} u(H) \) and thus there exists \( \varepsilon > 0 \) such that \( B_\infty(u(f^2), \varepsilon) \subseteq u(H) \). Let \( \beta^* \in (1/2, 1) \) be such that

\[
\left(\frac{1 - \beta^*}{\beta^*}\right)\left(\|u(f^2)\|_\infty + \|u(h^*)\|_\infty\right) < \varepsilon.
\]

By the choice of \( \beta^* \), there exists \( f^3 \in H \) such that

\[
u(f^3) = u(f^2) + \left(\frac{1 - \beta^*}{\beta^*}\right)u(f^2) - \frac{1 - \beta^*}{\beta^*}u(h^*).
\]

Especially, \( u(f^3) \in \text{int} u(H) \) and

\[
\beta^* u(f^3) + (1 - \beta^*)u(h^*) = u(f^2).
\]

Next, we show that \( f^2 \succeq f^* \): Assume, per contra, \( f^* \succ f^2 \). By applying Axiom 2 three times and by Lemma 31, there exist \( \alpha, \tilde{\alpha} \in (0, 1) \) such that for all \( \tilde{\alpha} \in [0, \alpha) \),

\[
\tilde{\alpha} f^* + (1 - \tilde{\alpha})g^* \succ \tilde{\alpha} f^1 + (1 - \tilde{\alpha})f^2 = (\tilde{\alpha} + (1 - \tilde{\alpha})\alpha^2)f^1 + (1 - \alpha^2)(1 - \tilde{\alpha})h^* \succ \tilde{\alpha} g^* + (1 - \tilde{\alpha})f^*.
\]

Thus by Conditions 2 and 3, for all \( \tilde{\alpha} \in [0, \alpha) \),

\[
u\left((\tilde{\alpha} + (1 - \tilde{\alpha})\alpha^2)f^1 + (1 - \alpha^2)(1 - \tilde{\alpha})h^*\right) \in \text{dom } I^A,
\]

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which contradicts (25). Thus \( f^2 \gtrsim f^* \). Especially, by Axiom 2,
\[
\alpha^2 > 0. 
\] (28)

Additionally, we have by Lemma 30,
\[
\beta^* f^3 + (1 - \beta^*) h^* \sim f^2 \gtrsim f^* \succ h^*. 
\]

Since by Lemma 53, \( h^* \) and \( g^* \) are equally crisp and by Lemma 32, we have for all \( \alpha \in [0, 1] \),
\[
\beta^* f^3 + (1 - \beta^*)(f^* \alpha g^*) \gtrsim \beta^* f^3 + (1 - \beta^*)g^* \succ \beta^* h^* + (1 - \beta^*)g^* \succ g^*. 
\] (29)

We show next that for all \( \alpha \in [0, \frac{1}{2}] \),
\[
\beta^* u(f^3) + (1 - \beta^*)u(f^* \alpha g^*) \in \text{dom } I^A. 
\] (30)

Let \( \alpha \in [0, \frac{1}{2}] \). Let \( 0 < \varepsilon^\alpha < \alpha^2 \) be such that
\[
0 < \varepsilon^\alpha < (1 - \beta^*)^{(1/2 - \alpha)} \min_{s \in S^p} |u_s(f_s^*) - u(g_s^*)| \|u(h^*) - u(f^1)\|_\infty \] (31)
and \( u((\alpha^2 - \varepsilon^\alpha)f^1 + (1 - \alpha^2 + \varepsilon^\alpha)h^*) \in \text{dom } I^A \) that exists by (25,28) and since \( u(h^*) \neq u(f^1) \) by Condition 3. Then we have for all \( s \in S^p \),
\[
u_s((\alpha^2 - \varepsilon^\alpha)f_s^1 + (1 - \alpha^2 + \varepsilon^\alpha)h_s^*) \\
\geq \alpha^2 u_s(f_s^1) + (1 - \alpha^2) u_s(h_s^*) - \varepsilon^\alpha \|u(h^*) - u(f^1)\|_\infty \\
\overset{(26,31)}{\geq} u_s(f^2) - \beta^*(1/2 - \alpha) \min_{s \in S^p} |u_s(f_s^2) - u_s(g_s^2)| \\
\overset{(27)}{\geq} u_s(\beta^* f_s^3 + (1 - \beta^*) h_s^*) - \beta^* \left( \frac{1}{2} - \alpha \right) (u_s(f_s^*) - u_s(g_s^*)) \\
= \beta^* u_s(f_s^3) + (1 - \beta^*) \left( \frac{1}{2} u_s(f_s^*) + \frac{1}{2} u_s(g_s^*) \right) + \beta^* \left( \alpha - \frac{1}{2} \right) (u_s(f_s^*) - u_s(g_s^*)) \\
= \beta^* u_s(f_s^3) + (1 - \beta^*) \left( \alpha u_s(f_s^*) + (1 - \alpha) u_s(g_s^*) \right) = u_s(\beta^* f_s^3 + (1 - \beta^*)(f_s^* \alpha g_s^*)). 
\]

By Lemma 30,
\[
(\alpha^2 - \varepsilon^\alpha)f^1 + (1 - \alpha^2 + \varepsilon^\alpha)g^* \gtrsim \beta^* f^3 + (1 - \beta^*)(f^* \alpha g^*) \overset{(29)}{\succ} g^*. 
\]

By Condition 2,
\[
\beta^* u(f^3) + (1 - \beta^*)u(f^* \alpha g^*) \in \text{dom } I^A 
\]

Next, we show that
\[
I^A(u(\beta^* f^3 + (1 - \beta^*)g^*)) + (1 - \beta^*) \frac{1}{2} = a_2 
\] (32)
and so especially $a_2 < \infty$. By Condition 5 and (30),

$$I^A(u(\beta^* f^3 + (1 - \beta^*)g^*) + (1 - \beta^*)\frac{1}{2} \leq a_2.$$

Assume, per contra, that there exists $k \in H$ such that $u(k) \in \text{dom } I^A$ and

$$I^A \circ u(k) > I^A(u(\beta^* f^3 + (1 - \beta^*)g^*)) + (1 - \beta^*)\frac{1}{2}.$$

Then by (30) and Conditions 4 and 5, for all $\alpha \in (0, 1/2)$,

$$k \succeq \beta^* f^3 + (1 - \beta^*)(f^* \alpha g^*).$$

Thus by Axiom 2,

$$k \succeq \beta^* f^3 + (1 - \beta^*)h^*.$$

By Condition 2, $\beta^* f^3 + (1 - \beta^*)h^* \in \text{dom } I^A$ which is a contradiction. This shows (32).

Denote

$$B = \{ \varphi \in \text{int } u(H) \mid \exists f \in H, \alpha \in (0, 1), u(f) = \varphi, f \sim \beta^* f^3 + (1 - \beta^*)(f^* \alpha g^*) \}.$$

We define an extension of $I^A$, $I^* : B \cup \text{dom } I^A \to \mathbb{R}$ as follows: for all $\varphi \in \text{dom } I^A$, $I^*(\varphi) = I^A(\varphi)$.

For all $f \in H$ such that $u(f) \notin \text{dom } I^A$, $u(f) \in \text{int } u(H)$, and there exists $\alpha \in [0, 1)$ such that $f \sim \beta^* f^3 + (1 - \beta^*)(f^* \alpha g^*)$, define $I^*(u(f)) = I^A(u(\beta^* f^3 + (1 - \beta^*)g^*)) + (1 - \beta^*)\alpha$.

First, by (32), $\text{Im } I^* \supseteq A$. Second, by Lemmas 30 and 32 and Conditions 4 and 5, $I^*$ is well-defined. Especially, we have for all $\alpha \in [0, 1/2)$, $I^A(u(\beta^* f^3 + (1 - \beta^*)(f^* \alpha g^*))) = I^A(u(\beta^* f^3 + (1 - \beta^*)g^*)) + (1 - \beta^*)\alpha$ by Condition 4 and (30). Thus for all $\alpha \in [0, 1)$,

$$I^*(u(\beta^* f^3 + (1 - \beta^*)(f^* \alpha g^*))) = I^A(u(\beta^* f^3 + (1 - \beta^*)g^*)) + (1 - \beta^*)\alpha. \quad (33)$$

We show that $I^*$ satisfies Conditions 1-5 for the interval $(a_1, a_2 + \frac{1}{2}(1 - \beta^*))$.

Condition 1: For $a \in (a_1, a_2)$, there exists $\varphi \in \text{dom } I^*$ such that $I^*(\varphi) = a$ since $I^*$ is an extension of $I^A$. Let $a \in [a_2, a_2 + \frac{1}{2}(1 - \beta^*)]$. We have $\frac{(a - a_2)}{(1 - \beta^*)} < \frac{1}{2}$ and so

$$I^* \circ u \left( \beta^* f^3 + (1 - \beta^*) \left[ f^* \left( \frac{1}{2} + \frac{(a - a_2)}{(1 - \beta^*)} \right) g^* \right] \right) \overset{(33)}{=} I^A(u(\beta^* f^3 + (1 - \beta^*)g^*)) + (1 - \beta^*) \left( \frac{1}{2} + \frac{(a - a_2)}{(1 - \beta^*)} \right) \overset{(32)}{=} a_2 + a - a_2 = a.$$

Condition 2: Let $f \in H$ be such that $u(f) \in \text{int } u(H)$ and there exist $f', g' \in H$ such that $u(f'), u(g') \in \text{dom } I^*$ and $f' \succ f \succ g'$. If $\beta^* f^3 + (1 - \beta^*)(f^* 1/2g^*) \succ f$, then by (30)
Then there exists \( u(f) \in \text{dom } I^A \subseteq \text{dom } I^* \). If \( f \gtrsim \beta^* f^3 + (1 - \beta^*)(f^* 1/2g^*) \), then \( u(f') \in B \). By Axiom 2, \( u(f) \in B \).

Condition 3: Follows directly since \( I^A \) satisfies Condition 3.

Condition 4: By Lemma 34 and Condition 4 for \( I^A, I^* \) satisfies Condition 4 in \( \text{dom } I^A \) and \( B \). So the only case left by Lemma 30 is that \( \varphi \in \text{dom } I^A \setminus B \) and \( \varphi + \alpha^0 \in B \setminus \text{dom } I^A \). Let \( \varphi^{-1}, f^\alpha \in H \) be such that \( u(\varphi^{-1}) = \varphi \) and \( u(f^\alpha) = \varphi + \alpha^0 I \). Since \( u(f^\alpha) \notin \text{dom } I^A, \varphi \notin B \), by (30) \( f^\alpha \gtrsim \beta^* f^3 + (1 - \beta^*)(f^* 1/4g^*) \) \( \gtrsim \varphi^{-1} \). Thus there exists \( \alpha^1 \in (0, \alpha^0) \) such that \( f^\alpha \alpha^1 \varphi^{-1} \sim \beta^* f^3 + (1 - \beta^*)(f^* 1/4g^*) \). Then we have by Condition 4 in \( \text{dom } I^A \) and \( B \),

\[
I^*(\varphi + \alpha^0 I) - I^*(\varphi) = I^*(\varphi + \alpha^0 I) - I^*(\varphi + \alpha^1 \alpha^0 I) + I^*(\varphi) = (1 - \alpha^1)\alpha^0 + I^A(\varphi + \alpha^1 \alpha^0 I) - I^A(\varphi) = (1 - \alpha^1)\alpha^0 + \alpha^1 \alpha^0 = \alpha^0.
\]

This shows Condition 4.

Condition 5: Let \( f, g \in H \) be such that \( u(f), u(g) \in \text{dom } I^* \). If \( u(f), u(g) \in \text{dom } I^A \) or \( u(f), u(g) \in B \), then the condition holds. So assume that \( u(f) \in \text{dom } I^* \setminus B \) and \( u(g) \in \setminus B \setminus I^A \). Then by (30) and the definition of \( B \),

\[
g \succ \beta^* f^3 + (1 - \beta^*)(f^* 1/4g^*) \succ f
\]

and

\[
I^*(u(g)) > I^*(\beta^* u(f^3) + (1 - \beta^*)u(f^* 1/4g^*)) > I^*(u(f)).
\]

The next result applies Zorn’s lemma for the desired representation.

**Proposition 38** Assume that \( S = S^P \). Let \( \succ \) satisfy Axioms 1-4,6, and for all \( s \in S^P \), let \( u_s : \Delta(X_s) \to \mathbb{R} \) be a weak affine representation for \( \succ_s \) such that \( u_s(f^*_s) - u_s(g^*_s) = 1 \). Then there exists \( I : \text{int } u(H) \to \mathbb{R} \) such that \( I \) is C-additive and for all \( f, g \in H \) such that \( u(f), u(g) \in \text{int } u(H) \),

\[
f \succ g \iff I(u(f)) \geq I(u(g)).
\]

**Proof.** Let \( f^*, g^* \) be equally crisp acts. By Lemma 29, for all \( s \in S^P \), there exist an affine \( u_s : \Delta(X_s) \to \mathbb{R} \) and continuous and monotonic \( H_s : \text{Im } u_s(\Delta(X_s)) \to \mathbb{R} \) such that for all \( x_s, y_s \in \Delta(X_s) \),

\[
x_s \succ_s y_s \iff H_s(u_s(x_s)) \geq H_s(u_s(y_s)).
\]
and $u_s(f_s^*) - u_s(g_s^*) = 1$. We construct $I$ that is C-additive and $I \circ u$ represents preferences. Let for all $\alpha \in [0, 1]$, $I^0 \circ u(\alpha f^* + (1 - \alpha)g^*) = \alpha$. For all $f \in H$ such that $u(f) \in \text{int} u(H)$ and there exists $\alpha \in [0, 1]$ such that $f \sim \alpha f^* + (1 - \alpha)g^*$, define $I^0 \circ u(f) = \alpha$. Next, we do a transfinite induction on the collection $A \subseteq \{(a_1, a_2) \subseteq \mathbb{R} | a_1, a_2 \in \mathbb{R} \cup \{-\infty, \infty\}, a_1 \leq a_2\}$ such that $A \in A$ if and only if

1. There exists $\bar{I}$ such that $\text{dom} \bar{I} \subseteq \text{int} u(H)$ and $\text{Im} \bar{I} = A$.
2. For all $f \in H$ such that $u(f) \in \text{int} u(H)$, if there exist $f', g' \in H$ such that $u(f'), u(g') \in \text{dom} \bar{I}$ and $f' \succ f \succ g'$, then $u(f) \in \text{dom} \bar{I}$.
3. For all $\alpha \in (0, 1)$, $u(\alpha f^* + (1 - \alpha)g^*) \in \text{dom} \bar{I}$ and \[\bar{I} \circ u(\alpha f^* + (1 - \alpha)g^*) = \alpha.\]
4. If $\varphi \in \text{dom} \bar{I}$ and $\alpha^1 > 0$ are such that $\varphi + \alpha^1 \bar{I} \in \text{dom} \bar{I}$, then $\bar{I}(\varphi + \alpha^1 \bar{I}) = \bar{I}(\varphi) + \alpha^1$.
5. For all $f, g \in H$ with $u(f), u(g) \in \text{dom} \bar{I}$, $f \succ g$ iff $\bar{I}(u(f)) \geq \bar{I}(u(g))$.

We order $A$ by $\subseteq$. By Lemma 35, $A \neq \emptyset$. By Lemma 36, every chain has an upper bound. By Zorn’s lemma, there exists a maximal element $A^* \in A$ and let $I^*$ be the associated function. By Lemma 37 and the maximality of $A^*$, $\text{dom} I^* = \text{int} u(H)$. This shows the claim.

The next result shows that if there exists a crisp act with utility 0, then the representation is positive homogeneous.

**Proposition 39** Assume that $S = S^P$. Let $\succcurlyeq$ satisfy Axioms 1-4,6, and for all $s \in S^P$, let $u_s : \Delta(X_s) \to \mathbb{R}$ be a weak affine representation for $\succcurlyeq_s$ such that $u_s(f_s^*) - u_s(g_s^*) = 1$, and $u_s(c_s) = 0$. Then there exists $I : \text{int} u(H) \to \mathbb{R}$ such that $I$ is C-additive, positive homogeneous, and for all $f, g \in H$ such that $u(f), u(g) \in \text{int} u(H)$,

$$f \succcurlyeq g \iff I(u(f)) \geq I(u(g)).$$

**Proof.** By Proposition 38, there exists $I : \text{int} u(H) \to \mathbb{R}$ such that $I$ is C-additive and for all $f, g \in H$ such that $u(f), u(g) \in \text{int} u(H)$,

$$f \succcurlyeq g \iff I(u(f)) \geq I(u(g))$$

and after adding a constant since $u(c) = \bar{0} \in u(H)$, $\lim_{u(H) \ni \varphi \to \bar{0}} I(\varphi) = 0$. We show that $I$ is positively homogeneous. Let $f \in H$ be such that $u(f) \in \text{int} u(H)$. Now for all $\alpha \in (0, 1]$, $\alpha \varphi \in \text{int} u(H)$. Denote the mapping $(0, 1) \ni \alpha \mapsto I(\alpha u(f))$. Since $I$ is 1-Lipschitz, $I$
is Lipschitz function and hence differentiable almost everywhere. Let the differentiability
domain of $J$ be $\Omega \subseteq (0, 1)$ and $\Omega$ is dense in $(0, 1)$.

We show that $\nabla J$ is locally constant in $\Omega$. If $u(f) = 0$, then the claim follows from
the normalization. Thus assume $u(f) \neq 0$. Let $x \in (0, 1)$. Then there exists $y \in \Omega$
such that $x < y$. Let $z \in \Omega$ such that $z < y$. We show that $\nabla J(y) = \nabla J(z)$. Since $y > 0$,
$u(yf + (1 - y)c) \in \text{int} u(H)$ and there exists $\varepsilon > 0$ such that $B_\infty(u(yf + (1 - y)c), \varepsilon) \subseteq \text{int} u(H)$.
Let $\gamma \in (0, 1)$ be such that $|y - \gamma| < \frac{\varepsilon}{\|u(f)\|_\infty} < \infty$. Since $I$ is C-additive and hence especially
1-Lipschitz, we have $|J(y) - J(\gamma)| \leq \|u(f)\|_\infty |y - \gamma| < \varepsilon$. Thus there exists $g \in H$ such that
$u(g) = u(yf + (1 - y)c) + \mathcal{I}(J(\gamma) - J(y)) \in \text{int} u(H)$ by C-additivity, we have
$$I(u(g)) = I\left(y u(f) + (1 - y)u(c) + \mathcal{I}(J(\gamma) - J(y))\right) \overset{C\text{-add.}}{=} J(y) + J(\gamma) - J(y)$$
$$= J(\gamma) = I(\gamma u(f) + (1 - \gamma)u(c))$$
By the representation and the crispness of $c$, for all $\alpha \in (0, 1)$,
$$g \sim \gamma f + (1 - \gamma)c \text{ and } \alpha g + (1 - \alpha)c \sim \alpha \left(\gamma f + (1 - \gamma)c\right) + (1 - \alpha)c.$$ 
By the representation,
$$I(\alpha u(g) + (1 - \alpha)u(c)) = I\left(\alpha y u(f) + \alpha \mathcal{I}(J(\gamma) - J(y))\right) \overset{C\text{-add.}}{=} J(\alpha y) + \alpha (J(\gamma) - J(y))$$
$$= I\left(\alpha \left(\gamma u(f) + (1 - \gamma)c\right) + (1 - \alpha)u(c)\right) = I(\alpha \gamma u(f)) = J(\alpha \gamma).$$
We have for all $\gamma \in (0, 1)$ such that $0 < |y - \gamma| < \frac{\varepsilon}{\|u(f)\|_\infty} < \infty$ and $\alpha \in (0, 1)$,
$$J(\alpha \gamma) = J(\alpha y) + \alpha (J(\gamma) - J(y)) \iff \frac{J(\alpha \gamma) - J(\alpha y)}{\alpha \gamma - \alpha y} = \frac{J(\gamma) - J(y)}{\gamma - y}.$$ 
By taking $\gamma \to y$ and $\alpha = \frac{z}{y} < 1$, we have since $y, z \in \Omega$
$$\nabla J(y) = \lim_{\gamma \to y} \frac{J(\gamma) - J(y)}{\gamma - y} = \lim_{\gamma \to y} \frac{J(z/\gamma) - J(z)}{z/\gamma - z} = \nabla J(z).$$
Since $z \in \Omega$ such that $z < y$ was arbitrary, $\nabla J$ is constant in the $(0,y)$ neighborhood of $x$.
Since $x \in (0, 1)$ was arbitrary, $\nabla J$ is locally constant. Since the interval $(0, 1)$ is connected,
by (Viro et al., 2008, Problem 12.2x) $\nabla J$ is constant on $\Omega$. Since $J$ is Lipschitz function and
by the normalization $\lim_{\alpha \to 0} J(\bar{\alpha}) = 0$, we have for all $\alpha \in (0, 1)$,
$$J(\alpha) - \lim_{\bar{\alpha} \to 0} J(\bar{\alpha}) = \int_{(0,\alpha)\cap \Omega} \nabla J(\alpha)d\alpha = \alpha \nabla J.$$
Thus $J$ is a linear function and by taking $\alpha = 1$, $\nabla J = J(1) = I(u(f))$. Thus for all $\alpha \in (0, 1]$, 
$$I(\alpha u(f)) = \alpha I(u(f)).$$

Since $f \in H$ such that $u(f) \in \text{int} u(H)$ was arbitrary, this shows the claim.

Finally, we put the above together and show that these representations are dual-self representations and extend the representation to the situation with non-proper states.

**Proposition 40** Assume that $S = S^P$. If $\succcurlyeq$ satisfies Axioms 1-6, then there exists $(u, P)$ that is a dual-self representation for $\succcurlyeq$.

**Proof.** By Lemma 29 and Proposition 39, for all $s \in S$, there exist affine $u_s : \Delta(X_s) \to \mathbb{R}$ and $I : \text{int} u(H) \to \mathbb{R}$ such that $I \circ u$ represents $\succcurlyeq$ in interior $u(H)$ and $I$ is C-additive and positive homogeneous. Thus by Chandrasekher et al.’s (2020) Lemma A.5 and the proof of Theorem 1, there exists $P \in \mathcal{K}(\Delta(S))$ such that for all $\varphi \in \text{int} u(H)$, $I(\varphi) = \max_{p \in P} \min_{p \in P} p \cdot \varphi$. Finally, the representation can be extended to $u(H)$, which gives the claim.

**Proposition 41** Assume that $S = S^P$. If $\succcurlyeq$ satisfies Axioms 1-4, 6, then there exists $(u, C)$ that is a dual-self variational representation for $\succcurlyeq$.

**Proof.** By Lemma 29 and Proposition 38, for all $s \in S$, there exist affine $u_s : \Delta(X_s) \to \mathbb{R}$ and $I : \text{int} u(H) \to \mathbb{R}$ such that $I \circ u$ represents $\succcurlyeq$ in interior $u(H)$ and $I$ is C-additive and $u(f^*) = \bar{1}, u(g^*) = \bar{0}$. Thus by Chandrasekher et al.’s (2020) Lemma A.5 and Lemma S.3.2, there exists $C \subseteq \{c : \Delta(S) \to \mathbb{R} \cup \{\infty\}|c \text{ is convex}\}$ such that for all $\varphi \in \text{int} u(H)$, $I(\varphi) = \max_{c \in C} \min_{p \in \Delta(S)} p \cdot \varphi + c(p)$. Finally, the representation can be extended to $u(H)$, which gives the claim.

**Proposition 42** If $\succcurlyeq$ satisfies Axioms 1-6, then there exists $(u, P)$ that is a dual-self representation for $\succcurlyeq$.

**Proof.** Denote $\widehat{H} := \bigtimes_{s \in S^P} \Delta(X_s)$ and define $\succcurlyeq \subseteq \widehat{H} \times \widehat{H}$ by for all $\hat{f}, \hat{g} \in \widehat{H}$,

$$\hat{f} \succcurlyeq \hat{g} \iff \forall h \in H(\hat{f}, h_{s \in S^P}) \succcurlyeq (\hat{g}, h_{s \in S^P}).$$

Now $\widehat{\succcurlyeq}$ satisfies Axioms 1-6 and it does not have null-states. By Proposition 40, there exist $(\hat{u}, \widehat{P})$ that is a dual-self representation for $\widehat{\succcurlyeq}$ and a crisp act $c \in \widehat{H}$ such that for all $s, s' \in S^P$,
$u_s(c_s) = u_{s'}(c_{s'}) = c^*$. For all $s \in S \setminus S^P$, define $u_s : \Delta(X_s) \to \mathbb{R}$ by for all $x_s \in \Delta(X_s)$, $u_s(x_s) = c^*$ and then $u_s$ is affine. For all $s \in S^P$, define $u_s = \tilde{u}_s$. Additionally, define

$$\mathbb{P} := \left\{ (p \in \Delta(S)) \left| (p_s)_{s \in S^P} \in \mathbb{P} \right| \mathbb{P} \in \hat{\mathbb{P}} \right\}.$$ 

Now for all $s \not\in S^P$ and $p \in P \in \mathbb{P}$, $p_s = 0$. Additionally, by the definition of null-states and $\succcurlyeq$, $(u, \mathbb{P})$ is a dual-self representation for $\succcurlyeq$. \hfill \square

**Proposition 43** If $\succcurlyeq$ satisfies Axioms 1-4, 6, then there exists $(u, \mathbb{C})$ that is a dual-self variational representation for $\succcurlyeq$.

**Proof.** Symmetrically to Proposition 42. \hfill \square

### C.4 Necessity of the Axioms for Representations

Finally, we show the necessity of our axioms. First, for state dependent dual-self variational expected utility and second for state dependent dual-self expected utility.

**Lemma 44** Let $\succcurlyeq$ have a state dependent dual-self variational representation with $(u, \mathbb{C})$. Then $u_s$ is a weak affine representation for $\succcurlyeq_s$.

**Proof.** Follows from the definition of $\succcurlyeq_s$ and the monotonicity of the dual-self representation. \hfill \square

**Lemma 45** Let $\succcurlyeq$ have a state dependent dual-self variational representation with $(u, \mathbb{C})$. Then $\succcurlyeq$ satisfies Axioms 1-4,6.

**Proof.** Define $I : u(H) \to \mathbb{R}$ by for all $f \in H$,

$$I(u(f)) = \max_{c \in \mathbb{C}} \min_{p \in \Delta(S)} p \cdot u(f) + c(p).$$

Now $I$ is $\mathbb{C}$-additive on the non-null states and monotonic. Thus especially it is 1-Lipschitz. Completeness and transitivity follow from representation. Nontriviality follows from the definition of dual-self variational representation: Since $\mathbb{C}$ is grounded, especially there exist $c \in \mathbb{C}$ and $p \in \Delta(S)$ such that $c(p) < \infty$. Since there exists $s \in S$ such that $p_s > 0$, by the definition of dual-self variational representation, $s$ is not null. Thus by the definition of null states, there exist $x_s, y_s \in \Delta(X_s), h \in H$ such that $(x_s, h_{-s}) \not\succcurlyeq (y_s, h_{-s})$. 71
Axiom 2: Let \( f, g, h \in H \). Define the mapping \([0, 1] \ni \alpha \mapsto I(\alpha u(f) + (1 - \alpha)u(g))\). Now \( J \) is continuous since \( I \) is continuous and thus the following sets are closed as preimages of closed sets of a continuous function
\[
J^{-1}\left[I(u(h)), \infty\right) = \{ \alpha \in [0, 1] | \alpha f + (1 - \alpha)g \succ h \} \text{ and } J^{-1}\left(-\infty, I(u(h))\right) = \{ \alpha \in [0, 1] | h \succ \alpha f + (1 - \alpha)g \}.
\]

Axiom 3: Follows from the monotonicity of \( I \).

Axiom 4: Follows from Lemmas 24 and 44.

Axiom 6: For all \( s \in S^P \), by above and Lemmas 20, 27, and 28, there exist \( x_s^*, x_{ss} \in \Delta(X_s) \), \( \alpha^* \in (0, 1) \) such that
\[
x_s^* \succ_s \alpha^*x_s^* + (1 - \alpha^*)x_{ss} \succ_s x_{ss}
\]
and either for all \( \alpha \in (\alpha^*, 1] \),
\[
\alpha x_s^* + (1 - \alpha)x_{ss} \succ_s \alpha^*x_s^* + (1 - \alpha^*)x_{ss}
\]
(34)
or for all \( \alpha \in [0, \alpha_s) \),
\[
\alpha^*x_s^* + (1 - \alpha^*)x_{ss} \succ_s \alpha x_s^* + (1 - \alpha)x_{ss}.
\]

Since \( u_s \) is a weak representation for \( \succ_s \), \( \alpha^*x_s^* + (1 - \alpha^*)x_{ss} \in \text{int} \, u_s\left(\Delta(X_s)\right) \). Thus there exists \( \alpha^* > 0 \) such that for all \( s \in S^P \), there exist \( f_s^*, g_s^* \) such that
\[
u_s(f_s^*) > u_s\left(\alpha x_s^* + (1 - \alpha)x_{ss}\right) > u_s(g_s^*)
\]
and \( u_s(f_s^*) - u_s(g_s^*) = \alpha^* \). We show that \( f_s^* \succ_s g_s^* \). Assume w.l.o.g. that for all \( \alpha \in (\alpha^*, 1] \),
\[
\alpha x_s^* + (1 - \alpha)x_{ss} \succ_s \alpha^*x_s^* + (1 - \alpha^*)x_{ss}.
\]
Then by the affinity of \( u_s \), there exists \( \alpha^1 \in (\alpha^*, 1] \) such that \( u_s(f_s^*) > u_s\left(\alpha^1x_s^* + (1 - \alpha^1)x_{ss}\right) \).

Since \( u_s \) is a weak representation for \( \succ_s \), we have
\[
f_s^* \succ \alpha^1x_s^* + (1 - \alpha^1)x_{ss} \overset{(34)}{\succ} \alpha^*x_s^* + (1 - \alpha^*)x_{ss} \succ g_s^*.
\]

Let \( \bar{h} \in H \) and define \( \tilde{f}^*, \tilde{g}^* \in H \) by for all \( s \in S^P \), \( \tilde{f}_s^* = f_s^* \), \( \tilde{g}_s^* = g_s^* \) and for all \( s \notin S^P \), \( \tilde{f}_s^* = \bar{h}_s = \tilde{g}_s^* \). We show that \( f^*, g^* \) are equally crisp. Let \( \alpha \in (0, 1] \), \( f, g \in H \), then we have since \( I \) is C-additive on the non-null states
\[
I(\alpha f + (1 - \alpha)\bar{g}^*) = I(\alpha f + (1 - \alpha)\tilde{f}^*) - (1 - \alpha)\alpha^*
\]
and

\[ I\left(u(\alpha g + (1 - \alpha)\tilde{g}^*)\right) = I\left(u(\alpha g + (1 - \alpha)\tilde{f}^*)\right) - (1 - \alpha)\alpha^*. \]

This shows that \( \tilde{f}^*, \tilde{g}^* \) are equally crisp acts such that for all \( s \in S^P, \tilde{f}^* \succeq_s \tilde{g}^* \). \( \Box \)

**Lemma 46** Let \( \succsim \) have a state dependent dual-self representation with \((u, P)\) such that assume that there exists \( c \in H \) such that for all \( p, q \in \bigcup_{P \in \mathbb{P}} P, \sum_{s \in S} p_s u_s(c_s) = \sum_{s \in S} q_s u_s(c_s) \). Then \( \succsim \) satisfies Axioms 1-6.

**Proof.** By Lemma 45, we need to only show non-triviality and Axiom 5. Non-triviality follows from that since \( \mathbb{P} \) is not empty there exists \( p \in P \in \mathbb{P}, \) and by the definition of dual-self representation, for all \( s \in S^P, p_s = 0 \) and thus there exists \( s \in S \) such that \( s \) is non-null. By the definition of null states, \( \succsim \) is non-trivial.

We show that \( c \) from the antecedent is crisp. Denote \( p \in \bigcup_{P \in \mathbb{P}} P \) and denote \( c^* := p \cdot u(c) \). Let \( f \in H, \alpha \in (0, 1) \). Then we have

\[ \max_{P \in \mathbb{P}} \min_{p \in P} p \cdot u\left(\alpha f + (1 - \alpha)c\right) = \max_{P \in \mathbb{P}} \alpha \cdot u(f) + (1 - \alpha)\min_{P \in \mathbb{P}} p \cdot u(c) = \alpha \max_{P \in \mathbb{P}} \min_{p \in P} p \cdot u(f) + (1 - \alpha)c^*. \]

This shows the crispness of \( c \). \( \Box \)

**C.5 State Independent Dual-Self Expected Utility**

Assume state independent setting with for all \( s \in S, X_s = X \). We start off by showing that under the conditions of Remark 2, all the state dependent utilities are positive affine transformations of each other. We show this by showing that there exists a state \( s \) such that the utilities for all the other states give a weak affine representation for \( \succsim_s \). Our first lemma shows that if there is a constant crisp act and another consequence better than the crisp consequence, then the statewise preferences are not censored above the crisp consequence and similarly for below.

**Lemma 47** Let \( \succsim \) be preferences on \( \Delta(X) \) such that there exist affine \( u : \Delta(X) \to \mathbb{R} \) and continuous and weakly increasing \( H : u(\Delta(X)) \to \mathbb{R} \) such that for all \( x, y \in \Delta(X) \),

\[ x \succsim y \iff H(u(x)) \geq H(u(y)). \]
Assume that there exists \( x^* \in \Delta(X) \) such that for all \( x, y \in \Delta(X), \alpha \in (0, 1), \)
\[
x \succ y \Rightarrow \alpha x + (1 - \alpha)x^* \succ \alpha y + (1 - \alpha)x^*.
\]

Then the following conditions hold.

1. If there exist \( x, y \in \Delta(X) \) such that \( x \succ y \succ x^* \), then for all \( \alpha, \beta \in [u(x^*), u(y)], \alpha > \beta \), we have \( H(\alpha) > H(\beta) \).

2. If there exist \( x, y \in \Delta(X) \) such that \( x^* \succ y \succ x \), then for all \( \alpha, \beta \in [u(y), u(x^*)], \alpha > \beta \), we have \( H(\alpha) > H(\beta) \).

**Proof.** We will show (1) since (2) follows symmetrically. Now \( u(x) > u(y) \) and \( H(u(x)) > H(u(y)) \) By continuity and weakly increasing of \( H \), there exists \( x^* \in \Delta(X) \) such that \( u(x^*) \leq u(x), H(u(x^*)) = H(u(x)) \) and for all \( \alpha \in u(\Delta(X)) \) such that \( \alpha < u(x^*) \), \( H(\alpha) < H(u(x^*)) \).

Now we have \( x^* \succ y \) and \( u(x^*) > u(y) \). Let \( \alpha, \beta \in [u(x^*), u(y)], \alpha > \beta \). Let \( x^* \in \Delta(X) \) be such that
\[
u(x^*) = \beta \frac{u(x^*) - u(x^*)}{\alpha - u(x^*)} + u(x^*) \frac{\alpha - u(x^*)}{\alpha - u(x^*)},
\]
that exists since \( u(\Delta(X)) \) is an interval and \( u(x^*) > u(x^*) \geq u(x^*) \). Now we have \( x^* \succ x^* \) by the representation and so by assumption,
\[
\frac{\alpha - u(x^*)}{u(x^*) - u(x^*)} x^* + \frac{u(x^*) - \alpha}{u(x^*) - u(x^*)} x^* \succ \frac{\alpha - u(x^*)}{u(x^*) - u(x^*)} x^* + \frac{u(x^*) - \alpha}{u(x^*) - u(x^*)} x^*.
\]
But by the affinity of \( u \), we have
\[
u\left(\frac{\alpha - u(x^*)}{u(x^*) - u(x^*)} x^* + \frac{u(x^*) - \alpha}{u(x^*) - u(x^*)} x^*\right) = \alpha \]
and
\[
u\left(\frac{\alpha - u(x^*)}{u(x^*) - u(x^*)} x^* + \frac{u(x^*) - \alpha}{u(x^*) - u(x^*)} x^*\right) = \frac{\alpha - u(x^*)}{u(x^*) - u(x^*)} \left(\beta \frac{u(x^*) - u(x^*)}{\alpha - u(x^*)} + u(x^*) \frac{\alpha - u(x^*)}{\alpha - u(x^*)}\right) + \frac{u(x^*) - \alpha}{u(x^*) - u(x^*)} u(x^*) = \beta.
\]
Thus by the representation, \( H(\alpha) > H(\beta) \).

\[ \square \]

The next lemma shows that if the constant crisp consequence is the best or the worst consequence in some state, then all other utilities provide a weak affine representation for the statewise preferences for this state.

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Lemma 48 Let $\succeq, \succeq'$ be preferences on $\Delta(X)$ that satisfy Axioms 1, 2, 4' and have censored EU representations with $(H, u)$ and $(H', u')$ respectively. If for all $x, y \in \Delta(X)$, 
\begin{equation}
    x \succ y \Rightarrow x \succeq' y \text{ and } x \succ' y \Rightarrow x \succeq y
\end{equation}
and there exists $x^* \in \Delta(X)$ such that for all $x', y' \in \Delta(X), \alpha \in (0, 1),$
\begin{equation}
    x' \succ y' \Rightarrow \alpha x' + (1-\alpha)x^* \succ \alpha y' + (1-\alpha)x^* \text{ and } x' \succ' y' \Rightarrow \alpha x' + (1-\alpha)x^* \succ' \alpha y' + (1-\alpha)x^*(36)
\end{equation}
and $u(x^*) = \sup u(\Delta(X))$ or $u(x^*) = \inf u(\Delta(X))$, then $u'$ is a weak affine representation for $\succeq$.

Proof. The case $u(x^*) = \inf u(\Delta(X))$ follows symmetrically to $u(x^*) = \sup u(\Delta(X))$ so we only consider the latter case. Since $\succeq$ is non-trivial, there exist $\tilde{x}^\dagger, y^\dagger \in \Delta(X)$ such that $\tilde{x}^\dagger \succ y^\dagger$. By Axiom 2, there exists $x^\dagger \in \Delta(X)$ such that $\tilde{x}^\dagger \succ x^\dagger \succ y^\dagger$. Now $u(x^*) \geq u(\tilde{x}^\dagger)$ and so by the weak affine representation, we have $x^* \succeq \tilde{x}^\dagger \succ x^\dagger \succ y^\dagger$. By Lemma 47, for all $\alpha, \beta \in [u(x^*), u(x^\dagger)], \alpha > \beta$, we have 
\begin{equation}
    H(\alpha) > H(\beta).
\end{equation}

Assume, per contra, that $u'$ is not a weak affine representation for $\succeq$. Thus there exist $\tilde{x}^\dagger \succ \tilde{y}^\dagger$ and $u'(\tilde{y}^\dagger) \geq u'(\tilde{x}^\dagger)$. By Axiom 1, $\succeq'$ is non-trivial and so for some $a, b \in \Delta(X)$, $u'(a) > u'(b)$. By (37) and by applying twice Axiom 2, there exist $x^\dagger, y^\dagger \in \Delta(X)$ such that $x^* \succ x^\dagger \succ y^\dagger$ and $u'(y^\dagger) > u'(x^\dagger)$.

We consider cases. First, assume that there exists $x \in \Delta(X)$ such that $x \succ' x^*$. By Axiom 2, there exists $\hat{x} \succ' x^*$ and $u(x^*) > u(\hat{x})$. By (37), $x^* \succ \hat{x}$ which contradicts (35).

By Axiom 1, there exists $x^\natural \in \Delta(X)$ such that $x^* \succ x^\natural$. By Axiom 2, there exists $y^\natural \in \Delta(X)$ such that $x^* \succ y^\natural \succ x^\natural$. By Lemma 47, $\alpha, \beta \in [u'(x^*), u'(y^\natural)], \alpha > \beta$, we have
\begin{equation}
    H'(\alpha) > H'(\beta).
\end{equation}

Let $z^\natural \in \Delta(X)$ be such that $u'(x^*) > u'(z^\natural) > u'(y^\natural)$. Now there exists $\alpha^0 \in (0, 1)$ such that
\begin{equation}
    u'(x^*) > u'(\alpha^0 y^\dagger + (1-\alpha^0)z^\natural) > u'(\alpha^0 x^\dagger + (1-\alpha^0)z^\natural) > u'(y^\natural).
\end{equation}

By (38),
\begin{equation}
    \alpha^0 y^\dagger + (1-\alpha^0)z^\natural \succ' \alpha^0 x^\dagger + (1-\alpha^0)z^\natural.
\end{equation}
Now \( u(x^*) > u(x) > u(y^1) \) and so
\[
u(x^*) > u(\alpha^0 x + (1 - \alpha^0)z^5) > u(\alpha^0 y^1 + (1 - \alpha^0)z^5).
\]

There exists \( \alpha^1 \in (0, 1) \) such that
\[
u(\alpha^1 [\alpha^0 x^1 + (1 - \alpha^0)z^5] + (1 - \alpha^1)x^*) > u(x^1).
\]

By (37), we have
\[
\alpha^1 [\alpha^0 x^1 + (1 - \alpha^0)z^5] + (1 - \alpha^1)x^* \succ \alpha^1 [\alpha^0 y^1 + (1 - \alpha^0)z^5] + (1 - \alpha^1)x^*.
\]

By (36, 39),
\[
\alpha^1 [\alpha^0 y^1 + (1 - \alpha^0)z^5] + (1 - \alpha^1)x^* \succ \alpha^1 [\alpha^0 x^1 + (1 - \alpha^0)z^5] + (1 - \alpha^1)x^*
\]

which contradicts (35).

The next lemma shows that if the crisp consequence is in the interior of statewise preferences in some state, then all other states’ utilities give a weak affine representation for the statewise preferences for this state.

**Lemma 49** Let \( \succsim, \succsim' \) be preferences on \( \Delta(X) \) that satisfy Axioms 1, 2, 4’ and have censored EU representations with \((H, u)\) and \((H', u')\) respectively. If for all \( x, y \in \Delta(X) \),
\[
x \succ y \Rightarrow x \succsim y \quad \text{and} \quad x \succsim' y \Rightarrow x \succ y
\]
and there exists \( x^* \in \Delta(X) \) such that for all \( x', y' \in \Delta(X), \alpha \in (0, 1), \)
\[
x' \succ y' \Rightarrow \alpha x' + (1 - \alpha)x^* \succ \alpha y' + (1 - \alpha)x^* \quad \text{and} \quad x' \succsim' y' \Rightarrow \alpha x' + (1 - \alpha)x^* \succsim \alpha y' + (1 - \alpha)x^*
\]
and there exist \( \tilde{x}, \tilde{y} \in \Delta(X) \) such that
\[
\tilde{x} \succ x^* \succ \tilde{y},
\]
then \( u' \) is a weak affine representation for \( \succsim \).

**Proof.** By Axiom 2, there exist \( \hat{x}, \hat{y} \in \Delta(X) \) such that
\[
\tilde{x} \succ \hat{x} \succ x^* \succ \hat{y} \succ \tilde{y}.
\]

By Lemma 47, for all \( \alpha, \beta \in [u(\hat{y}), u(\hat{x})], \alpha > \beta \), we have
\[
H(\alpha) > H(\beta).
\]

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Assume, per contra, that $u'$ is not a weak affine representation for $\succsim$. Thus there exist $\tilde{x} \succ \tilde{y}$ and $u'(\tilde{y}) \geq u'(\tilde{x})$. By Axiom 1, $\succsim'$ is non-trivial and so for some $a, b \in \Delta(X)$, $u'(a) > u'(b)$. By Axiom 2, there exist $\tilde{x}, \tilde{y} \in \Delta(X)$ such that $\tilde{x} \succ \tilde{y}$ and $u'(\tilde{y}) > u'(\tilde{x})$.

By Axiom 1, there exists $\tilde{x} \in \Delta(X)$ such that $x^* \succ \tilde{x}$. By Axiom 2, there exists $y \in \Delta(X)$ such that $x^* \succ y \succ x^*$. W.l.o.g. assume that $x^* \succ y \succ x^*$. By Lemma 47, for all $\alpha, \beta \in [u'(x^*), u'(y)], \alpha > \beta$, we have

$$H'(\alpha) > H'(\beta).$$

(44)

Let $z \in \Delta(X)$ be such that $u'(x^*) > u'(z) > u'(y)$.

Now there exists $\alpha^0 \in (0, 1)$ such that

$$u'(x^*) > u'(\alpha^0 y + (1 - \alpha^0) z) > u'(\alpha^0 x + (1 - \alpha^0) z) > u'(y).$$

By (44),

$$\alpha^0 y + (1 - \alpha^0) z \succ \alpha^0 x + (1 - \alpha^0) z.$$ 

(45)

Next, there exists $\alpha^1 \in (0, 1)$ such that

$$u(\tilde{x}) > u(\alpha^1[\alpha^0 y + (1 - \alpha^0) z] + (1 - \alpha^1)x^*) > u(\tilde{y}).$$

By (43), we have

$$\alpha^1[\alpha^0 x + (1 - \alpha^0) z] + (1 - \alpha^1)x^* \succ \alpha^1[\alpha^0 y + (1 - \alpha^0) z] + (1 - \alpha^1)x^*.$$ 

By (41,45),

$$\alpha^1[\alpha^0 y + (1 - \alpha^0) z] + (1 - \alpha^1)x^* \succ \alpha^1[\alpha^0 x + (1 - \alpha^0) z] + (1 - \alpha^1)x^*$$

which contradicts (40).

The following axiom summarizes the additional properties for the state independent Boolean expected utility from Remark 2.

**Axiom 10**  
(1) $\succsim$ satisfies monotonicity with between states comparisons: For all $s, s' \in S$, $f, g \in H, x, y \in \Delta(X),$

$$(x_s, f_{-s}) \succ (y_s, f_{-s}) \implies (x_{s'}, g_{-s'}) \succsim (y_{s'}, g_{-s'}).$$

(2) There exists a constant act $z \in \Delta(X)$ such that $z$ is a crisp act.
(3) There exist constant acts $x, y \in \Delta(X)$ such that $x \succ y$ and $x$ and $y$ are equally crisp.

The next lemma is a simple observation that connects crisp and equally crisp acts.

**Lemma 50** Assume that $\succsim$ has a state dependent Boolean expected utility representation with $(\mathbb{P}, u)$. If $f^* \in H$ is crisp and $g^*, f^*$ are equally crisp, then $g^*$ is crisp.

**Proof.** Let $f, g \in H$ and $\alpha \in (0, 1)$. Now we have by the definitions of crisp and equally crisp acts,
\[
  f \succsim g \iff \alpha f + (1 - \alpha)g^* \succsim \alpha f + (1 - \alpha)f^* \iff \alpha f + (1 - \alpha)g^* \succsim \alpha g + (1 - \alpha)f^*.
\]
This shows the claim. \hfill \qed

Next, by using the above lemmas, we can show that all the state dependent utilities are positive affine transformations of each other.

**Lemma 51** Assume that $\succsim$ satisfies Axioms 1-6 and 10 and has a state dependent Boolean expected utility representation with $(\mathbb{P}, u)$. Then there exist an affine $u^* : \Delta(X) \to \mathbb{R}$ and $a \in \mathbb{R}^S_+, b \in \mathbb{R}^S$ such that for all $s \in S^P$,
\[
u_s = a_s u^* + b_s.
\]

**Proof.** First, we observe that by Axioms 3 and 10, if $c \in \Delta(X)$ is a crisp act, then for all $s \in S, x, y \in \Delta(X), \alpha \in (0, 1),$
\[
x \succ y \Rightarrow \alpha x + (1 - \alpha)c \succ y \alpha y + (1 - \alpha)c.
\]
Additionally, by Axiom 10, for all $s, s' \in S, x, y \in \Delta(X),$
\[
x \succ y \Rightarrow x \succsim s' y.
\]

Let $z^* \in \Delta(X)$ be the crisp act from Axiom 10. We consider three cases. 1) $\exists s \in S^P, u_s(z^*) = \sup u_s(\Delta(X))$, 2) $\exists s \in S^P, u_s(z^*) = \inf u_s(\Delta(X))$, 3) $\forall s \in S^P, \sup u_s(\Delta(X)) \neq u_s(z^*) \neq \inf u_s(\Delta(X))$. The case 2) follows symmetrically to 1) so we only show 1).

Assume that there exists $s \in S^P$ such that $u_s(z^*) = \sup u_s(\Delta(X))$. Let $s' \in S^P$. By (46,47) and Lemma 48, $u_s'$ is a weak affine representation for $\succsim_s$. By Lemmas 27 and 28 and Axiom 4, $\succsim_s$ satisfies Axioms 1, 2, 4'. By Proposition 25, there exist $a_{s'} > 0, b_{s'} \in \mathbb{R}$ such that $u_{s'} = a_{s'}u_s + b_{s'}$. This shows the claim since $s' \in S^P$ was arbitrary.
Next, assume that for all \( s \in S^P \), \( \sup u_s(\Delta(X)) \neq u_s(z^*) \neq \inf u_s(\Delta(X)) \). By Axiom 10, let \( x^*, y^* \in \Delta(X) \) be such that \( x^* \succ y^* \) and \( x^* \) and \( y^* \) are equally crisp. By the case considered, \( \text{pr}_{S^P} u(z^*) \in \text{int} \text{pr}_{S^P} u(H) \). Thus there exist \( \beta \in (0, 1) \) and \( h \in H \) such that for all \( s \in S^P \),
\[
  u_s\left(\beta h_s + (1 - \beta)\left(\frac{1}{2}x^* + \frac{1}{2}y^*\right)\right) = u_s(z^*).
\]

By Lemma 61, \( \beta h + (1 - \beta)x^* \succ h + (1 - \beta)y^* \). By Lemmas 26, 30, and 50, \( \beta h + (1 - \beta)x^* \) and \( \beta h + (1 - \beta)y^* \) are crisp. By Lemma 26, there exists \( s^* \in S^P \) such that \( \beta h_s^* + (1 - \beta)x_{s^*} \succ_{s^*} \beta h_s^* + (1 - \beta)y_{s^*}^* \). By the crispness,
\[
  \beta h_s^* + (1 - \beta)x_{s^*} \succ_{s^*} \beta h_s^* + (1 - \beta)\left(\frac{1}{2}x^* + \frac{1}{2}y^*\right) \sim_{s^*} z^* \succ_{s^*} \beta h_s^* + (1 - \beta)y_{s^*}^* \quad (48)
\]

Let \( s' \in S^P \). By (46,47,48) and Lemma 49, \( u_{s'} \) is a weak affine representation for \( \succ_{s^*} \). Thus as above, by Proposition 25, there exist \( a_{s'} > 0, b_{s'} \in \mathbb{R} \) such that \( u_{s'} = a_{s'}u_{s^*} + b_{s'} \).

This shows the claim since \( s' \in S^P \) was arbitrary.

Finally, by putting above together, we have a state independent dual-self expected utility.

**Theorem 52** \( \succeq \) satisfies Axioms 1-6 and 10 iff \( \succeq \) has a state independent Boolean expected utility representation.

**Proof.** First, we show the sufficiency of the axioms. Let \( z^* \in \Delta(X) \) be a crisp act and \( x^*, y^* \in \Delta(X), x^* \succ y^* \) be equally crisp acts from 10. By Theorem 1, there exists a state dependent dual-self expected utility representation \((u, \mathbb{P})\) for \( \succeq \) such that for all \( s \in S \), \( u_s(z^*) = 0 \). By Lemma 51, there exist an affine \( u^*: \Delta(X) \rightarrow \mathbb{R} \) and \( a \in \mathbb{R}^S_{++}, b \in \mathbb{R}^S \) such that for all \( s \in S^P \),
\[
  u_s = a_s u^* + b_s.
\]

Since \( x^* \succ y^* \), \( u^*(x^*) > u^*(y^*) \). By Corollary 82 and the proofs of Propositions 39 and 40, there exists a state dependent dual-self expected utility \((\tilde{u}, \tilde{\mathbb{P}})\) for \( \succeq \) such that for all \( s \in S \),
\[
  \tilde{u}_s(z^*) = 0 \text{ and } \tilde{u}(x^*) - \tilde{u}(y^*) = 1. \quad (49)
\]

By Lemma 51, there exist an affine \( \tilde{u}^*: \Delta(X) \rightarrow \mathbb{R} \) and \( \tilde{a} \in \mathbb{R}^S_{++}, \tilde{b} \in \mathbb{R}^S \) such that for all \( s \in S^P \),
\[
  \tilde{u}_s = \tilde{a}_s \tilde{u}^* + \tilde{b}_s.
\]
By an affine transformation for \( u^* \), assume w.l.o.g. that \( \tilde{u}^*(z^*) = 0 \) and \( \tilde{u}^*(x^*) - \tilde{u}^*(y^*) = 1 \). By (49), we have for all \( s \in S_P \), \( \tilde{a}_s = 1 \) and \( \tilde{b}_s = 0 \). Now we can choose for all \( s \in S \setminus S_P \), \( u_s = \tilde{u}^* \) without affecting representation by the definition of state dependent dual-self expected utility. This gives a state independent dual-self expected utility.

Next, we show the necessity of the axioms. By Theorem 1, we only show Axiom 10. The properties (2) and (3) follow trivially since all constant acts are crisp and so equally crisp with each other and the preferences are non-trivial. Finally, (1) follows from the state independent utility and statewise monotonicity.

\[ \square \]

### D Dual-Self Identification

For this section, we assume that \( S = S_P \) since by the definition of dual-self and dual-self variational representations probabilities for null states are always identified and the utilities are not. Thus this is without loss of generality.

With an abuse of definitions we define following.

**Definition** Let \( \succsim \subseteq H \times H \). \((u, I)\) is a state dependent dual-self representation for \( \succsim \) if \( u = (u_s)_{s \in S} \) and for all \( s \in S \), \( u_s : \Delta(X_s) \to \mathbb{R} \) is affine, \( I : u(H) \to \mathbb{R} \) is C-additive, positive homogeneous, and monotonic and for all \( f, g \in H \),

\[
\quad f \succsim g \iff I(u(f)) \geq I(u(g)).
\]

**Definition** Let \( \succsim \subseteq H \times H \). \((u, I)\) is a state dependent variational dual-self representation for \( \succsim \) if \( u = (u_s)_{s \in S} \) and for all \( s \in S \), \( u_s : \Delta(X_s) \to \mathbb{R} \) is affine, \( I : u(H) \to \mathbb{R} \) is C-additive, and monotonic and for all \( f, g \in H \),

\[
\quad f \succsim g \iff I(u(f)) \geq I(u(g)).
\]

Since the state space is assumed to be finite \( \mathbb{R}^S \) is metrizable space and hence convergence is characterized by the convergence of sequences.

We start with preliminary results.
D.1 Dual-Self Variational Representation Additive Between Equally Crisp Acts

We first show that any dual-self variational representation is additive between any equally crisp acts. The next lemmas show that the set of equally crisp acts is convex.

Lemma 53 Let \( f^*, g^* \) and \( \tilde{f}^*, \tilde{g}^* \) be equally crisp acts and \( \alpha \in [0, 1] \) then \( \alpha f^* + (1 - \alpha)\tilde{f}^*, \alpha g^* + (1 - \alpha)\tilde{g}^* \) are equally crisp acts.

Proof. If \( \alpha \in \{0, 1\} \) then the claim follows directly. So assume \( \alpha \in (0, 1) \). Let \( f, g \in H \), \( \beta \in (0, 1) \). Now for \( h \in \{f, g\}, h^* \in \{f^*, g^*\}, \tilde{h}^* \in \{\tilde{f}^*, \tilde{g}^*\} \),

\[
\alpha h + (1 - \alpha)(h^* \beta \tilde{h}^*) \equiv (\alpha + \beta - \alpha \beta) \left( \frac{\alpha}{\alpha + \beta - \alpha \beta} h + \frac{\beta - \alpha \beta}{\alpha + \beta - \alpha \beta} h^* \right) + (1 - \alpha - \beta + \alpha \beta) \tilde{h}^* \tag{50}
\]

and

\[
\alpha h + (1 - \alpha)(h^* \beta \tilde{h}^*) \equiv (1 - \beta + \alpha \beta) \left( \frac{\alpha}{1 - \beta + \alpha \beta} h + \frac{1 - \alpha - \beta + \alpha \beta}{1 - \beta + \alpha \beta} \tilde{h}^* \right) + (1 - (1 - \beta + \alpha \beta)) h^*. \tag{51}
\]

Thus where equivalencies (*) follow from the assumptions that \( f^*, g^* \) and \( \tilde{f}^*, \tilde{g}^* \) are equally crisp.

\[
\alpha f + (1 - \alpha)(f^* \beta \tilde{f}^*) \succsim \alpha g + (1 - \alpha)(f^* \beta \tilde{f}^*)
\]

\[
\iff (\alpha + \beta - \alpha \beta) \left( \frac{\alpha}{\alpha + \beta - \alpha \beta} f + \frac{\beta - \alpha \beta}{\alpha + \beta - \alpha \beta} f^* \right) + (1 - \alpha - \beta + \alpha \beta) \tilde{f}^*
\]

\[
\succsim (\alpha + \beta - \alpha \beta) \left( \frac{\alpha}{\alpha + \beta - \alpha \beta} g + \frac{\beta - \alpha \beta}{\alpha + \beta - \alpha \beta} f^* \right) + (1 - \alpha - \beta + \alpha \beta) \tilde{f}^*
\]

\[
(\alpha + \beta - \alpha \beta) \left( \frac{\alpha}{\alpha + \beta - \alpha \beta} f + \frac{\beta - \alpha \beta}{\alpha + \beta - \alpha \beta} f^* \right) + (1 - \alpha - \beta + \alpha \beta) \tilde{g}^*
\]

\[
\succsim (\alpha + \beta - \alpha \beta) \left( \frac{\alpha}{\alpha + \beta - \alpha \beta} g + \frac{\beta - \alpha \beta}{\alpha + \beta - \alpha \beta} f^* \right) + (1 - \alpha - \beta + \alpha \beta) \tilde{g}^*
\]

\[
\iff \alpha f + (1 - \alpha)(f^* \beta \tilde{g}^*) \succsim \alpha g + (1 - \alpha)(f^* \beta \tilde{g}^*)
\]

\[
\iff (1 - \beta + \alpha \beta) \left( \frac{\alpha}{1 - \beta + \alpha \beta} f + \frac{1 - \alpha - \beta + \alpha \beta}{1 - \beta + \alpha \beta} \tilde{g}^* \right) + (1 - (1 - \beta + \alpha \beta)) f^*
\]

\[
\succsim (1 - \beta + \alpha \beta) \left( \frac{\alpha}{1 - \beta + \alpha \beta} g + \frac{1 - \alpha - \beta + \alpha \beta}{1 - \beta + \alpha \beta} \tilde{g}^* \right) + (1 - (1 - \beta + \alpha \beta)) f^*
\]

\[
(1 - \beta + \alpha \beta) \left( \frac{\alpha}{1 - \beta + \alpha \beta} f + \frac{1 - \alpha - \beta + \alpha \beta}{1 - \beta + \alpha \beta} \tilde{g}^* \right) + (1 - (1 - \beta + \alpha \beta)) g^*
\]

\[
\succsim (1 - \beta + \alpha \beta) \left( \frac{\alpha}{1 - \beta + \alpha \beta} g + \frac{1 - \alpha - \beta + \alpha \beta}{1 - \beta + \alpha \beta} \tilde{g}^* \right) + (1 - (1 - \beta + \alpha \beta)) g^*
\]
\((51)\) \(\iff \alpha f + (1 - \alpha) (g^\ast \beta g^\ast) \succneq \alpha g + (1 - \alpha) (g^\ast \beta g^\ast)\).

**Corollary 54** Let \(f^\ast, g^\ast\) be such that for all \(f, g \in H\) and \(\alpha \in (0, 1)\),
\[
\alpha f + (1 - \alpha) f^\ast \succneq \alpha g + (1 - \alpha) f^\ast \iff \alpha f + (1 - \alpha) g^\ast \succneq \alpha g + (1 - \alpha) g^\ast.
\]

Then for all \(\beta \in (0, 1)\),
\[
\alpha f + (1 - \alpha) f^\ast \succneq \alpha g + (1 - \alpha) f^\ast \iff \alpha f + (1 - \alpha) (f^\ast \beta g^\ast) \succneq \alpha g + (1 - \alpha) (f^\ast \beta g^\ast).
\]

**Proof.** Now \((f^\ast, g^\ast)\) and \((f^\ast, f^\ast)\) are equally crisp. Thus for \(\beta \in (0, 1)\), by Lemma 53, \(f^\ast\) and \(\beta f^\ast + (1 - \beta) g^\ast\) are equally crisp which shows the claim. \(\square\)

**Corollary 55** Let \(f^\ast, g^\ast\) be equally crisp acts and \(\alpha, \beta \in [0, 1]\) Then \(\alpha f^\ast + (1 - \alpha) g^\ast, \beta f^\ast + (1 - \beta) g^\ast\) are equally crisp acts.

**Proof.** Assume w.l.o.g. \(\alpha \geq \beta\). Now \((f^\ast, g^\ast)\) and \((g^\ast, g^\ast)\) are equally crisp by Lemma 53, \((g^\ast, \frac{\alpha - \beta}{1 - \beta} f^\ast + (1 - \frac{\alpha - \beta}{1 - \beta}) g^\ast)\) are equally crisp. Now since \((f^\ast, f^\ast)\) and \((g^\ast, \frac{\alpha - \beta}{1 - \beta} f^\ast + (1 - \frac{\alpha - \beta}{1 - \beta}) g^\ast)\) are equally crisp, by Lemma 53, \((\beta f^\ast + (1 - \beta) g^\ast, \alpha f^\ast + (1 - \alpha) g^\ast)\) are equally crisp since \(\beta f^\ast + (1 - \beta) \left(\frac{\alpha - \beta}{1 - \beta} f^\ast + (1 - \frac{\alpha - \beta}{1 - \beta}) g^\ast\right) \equiv \alpha f^\ast + (1 - \alpha) g^\ast\). \(\square\)

**Corollary 56** Let \(f^\ast, g^\ast\) be equally crisp acts and \(f \in H, \beta \in (0, 1)\) Then \(\beta f + (1 - \beta) f^\ast, \beta f + (1 - \beta) g^\ast\) are equally crisp acts.

**Proof.** Now \((f^\ast, g^\ast)\) and \((f, f)\) are equally crisp. Thus for \(\beta \in (0, 1)\), by Lemma 53, \(\beta f + (1 - \beta) f^\ast, \beta f + (1 - \beta) g^\ast\) are equally crisp which shows the claim. \(\square\)

The next two lemmas show that the dual-self variational representation is affine between equally crisp acts.

**Lemma 57** Let \((u, I)\) be a dual-self variational representation for \(\succeq\) and \(f^\ast, g^\ast\) equally crisp acts. Let \(f \in H, \beta \in (0, 1)\) be such that \(f \in \text{int} u(H)\). Then the mapping \((0, 1) \ni \alpha \mapsto I \circ u(\beta f + (1 - \beta) (f^\ast \alpha g^\ast))\) has locally constant derivative at the differentiability points.
Proof. Let \( \Omega \) be the differentiability points of the mapping \( J \). Assume, per contra, that there exists a point \( x \in (0,1) \) such that \( x \) does not have a neighborhood \( U \) such that for all \( y, z \in U \cap \Omega \), \( \nabla J(y) = \nabla J(z) \). First, since \( u(f) \in \text{int } u(H) \), there exists \( \varepsilon > 0 \) such that \( B_\infty(u(f), \varepsilon) \subseteq u(H) \). Second, since \( J \) is Lipschitz, by the counter assumption, there exist \( y, z \in (0,1) \) such that

\[
\left| y - z \right| < \frac{\beta x}{\|u(f^*) - u(g^*)\|_\infty} \quad \text{and} \quad \left| x - y \right|, \left| x - z \right| \leq \frac{\min\{x, 1-x\}}{3}, \quad \nabla J(y) > \nabla J(z).
\]

Since \( I \) is \( C \)-additive and hence especially \( 1 \)-Lipschitz, we have

\[
\frac{1}{\beta}\|u(f^*) - u(g^*)\|_\infty \left| y - z \right| < \varepsilon. \quad \text{Thus exists } g \in H \text{ such that } u(g) = u(f) + \frac{1}{\beta} I(J(z) - J(y)).
\]

By \( C \)-additivity, we have since \( \frac{1}{2}(1 + \beta) = \beta + \frac{(1-\beta)}{2} \) and \( 1 - \frac{1}{2}(1 + \beta) = \frac{1}{2}(1 - \beta) \),

\[
I \left( \frac{1}{2}(1 + \beta) \left( \frac{\beta}{1/(1 + \beta)} g + \frac{(1-\beta)/2}{1/(1 + \beta)} (f^*) y \right) + \left( 1 - \frac{1}{2}(1 + \beta) \right) (f^*) y \right) 
\]

\[
= I \left( \beta u(f) + \frac{1}{2}(1 - \beta) u(f^*) + \frac{1}{2}(1 - \beta) (f^*) y \right) = J(z).
\]

Thus by the representation,

\[
\frac{1}{2}(1 + \beta) \left( \frac{\beta}{1/(1 + \beta)} g + \frac{(1-\beta)/2}{1/(1 + \beta)} (f^*) y \right) + \left( 1 - \frac{1}{2}(1 + \beta) \right) (f^*) y 
\]

\[
\sim \frac{1}{2}(1 + \beta) \left( \frac{\beta}{1/(1 + \beta)} f + \frac{(1-\beta)/2}{1/(1 + \beta)} (f^*) y \right) + \left( 1 - \frac{1}{2}(1 + \beta) \right) (f^*) y.
\]

But since \( \nabla J(y) > \nabla J(z) \), there exists \( 0 < \delta < \frac{1-x}{3} \) such that

\[
\frac{J(y + \delta) - J(y)}{\delta} > \frac{J(z + \delta) - J(z)}{\delta} \quad \Rightarrow J(y + \delta) + J(z) - J(y) > J(z + \delta). \quad (53)
\]

Thus since \( y + 2\delta < 1 \)

\[
I \left( \frac{1}{2}(1 + \beta) \left( \frac{\beta}{1/(1 + \beta)} g + \frac{(1-\beta)/2}{1/(1 + \beta)} (f^*) y \right) + \left( 1 - \frac{1}{2}(1 + \beta) \right) (f^*) (y + 2\delta) \right) 
\]

\[
= I \left( \beta u(f) + \frac{1}{2}(1 - \beta) u(f^*) + \frac{1}{2}(1 - \beta) (f^*) (y + 2\delta) \right) + \beta^{1/\beta} I \times (J(z) - J(y)) \]

\[
= \text{C-add.} \quad J(y + \delta) + J(z) - J(y) \quad (53) \quad \Rightarrow J(y + \delta) + J(z) - J(y) > J(z + \delta)
\]

\[
= I \left( \beta u(f) + \frac{1}{2}(1 - \beta) u(f^*) (2z - y) + \frac{1}{2}(1 - \beta) (f^*) (y + 2\delta) \right)
\]
\[
I\left(\frac{1}{2}(1+\beta)\left(\frac{\beta}{1/2(1+\beta)}u(f) + \frac{(1-\beta)/2}{1/2(1+\beta)}u(f^*(2z-y)g^*)\right) + (1-1/2(1+\beta))u(f^*(y+2\delta)g^*)\right).
\]

Thus by the representation,
\[
\frac{1}{2}(1+\beta)\left(\frac{\beta}{1/2(1+\beta)}g + \frac{(1-\beta)/2}{1/2(1+\beta)}(f^*yg^*)\right) + (1-1/2(1+\beta))(f^*(y+2\delta)g^*)
\]
\[
> \frac{1}{2}(1+\beta)\left(\frac{\beta}{1/2(1+\beta)}f + \frac{(1-\beta)/2}{1/2(1+\beta)}(f^*(2z-y)g^*)\right) + (1-1/2(1+\beta))(f^*(y+2\delta)g^*).
\]

but this contradicts Corollary 54 and (52).

**Lemma 58** Let \((u, I)\) be a variational representation for \(\succsim\) and \(f^*, g^*\) equally crisp acts. Let \(f \in H, \beta \in (0, 1)\) be such that \(f \in \text{int} u(H)\). Then the mapping \([0, 1] \ni \alpha \mapsto I \circ u(\beta f + (1-\beta)(f^*g^*))\) is affine.

**Proof.** Since \(I\) is \(C\)-additive and hence \(1\)-Lipschitz, especially \(J\) is Lipschitz and it is differentiable almost everywhere. Let the set of differentiability points be \(\Omega\). Define
\[
\bar{J}(\alpha) = \begin{cases} 
\nabla J(\alpha), & \text{if } J \text{ is differentiable at } \alpha \\
\liminf_{\alpha \to \alpha, \alpha \in \Omega} \nabla J(\tilde{\alpha}), & \text{if } J \text{ is not differentiable at } \alpha
\end{cases}
\]

Since \(J\) is differentiable almost everywhere \(\bar{J}\) is well-defined. Let \(0 < \epsilon < 1/2\). We show that \(\bar{J}\) is constant in the set \((0, 1)\). By Lemma 57, for all \(\alpha \in (0, 1)\), there exists \(\epsilon_\alpha > 0, c \in \mathbb{R}\) such that for all \(x \in B_\infty(\alpha, \epsilon_\alpha) \cap \Omega, \nabla J(x) = c\). Thus by the definition of \(\bar{J}\), for all \(x \in B_\infty(\alpha, \epsilon_\alpha), \bar{J}(x) = c\). Thus \(\bar{J}\) is locally constant function in the set \((0, 1)\). Since the interval \((0, 1)\) is connected, by (Viro et al., 2008, Problem 12.2x) \(J'\) is constant on the set \((0, 1)\). Thus there exists \(c \in \mathbb{R}\) such that for all \(x \in (0, 1) \cap \Omega, \nabla J(x) = c\). Since \(J\) is Lipschitz, we have for all \(x \in [0, 1],\)
\[
J(x) - J(0) = \int_{[0,x] \cap \Omega} \nabla J(y) dy = \int_{(0,x) \cap \Omega} \nabla J(y) dy = \int_{(0,x) \cap \Omega} c dy = cx.
\]

Thus especially, the mapping \(J\) is affine. \hfill \Box

The next two lemmas show that the directional derivative of the representation to the direction \(u(g^*) - u(f^*)\) is constant always. First, we show that it is constant on each indifference level.

**Lemma 59** Let \((u, I)\) be a dual-self variational representation for \(\succsim\) and \(f^*, g^*\) equally crisp acts and denote \(h^* := f^*/2g^*\). Let \(f, g \in H, \beta, \alpha \in (0, 1)\) be such that \(f, g \in \text{int} u(H)\) and
\( \alpha f + (1-\alpha)h^* \sim \beta g + (1-\beta)h^* \). Then the affine mappings \([0, 1] \ni \gamma \mapsto \gamma J^f / (I \circ u(\alpha f + (1-\alpha)(f^* g^*)))\) and \([0, 1] \ni \gamma \mapsto \gamma J^g / (I \circ u(\beta g + (1-\beta)(f^* g^*)))\) with derivatives \(\nabla J^f\) and \(\nabla J^g\) respectively has \(\nabla J^f = \frac{1-\alpha}{1-\beta} \nabla J^g\).

**Proof.** Assume w.l.o.g. \(\alpha \geq \beta\). Then
\[
\alpha \left( \frac{\beta}{\alpha} g + \frac{\alpha - \beta}{\alpha} h^* \right) + (1-\alpha)h^* \equiv \beta g + (1-\beta)h^*.
\]
Define the mapping
\[
[0, 1] \ni \gamma \mapsto J^{gh^*} = I \circ u \left( \alpha \left( \frac{\beta}{\alpha} g + \frac{\alpha - \beta}{\alpha} h^* \right) + (1-\alpha)(f^* g^*) \right)
\]
Thus since \(f^*\) and \(g^*\) are equally crisp acts and by Corollary 54, we have by the representation, for all \(\gamma \in [0, 1]\),
\[
J^{gh^*}(\gamma) = I \circ u \left( \alpha \left( \frac{\beta}{\alpha} g + \frac{\alpha - \beta}{\alpha} h^* \right) + (1-\alpha)(f^* g^*) \right) = I \circ u(\alpha f + (1-\alpha)(f^* g^*)) = J^f(\gamma).
\]
Thus especially \(J^{gh^*}\) is affine with derivative \(\nabla J^{gh^*} = \nabla J^f\). But on the other hand for all \(\gamma \in [0, 1]\), we have
\[
\alpha \left( \frac{\beta}{\alpha} g + \frac{\alpha - \beta}{\alpha} h^* \right) + (1-\alpha)(f^* g^*) \equiv \beta g + (1-\beta) \left( f^* \frac{(\alpha-\beta)/2 + \gamma(1-\alpha)}{1-\beta} g^* \right)
\]
and so
\[
J^{gh^*}(\gamma) = I \circ u \left( \alpha \left( \frac{\beta}{\alpha} g + \frac{\alpha - \beta}{\alpha} h^* \right) + (1-\alpha)(f^* g^*) \right) = I \circ u(\beta g + (1-\beta) \left( f^* \frac{(\alpha-\beta)/2 + \gamma(1-\alpha)}{1-\beta} g^* \right)) = J^g \left( \frac{(\alpha-\beta)/2 + \gamma(1-\alpha)}{1-\beta} \right).
\]
Thus especially by the chain rule of derivatives, \(\nabla J^{gh^*} = \nabla J^g \frac{1-\alpha}{1-\beta}\) which shows the claim. \(\square\)

The next lemma shows that the directional derivative is constant at different indifference levels.

**Lemma 60** Let \((u, I)\) be a state dependent dual-self variational representation for \(\succeq\) and \(f^*, g^*\) equally crisp acts and denote \(h^* := f^*1/2g^*\). For all \(\alpha, \beta \in H\), define the mapping \([0, 1] \ni \gamma \mapsto J^f / (I \circ u(\beta f + (1-\beta)(f^* g^*)))\) with derivative \(\nabla J^f\). Define the mappings
\[K^*(\inf I(u(H)), \sup I(u(H))) \rightarrow \mathbb{R} \cup \{\infty\}\] and \(K_\alpha : (\inf I(u(H)), \sup I(u(H))) \rightarrow \mathbb{R} \cup \{\infty\}\)

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Thus since $\sup_\emptyset = \inf_\emptyset = \infty$. Then $K^*$ is constant and finite and $K^* = K_*$.

**Proof.** We show first that locally $K^*$ is constant and finite and $K^* = K_*$. Let $v \in (\inf I(\alpha H), \sup I(\alpha H))$. First, there exist $f_1, f_2 \in H$ such that $I(\alpha f_1) > v > I(\alpha f_2))$.

Since $S$ is finite, there exists $f_3 \in H$ such that $u(f_3) \in \text{int } u(H)$. By mixture continuity, there exist $\alpha_1, \alpha_2, \alpha_3 \in (0, 1)$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $I \circ u(\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3) = v$.

Denote $f_4 := \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$. Since $\alpha_3 > 0$ and $u(f_3) \in \text{int } u(H)$ and $u(H)$ is a convex set $u(f_4) \in \text{int } u(H)$. Thus there exists $\epsilon > 0$ such that $B_\epsilon (u(f_4), \epsilon) \subseteq u(H)$. Let $\beta \in (1/2, 1)$ be such that

$$\left(\frac{1}{\beta} - 1\right)\|u(f_4)\|_{\infty} + \frac{1-\beta}{\beta} \|u(h^*)\|_{\infty} < \epsilon.$$ 

Thus by the choice of $\beta$, there exists $f \in H$ such that

$$u(f) = u(f_4) + \left(\frac{1}{\beta} - 1\right)u(f_4) - \frac{1-\beta}{\beta}u(h^*).$$

Thus especially $u(f) \in \text{int } u(H)$ and

$$\beta u(f) + (1-\beta)u(h^*) = u(f_4)$$

and so $I(\beta u(f) + (1-\beta)u(h^*)) = v$. First, for all $\tilde{f} \in H, \tilde{\beta} \in (0, 1)$ such that $u(\tilde{f}) \in \text{int } u(H), I \circ u(\tilde{\beta}\tilde{f} + (1-\tilde{\beta})h^*) = v$, we have $\tilde{\beta}\tilde{f} + (1-\tilde{\beta})h^* \sim (1-\beta)f + (1-\beta)h^*$ and by Lemma 59,

$$\frac{1}{1-\beta} \nabla J_{\tilde{\beta}}^f = \frac{1}{(1-\beta)} \nabla J_{\tilde{\beta}}^f = \frac{1}{(1-\beta)} \nabla J_{\tilde{\beta}}^f = (54)$$

Thus since $\tilde{f} \in H, \tilde{\beta} \in (0, 1)$ were arbitrary such that $u(\tilde{f}) \in \text{int } u(H), I \circ u(\tilde{\beta}\tilde{f} + (1-\tilde{\beta})h^*) = \tilde{v}$,

$$K^*(v) \overset{(54)}{=} \frac{1}{(1-\beta)} \nabla J_{\tilde{\beta}}^f \overset{(54)}{=} K_*(v).$$

We consider two cases: 1) there exists $\alpha_0 \in (0, 1) \setminus \{1/2\}$ such that $\beta f + (1-\beta)h^* \sim \beta f + (1-\beta)f^*\alpha_0 g^*$. Now $J_{\tilde{\beta}}^f(1/2) = J_{\tilde{\beta}}^f(\alpha_0)$ and by Lemma 58, $J_{\tilde{\beta}}^f$ is affine and hence for all $\alpha \in (0, 1)$,

$$J_{\tilde{\beta}}^f(\alpha) = J_{\tilde{\beta}}^f(1/2).$$

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Thus \(K^*(v) = 0\). We show that there exists a neighborhood \(A\) of \(v\) such that for all \(\tilde{v} \in A\), \(K^*(\tilde{v}) = 0 = K_*(\tilde{v})\). Assume, per contra, that such a neighborhood \(A\) of \(v\) does not exist. Since \(u(f) \in \text{int} u(H)\), there exists \(\varepsilon > 0\) such that \(B_{\varepsilon}(u(f), \varepsilon) \subseteq u(H)\). Now there exists \(\tilde{v} \in B(v, \varepsilon)\) such that \(K^*(\tilde{v}) \neq 0\) or \(K_*(\tilde{v}) \neq 0\). Thus there exists \(\tilde{f} \in H\), \(\tilde{\beta} \in (0, 1)\) such that \(u(\tilde{f}) \in u(H)\), \(\partial u(\beta \tilde{f} + (1 - \beta)h^*) = \tilde{v}\), and \(\nabla J^f_{\tilde{\beta}} \neq 0\). Now \(u(f + (\tilde{v} - v)) \in u(H)\) and hence there exists \(\tilde{f}^5 \in H\) such that \(u(f^5) = u(f + 1/\beta(\tilde{v} - v))\). Thus \(u(\beta f^5 + (1 - \beta)h^*) = u(f + (1 - \beta)h^*) + (\tilde{v} - v)\) and by \(C\)-additivity, \(I \circ u(\beta f^5 + (1 - \beta)h^*) = \tilde{v}\). By Lemma 59, \(0 \neq \nabla J^f_{\tilde{\beta}} = \nabla J^5_{\tilde{\beta}}\) and thus especially there exists \(\alpha^1 \in (0, 1)\) such that

\[I \circ u(\beta f^5 + (1 - \beta)(f^* \alpha^1 g^*)) \neq \tilde{v} = I \circ u(\beta f^5 + (1 - \beta)h^*).
\]

However, then by \(C\)-additivity,

\[
I\left(\beta f + (1 - \beta)(f^* \alpha^1 g^*)\right) = I\left(\beta f + (1 - \beta)(f^* \alpha^1 g^*)\right) + (\tilde{v} - v) - (\tilde{v} - v)
\]

\[= I\left(\beta f^5 + (1 - \beta)(f^* \alpha^1 g^*)\right) - (\tilde{v} - v) \neq I\left(\beta f^5 + (1 - \beta)h^*\right) - (\tilde{v} - v)
\]

This contradicts (56). Thus there exists a neighborhood \(A\) of \(v\) such that for all \(\tilde{v} \in A\), \(K^*(\tilde{v}) = K^*(v) = K_*(\tilde{v})\).

2) There exists \(\alpha^0 \in (0, 1)\) such that \(\beta f + (1 - \beta)h^* \not\sim \beta f + (1 - \beta)f^* \alpha^0 g^*. \) Thus \(J^f_{\tilde{\beta}}(\alpha^0) \neq J^f_{\tilde{\beta}}(1/2)\). Since \(J^f_{\tilde{\beta}}\) is affine, we have

\[v^* := \max\{J^f_{\tilde{\beta}}(1/4), J^f_{\tilde{\beta}}(3/4)\} > J^f_{\tilde{\beta}}(1/2) > \min\{J^f_{\tilde{\beta}}(1/4), J^f_{\tilde{\beta}}(3/4)\} =: v_*.
\]

Assume w.l.o.g. \(v^* = I\left(\beta u(f) + (1 - \beta)u(f^* \alpha^0 g^*)\right)\) and \(v_* = I\left(\beta u(f) + (1 - \beta)u(f^* \alpha^0 g^*)\right)\) since the other case follows symmetrically. Let \(\tilde{v} \in [v_*, v^*]\). Then by Lemma 58,

\[I\left(\beta u(f) + (1 - \beta)u\left[f^*\left(\frac{1}{4} + \frac{1}{2} \frac{\tilde{v} - v_*}{v^* - v_*}\right)g^*\right]\right) = \tilde{v}
\]

Now since \(1/2(1 + \beta) = \beta + (1 - \beta)/2\) and \(1 - 1/2(1 + \beta) = 1/2(1 - \beta)
\]

\[
\frac{1}{2}(1 + \beta)\left(1 + \beta\frac{\beta}{1/2(1 + \beta)}\right) + \frac{1}{2}(1 - \beta)\frac{1}{2}(1 + \beta)\frac{\tilde{v} - v_*}{v^* - v_*}g^* + \frac{1}{2}(1 - \beta)\frac{\tilde{v} - v_*}{v^* - v_*}g^* + \left(1 - \frac{1}{2}(1 + \beta)\right)h^*
\]

\[\equiv \beta f + (1 - \beta)f^*\left(\frac{1}{4} + \frac{1}{2} \frac{\tilde{v} - v_*}{v^* - v_*}\right)g^*.
\]
Denote
\[
f^6 := \frac{\beta}{2(1 + \beta)} f + \frac{1}{2(1 + \beta)} (v - v_s) f^* + \frac{1}{2(1 + \beta)} v^* - \tilde{v} g^*.
\]
Then we have for all \( \gamma \in (0, 1) \), as above
\[
\frac{1}{2}(1 + \beta) f^6 + (1 - 1/2(1 + \beta))(f^* \gamma g^*) \equiv \beta f + (1 - \beta) f^* \left( \frac{\tilde{v} - v_s}{2(v^* - v_s)} + \frac{\gamma}{2} \right) g^*.
\]
Thus for all \( \gamma \in (0, 1) \),
\[
J_{f^6}^{\beta(1+\beta)}(\gamma) = I \circ u \left( \frac{1}{2}(1 + \beta) f^6 + (1 - 1/2(1 + \beta))(f^* \gamma g^*) \right)
\]
\[
= I \circ u \left( \beta f + (1 - \beta) f^* \left( \frac{\tilde{v} - v_s}{2(v^* - v_s)} + \frac{\gamma}{2} \right) g^* \right) = J_{f^6}^\beta \left( \frac{\tilde{v} - v_s}{2(v^* - v_s)} + \frac{\gamma}{2} \right).
\]
Thus by the chain rule of derivatives and by the affinity of \( J \) functions from Lemma 58, we have
\[
\nabla J_{f^6}^{\beta(1+\beta)} = 1/2 \nabla J_{f^6}^\beta
\]
and thus
\[
\frac{1}{1 - 1/2(1 + \beta)} \nabla J_{f^6}^{\beta(1+\beta)} = \frac{2}{1 - \beta} \nabla J_{f^6}^{\beta(1+\beta)} = \left( \frac{1}{1 - \beta} \nabla J_{f^6}^\beta \right). \tag{57}
\]
And finally for all \( \tilde{f} \in H, \tilde{\beta} \in (0, 1) \) such that \( u(\tilde{f}) \in \text{int} u(H), I \circ u(\tilde{\beta} f + (1 - \tilde{\beta}) h^*) = \tilde{v} \), we have \( \tilde{\beta} f + (1 - \tilde{\beta}) h^* \sim 1/2(1 + \beta) f^6 + (1 - 1/2(1 + \beta)) h^* \) and by Lemma 59,
\[
\frac{1}{1 - \beta} \nabla J_{f^6}^\beta = \frac{1}{1 - 1/2(1 + \beta)} \nabla J_{f^6}^{\beta(1+\beta)} = \frac{1}{1 - \beta} \nabla J_{f^6}^\beta \tag{55} \equiv K^*(v) = K_*(v) \tag{58}
\]
Thus since \( \tilde{f} \in H, \tilde{\beta} \in (0, 1) \) were arbitrary such that \( u(\tilde{f}) \in \text{int} u(H), I \circ u(\tilde{\beta} f + (1 - \tilde{\beta}) h^*) = \tilde{v} \),
\[
K^*(\tilde{v}) \tag{58} \equiv K^*(v) = K_*(v) \tag{58} \equiv K_*(\tilde{v}).
\]

Thus there exists a neighborhood of \( v \) such that \( K^* = K_* \) and \( K^* \) is constant and finite. Since \( v \in (\inf I(u(H)), \sup I(u(H))) \) was arbitrary, locally \( K^* = K_* \) and \( K^* \) is constant and finite. Since the interval \( (\inf I(u(H)), \sup I(u(H))) \) is connected, by (Viro et al., 2008, Problem 12.2x) \( K^* \) and \( K_* \) are constants and finite on the set \( (\inf I(u(H)), \sup I(u(H))) \) and \( K^* = K_*. \)

Finally, the next lemma shows that the constant directional derivative gives additivity of the representation between equally crisp acts.

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Lemma 61 Let \((u, I)\) be a state dependent dual-self variational representation for \(\succeq\), \(f^*, g^*\) are equally crisp acts iff for all \(\varphi \in u(H), \alpha \in (0, 1)\) such that \(\varphi + \alpha(u(f^*) - u(g^*)) \in u(H)\),
\[
I(\varphi + \alpha(u(f^*) - u(g^*))) = I(\varphi) + \alpha(I(u(f^*)) - I(u(g^*))).
\]

Proof. Let first \(f^*, g^*\) be equally crisp acts. Let \(\varphi \in \text{int } u(H), \alpha \in \mathbb{R}_{++}\) such that \(\varphi + \alpha(u(f^*) - u(g^*)) \in \text{int } u(H)\). Define the mapping \([0, \alpha] \ni \beta \mapsto I(\varphi + \beta(u(f^*) - u(g^*))).\) Now let \(\beta \in (0, \alpha)\). Then \(\varphi + \beta(u(f^*) - u(g^*)) \in \text{int } u(H)\). Thus there exists \(\varepsilon > 0\) such that \(B_\varepsilon(\varphi + \beta(u(f^*) - u(g^*)), \varepsilon) \subseteq u(H)\). Let \(\gamma \in (1/2, 1)\) be such that
\[
\left(\frac{1}{\gamma} - 1\right)\|\varphi + \beta(u(f^*) - u(g^*))\|_\infty + \frac{1-\gamma}{\gamma}\|u(h^*)\|_\infty < \varepsilon.
\]

Thus by the choice of \(\gamma\), there exists \(f \in H\) such that
\[
u(f) = \varphi + \beta(u(f^*) - u(g^*)) + \left(\frac{1}{\gamma} - 1\right)\varphi + \beta(u(f^*) - u(g^*)) - \frac{1-\gamma}{\gamma}u(h^*).
\]
Thus \(u(\gamma f + (1-\gamma)h^*) = \varphi + \beta(u(f^*) - u(g^*))\). Now for all \(\zeta \in (0, 1)\) such that \(\beta + (1-\gamma)(\zeta - 1/2) \geq 0\),
\[
I(\varphi + \beta(u(f^*) - u(g^*)) + (1-\gamma)\zeta g^*)) = J(\beta + (1-\gamma)(\zeta - 1/2)).
\]
Denote \(\zeta_* := \max\{0, 1/2 - \frac{\beta}{1-\gamma}\}\) and \(\zeta^* := \min\{1, 1/2 + \frac{\alpha-\beta}{1-\gamma}\}\). Define \([\zeta_*, \zeta^*] \ni \zeta \mapsto J^\beta(\beta + (1-\gamma)(\zeta - 1/2))\) where \(\zeta^* > 1/2 > \zeta_*\). Thus by Lemma 60, \(J^\beta\) is affine and \(\nabla J^\beta = I\left(u(f^*)\right) - I\left(u(g^*)\right)\).

Thus by the chain rule and (59),
\[
\nabla J(\beta) = \nabla J^\beta(1/2) = \frac{1}{1-\gamma} I\left(u(f^*)\right) - I\left(u(g^*)\right).
\]

Thus finally since \(I\) is 1-Lipschitz, we have
\[
I(\varphi + \alpha(u(f^*) - u(g^*)) - I(\varphi) = \int_{(0, \alpha)} \nabla J(\beta) d\beta \overset{(60)}{=} \alpha \left(I\left(u(f^*)\right) - I\left(u(g^*)\right)\right)\).
\]
Thus we have for all \(\varphi \in \text{int } u(H), \alpha \in (0, 1)\) such that \(\varphi + \alpha(u(f^*) - u(g^*)) \in \text{int } u(H)\),
\[
I(\varphi + \alpha(u(f^*) - u(g^*))) = I(\varphi) + \alpha(I(u(f^*)) - I(u(g^*))).
\]
Finally, since the mapping \(I\) is continuous, we have for all \(\varphi \in u(H), \alpha \in (0, 1)\) such that \(\varphi + \alpha(u(f^*) - u(g^*)) \in u(H)\),
\[
I(\varphi + \alpha(u(f^*) - u(g^*))) = I(\varphi) + \alpha(I(u(f^*)) - I(u(g^*)))).
\]
Next, assume that \( f^*, g^* \in H \) are such that for all \( \varphi \in u(H), \alpha \in (0,1] \) such that \( \varphi + \alpha(u(f^*) - u(g^*)) \in u(H) \),

\[
I(\varphi + \alpha(u(f^*) - u(g^*))) = I(\varphi) + \alpha(I(u(f^*)) - I(u(g^))). \tag{61}
\]

We show that \( f^* \) and \( g^* \) are equally crisp. Let \( f, g \in H, \alpha \in (0,1) \). Then we have for \( h \in \{f, g\} \),

\[
I(\alpha u(h) + (1 - \alpha) g^*) \overset{(61)}{=} I(\alpha u(h) + (1 - \alpha) f^*) - I(u(h)) + I(u(g^*)).
\]

Thus

\[
I(\alpha u(h) + (1 - \alpha) g^*) \geq I(\alpha u(h) + (1 - \alpha) f^*) \Leftrightarrow I(\alpha u(h) + (1 - \alpha) f^*) \geq I(\alpha u(h) + (1 - \alpha) g^*),
\]

which proves the claim since \( I \circ u \) represents \( \succeq \).

\[ \square \]

**D.2 Behavioral Directional Derivatives**

By the previous part, dual-self variational representation is additive between equally crisp acts. Next we show that this allows us to capture the derivatives of the representation at differentiability points. These derivatives will give us the probabilities of the decision maker.

First, we define how to capture behaviorally directional derivatives. The next definition gives the level of change of preferences at \( f \) to the direction of \( g \) at mixture level \( \alpha \) using the \( \beta \) mixture of equally crisp acts as the measuring tool. Due to non-convex preferences, we focus on the smallest amount of change with \( C \). We keep the crisp acts \( f^*, g^* \) such that for all \( s \in \mathcal{S}^P, f^*_s \succ s g^*_s \) as fixed and denote \( h^* := \frac{1}{2} f^* + \frac{1}{2} g^* \).

**Definition** For \( f, g \in H, \beta \in (0,1) \), define for all \( \alpha \in (0,1) \),

\[
A_{f,\beta}^{g,\alpha} := \left\{ \hat{\alpha} \in [-\frac{1}{2\alpha}, \frac{1}{2\alpha}] | \beta f + (1 - \beta) \left( (\alpha \hat{\alpha} + \frac{1}{2}) f^* + (\frac{1}{2} - \alpha \hat{\alpha}) g^* \right) \sim \alpha g + (1 - \alpha) \left( \beta f + (1 - \beta) h^* \right) \right\}
\]

and

\[
C_{f,\beta}^{g,\alpha} = \begin{cases} 
\arg \min \left\{ |\hat{\alpha}| | \hat{\alpha} \in A_{f,\beta}^{g,\alpha} \right\}, & \text{if } A_{f,\beta}^{g,\alpha} \neq \emptyset \\
\infty, & \text{if } A_{f,\beta}^{g,\alpha} = \emptyset
\end{cases}
\]

where \( \sup \emptyset = \infty = \inf \emptyset \).
The next lemmas show that \( \lim_{\alpha \to 0} C_{f,\beta}^{g,\alpha} \) captures the directional derivative of the representation at \( f \) to the direction \( g \).

The first lemma shows that for a small enough \( \alpha \), \( C_{f,\beta}^{g,\alpha} \) exists and the argmin in it is well-defined. First, we observe that the argmin is always at most a singleton by the affinity of the representation between equally crisp acts, Lemma 61.

**Lemma 62** Let \( \preceq \) have a state dependent dual-self variational representation. Let \( f, g \in H \) and \( \beta \in (0, 1) \). Let \((\alpha_i)_{i=1}^\infty\) and \( \lim_{i \to \infty} \alpha_i = 0 \). Then there exists \( i^1 \) such that for all \( i > i^1 \), \( C_{f,\beta}^{g,\alpha_i} \in \mathbb{R} \).

**Proof.** Assume that \( \alpha^0 \in (0, \frac{1}{2}] \) since the other case follows symmetrically. First, there exists \( i^1 \in \mathbb{N} \) such that for all \( i > i^1 \), \( C_{f,\beta}^{g,\alpha_i} \in \mathbb{R} \). Since the sets
\[
\begin{align*}
\{ \alpha \in [0, 1] \mid \alpha g + (1 - \alpha) (\beta f + (1 - \beta) h^*) \succ (\beta f + (1 - \beta) g^*) \} \\
\{ \alpha \in [0, 1] \mid \beta f + (1 - \beta) f^* \succ (1 - \alpha) (\beta f + (1 - \beta) h^*) \}
\end{align*}
\]
are open by Axiom 2 and include 0, there exists \( \varepsilon > 0 \) such that for all \( \alpha \in (0, \varepsilon) \),
\[
\beta f + (1 - \beta) f^* \succ \alpha g + (1 - \alpha) (\beta f + (1 - \beta) h^*) \succ \beta f + (1 - \beta) g^*.
\]
Thus especially for all \( \alpha \in (0, \varepsilon) \), \( C_{f,\beta}^{g,\alpha_i} \in \mathbb{R} \). Since \( \lim_{i \to \infty} \alpha_i = 0 \), there exists \( i^1 \in \mathbb{N} \) such that for all \( i > i^1 \), \( \alpha_i \in [0, \varepsilon) \). \( \square \)

**Lemma 63** Assume that \( \preceq \) has a state dependent dual-self variational representation. Let \( f, g \in H \) and \( \beta, \alpha \in (0, 1) \). If \( C_{f,\beta}^{g,\alpha} \in \mathbb{R} \), then
\[
\beta f + (1 - \beta) \left( (\alpha C_{f,\beta}^{g,\alpha} + \frac{1}{2}) f^* + (\frac{1}{2} - \alpha C_{f,\beta}^{g,\alpha}) g^* \right) \sim \alpha g + (1 - \alpha) \left( \beta f + (1 - \beta) h^* \right)
\]

**Proof.** Assume, per contra,
\[
\beta f + (1 - \beta) \left( (\alpha C_{f,\beta}^{g,\alpha} + \frac{1}{2}) f^* + (\frac{1}{2} - \alpha C_{f,\beta}^{g,\alpha}) g^* \right) \not\prec \alpha g + (1 - \alpha) \left( \beta f + (1 - \beta) h^* \right).
\]
Assume w.l.o.g.
\[
\beta f + (1 - \beta) \left( (\alpha C_{f,\beta}^{g,\alpha} + \frac{1}{2}) f^* + (\frac{1}{2} - \alpha C_{f,\beta}^{g,\alpha}) g^* \right) \succ \alpha g + (1 - \alpha) \left( \beta f + (1 - \beta) h^* \right)
\]
since the other case is symmetric. Now the set
\[
A := \left\{ \alpha \in [0, 1] \mid \alpha \left( \beta f + (1 - \beta) f^* \right) + (1 - \alpha) \left( \beta f + (1 - \beta) g^* \right) \succ \alpha g + (1 - \alpha) \left( \beta f + (1 - \beta) h^* \right) \right\}
\]
is open by Axiom 2 and includes \( \alpha C_{f,\beta}^{g,\alpha} + 1/2 \). Thus there exists \( \varepsilon > 0 \) such that for all \( \gamma \in (-\varepsilon, \varepsilon) \), \( \alpha C_{f,\beta}^{g,\alpha} + 1/2 + \gamma \in A \). Thus for all \( \gamma \in (-\varepsilon/\alpha, \varepsilon/\alpha) \),

\[
\beta f + (1 - \beta)
\left((\alpha(C_{f,\beta}^{g,\alpha} + \gamma) + 1/2)\tilde{g}^* + \left(1/2 - \alpha(C_{f,\beta}^{g,\alpha} + \gamma)\right)\tilde{g}^*\right)
\equiv (\alpha C_{f,\beta}^{g,\alpha} + 1/2 + \alpha \gamma)\left(\beta f + (1 - \beta)\tilde{f}^*\right) + (1 - \alpha C_{f,\beta}^{g,\alpha} + 1/2 + \alpha \gamma)\left(\beta f + (1 - \beta)\tilde{g}^*\right)
\succ \alpha g + (1 - \alpha)\left(\beta f + (1 - \beta)\tilde{h}^*\right).
\]

However, this contradicts the definition of \( C_{f,\beta}^{g,\alpha} \) since there exists \( \tilde{\alpha} \in [-1/2\alpha, 1/2\alpha] \) such that

\[|C_{f,\beta}^{g,\alpha} - \tilde{\alpha}| < \frac{\varepsilon}{\alpha}\]

and

\[
\beta f + (1 - \beta)\left((\alpha\tilde{\alpha} + 1/2)f^* + \left(1/2 - \alpha\tilde{\alpha}\right)g^*\right) \sim \alpha g + (1 - \alpha)\left(\beta f + (1 - \beta)h^*\right)
\]

The next two lemmas show how \( \lim_{\alpha \to 0} C_{f,\beta}^{g,\alpha} \) captures the directional derivative to direction \( g \) at differentiability points.

**Lemma 64** Let \( (u, I) \) be a state dependent dual-self variational representation and \( f, g \in H, \beta \in (0, 1) \). Let \( (\alpha_i)_{i=1}^{\infty} \subset (0, 1) \) be such that \( \lim_{i \to \infty} \alpha_i = 0 \). Then there exists \( i^1 \) such that for all \( i > i^1 \),

\[
I\left(\alpha_i u(g) + (1 - \alpha_i)u\left(\beta f + (1 - \beta)h^*\right)\right)
\alpha_i
= (1 - \beta)C_{f,\beta}^{g,\alpha_i}\left(I\left(\tilde{u}^*\right) - I\left(u\tilde{g}^*\right)\right).
\]

**Proof.** By Lemmas 62 and 63, there exists \( i^1 \in \mathbb{N} \) such that for all \( i > i^1 \),

\[
I\left(\alpha_i u(g) + (1 - \alpha_i)u\left(\beta f + (1 - \beta)h^*\right)\right)
= I\left(\beta u(f) + (1 - \beta)\left((\alpha_i C_{f,\beta}^{g,\alpha_i} + 1/2)u(f^* + (1/2 - \alpha_i C_{f,\beta}^{g,\alpha_i})u(g^*))\right)\right).
\]

Let \( i > i^1 \). We have by Lemmas 61 and 63,

\[
I\left(\alpha_i u(g) + (1 - \alpha_i)u\left(\beta f + (1 - \beta)h^*\right)\right)
= I\left(\beta u(f) + (1 - \beta)\left((\alpha_i C_{f,\beta}^{g,\alpha_i} + 1/2)u(f^* + (1/2 - \alpha_i C_{f,\beta}^{g,\alpha_i})u(g^*))\right)\right)
= I\left(\beta f + (1 - \beta)g^*\right) + (1 - \beta)(1/2 + \alpha_i C_{f,\beta}^{g,\alpha_i})\left(I(f^*) - I(g^*)\right)
= I\left(\beta f + (1 - \beta)h^*\right) + (1 - \beta)(\alpha_i C_{f,\beta}^{g,\alpha_i})\left(I(f^*) - I(g^*)\right).
\]

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The claim follows by subtracting $I\left(\beta f + (1 - \beta)h^*\right)$ from both sides and dividing both sides by $\alpha_i$. 

**Lemma 65** Let $S$ be finite and $(u, I)$ be a state dependent dual-self representation $f, g \in H$, $\beta \in (0, 1)$, and $(\alpha_i)_{i=1}^\infty \subset (0, 1)$ be such that $\lim_{i \to \infty} \alpha_i = 0$. If

$$\lim_{i \to \infty} C_{f,\beta}^{g,\alpha_i} \in \mathbb{R} \text{ or } \lim_{i \to \infty} \frac{I\left(\alpha_i u(g) + (1 - \alpha_i)u\left(\beta f + (1 - \beta)h^*\right)\right) - I\left(u\left(\beta f + (1 - \beta)h^*\right)\right)}{\alpha_i} \in \mathbb{R},$$

then both limits exist and

$$\lim_{i \to \infty} \frac{I\left(\alpha_i u(g) + (1 - \alpha_i)u\left(\beta f + (1 - \beta)h^*\right)\right) - I\left(u\left(\beta f + (1 - \beta)h^*\right)\right)}{\alpha_i} = (1 - \beta)\left(I\left(u(f^*)\right) - I\left(u(g^*)\right)\right) \lim_{i \to \infty} C_{f,\beta}^{g,\alpha_i}.$$

**Proof.** By Lemma 64, there exists $i^1$ such that for all $i > i^1$,

$$\frac{I\left(\alpha_i u(g) + (1 - \alpha_i)u\left(\beta f + (1 - \beta)h^*\right)\right) - I\left(u\left(\beta f + (1 - \beta)h^*\right)\right)}{\alpha_i} = (1 - \beta)C_{f,\beta}^{g,\alpha_i}\left(I\left(u(f^*)\right) - I\left(u(g^*)\right)\right).$$

Thus taking the limit of $i \to \infty$ shows the claim since either the limit on the left or the right hand side exists. 

The next lemma shows how interior points are the same for all representations. 

**Lemma 66** Let $(u, I)$ and $(\tilde{u}, \tilde{I})$ be state dependent dual-self variational representations for $\succeq$. Let $f \in H$. Then $u(f) \in \text{int } u(H)$ iff $\tilde{u}(f) \in \text{int } \tilde{u}(H)$

**Proof.** We show w.l.o.g. that if $u(f) \in \text{int } u(H)$, then $\tilde{u}(f) \in \text{int } \tilde{u}(H)$. By Lemma 44 and Proposition 25, for all $s \in S^P$, there exist $A_s > 0, B_s \in \mathbb{R}$ such that $\tilde{u}_s = A_s u_s + B_s$. Thus if $u(f) \in \text{int } u(H)$, there exists $\varepsilon > 0$ such that $B_\infty(u(f), \varepsilon) \subseteq \text{int } u(H)$ and so $B_\infty(u(f), \varepsilon) \subseteq \text{int } u(H)$. Thus by above, $B_\infty(\tilde{u}(f), \min_{A_s \mid s \in S^P} \varepsilon) \subseteq \text{int } \tilde{u}(H)$, which shows the claim.

The next lemma shows how we can relate directional derivatives of different representations to each other using the behavioral directional derivatives.
Lemma 67 Let \((u, I)\) and \((\tilde{u}, \tilde{I})\) be state dependent dual-self variational representations for \(\succsim\). Let \(A \in \mathbb{R}^{S^\tau}_+\), \(B \in \mathbb{R}^S\) be such that for all \(s \in S\), \(\tilde{u}_s = A_s u_s + B_s\). Denote \(A^{-1} := (A_s^{-1})_{s \in S}\). Let \(f \in H\) be such that \(u(f) \in \text{int} \, (u(H))\). If \(I\) is differentiable at \(u(f)\) with derivative \(\nabla I(u(f))\), then \(\tilde{I}\) is differentiable at \(\tilde{u}(f)\) with derivative

\[
\frac{\tilde{I}(\tilde{u}(f^*)) - \tilde{I}(\tilde{u}(g^*))}{I(u(f^*)) - I(u(g^*))} A^{-1} \nabla I(u(f)).
\]

Proof. Let the derivative of \(I\) at \(u(f)\) be \(\nabla I(u(f))\). By Lemma 44 and Proposition 25, for all \(s \in S^\tau\), there exist \(A_s > 0, B_s \in \mathbb{R}\) such that

\[\tilde{u}_s = A_s u_s + B_s.\]  \hspace{2cm} (62)

By Lemma 66, \(\tilde{u}(f) \in \text{int} \, \tilde{u}(H)\). Let \(v \in \mathbb{R}^S\). Then there exists \(\alpha^* > 0\) such that \(\tilde{u}(f) + \alpha^* v, \tilde{u}(f) - \alpha^* v \in \tilde{u}(H)\). Let \(g, \tilde{g} \in H\) be such that \(\tilde{u}(g) = \tilde{u}(f) + \alpha^* v\) and \(\tilde{u}(\tilde{g}) = \tilde{u}(f) - \alpha^* v\). Let \(\beta \in (0, 1)\) be such that \(\frac{1}{\beta} \tilde{u}(f) - \frac{1 - \beta}{\beta} u(h^*) \in \tilde{u}(H)\). Let \(\tilde{f} \in H\) be such that \(u(\tilde{f}) = \frac{1}{\beta} \tilde{u}(f) - \frac{1 - \beta}{\beta} u(h^*)\). Then

\[
\tilde{u}(\beta \tilde{f} + (1 - \beta) h^*) = \tilde{u}(f).
\]  \hspace{2cm} (63)

Thus by (62) \(u(\beta \tilde{f} + (1 - \beta) h^*) = u(f)\). Let \((\alpha_i)_{i=1}^\infty \subseteq (0, 1)\) be such that \(\alpha_i \to 0\). Since \(I\) is differentiable at \(u(\beta \tilde{f} + (1 - \beta) h^*) = u(f)\) and so

\[
\lim_{i \to \infty} \frac{I(\alpha_i u(g) + (1 - \alpha_i) u(f)) - I(u(f))}{\alpha_i} = \nabla I(u(f)) \cdot (u(g) - u(f)) \in \mathbb{R}.
\]

Thus by Lemma 65, \(\lim_{i \to \infty} C_{f, \beta}^{g, \alpha_i}\) exists and

\[
(1 - \beta) \left( I(u(f^*)) - I(u(g^*)) \right) \lim_{i \to \infty} C_{f, \beta}^{g, \alpha_i} = \nabla I(u(f)) \cdot (u(g) - u(f)).
\]

Thus by Lemma 65 and (63),

\[
\lim_{i \to \infty} \frac{I(\alpha_i u(g) + (1 - \alpha_i) u(f)) - I(u(f))}{\alpha_i} = (1 - \beta) \left( \tilde{I}(\tilde{u}(f^*)) - \tilde{I}(\tilde{u}(g^*)) \right) \lim_{i \to \infty} C_{f, \beta}^{g, \alpha_i}
\]  \hspace{2cm} (64)

\[
= \frac{\tilde{I}(\tilde{u}(f^*)) - \tilde{I}(\tilde{u}(g^*))}{I(u(f^*)) - I(u(g^*))} \nabla I(u(f)) \cdot (u(g) - u(f)).
\]

Symmetrically for \(\tilde{g}\), we have

\[
\lim_{i \to \infty} \frac{I(\alpha_i \tilde{u}(\tilde{g}) + (1 - \alpha_i) \tilde{u}(f)) - I(\tilde{u}(f))}{\alpha_i} = (1 - \beta) \left( \tilde{I}(\tilde{u}(f^*)) - \tilde{I}(\tilde{u}(g^*)) \right) \lim_{i \to \infty} C_{f, \beta}^{\tilde{g}, \alpha_i}
\]  \hspace{2cm} (65)

\[
= \frac{\tilde{I}(\tilde{u}(f^*)) - \tilde{I}(\tilde{u}(g^*))}{I(u(f^*)) - I(u(g^*))} \nabla I(u(f)) \cdot (u(\tilde{g}) - u(f)).
\]
Next by (62), we have
\[
\nabla I(u(f)) \cdot (u(g) - u(f)) = \sum_{s \in S} \nabla I(u(f))_s A_s^{-1}(\tilde{u}(g) - \tilde{u}(f))
\]
\[
= \sum_{s \in S} A_s^{-1}\nabla I(u(f))_s \alpha^* v_s = A^{-1} \nabla I(u(f)) \cdot (\alpha^* v).
\]
Since \(\tilde{u}(g) - \tilde{u}(f) = - (\tilde{u}(\tilde{g}) - \tilde{u}(f))\), by (62), we have \(u(g) - u(f) = - (u(\tilde{g}) - u(f))\). Thus by (64,65) and the choice of \(g\), we have
\[
\lim_{i \to \infty} \frac{I(\tilde{u}(f) + \alpha_i \alpha^* v)}{\alpha^* \alpha_i} - I(\tilde{u}(f)) = \frac{1}{\alpha^*} \frac{I(\tilde{u}(f^*)) - I(\tilde{u}(g^*))}{I(u(f^*)) - I(u(g^*))} A^{-1} \nabla I(u(f)) \cdot (\alpha^* v)
\]
\[
= \lim_{i \to \infty} \frac{I(\tilde{u}(f) - \alpha_i \alpha^* v)}{- \alpha^* \alpha_i} - I(\tilde{u}(f))
\]
Since the sequence \((\alpha_i)_{i=1}^\infty \subseteq (0,1)\) such that \(\alpha_i \to 0\) was arbitrary, we have that
\[
\lim_{\alpha \to 0} \frac{I(\tilde{u}(f) + \alpha v) - I(\tilde{u}(f))}{\alpha} = \frac{I(\tilde{u}(f^*)) - I(\tilde{u}(g^*))}{I(u(f^*)) - I(u(g^*))} A^{-1} \nabla I(u(f)) \cdot v.
\]
Since \(v \in \mathbb{R}^S\) was arbitrary, \(I\) is differentiable at \(\tilde{u}(f)\) with derivative
\[
\frac{I(\tilde{u}(f^*)) - I(\tilde{u}(g^*))}{I(u(f^*)) - I(u(g^*))} A^{-1} \nabla I(u(f)).
\]

Finally, we have some useful simple lemmas. The first lemma relates Clarke derivatives to standard derivatives.

**Lemma 68** Let \(S\) be finite and \((u, I)\) be a state dependent dual-self variational representation for \(\succcurlyeq\). Then for all \(f \in H\) such that \(u(f) \in \text{int} u(H)\),
\[
\partial I(u(f)) = \text{convex}\{\lim_{i \to \infty} \nabla I(\varphi_i)(\varphi_i)_{i=1}^\infty \subseteq u(H), \lim_{i \to \infty} \varphi_i = u(f), \forall i \in \mathbb{N}, I\text{ differentiable at }\varphi_i\}.
\]

**Proof.** Since \(I\) is Lipschitz on \(u(H)\), this follows directly from Clarke’s (1983) Theorem 2.5.1. \(\square\)

The next lemma shows how for dual-self representation the derivative gives the value of the representation.

**Lemma 69** Let \(S\) be finite and \((u, I)\) be a state dependent dual-self representation for \(\succcurlyeq\). Let \(f \in H\) be such that \(u(f) \in \text{int}_\infty u(H)\). If \(I\) is differentiable at \(u(f)\) with derivative \(p\), then \(I(u(f)) = p \cdot u(f)\).
Proof. Since \( u(f) \in \text{int}_\infty u(H) \), there exists \( \alpha^0 > 0 \) such that \((1+\alpha^0)u(f), (1-\alpha^0)u(f) \in u(H)\). Since \( u(H) \) is convex and \( I \) is positive homogeneous, we have for all \( \alpha \in (-\alpha^0, \alpha^0) \setminus \{0\} \),

\[
I\left((1+\alpha)u(f)\right) = (1+\alpha)I\left(u(f)\right) \Rightarrow \frac{I(\alpha u(f) + u(f)) - I(u(f))}{\alpha} = I(u(f)).
\]

Thus by taking the limit \( \alpha \to 0 \), we have by the differentiability of \( I \) at \( u(f) \),

\[
I\left(u(f)\right) = \lim_{\alpha \to 0} \frac{I(\alpha u(f) + u(f)) - I(u(f))}{\alpha} = p \cdot u(f).
\]

The last lemma is an observation of separating any vector that has a constant value for all probabilities to a constant number and a vector perpendicular to the probabilities \( y \).

**Lemma 70** Let \( C \subseteq \Delta(S) \). If there exist \( x \in \mathbb{R}^S_+ \) and \( \beta \in \mathbb{R}_+ \) such that for all \( p \in C \),

\[
\sum_{s \in S} x_s p_s = \beta
\]

then there exists \( y \in C^\perp \) such that

\[
x = \mathbb{1}_S \beta + y
\]

Proof. Let \( x \in \mathbb{R}^S_+ \) be as in the antecedent. Since \( \sum_{s \in S} p_s = 1 \), we have for all \( p \in \tilde{C} \),

\[
\sum_{s \in S} (x - \beta \mathbb{1}_S)_s p_s = 0.
\]

Thus

\[
(x - \beta \mathbb{1}_S) \in \tilde{C}^\perp := \{y \in \mathbb{R}^S | \forall \ p \in \tilde{C}, \sum_{s \in S} y_s p_s = 0\}.
\]

So there exists \( y \in \tilde{C}^\perp \) such that

\[
y = x - \beta \mathbb{1}_S \Rightarrow \beta \mathbb{1}_S + y = x.
\]

\( \square \)

### D.3 Partial Identification

The next lemma shows the general identification for dual-self variational representation’s utilities and probabilities.
Lemma 71 Let \((u, I)\) and \((\tilde{u}, \tilde{I})\) be state dependent dual-self variational representations for \(\succsim\). Then there exist \(x \in \left( \bigcup_{\varphi \in \text{int}(u(H))} \partial I(\varphi) \right)^{\perp} \), \(\alpha > 0\), \(B \in \mathbb{R}^S \beta \in \mathbb{R}\) such that when we denote the mapping for all \(\mu \in \Delta(S)\),

\[
(1 + x)(\mu) := \left((1 + x_s)\mu_s \right)_{s \in S}
\]

we have

\[
\bigcup_{\varphi \in \text{int}(\tilde{u}(H))} \partial \tilde{I}(\varphi) = (1 + x) \bigcup_{\varphi \in \text{int}(u(H))} \partial I(\varphi).
\]

and for all \(S^p\),

\[
(\tilde{u}_s)_{s \in S^p} = \left(\frac{\alpha}{1 + x_s}u_s + B_s \right)_{s \in S^p}.
\]

Proof. By Lemma 44 and Proposition 25, there exist \(A \in \mathbb{R}^S_{++}, B \in \mathbb{R}^S_{++}\) such that for all \(s \in S^p\),

\[
\tilde{u}_s = A_s u_s + B_s.
\]

Denote

\[
\alpha := \frac{\tilde{I}(\tilde{u}(f^*)) - \tilde{I}(\tilde{u}(g^*))}{I(u(f^*)) - I(u(g^*))} > 0.
\]

By Lemmas 66 and 67, \(I\) is differentiable at \(\varphi \in \text{int}(u(H))\) with derivative \(\nabla I(\varphi)\) if and only if \(\tilde{I}\) is differentiable at \(A\varphi + B \in \text{int}(\tilde{u}(H))\) with derivative

\[
\alpha A^{-1} \nabla I(u(f)).
\]

Thus for all \(\varphi \in \text{int}(u(H))\), by Lemma 68,

\[
\partial \tilde{I}(A\varphi + B) = \alpha A^{-1} \partial I(\varphi).
\]

Thus by Lemma 66,

\[
\bigcup_{\varphi \in \text{int}(\tilde{u}(H))} \partial \tilde{I}(\varphi) = \bigcup_{\varphi \in \text{int}(u(H))} \alpha A^{-1} \partial I(\varphi).
\]

By continuity of \(A^{-1}\) operator, we have

\[
\bigcup_{\varphi \in \text{int}(\tilde{u}(H))} \partial \tilde{I}(\varphi) = \alpha A^{-1} \bigcup_{\varphi \in \text{int}(u(H))} \partial I(\varphi).
\]

Thus for all \(p \in \bigcup_{\varphi \in \text{int}(u(H))} \partial I(\varphi)\), we have

\[
\alpha A^{-1} p \in \Delta(S).
\]
Thus by Lemma 70, there exists \( x \in \left( \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \right)^\perp \) such that
\[
1 + x = \alpha A^{-1}.
\]

Thus especially for all \( s \in S \),
\[
A_s = \frac{\alpha}{1 + x_s}.
\]
This proves the claim.

The next lemma shows that the values of the dual-self variational representation are unique up to a positive affine transformation.

**Lemma 72** Let \((u, I)\) and \((\bar{u}, \bar{I})\) be state dependent dual-self variational representations for \( Z \). Then there exist \( \alpha > 0, \beta \in \mathbb{R} \) such that for all \( f \in H \),
\[
\bar{I}(\bar{u}(f)) = \alpha I(u(f)) + \beta
\]
where
\[
\alpha = \frac{\bar{I}(\bar{u}(f^*)) - \bar{I}(\bar{u}(g^*))}{I(u(f^*)) - I(u(g^*))} \quad \text{and} \quad \beta = \bar{I}(\bar{u}(g^*)) - \alpha I(u(g^*)).
\]

**Proof.** Let \( \alpha \) and \( \beta \) be as above. Define the mapping \( K: (\inf I(u(H)), \sup I(u(H))) \to \mathbb{R} \cup \{\infty\} \)
\[
K(v) = \sup \left\{ \left| \bar{I}(\bar{u}(f)) - \alpha I(u(f)) + \beta \right|; f \in H, I \circ u(f) = v \right\}
\]
where \( \sup \mathcal{D} = \infty \). We show that \( K \) is locally constant. Let \( v \in (\inf I(u(H)), \sup I(u(H))) \).

First, there exist \( f^1, f^2 \in H \) such that \( I(u(f^1)) > v > I(u(f^2)) \). Since \( S \) is finite there exists \( f^3 \in H \) such that \( u(f^3) \in \text{int} u(H) \). By mixture continuity, there exist \( \alpha^1, \alpha^2, \alpha^3 \in (0, 1) \) such that \( \alpha^1 + \alpha^2 + \alpha^3 = 1 \) and \( I \circ u(\alpha^1 f^1 + \alpha^2 f^2 + \alpha^3 f^3) = v \). Denote \( f^4 := \alpha^1 f^1 + \alpha^2 f^2 + \alpha^3 f^3 \).

Since \( \alpha^3 > 0 \) and \( u(f^3) \in \text{int} u(H) \) and \( u(H) \) is a convex set \( u(f^4) \in \text{int} u(H) \). Thus there exists \( \varepsilon > 0 \) such that \( B_{\infty}(u(f^4), \varepsilon) \subseteq u(H) \). Let \( \beta \in (1/2, 1) \) be such that
\[
\left( \frac{1}{\beta} - 1 \right) \| u(f^4) \|_{\infty} + \frac{1 - \beta}{\beta} \| u(h^*) \|_{\infty} < \varepsilon.
\]
Thus by the choice of \( \beta \), there exists \( f \in H \) such that
\[
u(f) = u(f^4) + \left( \frac{1}{\beta} - 1 \right) u(f^4) - \frac{1 - \beta}{\beta} u(h^*).
\]
Thus especially \( u(f) \in \text{int} u(H) \) and
\[
\beta u(f) + (1 - \beta) u(h^*) = u(f^4)
\]
and so \(I\left(\beta u(f) + (1 - \beta)u(h^*)\right) = v\). By Lemma 60, for all \(\alpha \in (0, 1)\), we have

\[
I\left(\beta u(f) + (1 - \beta)u(f^*\alpha g^*)\right) = I\left(\beta u(f) + (1 - \beta)u(h^*)\right) + (1 - \beta)(\alpha - 1/2)\left(I\left(u(f^*)\right) - I\left(u(g^*)\right)\right) \tag{66}
\]

and

\[
\bar{I}\left(\beta \bar{u}(f) + (1 - \beta)\bar{u}(f^*\alpha g^*)\right) = \bar{I}\left(\beta \bar{u}(f) + (1 - \beta)\bar{u}(h^*)\right) + (1 - \beta)(\alpha - 1/2)\left(\bar{I}\left(\bar{u}(f^*)\right) - \bar{I}\left(\bar{u}(g^*)\right)\right) \tag{67}
\]

Thus let

\[
\bar{v} \in \left(v - 1/2(1 - \beta)\left(I\left(u(f^*)\right) - I\left(u(g^*)\right)\right), v + 1/2(1 - \beta)\left(I\left(u(f^*)\right) - I\left(u(g^*)\right)\right)\right).
\]

and let \(\bar{\alpha} \in (0, 1)\) be such

\[
\bar{v} = I\left(\beta u(f) + (1 - \beta)u(h^*)\right) + (1 - \beta)(\bar{\alpha} - 1/2)\left(I\left(u(f^*)\right) - I\left(u(g^*)\right)\right).
\]

Let \(\bar{f} \in H\) be such that \(I \circ u(\bar{f}) = \bar{v}\). Then \(\bar{f} \sim \beta \bar{u}(f) + (1 - \beta)\bar{u}(f^*\bar{\alpha}g^*)\). Thus by (66,67),

\[
I\left(u(\bar{f})\right) = \bar{I}\left(\beta \bar{u}(f) + (1 - \beta)u(f^*\bar{\alpha}g^*)\right) = I\left(\beta u(f) + (1 - \beta)u(h^*)\right) + (1 - \beta)(\bar{\alpha} - 1/2)\left(I\left(u(f^*)\right) - I\left(u(g^*)\right)\right)\]

and

\[
\bar{I}\left(\bar{u}(\bar{f})\right) = \bar{I}\left(\beta \bar{u}(f) + (1 - \beta)\bar{u}(f^*\bar{\alpha}g^*)\right) = \bar{I}\left(\beta \bar{u}(f) + (1 - \beta)\bar{u}(h^*)\right) + (1 - \beta)(\bar{\alpha} - 1/2)\left(\bar{I}\left(\bar{u}(f^*)\right) - \bar{I}\left(\bar{u}(g^*)\right)\right).
\]

Hence

\[
\bar{I}\left(\bar{u}(\bar{f})\right) - \alpha I\left(u(\bar{f})\right) - \beta
\]

\[
= \bar{I}\left(\beta \bar{u}(f) + (1 - \beta)\bar{u}(h^*)\right) + (1 - \beta)(\bar{\alpha} - 1/2)\left(\bar{I}\left(\bar{u}(f^*)\right) - \bar{I}\left(\bar{u}(g^*)\right)\right)
\]

\[
- \frac{\bar{I}\left(\bar{u}(f^*)\right) - \bar{I}\left(\bar{u}(g^*)\right)}{I\left(u(f^*)\right) - I\left(u(g^*)\right)} \left(I\left(\beta u(f) + (1 - \beta)u(h^*)\right) + (1 - \beta)(\bar{\alpha} - 1/2)\left(I\left(u(f^*)\right) - I\left(u(g^*)\right)\right)\right) - \beta
\]

\[
= \bar{I}\left(\beta \bar{u}(f) + (1 - \beta)\bar{u}(h^*)\right) - \alpha I\left(\beta u(f) + (1 - \beta)u(h^*)\right) - \beta.
\]

Since the last line does not depend on \(\bar{f}, \bar{\alpha}, \bar{v}\), we have for all

\[
\bar{v} \in \left(v - 1/2(1 - \beta)\left(I\left(u(f^*)\right) - I\left(u(g^*)\right)\right), v + 1/2(1 - \beta)\left(I\left(u(f^*)\right) - I\left(u(g^*)\right)\right)\right),
\]

\[
K(\bar{v}) = \left|\bar{I}\left(\beta \bar{u}(f) + (1 - \beta)\bar{u}(h^*)\right) - \alpha I\left(\beta u(f) + (1 - \beta)u(h^*)\right) - \beta\right|.
\]
Thus $K$ is constant in a neighborhood of $v$. Since $v \in (\inf I(u(H)), \sup I(u(H)))$ was arbitrary, $K$ is locally constant function. Since the interval $(\inf I(u(H)), \sup I(u(H)))$ is connected, by (Viro et al., 2008, Problem 12.2x) $K$ is constant function. Finally, let $f \in H$ be such that $I \circ u(f) = I \circ u(g^*)$. Then we have $f \sim g^*$ and so
\[
\tilde{I}(\tilde{u}(f)) - \alpha I(u(f)) - \beta = \tilde{I}(\tilde{u}(g^*)) - \alpha I(u(g^*)) - \tilde{I}(\tilde{u}(g^*)) + \alpha I(u(g^*)) = 0.
\]
Since $f$ such that $I \circ u(f) = I \circ u(g^*)$ was arbitrary, we have $K(I \circ u(g^*)) = 0$. Thus $K$ is a constant function at 0. Finally, by the continuity of $I$ and $\tilde{I}$, we have for all $f \in H$,
\[
\tilde{I}(\tilde{u}(f)) = \alpha I(u(f)) + \beta.
\]

\[\Box\]

These results characterize the identification for dual-self variational representation.

**Corollary 73** Let $(u, \mathbb{C})$ be a state dependent tight dual-self variational representation for $\succsim$, and $(\tilde{u}, \tilde{\mathbb{C}})$ be a utility-cost set pair. Then $(\tilde{u}, \tilde{\mathbb{C}})$ is a state dependent tight dual-self variational representation for $\succsim$ if and only if there exist $x \in \mathbb{R}^{S^+}, B \in \mathbb{R}^S, \beta \in \mathbb{R}$ such that for all $p, q \in \bigcup_{c \in \mathbb{C}} \text{dom } c$,
\[
\sum_{s \in S} x_sp_s = \sum_{s \in S} x_sq_s =: \alpha,
\]
for all $s \in S^p$,
\[
\tilde{u}_s = \frac{1}{x_s} u_s + B_s,
\]
and for all $f \in H$,
\[
\max_{c \in \mathbb{C}} \min_{p \in \Delta(S)} \sum_{s \in S} p_s \tilde{u}_s(f_s) + c(p) = \max_{c \in \mathbb{C}} \min_{p \in \Delta(S)} \sum_{s \in S} p_s \tilde{u}_s(f_s) + \alpha c(x/\alpha) - \sum_{s \in S} B_sp_s + \beta.
\]
Especially $(\tilde{u}, \alpha(\mathbb{C} \circ x/\alpha) - dB + \beta)$, where multiplication and subtractions are done elementwise, is another state dependent tight dual-self variational representation for $\succsim$ and
\[
\bigcup_{\tilde{c} \in \tilde{\mathbb{C}}} \text{dom } \tilde{c} = \bigcup_{c \in \mathbb{C}} \frac{x}{\alpha} \text{ dom } c.
\]

**Proof.** This follows symmetrically to Corollary 75 since for $I: u(H) \to \mathbb{R}$ by defining
\[
I(u(f)) = \max_{c \in \mathbb{C}} \min_{p \in \Delta(S)} p \cdot u(f) + c(p),
\]
we have by Chandrasekher et al.’s (2020) Section S.3,
\[
\bigcup_{\varphi \in u(H)} \partial I(\varphi) = \bigcup_{c \in \mathbb{C}} \text{ dom } c.
\]
We move onto dual-self representation. The next lemma shows the identification for probabilities and utilities.

**Lemma 74** Let \((u, I)\) and \((\tilde{u}, \tilde{I})\) be state dependent dual-self representations for \(\succeq\). Then there exist \(x, y \in \left(\bigcup_{\varphi \in u(H)} \partial I(\varphi)\right)^\perp\), \(\alpha > 0, \beta \in \mathbb{R}\) such that when we denote the mapping for all \(\mu \in \Delta(S)\),

\[
(1 + x)(\mu) := (1 + x_s)\mu_s
\]

we have

\[
\bigcup_{\varphi \in \text{int } \tilde{u}(H)} \partial \tilde{I}(\varphi) = (1 + x) \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi).
\]

and for all \(S^p\),

\[
(\tilde{u}_s)_{s \in S^p} = \left(\frac{\alpha}{1 + x_s}(u_s + y_s) + \beta\right)_{s \in S^p}.
\]

**Proof.** By the proof of Lemma 71, there exist \(x \in \left(\bigcup_{\varphi \in u(H)} \partial I(\varphi)\right)^\perp\), \(\alpha > 0, B \in \mathbb{R}^S\beta \in \mathbb{R}\) such that

\[
(\tilde{u}_s)_{s \in S^p} = \left(\frac{\alpha}{1 + x_s}(u_s + y_s) + B_s\right)_{s \in S^p}
\]

where

\[
\alpha = \frac{\tilde{I}(\tilde{u}(f^*)) - \tilde{I}(\tilde{u}(g^*))}{I(u(f^*)) - I(u(g^*))}.
\]

Let \(\beta := \tilde{I}(\tilde{u}(g^*)) - \alpha I(u(g^*))\).

By Lemmas 66 and 67, \(I\) is differentiable at \(\varphi \in \text{int } u(H)\) with derivative \(\nabla I(\varphi)\) if and only if \(\tilde{I}\) is differentiable at \(A\varphi + B \in \text{int } \tilde{u}(H)\) with derivative

\[
(1 + x)\nabla I(u(f)).
\]

Let \(f \in H\) be such that \(u(f) \in \text{int } u(H)\) and \(u(f)\) is a differentiability point of \(I\). Then by Lemma 69, \(I(u(f)) = \nabla I(u(f)) \cdot u(f)\) and \(\tilde{I}(\tilde{u}(f)) = (1 + x)\nabla I(u(f)) \cdot \tilde{u}(f)\). Thus we have by (68)

\[
\tilde{I}(\tilde{u}(f)) = (1 + x)\nabla I(u(f)) \cdot \frac{\alpha}{1 + x} u(f) + B
\]

\[
= \alpha \nabla I(u(f)) \cdot u(f) + (1 + x)\nabla I(u(f)) \cdot B = \alpha I(u(f)) + (1 + x)\nabla I(u(f)) \cdot B.
\]
Thus by Lemma 72,
\[(1 + x)\nabla I(u(f)) \cdot B = \bar{I}(\bar{u}(f)) - \alpha I(u(f)) = \beta\]
and so
\[(1 + x)\nabla I(u(f)) \cdot B = \bar{I}(\bar{u}(f)) - \alpha I(u(f)) = \beta\]

Since \(f\) was arbitrary such that \(u(f) \in \text{int} u(H)\) and \(u(f)\) is a differentiability point of \(I\), we have by Lemma 68, for all \(p \in \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi)\),
\[
\beta = (1 + x)p \cdot B = \sum_{s \in S} ((1 + x_s)B_s)p_s
\]
and since \((1 + x)p \in \Delta(S)\)
\[
0 = \sum_{s \in S} ((1 + x_s)B_s - \beta(1 + x_s))p_s.
\]

Thus by Lemma 70, there exists \(\alpha y \in \left(\bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi)\right)^\perp\) such that
\[
\alpha y = (1 + x)B - (1 + x)\beta \iff B = \frac{\alpha y}{1 + x} + \beta.
\]

Thus since \(\left(\bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi)\right)^\perp\) is a linear space \(y \in \left(\bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi)\right)^\perp\) and for all \(s \in S^P\),
\[
(\bar{u}_s)_{s \in S^P} = \left(\frac{\alpha}{1 + x_s}(u_s + y_s) + \beta\right)_{s \in S^P}.
\]

Now for all differentiability points, \(p \cdot u(f) = I(u(f))\) and \((1 + x)p \cdot \bar{u}(f) = \bar{I}(\bar{u}(f))\). Thus
\[
(1 + x)p \cdot \left(\frac{\alpha}{1 + x}(1 + x)u(f) + B\right) = \bar{I}(\bar{u}(f))
\]
and so
\[
(1 + x)p \cdot B = \bar{I}(\bar{u}(f)) - \alpha I(u(f)).
\]

Since this holds for all differentiability points, this shows the claim. \(\square\)

The next lemma shows the identification for dual-self representation.

**Corollary 75** Let \((u, \mathbb{P})\) be a state dependent tight dual-self representation for \(\succsim\), and \((\bar{u}, \bar{\mathbb{P}})\) be a utility-probability set collection pair. Then \((\bar{u}, \bar{\mathbb{P}})\) is a state dependent tight dual-self representation for \(\succsim\) if and only if there exist \(x \in \mathbb{R}^S_{++}, y \in \mathbb{R}^S\) such that for all \(p, q \in \bigcup_{P \in \mathbb{P}} P\),
\[
\sum_{s \in S} x_sp_s = \sum_{s \in S} x_sq_s =: \alpha \quad \text{and} \quad \sum_{s \in S} y_sp_s = \sum_{s \in S} y_sq_s.
\]
for all $s \in S^P$,
\[
\tilde{u}_s = \frac{1}{x_s}(u_s + y_s),
\]
and for all $f \in H$,
\[
\max \min \sum_{P \in P} \tilde{p}_s \tilde{u}_s(f_s) = \max \min \sum_{P \in P} p_s \tilde{u}_s(f_s).
\]
Especially $(\tilde{u}, \frac{\partial}{\partial \alpha})$ is another state dependent tight dual-self representation for $\approx$ and
\[
\tilde{v} \cup \bar{P} = \tilde{v} \cup \bar{P}.
\]

Proof. Let $(\tilde{u}, \tilde{P})$ be a state dependent tight dual-self representation for $\approx$. Define $I : u(H) \to \mathbb{R}$ and $\bar{I} : \bar{u}(H) \to \mathbb{R}$ by for all $f \in H$,
\[
I(u(f)) = \max \min p \cdot u(f) \text{ and } \bar{I}(\bar{u}(f)) = \max \min \bar{p} \cdot \bar{u}(f).
\]
Now $(u, I)$ and $(\tilde{u}, \bar{I})$ are state dependent dual-self representations for $\approx$. By Chandrasekher et al.’s (2020) Lemma B.3
\[
\tilde{v} \cup \partial I(\varphi) = \tilde{v} \cup P \text{ and } \bar{v} \cup \partial \bar{I}(\bar{\varphi}) = \bar{v} \cup \bar{P}.
\]
Thus by Lemma 74, there exist $\bar{x}, \bar{y} \in \left( \bigcup_{P \in P} \tilde{P} \right)^\perp$, $\bar{\alpha} > 0$, $\beta \in \mathbb{R}$ such that
\[
\bigcup_{\varphi \in \text{int } \bar{u}(H)} \partial \bar{I}(\bar{\varphi}) = (1 + \bar{x}) \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi).
\]
and for all $S^P$,
\[
(\tilde{u}_s)_{s \in S^P} = \left( \frac{\alpha}{1 + \bar{x}}(u_s + \bar{y}_s) + \beta \right)_{s \in S^P}.
\]
Thus define $y = \bar{y} + \bar{\alpha}^{-1} \beta$, $x = \bar{\alpha}^{-1}(1 + \bar{x})$, and $\alpha = \bar{\alpha}^{-1}$ which gives the utility identification. Additionally, we have for all $f \in H$,
\[
\max \min \tilde{p} \cdot \tilde{u}(f) = \alpha \max \min p \cdot u(f) + \beta.
\]
And $\frac{\partial}{\partial \alpha} \in \mathcal{K}(\Delta(S))$ since $\|\frac{\partial}{\partial \alpha}\|_\infty < \infty$. Thus $(\tilde{u}, \frac{\partial}{\partial \alpha})$ is a state dependent dual-self representation for $\approx$ and thus by Lemma 72, for all $f \in H$,
\[
\max \min \sum_{P \in P} \tilde{p}_s \tilde{u}_s(f_s) = \max \min \sum_{P \in P} p_s \tilde{u}_s(f_s).
\]
\[
\square
\]
D.4 Full Identification

Next, we move on to full-identification and show how Axiom 7 characterizes the full identification. The next lemma shows that instead of utility trade-off for acts in Axiom 7, we can use statewise preference trade-offs for them.

Lemma 76  Let $\succsim$ have a state dependent dual-self variational representation with $(u, I)$. Let $f, g \in H$ be equally crisp acts such that there exist $s, s' \in S^P$ such that $u_s(f_s) > u_s(g_s)$ and $u_{s'}(g_{s'}) > u_{s'}(f_{s'})$. Then there exist equally crisp acts $f^1, g^1$ and $s^1, s^2 \in S^P$ such that $f^1 \sim g^1, f^1 \succ s^1, g^2$, and $g^1 \succ s^2, f^2$.

**Proof.** Assume that there exist $\tilde{f}^*, \tilde{g}^*, s, s' \in S$ such that $u_s(\tilde{f}^*_s) > u_s(\tilde{g}^*_s)$ and $u_{s'}(\tilde{g}^*_{s'}) > u_{s'}(\tilde{f}^*_{s'})$ and $\tilde{f}^*, \tilde{g}^*$ are equally crisp. Assume without loss of generality $\tilde{f}^* \succsim \tilde{g}^*$. By Lemma 61, for all $\varphi \in u(H), \alpha \in (0, 1]$ such that $\varphi + \alpha(u(\tilde{f}^*) - u(\tilde{g}^*)) \in u(H)$,

$$I(\varphi + \alpha(u(\tilde{f}^*) - u(\tilde{g}^*)) = I(\varphi) + \alpha(I(u(\tilde{f}^*)) - I(u(\tilde{g}^*))) \quad (69)$$

Let $g^0 \in H$ be such that $u(f^0) \in \text{int } u(H)$. Now there exists $\alpha^* > 0$ such that $u(f^0) + \alpha^*(u(\tilde{f}^*) - u(\tilde{g}^*))$, $u(f^0) - \alpha^* \bar{I}(I(u(\tilde{f}^*)) - I(u(\tilde{g}^*))) \in u(H)$. Let $f^0, f^1 \in H$ be such that

$$u(f^0) = u(f^0) + \alpha^*(u(\tilde{f}^*) - u(\tilde{g}^*))$$

and

$$u(f^1) = u(f^0) - \alpha^* \bar{I}(I(u(\tilde{f}^*)) - I(u(\tilde{g}^*)))$$

By Equation (69) and C-additivity of $I$, $(g^0, f^0)$ and $(g^0, f^1)$ are equally crisp acts. Then by Lemma 53, $(g^0, 1/2f^0 + 1/2f^1)$ are equally crisp acts. Denote $f^2 := 1/2f^0 + 1/2f^1$. Now we have since $u(H)$ is a convex set

$$I(u(f^2)) = I(u(f^0) + 1/2\alpha^*(u(\tilde{f}^*) - u(\tilde{g}^*)) - 1/2\alpha^* \bar{I}(I(u(\tilde{f}^*)) - I(u(\tilde{g}^*))) \quad (70)$$

$$\overset{\text{C-add.}}{=} I(u(f^0) + 1/2\alpha^*(u(\tilde{f}^*) - u(\tilde{g}^*))) - 1/2\alpha^* \bar{I}(I(u(\tilde{f}^*)) - I(u(\tilde{g}^*)))$$

$$\overset{(69)}{=} I(u(f^0)) + 1/2\alpha^*(I(u(\tilde{f}^*)) - I(u(\tilde{g}^*))) - 1/2\alpha^* \bar{I}(I(u(\tilde{f}^*)) - I(u(\tilde{g}^*))) = I(u(f^0)).$$

Thus $f^2 \sim f^0$. First, since $u_{s'}(\tilde{g}^*_{s'}) > u_{s'}(\tilde{f}^*_{s'})$, we have

$$u_{s'}(f^2_{s'}) = u_{s'}(f^0_{s'}) + 1/2\alpha^*(u_{s'}(\tilde{f}^*_{s'}) - u_{s'}(\tilde{g}^*_{s'})) + 1/2(I(u(\tilde{g}^*)) - I(u(\tilde{f}^*))) < u_{s'}(f^0_{s'}).$$
Since \( s' \in S^p \), by Lemma 68, there exists \( \varphi \in \text{int } u(H) \) such that \( I \) is differentiable at \( \varphi \) and \( \nabla I(\varphi)_{s'} > 0 \). By Lemma 61, we have
\[
\nabla I(\varphi) \cdot \left( u(f^2) - u(f^0) \right) = 0.
\]
Thus there exists \( s'' \) such that \( u_{s''}(f^2_{s''}) > u_{s''}(g^0_{s''}) \) and \( \nabla I(\varphi)_{s''} > 0 \). Let \( g_3, f^3, \alpha^3 \) be such that \( u(g^3) = \varphi, u(f^3) = \varphi + \alpha^3 \left( u(f^2) - u(g^0) \right) \). By Lemma 61, \( (g^3, f^3) \) are equally crisp acts and by (70) \( g^3 \sim f^3 \) and
\[
\begin{align*}
&u_{s''}(f^3_{s''}) - u_{s''}(g^3_{s''}) > 0 > u_{s''}(f^3_{s''}) - u_{s''}(g^3_{s''}).
\end{align*}
\]
Additionally, by the differentiability of \( I \) at \( u(g^3) \), there exists
\[
0 < \varepsilon < \min \{ u_{s''}(f^3_{s''}) - u_{s''}(g^3_{s''}), u_{s''}(g^3_{s''}) - u_{s''}(f^3_{s''}) \}
\]
such that
\[
I(u(g^3) + \text{pr}_{s''}, \varepsilon) > I(u(g^3)) + \nabla I(\varphi)_{s''} \cdot \varepsilon \quad \text{and} \quad I(u(g^3) - \text{pr}_{s''}, \varepsilon) < I(u(g^3)) - \nabla I(\varphi)_{s''} \cdot \varepsilon.
\]
Thus
\[
I(u(f^3_{s''}, g^3_{s''})) \leq I(u(g^3) - \text{pr}_{s''}, \varepsilon) < I(u(g^3)) \quad \text{and} \quad I(u(f^3_{s''}, g^3_{s''})) \geq I(u(g^3) + \text{pr}_{s''}, \varepsilon) > I(u(g^3))
\]
Thus \( f^3_{s''} \succ_{s''} g^3_{s''} \) and \( g^3_{s''} \succ_{s''} f^3_{s''} \) and \( (g^3, f^3) \) are equally crisp.

The next lemma shows the equivalency between Axiom 7 and equally crisp acts being statewise ranked.

**Lemma 77** Assume that \( \succsim \) has a state dependent dual-self variational representation with \((u, I)\). Then the following are equivalent:

(I) If \( f, g \) are equally crisp acts, then for all \( s \in S, f_s \sim_s g_s \) or for all \( s \in S, g_s \sim_s f_s \).

(II) For all \( f, g \) and \( s, s' \in S \) such that \( f \sim g, f_s \succ_s g_s \), and \( f_{s'} \succ_{s'} g_{s'} \), there exist \( h \in H, \alpha \in (0, 1) \) such that \( \alpha h + (1 - \alpha) f \not\sim \alpha h + (1 - \alpha) g \).

**Proof.** Assume that (II) does not hold and we show that (I) does not hold. By the counter assumption, there exist \( f, g \in H, s, s' \in S \) such that \( f \sim g, f_s \succ_s g_s \), and \( f_{s'} \succ_{s'} g_{s'} \) and for all \( h \in H, \alpha \in (0, 1) \), \( \alpha h + (1 - \alpha) f \not\sim \alpha h + (1 - \alpha) g \). Now we have for all \( h, h', \alpha \in (0, 1) \),
\[
\alpha h + (1 - \alpha) f \sim \alpha h + (1 - \alpha) g \quad \text{and} \quad \alpha h' + (1 - \alpha) f \sim \alpha h' + (1 - \alpha) g
\]
Thus
\[ \alpha h + (1 - \alpha)f \gtrless \alpha h' + (1 - \alpha)f \iff \alpha h + (1 - \alpha)g \gtrless \alpha h' + (1 - \alpha)g. \]

Hence \( f, g \) are equally crisp and \( (I) \) does not hold.

Assume that \( (I) \) does not hold and we show that \( (II) \) does not hold. Assume that there exist \( \tilde{f}^*, \tilde{g}^*, s, s' \in S \) such that \( \tilde{f}_s^* \succ_s \tilde{g}_s^* \) and \( \tilde{g}_{s'}^* \succ_{s'} \tilde{f}_{s'}^* \) and \( \tilde{f}^*, \tilde{g}^* \) are equally crisp. First, by Lemma 44, \( u_s(\tilde{f}^*)_s > u_s(\tilde{g}^*)_s \) and \( u_{s'}(\tilde{g}^*_{s'}) > u_{s'}(\tilde{f}^*_{s'}) \). Thus by Lemma 76, there exist equally crisp acts \( f^1, g^1 \) and \( s^1, s^2 \in S^P \) such that \( f^1 \sim g^1, f^1_s \succ s^1 g^1_s, \) and \( g^1_{s^2} \succ s^2 f^1_{s^2}. \) However, for all \( h \in H, \alpha \in (0, 1), \) by Lemma 61,
\[
I(\alpha u(h) + (1 - \alpha)u(f^1)) = I(\alpha u(h) + (1 - \alpha)u(g^1)) + (1 - \alpha) I(u(f^1)) - I(u(g^1))
\]
and hence \( \alpha h + (1 - \alpha)f^1 \sim \alpha h + (1 - \alpha)g^1. \) This contradicts \( (II) \). \( \square \)

The next lemma shows that Axiom 7 characterizes the full-dimensional uncertainty that gives the identification by previous partial identification.

**Lemma 78** Assume that \( \gtrless \) has a state dependent dual-self variational representation with \((u, I)\). Then the following are equivalent:

(I) \( \gtrless \) satisfies Axiom 7.

(II) \( \text{pr}_{S^P} \overline{\bigcup}_{\varphi \in \text{int} u(H)} \partial I(\varphi) \) has a non-empty interior in \( \Delta(S^P) \).

**Proof.** Enumerate \( S^P = \{ \tilde{s}_1, \ldots, \tilde{s}_n \} \) where \( n = |S^P| \). Assume that (I) holds. Then if \( f, g \) are equally crisp acts, by Lemmas 76 and 77, for all \( s, s' \in S^P, u_s(f^1) - u_s(g_s) = u_{s'}(f^1) - u_{s'}(g_{s'}). \) Thus \( \left( \text{pr}_{S^P} \overline{\bigcup}_{\varphi \in \text{int} u(H)} \partial I(\varphi) \right)^\perp = \{0\}. \) Hence \( \text{pr}_{S^P} \overline{\bigcup}_{\varphi \in \text{int} u(H)} \partial I(\varphi) \) contains \( |S^P| = n \) linearly independent probabilities \( \{p^1, \ldots, p^n\} \). Consider the set \( \text{pr}_{S^P} \overline{\bigcup}_{\varphi \in \text{int} u(H)} \partial I(\varphi) - n \) where the subtraction is vector subtraction in \( \mathbb{R}^{S^P} \). Now the set \( \text{pr}_{S^P} \overline{\bigcup}_{\varphi \in \text{int} u(H)} \partial I(\varphi) - p^n \supset \{ p^1 - p^n, \ldots, p^{n-1} - p^n \} \) is linearly independent. Consider the matrix \( P = [p^i - p^n]_{1 \leq i \leq n-1} \). First, the rank of \( P \) is \( n - 1 \) by above. Second, we have for all \( 1 \leq i \leq n - 1, \sum_{j=1}^n p^i_j - p^i = 0. \) Third, since the row rank of \( P \) is \( n - 1 \) and the last row \( \tilde{s}_n \) is linearly dependent on rows \( \{ \tilde{s}_1, \ldots, \tilde{s}_{n-1} \}, \) the row vectors of \( P \) for rows \( \{ \tilde{s}_1, \ldots, \tilde{s}_{n-1} \} \) are linearly independent. Next, consider the matrix \( \tilde{P} = [\text{pr}_{S^P \setminus \{ \tilde{s}_n \}} p^i - p^n]_{1 \leq i \leq n-1}. \) This is collecting row vectors of \( P \) for rows \( \{ \tilde{s}_1, \ldots, \tilde{s}_{n-1} \} \) and thus by above, the rank of \( \tilde{P} \) is \( n - 1 \). Thus
especially \( \{ \text{pr}_{SP\backslash\{\tilde{s}_n\}} p^1 - p^n, \ldots, \text{pr}_{SP\backslash\{\tilde{s}_n\}} p^{n-1} - p^n \} \) are linearly independent. Thus the set \( \text{pr}_{SP\backslash\{\tilde{s}_n\}} \text{co} \bigcup_{\varphi \in \text{int}(H)} \partial I(\varphi) - p^n \) contains the zero vector and \( n-1 \) linearly independent vectors and is a convex set. Hence the set \( \text{pr}_{SP\backslash\{\tilde{s}_n\}} \text{co} \bigcup_{\varphi \in \text{int}(H)} \partial I(\varphi) - p^n \) cannot be contained in any \( n - 2 \)-dimensional set and thus it has a non-empty interior in \( \mathbb{R}^{SP\backslash\{\tilde{s}_n\}} \) (Boyd and Vandenberghe, 2004, Section 2.5.2). Thus especially the set \( \text{pr}_{SP\backslash\{\tilde{s}_n\}} \text{co} \bigcup_{\varphi \in \text{int}(H)} \partial I(\varphi) \) has a non-empty interior in \( \Delta(SP) \) as a translation. Thus by the choice of topology for \( \Delta(SP) \), \( \text{pr}_{SP\backslash\{\tilde{s}_n\}} \text{co} \bigcup_{\varphi \in \text{int}(H)} \partial I(\varphi) \) has a non-empty interior in \( \Delta(SP) \).

Assume that (II) holds. Then \( \text{pr}_{SP} \text{co} \bigcup_{\varphi \in \text{int}(H)} \partial I(\varphi) \) contains \( |SP| \) linearly independent probabilities. Let \( f, g \) be equally crisp acts. Assume w.l.o.g. that there exists \( s \in SP \) such that \( u_s(f_s) > u_s(g_s) \) or for all \( s \in SP \), \( u_s(f_s) = u_s(g_s) \). Then by Lemmas 61 and 68, for all \( p, q \in \text{co} \bigcup_{\varphi \in \text{int}(H)} \partial I(\varphi) \),

\[
p \cdot (u(f) - u(g)) = q \cdot (u(f) - u(g)) := \alpha.
\]

By Lemma 70, there exists \( y \in \left( \text{co} \bigcup_{\varphi \in \text{int}(H)} \partial I(\varphi) \right)^\perp \) such that for all \( s \in S \),

\[
\alpha + y = u(f) - u(g).
\]

Since \( \left( \text{co} \bigcup_{\varphi \in \text{int}(H)} \partial I(\varphi) \right)^\perp = \{ \bar{0} \} \), we have for all \( s \in SP \), \( u_s(f_s) - u_s(g_s) = \alpha \). Now \( \alpha \geq 0 \) by the above assumption. Thus for all \( s \in SP \), \( u_s(f_s) \geq u_s(g_s) \). Thus by Lemma 44, for all \( s \in SP \), \( f_s \succeq_s g_s \). This shows (I) by Lemma 77.

\[\square\]

### D.5 Identification Characterization with Crisp Acts and Equally Crisp Acts

Next, we show Corollaries 6 and 11 and characterize the identification with crisp and equally crisp acts.
**D.5.1 Preliminary Results**

We first show some simple preliminary results. The first result shows for a dual-self expected utility taking a convex combination with a crisp act does not affect the derivative of the representation.

**Lemma 79** Let \((u, I)\) be a state dependent dual-self representation for \(\succeq\). If \(c \in H\) is crisp, \(\alpha \in (0, 1)\) and \(h \in H\) is such that \(u(h) \in \text{int} u(H)\) and \(I\) is differentiable at \(u(h)\), then \(I\) is differentiable at \(\alpha u(c) + (1 - \alpha)u(h)\) with

\[
\nabla I\left(\alpha u(c) + (1 - \alpha)u(h)\right) = \nabla I(u(h)).
\]

**Proof.** Let \(\alpha^* \in (0, 1)\). As in the proof of Lemma 45, there exist equally crisp \(f^*, g^* \in H, a > 0\) such that for all \(s \in S^P\), \(u_s(f^*_s) - u_s(g^*_s) = a\) and \(f^*_s \succ_s g^*_s\). Denote \(h^* = \frac{1}{2}f^* + \frac{1}{2}g^*\).

We use notation from Section D from using equally crisp acts \(f^*, g^*\). Since \(u(h) \in \text{int} u(H)\), there exist \(\hat{h} \in H, \beta \in (0, 1)\) such that for all \(s \in S^P\), \(u_s(\beta \hat{h} + (1 - \beta)h^*) = u_s(h)\).

Let \(g \in H\). Denote \(\hat{h} = \frac{\alpha^*}{\alpha^* + (1 - \alpha^*)} c + \frac{(1 - \alpha^*)}{\alpha^* + (1 - \alpha^*)} \beta \hat{h}, \tilde{\beta} = \alpha^* + (1 - \alpha^*)\beta, \tilde{g} = \alpha^* c + (1 - \alpha^*)g\).

We will show that for all \(\alpha \in (0, 1)\),

\[
C_{\hat{h}, \beta}^{\alpha, \alpha^*} = C_{\tilde{h}, \tilde{\beta}}^{\alpha, \alpha^*}.
\]

This follows from the definition of \(C_{\hat{h}, \beta}^{\alpha, \alpha^*}\) and the observation that for all \(\tilde{\alpha} \in [-1/2\alpha, 1/2\alpha]\), we have

\[
\beta \hat{h} + (1 - \beta)\left( (\alpha \tilde{\alpha} + 1/2) f^* + (1/2 - \alpha \tilde{\alpha}) g^* \right) \sim \alpha g + (1 - \alpha) \left( \beta \hat{h} + (1 - \beta) h^* \right)
\]

\[
\iff \alpha^* c + (1 - \alpha^*) \left( \beta \hat{h} + (1 - \beta) \left( (\alpha \tilde{\alpha} + 1/2) f^* + (1/2 - \alpha \tilde{\alpha}) g^* \right) \right) \sim \alpha^* c + (1 - \alpha^*) \left( \alpha g + (1 - \alpha) \left( \beta \hat{h} + (1 - \beta) h^* \right) \right)
\]

\[
\iff \tilde{\beta} \tilde{h} + (1 - \tilde{\beta}) \left( (\alpha \tilde{\alpha} + 1/2) f^* + (1/2 - \alpha \tilde{\alpha}) g^* \right) \sim \left( \alpha \tilde{g} + (1 - \alpha) \left( \tilde{\beta} \tilde{h} + (1 - \tilde{\beta}) h^* \right) \right)
\]

where the second equivalence follows as an identity.

Since \(I\) is differentiable at \(u(h)\), the differentiability follows from Lemma 65 as in the proof of Lemma 67 since for all \(v \in \mathbb{R}^{S^P}\), there exist \(g \in H, \alpha^* > 0\) such that for all \(s \in S^P\), \(u_s(g) = u_s(h) + \alpha^* v\) and so

\[
u_s(\alpha^* c + (1 - \alpha^*)g) = u_s\left( \tilde{\beta} \tilde{h} + (1 - \tilde{\beta}) h^* \right) + (1 - \alpha^*) \alpha^* v.
\]

Additionally, the derivative being the same follows from Lemma 65.
Next, we show that the expected utility of crisp acts is the same for all probabilities.

**Lemma 80** Let \((u, I)\) be a state dependent dual-self representation for \(\succeq\). If \(c \in H\) is crisp, then for all \(p \in \bigcup_{\varphi \in u(H)} \partial I(\varphi)\),
\[
p \cdot u(c) = I(u(c)).
\]

*Proof.* Let \(h \in H\) be such that \(u(h) \in \text{int} u(H)\) and \(I\) is differentiable at \(u(h)\). Then by Lemmas 69 and 79, for all \(\alpha \in [0, 1)\),
\[
I\left(u(\alpha c + (1-\alpha)h)\right) = \nabla I\left(u(h)\right) \cdot u(\alpha c + (1-\alpha)h).
\]
Thus by the continuity of \(I\) and by taking limit \(\alpha \to 1\),
\[
I\left(u(c)\right) = \nabla I\left(u(h)\right) \cdot u(c).
\]
Finally, the claim follows from Lemma 68. \(\square\)

**D.5.2 Identification with Crisp and Equally Crisp Acts**

**Corollary 81** Let \((u, P)\) be a state dependent dual-self representation for \(\succeq\) and \(x \in \mathbb{R}^S_{++}, y \in \mathbb{R}^S\). The following two conditions are equivalent:

1. \((\bar{u}, \bar{P})\) is a state dependent dual-self representation for \(\succeq\) such that for all \(s \in S^P\),
   \[
   \bar{u}_s = \frac{1}{x_s} \left(u_s + y_s\right)
   \]
   and
   \[
   0, 1 \in \bigcap_{s \in S^P} \bar{u}_s\left(\Delta(X_s)\right).
   \]
2. There exist crisp acts \(f^*, g^*\) such that \(u_s(g^*_s) = -y_s\) and \(u_s(f^*_s) = -y_s + x_s\).

*Proof.* Follows from Lemma 80 and Proposition 3 since \(x \in \mathbb{R}^S_{++}\). \(\square\)

**Corollary 82** Let \((u, C)\) be a state dependent dual-self variational representation for \(\succeq\) and \(x \in \mathbb{R}^S_{++}, B \in \mathbb{R}^S\). The following two conditions are equivalent:

1. \((\bar{u}, \bar{C})\) is a state dependent dual-self variational representation for \(\succeq\) such that for all \(s \in S^P\),
   \[
   \bar{u}_s = \frac{1}{x_s} u_s + \frac{B_s}{x_s}
   \]
and
\[ 0, 1 \in \bigcap_{s \in S^P} \bar{u}_s(\Delta(X_s)). \]

(2) There exist equally crisp acts \( f^*, g^* \) such that \( u_s(g^*_s) = -B_s \) and \( u_s(f^*_s) = -B_s + x_s \).

Proof. Follows from Lemmas 61 and 85 and Theorem 9 since \( x \in \mathbb{R}^{S^+}_+ \).

\[ \square \]

\section*{D.6 Relative Likelihood and ProbabilityCharacterizations}

Next, we move on to proving Propositions 3 and 4. We show these results by characterizing the possible transformations for probabilities behaviorally and in terms of the set of probabilities: First, for the relative likelihood identification, we show when all the possible probability transformations change the probabilities for states \( s \) and \( s' \) similarly that is when for all \( x \in \left( \bigcup_{\varphi \in \text{int} u(H) \partial I(\varphi)} \right)^\perp, x_s = x_{s'} \). Second, for the probability identification, we show when all the possible probability transformations keep the probability for state \( s \) the same that is when for all \( x \in \left( \bigcup_{\varphi \in \text{int} u(H) \partial I(\varphi)} \right)^\perp, x_s = 0 \). We show these results for state dependent dual-self expected utility.

\section*{D.6.1 Preliminary Results}

We start with some simple lemmas. The next lemma is a trivial one showing that non-null states are characterized by zero probability.

\textbf{Lemma 83} Let \((u, I)\) be a state dependent dual-self variational representation for \( \succeq \). If \( s \in S^P \), then there exists \( p \in \bigcup_{\varphi \in \text{int} u(H) \partial I(\varphi)} \) such that \( p_s > 0 \).

Proof. Follows from the definition of \( S^P \) and the definition of Clarke derivatives. \( \square \)

The next lemma shows that any utility trade-off between two states can be represented with acts that have statewise preference trade-offs in those two states.

\textbf{Lemma 84} Let \((u, I)\) be a state dependent dual-self variational representation for \( \succeq \). Let \( x \in \mathbb{R}^S \) and \( s, s' \in S^P \) be such that \( x_s > 0 > x_{s'} \) and for all \( \tilde{s} \notin S^P, x_{\tilde{s}} = 0 \). Then there exist \( f, g \in H, a > 0 \) such that \( f_s \succ_s g_s \) and \( g_{s'} \succ_{s'} f_{s'}, u(f), u(g) \in \text{int} u(H) \), and \( u(f) - u(g) = ax \).
Proof. Since \( s, s' \in S^P \) by Lemmas 68 and 83, there exist \( f^s, f^{s'} \in H \) such that \( u(f^s), u(f^{s'}) \in \text{int } u(H) \), \( I \) is differentiable at \( u(f^s), u(f^{s'}) \) and
\[
\nabla I(u(f^s))_s, \nabla I(u(f^{s'}))_{s'} > 0.
\]

(72)

Let \( f \in H \) be such that \( u(f) \in \text{int } u(H) \), \( u_s(f_s) = \phi_s \), and \( u_s'(f_s') = \phi_s' \). Since \( u(f) \in \text{int } u(H) \) and for all \( \tilde{s} \notin S^P \), \( x_{\tilde{s}} = 0 \), there exist \( a > 0, g \in H \) such that \( u(f) - u(g) = ax \). Additionally, since \( I \) is monotonic by (72), we have
\[
(f_s, f_{s}^{-}) \succ (g_s, f_{s}^{-}) \quad \text{and} \quad (g_{s}', f_{s}'^{-}) \succ (f_{s}', f_{s}'^{-}).
\]

The last lemma shows that if \( x \) has a constant expected utility for all the probabilities, then the representation is additive in adding utility \( x \) to any utility \( \phi \).

Lemma 85 Let \((u, I)\) be a state dependent dual-self variational representation for \( \succcurlyeq \). Let \( x \in \mathbb{R}^S \) and \( c \in \mathbb{R} \). The following two are equivalent:

1. \( (x - \overline{1}c) \in \left( \bigcup_{\phi \in \text{int } u(H)} \partial I(\phi) \right)^\perp. \)
2. For all \( \phi \in \text{int } u(H) \) such that \( \phi + x \in \text{int } u(H) \), we have
\[
I(\phi + x) = I(\phi) + c.
\]

Proof. We first show that \( (1) \Rightarrow (2) \). By Chandrasekher et al.’s (2020) Section S.3,
\[
\bigcup_{\phi \in \text{int } u(H)} \partial I(\phi) \subseteq \Delta(S).
\]

Since \( \text{int } u(H) \) is convex, for all \( \alpha \in [0, 1] \), \( \psi + \alpha x \in \text{int } u(H) \) Define a mapping \( [0, 1] \ni \alpha \mapsto I(\psi + \alpha x) \). Since \( I \) is C-additive, it is 1-Lipschitz. So especially \( J \) is Lipschitz and so differentiable almost everywhere. Now if \( J \) is differentiable at \( \alpha \in (0, 1) \), we have by Lemma 68,
\[
J'(\alpha) \leq \sup_{\psi \in \partial I(\psi + \alpha x)} \psi \cdot x = \sup_{\psi \in \partial I(\psi + \alpha x)} \psi \cdot (x - \overline{1}c) + \psi \cdot \overline{1}c = \sup_{\psi \in \partial I(\psi + \alpha x)} \psi \cdot \overline{1}c = c
\]
and symmetrically
\[
J'(\alpha) \geq \inf_{\psi \in \partial I(\psi + \alpha x)} \psi \cdot x = c.
\]
Let $\Omega \subseteq (0, 1)$ be the set of differentiability points for $J$. By the Fundamental theorem of calculus, we have

$$I(\varphi + x) = J(0) + J(1) - J(0) = I(\varphi) + \int_{\Omega} J'(x) \, dx = I(\varphi) + c.$$ 

Next, we show that $(2) \Rightarrow (1)$. Let $\varphi \in \text{int} \, u(H)$ be such that $I$ is differentiable at $\varphi$. Now we have for all $\alpha \in (0, 1)$ such that $\varphi + \alpha(x - \bar{1}c), \varphi + \alpha x \in \text{int} \, u(H)$, by C-additivity and (2),

$$I(\varphi + \alpha(x - \bar{1}c)) = I(\varphi + \alpha(x - \bar{1}c)) = I(\varphi + \alpha x) - \alpha c = I(\varphi) + \alpha c - \alpha c = I(\varphi).$$

Now there exists $\alpha^* \in (0, 1)$ such that for all $\alpha \in (0, 1), \varphi + \alpha(x - \bar{1}c), \varphi + \alpha x \in \text{int} \, u(H)$ and thus by the differentiability of $I$,

$$\nabla I(\varphi) \cdot (x - \bar{1}c) = \lim_{\alpha \to 0} \frac{I(\varphi + \alpha(x - \bar{1}c)) - I(\varphi)}{\alpha} = 0.$$ 

Thus since $\varphi$ was arbitrary, we have by Lemma 68, for all $p \in (\bigcup_{\varphi \in \text{int} \, u(H)} \partial I(\varphi)), p \cdot (x - \bar{1}c) = 0$.

This shows (1). 

D.6.2 Relative Likelihood Identification

Our first lemma for relative likelihood identification shows that our behavioral assumption for relative likelihood identification characterizes their identification. This follows as a corollary of this lemma and our previous partial identification result.

**Lemma 86** Let $(u, I)$ be a state dependent dual-self variational representation for $\succsim$ and $s, s' \in S^p, s \neq s'$. The following two are equivalent:

1. For all $f, g \in H$ such that $f_s \succ g_s$ and $g_{s'} \succ f_{s'}$, there exist $h, h' \in H$ and $\alpha \in (0, 1)$ such that
   $$\alpha h + (1 - \alpha) f \succ \alpha h' + (1 - \alpha) f \text{ and } \alpha h + (1 - \alpha) g \prec \alpha h' + (1 - \alpha) g.$$ 

2. For all $x \in \left(\bigcup_{\varphi \in \text{int} \, u(H)} \partial I(\varphi)\right)^\perp, x_s = x_{s'}$.

**Proof.** We will first show that $(1) \Rightarrow (2)$. Assume, per contra, there exists $x \in \left(\bigcup_{\varphi \in \text{int} \, u(H)} \partial I(\varphi)\right)^\perp, x_s \neq x_{s'}$. Assume w.l.o.g. $x_s > x_{s'}$. Let $c = -\frac{1}{2} x_s - \frac{1}{2} x_{s'}$. By (84), there
exist \( f, g \in H, a > 0 \) such that \( f_s \succ_g s \) and \( g_{s'} \succ_{f'} f_{s'} \) and \( u(f) - u(g) = a(x + c) \). Let \( h \in H, \alpha \in (0, 1) \) with \( u(h) \in \text{int} u(H) \). By Lemma 85, we have
\[
I(\alpha u(h) + (1 - \alpha)u(f)) = I(\alpha u(h) + (1 - \alpha)u(g) + (1 - \alpha)(u(f) - u(g)))
\]
\[
= I(\alpha u(h) + (1 - \alpha)u(g) + (1 - \alpha)a(x + c)) = I(\alpha u(h) + (1 - \alpha)u(g)) + (1 - \alpha)ac.
\]
Since \( I \) is continuous, we have for all \( h \in H, \alpha \in (0, 1) \)
\[
I(\alpha u(h) + (1 - \alpha)u(f)) = I(\alpha u(h) + (1 - \alpha)u(g)) + (1 - \alpha)ac.
\]
Thus we have for all \( h, h' \in H, \alpha h + (1 - \alpha)f \succeq \alpha h' + (1 - \alpha)f \iff I(\alpha u(h) + (1 - \alpha)u(f)) \geq I(\alpha u(h') + (1 - \alpha)u(f))
\]
\[
\iff I(\alpha u(h) + (1 - \alpha)u(f)) + (1 - \alpha)ac \geq I(\alpha u(h') + (1 - \alpha)u(f)) + (1 - \alpha)ac
\]
\[
\iff I(\alpha u(h) + (1 - \alpha)u(g)) \geq I(\alpha u(h') + (1 - \alpha)u(g)) \iff \alpha h + (1 - \alpha)f \succeq \alpha h' + (1 - \alpha)f.
\]
This contradicts (1).

Second, we show that (2) \( \Rightarrow \) (1). Assume, per contra, that there exist \( f, g \in H \) such that \( f_s \succ_g s \) and \( g_{s'} \succ_{f'} f_{s'} \) and for all \( h, h', \alpha \in (0, 1), \alpha h + (1 - \alpha)f \succeq \alpha h' + (1 - \alpha)f \Rightarrow \alpha h + (1 - \alpha)g \succeq \alpha h' + (1 - \alpha)g. \)

We show first that for all \( h, h', \in H \) with \( u(h), u(h') \in \text{int} u(H), \alpha \in (0, 1), \alpha h + (1 - \alpha)f \succeq \alpha h' + (1 - \alpha)f \Rightarrow \alpha h + (1 - \alpha)g \succeq \alpha h' + (1 - \alpha)g. \)

By the continuity of \( I \) and since \( h \in \text{int} u(H) \), there exist \( h^\dagger \in H, a > 0 \) such that \( u(h^\dagger) = u(h) - la \) and \( \alpha h + (1 - \alpha)f \succeq \alpha h' + (1 - \alpha)f \). So by assumption and C-additivity of \( I \), we have
\[
\alpha h + (1 - \alpha)g \succeq \alpha h^\dagger + (1 - \alpha)g \succeq \alpha h' + (1 - \alpha)g.
\]
Symmetrically by the negation of the assumption, we have for all \( h, h', \alpha \in (0, 1), \alpha h + (1 - \alpha)g \succeq \alpha h' + (1 - \alpha)g \Rightarrow \alpha h + (1 - \alpha)f \succeq \alpha h' + (1 - \alpha)f \)

and so for all \( h, h', \in H \) with \( u(h), u(h') \in \text{int} u(H), \alpha \in (0, 1), \alpha h + (1 - \alpha)g \succeq \alpha h' + (1 - \alpha)g \Rightarrow \alpha h + (1 - \alpha)f \succeq \alpha h' + (1 - \alpha)f. \)

So for all \( h, h', \in H \) with \( u(h), u(h') \in \text{int} u(H), \alpha \in (0, 1), \alpha h + (1 - \alpha)g \succeq \alpha h' + (1 - \alpha)g \Rightarrow \alpha h + (1 - \alpha)f \succeq \alpha h' + (1 - \alpha)f \).

So for all \( h, h', \in H \) with \( u(h), u(h') \in \text{int} u(H), \alpha \in (0, 1), \alpha h + (1 - \alpha)g \succeq \alpha h' + (1 - \alpha)g. \)

(73)
Thus \( f \) and \( g \) are equally crisp acts in interior of \( u(H) \). By Lemma 61, we have for all \( \varphi \in \text{int} \, u(H), \alpha \in (0, 1) \) such that \( \varphi + \alpha(u(f) - u(g)) \in \text{int} \, u(H) \),
\[
I\left(\varphi + \alpha(u(f) - u(g))\right) = I(\varphi) + \alpha\left(I(u(f)) - I(u(g))\right).
\]
Thus for all differentiability points \( \varphi \) of \( I \) in \( \text{int} \, u(H) \), we have
\[
\nabla I(\varphi) \cdot (u(f) - u(g)) = I(u(f)) - I(u(g)).
\]
By Chandrasekher et al.’s (2020) Section S.3
\[
\bigcup_{\varphi \in \text{int} \, u(H)} \partial I(\varphi) \subseteq \Delta(S).
\]
Thus we have for all differentiability points \( \varphi \) of \( I \) in \( \text{int} \, u(H) \), we have
\[
\nabla I(\varphi) \cdot (u(f) - u(g) - \bar{1}(I(u(f)) - I(u(g)))) = I(u(f)) - I(u(g)) - \left(I(u(f)) - I(u(g))\right) = 0.
\]
Additionally, by Lemma 44,
\[
u_{s}(f_{s}) > u_{s}(g_{s}) \text{ and } u_{s'}(g_{s'}) > u_{s'}(f_{s'})
\]
and so
\[
u_{s}(f_{s}) - u_{s}(g_{s}) - \left(I(u(f)) - I(u(g))\right) \neq u_{s'}(f_{s'}) - u_{s'}(g_{s'}) - \left(I(u(f)) - I(u(g))\right).
\]
By Lemma 68,
\[
u(f) - u(g) - \bar{1}(I(u(f)) - I(u(g))) \in \left(\bigcup_{\varphi \in \text{int} \, u(H)} \partial I(\varphi)\right)^{\perp},
\]
which is a contradiction with (2).
\[\square\]

The next lemma characterizes in terms of probabilities when the relative likelihood for states is identified.

**Lemma 87** Let \( P \subseteq \Delta(S), P \neq \emptyset, s, s' \in S, s \neq s' \). The following two are equivalent:

1. For all \( x \in P^{\perp}, x_{s} = x_{s'} \).
2. There exist \( p, q \in P \) such that for all \( \bar{s} \in S \setminus \{s, s'\}, p_{\bar{s}} = q_{\bar{s}} \) and \( \frac{p_{s}}{p_{s'}} \neq \frac{q_{s}}{q_{s'}} \).

**Proof.** We will first show that (2) \( \Rightarrow \) (1). Assume, per contra, that there exists \( x \in P^{\perp} \) such that \( x_{s} \neq x_{s'} \). Let \( p, q \) be as in (2). Then
\[
\sum_{\bar{s} \in S} x_{\bar{s}} p_{\bar{s}} = 0 = \sum_{\bar{s} \in S} x_{\bar{s}} q_{\bar{s}}
\]
Since for all $\bar{s} \in S \setminus \{s, s'\}$, $p_{\bar{s}} = q_{\bar{s}}$ and so $p_s + p_{s'} = q_s + q_{s'}$,

$$(x_s - x_{s'})p_s + x_s(p_s + p_{s'}) = x_s p_s + x_{s'}p_{s'} = x_sq_s + x_{s'}q_{s'} = (x_s - x_{s'})q_s + x_s(q_s + q_{s'}).$$

Hence,

$$(x_s - x_{s'})(p_s - q_s) = 0.$$ 

Since $x_s \neq x_{s'}$ by the counter assumption, we have $p_s = q_s$ which is a contradiction.

Next, we show that (1) $\Rightarrow$ (2). Denote $n^+ = \dim P^\perp$ and $n = \dim P$. Now $n^+ + n = |S|$. There exist $(c^i)^{n^+}_{i=1} \subseteq P^\perp$ linearly independent vectors and $(p^i)^n_{i=1} \subseteq P$ linearly independent vectors. First, we show that $n \geq 2$. Since $P$ is non-empty, $n \geq 1$. Assume, per contra, $n = 1$. Thus $P$ is a singleton $P = \{p\}$. If $p_s = 0$, then $(1_s, 0_{-s}) \in P^\perp$ which is a contradiction. Similarly $p_{s'} \neq 0$. Now $(\frac{1}{p_{s'}})^n_{s', s}\{0_{-s}, s'\} \in P^\perp$ that is a contradiction. So $n \geq 2$.

We consider two cases. First, assume that for all $i \in \{1, \ldots, n^\perp\}$,

$$c^i_s = 0. \quad (74)$$

Now we can consider the matrix $C$ formed by row vectors $(c^i)^{n^+}_{i=1}$ and each column in $\bar{s} \in S$. By (74), we can reduce $C$ into Smith normal form $\bar{C}$ formed by row vectors $(\bar{c}^i)^{n^+}_{i=1}$ such that for each $i \in \{1, \ldots, n^\perp\}$, there exists $s^i \in S \setminus \{s, s'\}$ such that $\bar{c}^i_{s^i} = -1$, for all $j \in \{1, \ldots, n^\perp\} \setminus \{i\}$, $\bar{c}^i_{sj} = 0$, and for all $k, l \in \{1, \ldots, n^\perp\}$, $k \neq l$, $s^k \neq s^l$. Now $(\bar{c}^i)^{n^+}_{i=1} \subseteq P^\perp$ are linearly independent and for all $i \in \{1, \ldots, n^\perp\}$,

$$\bar{c}^i_s = \bar{c}^i_{s'} = 0. \quad (75)$$

Denote $S^\perp = \{s^1, \ldots, s^{n^\perp}\}$. Now we have for all $i \in \{1, \ldots, n^\perp\}$ and $p \in P$,

$$p_{s^i} = \sum_{\bar{s} \in S \setminus S^\perp} \bar{c}^i_{s^i P_{\bar{s}}} \quad (76)$$

Let $\pi : \{1, \ldots, n - 2\} \to S \setminus (S^\perp \cup \{s, s'\})$ be a one-to-one function. We show by induction that for each $i \in \{0, \ldots, n - 2\}$, there exists a linearly independent collection of probabilities $(\bar{p}^{j, i})^{n-i}_{j=1}$ such that for all $j, k \in \{1, \ldots, n - i\}$, $1 \leq l \leq i$,

$$p_{\pi(l)}^{j, i} = p_{\pi(l)}^{k, i}. \quad (77)$$

For the first step $i = 0$, define for all $j \in \{1, \ldots, n\}$, $\bar{p}^{j, i} = p^j$. For the induction step, assume that for $0 \leq i \leq n - 2$, there exists $(\bar{p}^{j, i})^{n-i}_{j=1}$ that satisfy (77). First, we show that

$$\min\{p_{\pi(i+1)}^{j, i} | j \in \{1, \ldots, n - i\}\} \neq \max\{p_{\pi(i+1)}^{j, i} | j \in \{1, \ldots, n - i\}\} \quad (78)$$

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Assume, per contra, that for all \( j, k \in \{1, \ldots, n-1\} \), \( p_{\pi(\ell+1)}^{j,i} = p_{\pi(\ell+1)}^{k,i} \). We collect \((\tilde{p}_{j,i})^{n-i}_{j=1}\) into a matrix \( P^i \) where each \( p_{j,i} \) is a column vector. The column rank of \( P^i \) is \( n-i \) since each column is linearly independent. Thus the row rank of \( P^i \) is \( n-i \). However, each row \( \tilde{s} \in S^\perp \) is linearly dependent on rows \( S \setminus S^\perp \) by (76) and each row \( \tilde{s} \in \{\pi(1), \ldots, \pi(i), \pi(i+1)\} \) is constant and so since the rows sum to 1, linearly dependent on the rows \( S \setminus \{\pi(1), \ldots, \pi(i), \pi(i+1)\} \). Thus the maximum row rank for \( P^i \) is \( |S| - n^\perp - i - 1 = n - i - 1 \) which is a contradiction. This shows (78).

Let

\[
j^* \in \arg\max \left\{ p_{\pi(\ell+1)}^{j,i} \middle| j \in \{1, \ldots, n-i\} \right\} \quad \text{and} \quad j_* \in \arg\min \left\{ p_{\pi(\ell+1)}^{j,i} \middle| j \in \{1, \ldots, n-i\} \right\}.
\]

By (78), there exists \( \beta^i \in (0, 1) \) such that for all \( j \in \{1, \ldots, n-i\} \setminus j_* \),

\[
\frac{1}{2} p_{\pi(\ell+1)}^{j,i} + \frac{1}{2} p_{\pi(\ell+1)}^{j^*,i} > \beta^i > p_{\pi(\ell+1)}^{j^*,i}.
\]

Thus for each \( j \in \{1, \ldots, n-i\} \setminus j_* \), there exists \( \alpha^j \in (0, 1) \) such that

\[
\alpha^j \left( \frac{1}{2} p_{\pi(\ell+1)}^{j,i} + \frac{1}{2} p_{\pi(\ell+1)}^{j^*,i} \right) + (1 - \alpha^j) p_{\pi(\ell+1)}^{j^*,i} = \beta^i.
\]

Denote for \( j \in \{1, \ldots, n-i-1\} \setminus j_* \),

\[
\tilde{p}^{(j-1)(j > j_*)}(i+1) = \alpha^j \left( \frac{1}{2} p^{j,i} + \frac{1}{2} p^{j^*,i} \right) + (1 - \alpha^j) p^{j^*,i},
\]

where \( I(j > j_*) \) is an indicator function for \( j > j_* \). Now \((\tilde{p}^{j,i})^{n-i-1}_{j=1}\) are linearly independent since they have been created using elementary column operations using the above \( P^i \). Additionally, it satisfies (77). This shows the induction step and concludes the induction.

Since \( n \geq 2 \), by the induction, there exist \( p^*, p^\dagger \in P \) that are linearly independent and for all \( \tilde{s} \in S \setminus (S^\perp \cup \{s, s'\}) \), \( p^\dagger_\tilde{s} = p^\dagger_\tilde{s} \).

By (75,76), for all \( \tilde{s} \in S \setminus \{s, s'\} \),

\[
p^\dagger_\tilde{s} = p^\dagger_\tilde{s}.
\]

This shows the claim since \( p^\dagger \) and \( p^* \) are linearly independent.

Second, we consider the case that there exists \( i \in \{1, \ldots, n^\perp\} \) such that \( c^i_\tilde{s} \neq 0 \). First, if a vector \( \tilde{c} \in \mathbb{R}^S \) is such that \( \tilde{c}_\tilde{s} = \tilde{c}_{s'} \neq 0 \) and for all \( \tilde{s} \in S \setminus \{s, s'\} \), \( \tilde{c}_\tilde{s} = 0 \), then

\[
\tilde{c} \notin P^\perp
\]

since each \( p \in P \) is non-negative.
Now we can consider the matrix $C$ formed by row vectors $(c^i)_{i=1}^{n^+}$ and each column in $\tilde{s} \in S$. Denote $s^{n^+} = s, s^{n^++1} = s'$. By (79), we can reduce $C$ into Smith normal form $\tilde{C}$ formed by row vectors $(\tilde{c}^i)_{i=1}^{n^+}$ such that for each $i \in \{1, \ldots, n^+ - 1\}$, there exists $s' \in S \setminus \{s, s'\}$ such that for each $\tilde{i} \in \{1, \ldots, n^+\}$, $\tilde{c}^\tilde{i}_{s'} = -1$, for all $j \in \{1, \ldots, n^\perp\} \setminus \{i\}$, $\tilde{c}^\tilde{j}_{s'} = 0$, and for all $k,l \in \{1, \ldots, n^\perp\}, k \neq l, s^k \neq s^l$. Now $(\tilde{c}^i)_{i=1}^{n^+} \subseteq P^\perp$ are linearly independent and for all $i \in \{1, \ldots, n^+ - 1\}$,

$$\tilde{c}^1_i = \tilde{c}^1_{s'} = 0 \text{ and } \tilde{c}^n_{s'} = \tilde{c}^n_{s'} = -1.$$  

Denote $S^\perp = \{s^1, \ldots, s^{n^\perp}, s^{n^++1}\}$. Now we have for all $i \in \{1, \ldots, n^\perp - 1\}$ and $p \in P$,

$$p_{s^i} = \sum_{\tilde{s} \in S \setminus S^\perp} \tilde{c}^1_{s^i} p_{\tilde{s}} \text{ and } p_{s^{n^\perp}} + p_{s^{n^++1}} = \sum_{\tilde{s} \in S \setminus S^\perp} \tilde{c}^n_{s^i} p_{\tilde{s}}. \quad (80)$$

Next, we show that there exists $s^i \in S \setminus S^\perp$ such that

$$\sum_{i=1}^{n^\perp} \tilde{c}^i_{s^i} \neq -1. \quad (81)$$

Assume, per contra, for all $\tilde{s} \in S \setminus S^\perp$,

$$\sum_{i=1}^{n^\perp} \tilde{c}^i_{\tilde{s}} = -1.$$  

Then, we have for $p \in P$,

$$1 = \sum_{\tilde{s} \in S} p_{\tilde{s}}^{(80)} = \sum_{\tilde{s} \in S \setminus (S^\perp \cup \{s^i\})} \left(1 + \sum_{i=1}^{n^\perp} \tilde{c}^i_{\tilde{s}}\right) p_{\tilde{s}} = 0$$

that is a contradiction which shows (81).

Let $\pi : \{1, \ldots, n - 2\} \rightarrow S \setminus (S^\perp \cup \{s^i\})$ be a one-to-one function. By an induction as in the previous case for each $i \in \{0, \ldots, n - 2\}$, there exists a collection of linearly independent probabilities $(\tilde{p}^{j,i})_{j=1}^{n-i}$ such that for all $j,k \in \{1, \ldots, n - i\}, 1 \leq l \leq i$,

$$p^{j,i}_{\pi(l)} = p^{k,i}_{\pi(l)}. \quad (82)$$

Since $n \geq 2$, by the induction, there exist $p^\dagger, p^\ast \in P$ that are linearly independent and for all $\tilde{s} \in S \setminus (S^\perp \cup \{s^i\}), p^\dagger_{\tilde{s}} = p^\ast_{\tilde{s}}$.

Now we have for $p \in \{p^\ast, p^\dagger\}$,

$$1 = \sum_{\tilde{s} \in S} p_{\tilde{s}}^{(80)} = \sum_{\tilde{s} \in S \setminus (S^\perp)} \left(1 + \sum_{i=1}^{n^\perp} \tilde{c}^i_{s^i}\right) p_{\tilde{s}} \Rightarrow p_{s^i} = \frac{1 - \sum_{\tilde{s} \in S \setminus (S^\perp \cup \{s^i\})} \left(1 + \sum_{i=1}^{n^\perp} \tilde{c}^i_{s^i}\right) p_{\tilde{s}}}{1 + \sum_{i=1}^{n^\perp} \tilde{c}^i_{s^i}}.$$

Thus $p^\ast_{s^i} = p^\dagger_{s^i}$.
Now for all \( \bar{s} \in S \setminus S^\perp \), \( p^*_s = p^\dagger_s \). By (80), for all \( \bar{s} \in S \setminus \{s, s'\} \),

\[
p^*_s = p^\dagger_s \quad \text{and} \quad p^*_s + p^*_{s'} = p^\dagger_s + p^\dagger_{s'}.
\]

This shows the claim since \( p^\dagger \) and \( p^* \) are linearly independent.

The previous two lemmas give the relative likelihood identification characterization.

**Proposition 88 (Relative Likelihood Identification)** Let \((u, P)\) be a state dependent tight dual-self representation for \(\succsim\) and \(s, s' \in S^P\). The following four conditions are equivalent:

1. If \(f, g \in H\) are such that and \(f_s \succ f_{s'}\), then there exist \(h, h' \in H\) and \(\alpha \in (0, 1)\) such that
   \[
   ah + (1 - \alpha)f \succ ah + (1 - \alpha)g \quad \text{and} \quad ah' + (1 - \alpha)f < ah' + (1 - \alpha)g.
   \]

2. There exist \(p, q \in \overline{\mathbb{C}} \cup \bigcup_{P \in \mathbb{P}} P\) such that \(\frac{p_s}{p_{s'}} \neq \frac{q_s}{q_{s'}}\) and for all \(\bar{s} \in S \setminus \{s, s'\}\), \(p_{\bar{s}} = q_{\bar{s}}\).

3. If \((\bar{u}, \overline{P})\) is a state dependent tight dual-self representation for \(\succsim\), then there exist \(\alpha \in \mathbb{R}_+\) and \(\beta \in \mathbb{R}\) such that
   \[
   \bar{u}_s = \alpha u_s + \beta \quad \text{and} \quad \bar{u}_{s'} = \alpha u_{s'} + \beta.
   \]

4. If \((\bar{u}, \overline{P})\) is a state dependent tight dual-self representation for \(\succsim\), then
   \[
   \left\{ \frac{\bar{p}_s}{\bar{p}_{s'}} \mid \bar{p} \in \overline{\mathbb{C}} \bigcup \bigcup_{P \in \mathbb{P}} P \right\} = \left\{ \frac{P_s}{P_{s'}} \mid \bar{p} \in \overline{\mathbb{C}} \bigcup \bigcup_{P \in \mathbb{P}} P \right\}.
   \]

**Proof.** (1) \(\iff\) (2) follows from Lemmas 86 and 87. (2) \(\iff\) (3) and (2) \(\iff\) (4) follows from Corollary 75 and Lemma 87.

**D.6.3 Probability Identification**

The identification for a probability is similar to the identification for the relative likelihood of some states. First, we show that our behavioral assumption for probability identification characterizes its identification.

**Lemma 89** Let \((u, I)\) be a state dependent dual-self variational representation for \(\succsim\) and \(s \in S^P\). The following two are equivalent:
(1) For all $f, g \in H$ such that $f \sim g$, $f_s \succ_s g_s$, and for some $s' \in S$, $g_{s'} \succ_{s'} f_{s'}$, there exist $h \in H$ and $\alpha \in (0, 1)$ such that

$$\alpha h + (1 - \alpha) f \not\sim \alpha h + (1 - \alpha) g.$$  

(2) For all $x \in (\bigcup_{\varphi \in \text{int}(H)} \partial I(\varphi))^\perp$, $x_s = 0$.

**Proof.** We will first show that $(1) \Rightarrow (2)$. Assume, per contra, there exists $\tilde{x} \in (\bigcup_{\varphi \in \text{int}(H)} \partial I(\varphi))^\perp$, $\tilde{x}_s \neq 0$. Assume w.l.o.g. $\tilde{x}_s > 0$. Since for all $p \in (\bigcup_{\varphi \in \text{int}(H)} \partial I(\varphi))^\perp$, $s \notin S^p$, we have $p_s = 0$ so $x = (\tilde{x}_s, 0_{-S^p}) \in (\bigcup_{\varphi \in \text{int}(H)} \partial I(\varphi))^\perp$. We show that there exists $s' \in S^p$ such that $x_{s'} < 0$. Assume, per contra, for all $s' \in S^p, x_{s'} \geq 0$. Since $s \in S^p$, by Lemma 83, there exists $p \in (\bigcup_{\varphi \in \text{int}(H)} \partial I(\varphi))$ such that $p_s > 0$. So by the counterassumption and since $p$ is non-negative,

$$\sum_{s \in S} x_sp_s \geq x_sp_s > 0$$

which is a contradiction since $x \in (\bigcup_{\varphi \in \text{int}(H)} \partial I(\varphi))^\perp$.

By Lemma 84, there exist $f, g \in H$, $a > 0$ such that $f_s \succ_s g_s$ and $g_{s'} \succ_{s'} f_{s'}$, $u(f), u(g) \in \text{int} u(H)$, and $u(f) - u(g) = ax$. Let $h \in H$ and $\alpha^* \in [0, 1)$. Consider a mapping $J : [0, 1] \to \mathbb{R}$, defined by for all $\alpha \in [0, 1]$,

$$J(\alpha) = I \circ u \left( \alpha^* h + (1 - \alpha^*) \left( (1 - \alpha) f + (1 - \alpha) g \right) \right) = I \left( \alpha^* u(h) + (1 - \alpha^*) u(g) + (1 - \alpha^*) ax \right).$$

Since $u(f), u(g) \in \text{int} u(H)$, we have by the definition of Clarke derivative and since $x \in (\bigcup_{\varphi \in \text{int}(H)} \partial I(\varphi))^\perp$, for all differentiability points $\alpha$ of $J$,

$$J'(\alpha) = 0.$$  

Since $J$ is differentiable almost everywhere, we have by the fundamental theorem of calculus,

$$I \circ u \left( \alpha^* h + (1 - \alpha^*) g \right) = J(0) = J(1) = I \circ u \left( \alpha^* h + (1 - \alpha^*) f \right)$$

and so $\alpha^* h + (1 - \alpha^*) g \sim \alpha^* h + (1 - \alpha^*) f$. Especially $f \sim g$ and this contradicts $(1)$.

Second, we show that $(2) \Rightarrow (1)$. Assume, per contra, that there exist $f, g \in H, s' \in S$ such that $f \sim g$, $f_s \succ_s g_s$ and $g_{s'} \succ_{s'} f_{s'}$ and for all $h \alpha \in (0, 1)$,

$$\alpha h + (1 - \alpha) f \sim \alpha h + (1 - \alpha) g.$$
Let $\varphi \in \text{int} u(H)$ be a differentiability point of $I$. Now there exist $h \in H$ with $u(h) \in \text{int} u(H)$ and $\alpha^* \in (0, 1)$ such that
\[
u(\alpha^* h + (1 - \alpha^*) g) = \varphi.
\]
Now by the counter assumption, for all $\alpha \in [0, 1], \alpha^* h + (1 - \alpha^*) g \sim \alpha^* h + (1 - \alpha^*)(\alpha f + (1 - \alpha) g)$.

Thus
\[
(1 - \alpha^*) \Delta I(\varphi) \cdot \left( u(f) - u(g) \right) = \lim_{\alpha \to 0} I[\nu(\alpha^* h + (1 - \alpha^*) g) + \alpha(1 - \alpha^*)(u(f) - u(g))] - I[\nu(\alpha^* h + (1 - \alpha^*) g)] = 0.
\]
By Lemma 68, $u(f) - u(g) \in \left( \bigcup_{\varphi \in \text{int} u(H)} \partial I(\varphi) \right)^\bot$. But by Lemma 44, $u_s(f_s) - u_s(g_s) \neq 0$ which is a contradiction with (2).

The second lemma characterizes in terms of probabilities when the probabilities for a single state are identified.

Lemma 90 Let $P \subseteq \Delta(S)$, $P \neq \emptyset$, $s \in S$. The following two are equivalent:

1. For all $x \in P^\perp$, $x_s = 0$ and there exist $s' \in S, s \neq s', p \in P$ such that $p_{s'} > 0$.
2. There exist $p, q \in P$ such that $p_s \neq q_s$ and for all $\bar{s} \in S \setminus \{s\},$
\[
\frac{p_{\bar{s}}}{1 - p_s} = \frac{q_{\bar{s}}}{1 - q_s}.
\]

Proof. We show first that (2) $\Rightarrow$ (1). First, by (2), $|P| \geq 2$ and so there exist $s' \in S, s \neq s', p \in P$ such that $p_{s'} > 0$. Assume, per contra, that there exists $x \in P^\perp$ such that $x_s \neq 0$. Let $p, q$ be as in (2). Since $p_s \neq 1$ or $q_s \neq 1$, we have by (83) since $|S| \geq 2,$
\[
p_s \neq 1 \text{ and } q_s \neq 1.
\]
Since $p, q \in P^\perp,$ we have \[
\sum_{\bar{s} \in S} x_{\bar{s}} p_{\bar{s}} = 0 \text{ and } \sum_{\bar{s} \in S} x_{\bar{s}} q_{\bar{s}} = 0.
\]
By (84) and by multiplying the first equation by $(1 - p_s)^{-1}$ and the second equation by $(1 - q_s)^{-1}$, we have \[
\sum_{\bar{s} \in S \setminus \{s\}} x_{\bar{s}} \frac{p_{\bar{s}}}{1 - p_s} + x_s \frac{p_s}{1 - p_s} = 0 = \sum_{\bar{s} \in S \setminus \{s\}} x_{\bar{s}} \frac{q_{\bar{s}}}{1 - q_s} + x_s \frac{q_s}{1 - q_s}.
\]
By (83), we have
\[ x_s \frac{p_s}{1 - p_s} = x_s \frac{q_s}{1 - q_s} \quad \text{for } x_s \neq 0, \Rightarrow \quad \frac{p_s}{1 - p_s} = \frac{q_s}{1 - q_s}. \]

Now \( x \mapsto \frac{x}{1 - x} \) is strictly increasing function for \( x \in [0, 1) \) and so by (84),
\[ \frac{p_s}{1 - p_s} = \frac{q_s}{1 - q_s} \Rightarrow p_s = q_s. \]

This contradicts (2).

Next, we show that (1) \( \Rightarrow \) (2). Denote \( n^+ = \dim P^\perp \) and \( n = \dim P \). Now \( n^+ + n = |S| \).

There exist \( (c^i)_{i=1}^{n^+} \subseteq P^\perp \) linearly independent vectors and \( (p^i)_{i=1}^{n} \subseteq P \) linearly independent vectors.

First, we show that \( n \geq 2 \). Since \( P \) is non-empty, \( n \geq 1 \). Assume, per contra, \( n = 1 \).

Thus \( P \) is a singleton \( P = \{ p \} \). If \( p_s = 0 \), then \( (1, 0_{-s}) \in P^\perp \) which is a contradiction. By assumption, there exists \( s' \in S, s' \neq s \) such that \( p_{s'} > 0 \). Now \( \left( \frac{1}{p_s}, (\frac{1}{p_s}), 0_{-s, s'} \right) \in P^\perp \) that is a contradiction. So \( n \geq 2 \).

Now we can consider the matrix \( C \) formed by row vectors \( (c^i)_{i=1}^{n^+} \) and each column in \( \tilde{s} \in S \). We can reduce \( C \) into Smith normal form \( \tilde{C} \) formed by row vectors \( (\tilde{c}^i)_{i=1}^{n^+} \) such that for each \( i \in \{1, \ldots, n^+\} \), there exists \( s' \in S \setminus \{s\} \) such that \( c_{i,s'} = -1 \), for all \( j \in \{1, \ldots, n^+\} \setminus \{i\}, \tilde{c}_{i,s'} = 0 \), and for all \( k, l \in \{1, \ldots, n^+\}, k \neq l, s^k \neq s^l \). Now \( (\tilde{c}^i)_{i=1}^{n^+} \subseteq P^\perp \) are linearly independent and for all \( i \in \{1, \ldots, n^+\} \), \( \tilde{c}^i_s = 0 \).

Denote \( S^\perp = \{ s^1, \ldots, s^{n^+} \} \). Now we have for all \( i \in \{1, \ldots, n^+\} \) and \( p \in P \),
\[ p_{s^i} = \sum_{\tilde{s} \in S \setminus (S^\perp \cup \{s\})} \tilde{c}_{i, \tilde{s}}^i \tilde{p}_{\tilde{s}}. \]  

(85)

Let \( \pi : \{1, \ldots, n-1\} \to S \setminus (S^\perp \setminus \{s\}) \) be a one-to-one function such that \( \pi(n) = s \) and \( \pi(n - 1) = s^1 \). We show by induction that for each \( i \in \{0, \ldots, n\} \), there exists a collection of linearly independent probabilities \( (\tilde{p}_{\pi(j)}^{j,i})_{j=1}^{n} \) such that for all \( 1 \leq m \leq i \), and \( j, k \in \{1, \ldots, n\} \setminus \{m\} \),
\[ p_{\pi(m)}^{j,i} = p_{\pi(m)}^{k,i} > p_{\pi(m)}^{m,i}. \]  

(86)

This follows symmetrically to the proof in Lemma 87 since at each step for \( i < n - 1 \),
\[ \min \left\{ p_{\pi(j)}^{j,i} \bigg| j \in \{i + 1, \ldots, n\} \right\} \neq \max \left\{ p_{\pi(j)}^{j,i} \bigg| j \in \{i + 1, \ldots, n\} \right\} \]
by linear independence and since for all \( k \leq i \) and \( j, l \in \{i + 1, \ldots, n\} \), \( p_{\pi(k)}^{j,i} = p_{\pi(k)}^{l,i} \).
Since $n \geq 2$, by the induction and by considering a convex combination of $(p^{i,n-1})_{j=1}^{n-1}$, there exist $a > 1$ and $p^*, p^\dagger \in P$ that are linearly independent and for all $\bar{s} \in S \setminus (S^\perp \cup \{s\})$, $p^*_{\bar{s}} = ap^\dagger_{\bar{s}} > 0$. Let $\bar{s}^\dagger \in S \setminus (S^\perp \cup \{s\})$. Now we have for all $\bar{s} \in S \setminus (S^\perp \cup \{s\})$,

$$\frac{p^*_{\bar{s}}}{p^\dagger_{\bar{s}}} = \frac{ap^\dagger_{\bar{s}}}{ap^\dagger_{\bar{s}}} = \frac{p^\dagger_{\bar{s}}}{p^\dagger_{\bar{s}}}.$$

By (85), we have for all $i \in \{1, \ldots, n^\perp\}$ and $p \in P$,

$$\frac{p^i_s}{p^i_s} = \sum_{\bar{s} \in S \setminus (S^\perp \cup \{s\})} \bar{c}_{\bar{s}} \frac{p^s_{\bar{s}}}{p^s_{\bar{s}}}.$$

Thus for all $\bar{s} \in S \setminus \{s\}$,

$$\frac{p^*_{\bar{s}}}{p^\dagger_{\bar{s}}} = \frac{p^i_s}{p^i_s}. \quad (87)$$

By taking a sum over $\bar{s}$, we have

$$\frac{1 - p_s}{p^*_{\bar{s}}} = \frac{1 - p_s}{p^\dagger_{\bar{s}}}.$$

Thus by multiplying (87) by $\frac{p^s_{\bar{s}}}{1 - p_s}$, we have for all $\bar{s} \in S \setminus \{s\}$,

$$\frac{p^s_{\bar{s}}}{1 - p_s} = \frac{p^i_s}{1 - p^\dagger_{\bar{s}}}. \quad (88)$$

Finally, we show that $p^*_{\bar{s}} \neq p^\dagger_{\bar{s}}$. Assume, per contra, $p^*_{\bar{s}} = p^\dagger_{\bar{s}}$. Then by (88) for all $\bar{s} \in S$, $p^*_{\bar{s}} = p^\dagger_{\bar{s}}$, which contradicts, that $p^*$ and $p^\dagger$ are linearly independent. Thus $p^*_{\bar{s}} \neq p^\dagger_{\bar{s}}$ which shows the claim. \hfill \Box

The previous two lemmas give the probability dentification characterization.

**Proposition 91 (Probability Identification)** Let $(u, \mathbb{P})$ be a state dependent tight dual-self representation for $\succsim$ and $s \in S^\mathbb{P}$. The following three conditions are equivalent:

1. If $f, g \in H$ are such that $f \sim g$, $f_s \succ g_s$, and there exists $s' \in S$ such that $g_{s'} \succ s' f_{s'}$, then there exist $h \in H$ and $\alpha \in (0, 1)$ such that

$$\alpha h + (1 - \alpha)f \not\succsim \alpha h + (1 - \alpha)g.$$

2. $S^\mathbb{P} = \{s\}$ or there exist $p, q \in \mathbb{P} \cup \bigcup_{P \in \mathbb{P}} P$ such that $p \not\succsim q$ and for all $\bar{s} \in S \setminus \{s\}$,

$$\frac{p_{\bar{s}}}{1 - p_{\bar{s}}} = \frac{q_{\bar{s}}}{1 - q_{\bar{s}}}.$$

3. If $(\bar{u}, \bar{\mathbb{P}})$ is a state dependent tight dual-self representation for $\succsim$, then

$$\left\{ \bar{p} \mid \bar{p} \in \mathbb{P} \bigcup \bigcup_{\bar{P} \in \bar{\mathbb{P}}} \bar{P} \right\} = \left\{ p \mid p \in \mathbb{P} \bigcup \bigcup_{P \in \mathbb{P}} P \right\}.$$

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Then there exists

then for all

We will show that there does not exist

Proof. (1) \iff (2): We have two cases. If \( S^p = \{s\} \), then \( \cup_{p \in P} P = \{\delta_s\} \) and so the claim follows Lemma 89 and since for all \( x \in \mathbb{R}^S \), \( \delta_s \cdot x = 0 \) iff \( x_s = 0 \). If \( S^p \neq \{s\} \), then there exist \( s' \in S, s \neq s' \) and \( p \in \cup_{p \in P} P \) such that \( p_{s'} > 0 \). Thus the equivalence follows from Lemmas 89 and 90. (2) \iff (3) follows from Corollary 75 and Lemma 90 \( \Box \)

E Social Choice Theory: Symmetric State Dependent MaxMin

We first show that non-negative symmetric set is either full dimensional or a singleton.

Lemma 92 Let \( P \subseteq \mathbb{R}^S_+ \) be non-empty and \( \bar{0} \notin P \) such that if \( p, q \in P \) are such that \( p \neq q \), then for all \( a > 0, ap \neq q \) and for all permutations \( \pi : S \rightarrow S \) and \( p \in P \),

\[
(p_{\pi(s)})_{s \in S} \in P. \tag{89}
\]

Then there exists \( c \in R_{++} \) such that \( P = \{(c)_{s \in S}\} \) or \( P^\perp = \{0\} \).

Proof. We will show that there does not exist \( p \in P, q \in P^\perp, s_1, s_2, s_3, s_4 \in S \) such that \( p_{s_1} \neq p_{s_2} \) and \( q_{s_3} \neq q_{s_4} \). Let \( \pi^1 : S \rightarrow S \) and \( \pi^2 : S \rightarrow S \) be two permutations such that \( \pi^1(s_1) = s_3, \pi^1(s_2) = s_4, \pi^2(s_1) = s_4, \pi^2(s_2) = s_3 \), and for all \( s \notin \{s_1, s_2\}, \pi^1(s) = \pi^2(s) \). Now by (89) and \( q \in P^\perp \), we have

\[
\sum_{s \in S} p_{\pi^1(1^{-1}(s))} q_s = 0 = \sum_{s \in S} p_{\pi^2(1^{-1}(s))} q_s.
\]

Since for all \( s \notin \{s_3, s_4\}, (\pi^1)^{-1}(s) = (\pi^2)^{-1}(s) \), we have

\[
p_{s_1}q_{s_3} + p_{s_2}q_{s_4} = p_{s_2}q_{s_3} + p_{s_1}q_{s_4} \Rightarrow (p_{s_1} - p_{s_2})(q_{s_4} - q_{s_3}) = 0.
\]

Thus \( p_{s_1} = p_{s_2} \) or \( q_{s_4} = q_{s_3} \).

First, assume that \( P \) is a singleton with \( p \in P \). Then by (89) and since \( P \) is a singleton, for all \( s, s' \in S, p_s = p_{s'} \). This shows the claim since \( \bar{0} \notin P \).

Assume next that \( P \) is not a singleton. Thus there exist \( p, q \in P \) such that \( p \neq q \). Now for all \( a \in R_{++}, ap \neq q \). Especially, there exist \( p \in P, s_1, s_2 \in S \) such that \( p_{s_1} \neq p_{s_2} \). Thus by above, for all \( q \in P^\perp, s_3, s_4 \in S, q_{s_3} = q_{s_4} \). Since for all \( s \in S, p_s \geq 0 \) and for some \( s' \in S, p_{s'} > 0 \), this is possible only if \( P^\perp = \{0\} \). This shows the claim. \( \Box \)
The above lemma gives almost the characterization for symmetric state dependent maxmin. The next lemma shows that our behavioral symmetry assumption, Axiom 9, characterizes a symmetric set of probabilities. This shows the characterization.

**Proposition 93 (Symmetric State Dependent MaxMin)** \(\succeq\) satisfies Axioms 1-6, 8, 9 if and only if (1) or (2) of the following conditions holds:

1. For all \(p \in \text{int} \Delta(S)\), there exists \(u = (u_s)_{s \in S}\) such that for all \(s \in S\), \(u_s : \Delta(X_s) \to \mathbb{R}\) is affine and \(\text{int} u_s(\Delta(X_s)) \neq \emptyset\) and for all \(f, g \in H\),
   \[
   f \succeq g \iff \sum_{s \in S} p_s u_s(f_s) \geq \sum_{s \in S} p_s u_s(g_s).
   \]

2. There exists \((u, P)\) that is a state dependent maxmin representation for \(\succeq\) such that \(\text{int} P \neq \emptyset\), for all \(p \in P\), permutations \(\pi : S \to S\), \((p_{\pi(s)})_{s \in S} \in P\), and
   \[
   \text{int} \bigcap_{s \in S} u_s(\Delta(X_s)) \neq \emptyset.
   \]

Additionally, if (2) holds and \((\tilde{u}, \tilde{P})\) is another state dependent maxmin representation for \(\succeq\), then there exist \(\alpha > 0, \beta \in \mathbb{R}\) such that \(\tilde{P} = P\) and for all \(s \in S\), \(\tilde{u}_s = \alpha u_s + \beta\).

**Proof.** We show first the if direction.

By an assumption, \(\succeq\) satisfies Axioms 1-6, 8, there exists a concave state dependent Boolean representation for \(\succeq\). This corresponds to a state dependent maxmin representation \((P, u)\) with probabilities \(P \subseteq \Delta(S)\) and affine utility \(u = (u_s)_{s \in S}\).

If \(|S| = 1\), then by non-triviality, (1) holds. So assume that \(|S| > 1\).

Since \(f^* \succeq g^*\), there exists \(s^* \in S^P\) such that \(u_{s^*}(f_{s^*}^*) > u_{s^*}(g_{s^*}^*)\). By the maxmin representation, \(s^* \in S^P\), and Lemma 80, since the set of crisp acts is convex, we have

\[
\frac{1}{|S|-1} f^* + \frac{|S|-2}{|S|-1} g^* > \frac{1}{|S|-1} (g_{s^*}^*, f_{s^*}^*) + \frac{|S|-2}{|S|-1} g^* = \left( g_{s^*}^*, \frac{1}{|S|-1} f_{s^*}^* + \frac{|S|-2}{|S|-1} g_{s^*}^* \right).
\]

By Axiom 9, for all \(s \in S\),

\[
\frac{1}{|S|-1} f^* + \frac{|S|-2}{|S|-1} g^* > \left( g_{s}^*, \frac{1}{|S|-1} f_{s}^* + \frac{|S|-2}{|S|-1} g_{s}^* \right).
\]

Thus \(S = S^P\) and for all \(s \in S\), \(u_s(f_s^*) > u_s(g_s^*)\). Denote for each \(s \in S\),

\[
\tau_s = u_s(f_s^*) - u_s(g_s^*) \in \mathbb{R}_{++}.
\]
Next assume that \( h \in H, u(h) \in \text{int}(u(H)) \) is such that \( I \) is differentiable at \( u(h) \). Let \( \pi : S \to S \) be a permutation. We will show that \( \left( \nabla I(u(h))_{\pi(s)} \right)_{s \in S} \in \mathcal{P} \).

First, since \( |S| > 1 \), there exists \( \alpha^1 \in (0, 1) \) such that for all \( s \in S \),
\[
\alpha u_s(f^*_s) + (1 - \alpha^1)u_s(h_s) > \frac{1}{|S|}u_s(f^*_s) + \frac{|S| - 1}{|S|}u_s(g^*_s).
\]

Thus there exists \( \theta \in (0, 1)^S \) such that for each \( s \in S \),
\[
u_s(\theta_s f^*_s + (1 - \theta_s)g^*_s) = \alpha^1 u_s(f^*_s) + (1 - \alpha^1)u_s(h_s).
\]

Now especially \( \sum_{s \in S} \theta_s > 1 \) and so \( \alpha^2 = \left( \sum_{s \in S} \theta_s \right)^{-1} \in (0, 1) \). Denote
\[
\hat{h} = \alpha^2 \alpha^1 (3/4f^* + 1/4g^*) + (1 - \alpha^1)u_s(h_s) + (1 - \alpha^2)g^*.
\]

By Lemma 79, \( I \) is differentiable at \( u(\hat{h}) \) and \( \nabla I(\hat{h}) = \nabla I(h) \). Denote for all \( s \in S \), \( \gamma_s = \theta_s \alpha^2 \).

Now \( \sum_{s \in S} \gamma_s = 1 \). Especially, \( I \) is differentiable at \( \left( \gamma_s u_s(f^*_s) + (1 - \gamma_s)u_s(g^*_s) \right)_{s \in S} \) with derivative \( \nabla I(h) \).

We will show that \( I \) is differentiable at \( \left( \gamma_{\pi(s)} u_s(f^*_s) + (1 - \gamma_{\pi(s)}) u_s(g^*_s) \right)_{s \in S} \) with a derivative \( \left( \nabla I(u(h))_{\pi(s)} \right)_{s \in S} \).

Let \( v \in \mathbb{R}^S \) and \( a > 0 \) be such that for all \( s \in S \),
\[
u_s(f^*_s) > u_s(\hat{h}_s) + av_s > u_s(g^*_s).
\]

Thus there exists \( \tilde{\theta} \in (0, 1)^S \) such that for each \( s \in S \),
\[
u_s(\tilde{\theta}_s f^*_s + (1 - \tilde{\theta}_s)g^*_s) = u_s(\hat{h}_s) + av_s.
\]

Now for each \( s \in S \),
\[
\tilde{\theta}_s - \gamma_s = av_s(\tau_s^{-1}).
\]

Assume w.l.o.g. \( \sum_{s \in S} v_s \geq 0 \) since the other case follows similarly. Now \( \sum_{s \in S} \tilde{\theta}_s \geq 1 \). Denote \( \tilde{\alpha} = (\sum_{s \in S} \tilde{\theta}_s)^{-1} \). Now \( \tilde{\alpha} \tilde{\theta} \in \Delta(S) \). Thus by Axiom 9,
\[
\tilde{\alpha}(\tilde{\theta}_s f^*_s + (1 - \tilde{\theta}_s)g^*_s)_{s \in S} + (1 - \tilde{\alpha})g^* = (\tilde{\alpha}\tilde{\theta}_s f^*_s + (1 - \tilde{\alpha}\tilde{\theta}_s)g^*_s)_{s \in S}
\]
\[
\sim (\tilde{\alpha}\tilde{\theta}_{\pi(s)} f^*_s + (1 - \tilde{\alpha}\tilde{\theta}_{\pi(s)})g^*_s)_{s \in S} = \tilde{\alpha}(\tilde{\theta}_{\pi(s)} f^*_s + (1 - \tilde{\theta}_{\pi(s)})g^*_s)_{s \in S} + (1 - \tilde{\alpha})g^*.
\]

Since \( g^* \) is crisp, we have
\[
(\tilde{\theta}_s f^*_s + (1 - \tilde{\theta}_s)g^*_s)_{s \in S} \sim (\tilde{\theta}_{\pi(s)} f^*_s + (1 - \tilde{\theta}_{\pi(s)})g^*_s)_{s \in S}.
\]
Thus especially, by subtracting
by the representation,
\[
I(u(\hat{h}) + av) = I\left(\left(\gamma_p(s)u_s(f^*_s) + (1 - \gamma_p(s))u_s(g^*_s)\right)_{s \in S} + (av_{\pi(s)}\tau^{-1}_{\pi(s)}\tau_s)_{s \in S}\right)
\]
Thus especially, by subtracting \(I(u(\hat{h})) = I\left(\left(\gamma_p(s)u_s(f^*_s) + (1 - \gamma_p(s))u_s(g^*_s)\right)_{s \in S}\right)\) from both
hand sides, dividing by \(a\) and taking \(a \to 0\), we have
\[
\nabla I(u(h)) \cdot v = \lim_{a \to 0} \frac{I\left(\left(\gamma_p(s)u_s(f^*_s) + (1 - \gamma_p(s))u_s(g^*_s) + av_{\pi(s)}\tau^{-1}_{\pi(s)}\tau_s\right)_{s \in S}\right) - I\left(\left(\gamma_p(s)u_s(f^*_s) + (1 - \gamma_p(s))u_s(g^*_s)\right)_{s \in S}\right)}{a}.
\]
Since \(v\) was arbitrary, \(I\) is differentiable at \(\left(\gamma_p(s)u_s(f^*_s) + (1 - \gamma_p(s))u_s(g^*_s)\right)_{s \in S}\) and
\[
\nabla I\left(\left(\gamma_p(s)u_s(f^*_s) + (1 - \gamma_p(s))u_s(g^*_s)\right)_{s \in S}\right) = \left(\frac{\tau^{-1}_{\pi(s)}\nabla I(u(h))_{\pi^{-1}(s)}\right)_{s \in S}.
\]
That is for all \(p \in P\),
\[
p = \left(\frac{\tau_{\pi(s)} p_{\pi(s)}}{\tau_s}\right)_{s \in S} \iff (\tau_s p_s)_{s \in S} = (\tau_{\pi(s)} p_{\pi(s)})_{s \in S}. \quad (91)
\]
Define the set
\[
\tau P = \{(\tau_s p_s)_{s \in S} \mid p \in P\}.
\]
By (90,91), \(\tau P \subseteq \mathbb{R}_{++}\) and for all permutations \(\pi : S \to S\) and \(p \in P\), \((\tau_{\pi(s)} p_{\pi(s)})_{s \in S} \in \tau P\). Additionally, let \(p,q \in P, p \neq q\) and assume, per contra, that there exists \(a > 0\) such that
\((\tau_s p_s)_{s \in S} = (a\tau_s q_s)_{s \in S}\). Then \(p = aq\), which is a contradiction since \(p\) and \(q\) are different
probabilities that sum to 1.

By Lemma 92, there exists \(c \in \mathbb{R}_{++}\) such that \(\tau P = \{(c)_{s \in S}\} \) or \((\tau P)^{\perp} = \{\hat{0}\}\).

We first consider the case that there exists \(c \in \mathbb{R}_{++}\) such that \(\tau P = \{(c)_{s \in S}\}\) and show that in this case (1) holds. This follows from the observation that since \(\tau P\) is a singleton, \(P\) is a singleton. Thus the claim follows from \(\tau \in \mathbb{R}^S_{++}\) and Theorem 5.

Second, we consider the case that \((\tau P)^{\perp} = \{\hat{0}\}\) and show that (2) holds. First, \(|S| > 1\)
and so the first case does not hold. First, we show that \(P^{\perp} = \{\hat{0}\}\). Assume, per contra there
exists \(x \in P^{\perp}, x \neq \hat{0}\). Now for all \(p \in P\), since \(\tau \in \mathbb{R}^S_{++}\),
\[
0 = x \cdot p = \sum_{s \in S} \frac{x_s}{\tau_s} \tau_s p_s.
\]
So \( \left( \frac{-a}{\varepsilon} \right)_{s \in S} \in (\tau P)^\perp \) and by (90) \( \left( \frac{-a}{\varepsilon} \right)_{s \in S} \neq \bar{0} \) which is a contradiction. Thus \( P^\perp = \{ \bar{0} \} \). By Lemma 80, for all \( s \in S \),

\[
u_s(f^*_s) = I(u(f^*)) \quad \text{and} \quad \nu_s(g^*_s) = I(u(g^*)).
\]

Thus especially for all \( s \in S \), \( \tau_s = I(u(f^*)) - I(u(g^*)) \). By (91), for all \( p \in P \) and permutations \( \pi : S \to S \), \( (p_{\pi(s)})_{s \in S} \in P \). By the proof of Lemma 78, \( \text{int} P \neq \emptyset \). This shows (2) and the additional claim follows from Theorem 2 since if \( \text{int} \bigcap_{s \in S} \nu_s(\Delta(X_s)) \neq \emptyset \), then every state dependent maxmin representation is tight.

Next, we show the only if direction. By Theorem 1, we only need to show Axiom 9. First, assume that (1) holds. Then \( \succeq \) satisfies independence axiom and every act is crisp. Let \( p \in \text{int} \Delta(S) \) and \( (u_s)_{s \in S} \) be the corresponding utility functions. Since for all \( s \in S \), \( \text{int} u_s(\Delta(X_s)) \neq \emptyset \), there exist \( g^*, f^* \in H, a > 0 \) such that for all \( s \in S \), \( u_s(f^*_s) = u_s(g^*_s) + \frac{a}{p_s} \). Let \( \gamma \in \Delta(S) \) and \( \pi : S \to S \) be a permutation. Now we have

\[
\sum_{s \in S} p_s u_s(\gamma_s f^*_s + (1 - \gamma_s) g^*_s) = \sum_{s \in S} p_s \left( u_s(g^*_s) + \frac{\gamma_s a}{p_s} \right) = \sum_{s \in S} p_s u_s(g^*_s) + a \sum_{s \in S} \gamma_s
\]

\[
= \sum_{s \in S} p_s u_s(g^*_s) + a \sum_{s' \in S} \gamma_{\pi(s')} = \sum_{s \in S} p_s \left( u_s(g^*_s) + \frac{a\gamma_{\pi(s)}}{p_s} \right) = \sum_{s \in S} p_s u_s(\gamma_{\pi(s)} f^*_s + (1 - \gamma_{\pi(s)}) g^*_s).
\]

This shows Axiom 9.

Second, assume that (2) holds. Since \( \text{int} \bigcap_{s \in S} u_s(\Delta(X_s)) \neq \emptyset \), there exist \( f^*, g^* \in H, c^f, c^g \in \mathbb{R}, c^f > c^g \) such that for all \( s \in S \), \( u_s(f^*_s) = c^f \) and \( u_s(g^*_s) = c^g \). As constant utility acts \( f^*, g^* \) are crisp acts. First, for all permutations \( \pi : S \to S \) and \( \gamma \in \Delta(S) \), we have

\[
p \cdot u \left( \left( \gamma_s f^*_s + (1 - \gamma_s) g^*_s \right)_{s \in S} \right) = \sum_{s \in S} p_s \cdot c^f + \sum_{\delta \in S} p_s \gamma_s (c^f - c^g)
\]

\[
= \sum_{s \in S} p_{\pi(s)} \cdot c^f + \sum_{\delta \in S} p_{\pi(s)} \gamma_{\pi(s)} (c^f - c^g) = (p_{\pi(s)})_{s \in S} \cdot \arg \min \left\{ q \cdot u \left( \left( \gamma_{\pi(s)} f^*_s + (1 - \gamma_{\pi(s)}) g^*_s \right)_{s \in S} \right) \right\}.
\]

Let \( \gamma \in \Delta(S) \) and permutation \( \pi : S \to S \). First, we show that

\[
p \in \arg \min \{ q \cdot u \left( \left( \gamma_s f^*_s + (1 - \gamma_s) g^*_s \right)_{s \in S} \right) \} \Rightarrow (p_{\pi(s)})_{s \in S} \in \arg \min \{ q \cdot u \left( \left( \gamma_{\pi(s)} f^*_s + (1 - \gamma_{\pi(s)}) g^*_s \right)_{s \in S} \right) \}.
\]

(93)

Assume, per contra, that there exists \( q \in P \) such that

\[
q \cdot u \left( \left( \gamma_{\pi(s)} f^*_s + (1 - \gamma_{\pi(s)}) g^*_s \right)_{s \in S} \right) < (p_{\pi(s)})_{s \in S} \cdot u \left( \left( \gamma_{\pi(s)} f^*_s + (1 - \gamma_{\pi(s)}) g^*_s \right)_{s \in S} \right).
\]
Then by Equation (92) and by using the permutation $\pi^{-1}$,

$$(q_{\pi^{-1}(s)}(s)\in S \cdot u\left((\gamma_s f^*_s + (1 - \gamma_s) g^*_s)_{s\in S}\right) < \min_{p\in P} \cdot u\left((\gamma_s f^*_s + (1 - \gamma_s) g^*_s)_{s\in S}\right)$$

which is a contradiction by the symmetry of $P$.

Now we have by (92,93),

$$\min_{p\in P} \cdot u\left((\gamma_s f^*_s + (1 - \gamma_s) g^*_s)_{s\in S}\right) = \min_{p\in P} \cdot u\left((\gamma_{\pi(s)} f^*_s + (1 - \gamma_{\pi(s)}) g^*_s)_{s\in S}\right)$$

which shows Axiom 9.

\[\square\]

## F State Dependent Bewley Representation

This section axiomatizes the existence of a state dependent Bewley representation and characterizes its uniqueness.

First, we present the five characterizing axioms. The first axiom is the standard assumption that $\succsim$ is a nontrivial partial order.

**Axiom 1** $\succsim$ is nontrivial, reflexive and transitive.

The next one is the continuity axiom as before.

**Axiom 2** For all $f, g, h \in H$, the sets $\{\alpha \in [0, 1]|\alpha f + (1 - \alpha)g \succsim h\}$ and $\{\alpha \in [0, 1]|h \succsim \alpha f + (1 - \alpha)g\}$ are closed in $\mathbb{R}$.

The next axiom assumes that there is no uncertainty about tastes for the consequences but instead when the acts differ only in one state they can always be compared. This also assumes that there is no ambiguity about a single state.

**Axiom 3** For all $s \in S, f \in H, x_s, y_s \in \Delta(X_s)$,

$$(x_s, f_{-s}) \succsim (y_s, f_{-s}) \text{ or } (y_s, f_{-s}) \succsim (x_s, f_{-s}).$$

The next axiom is the second part of the assumption that there is no ambiguity about a single state. It assumes that preferences over consequences in a single state does not depend on what common act is received in other states. This axiom guarantees that statewise preferences are well-defined.
Axiom 4B For all \( s \in S, f, g \in H, x_s, y_s \in \Delta(X_s) \),
\[
(x_s, f_{s-s}) \succ (y_s, f_{s-s}) \implies (x_s, g_{s-s}) \succeq (y_s, g_{s-s}).
\]
The last axiom is the standard independence axiom.

Axiom 5B For all \( f, g \in H, \alpha \in (0, 1) \),
\[
f \succeq g \iff \alpha f + (1 - \alpha)h \succeq \alpha g + (1 - \alpha)h.
\]

The next result states that the previous five axioms characterize the existence of a state
dependent Bewley representation and characterizes the uniqueness of the representation. The
uniqueness part states that every statewise positive affine transformations for the utilities
are allowed but for each utility function there is a unique set of probabilities.

Theorem 94 (Bewley Representation) \( \succsim \) satisfies Axioms 1B-5B if and only if there
exists \((u, C)\) that is a state dependent Bewley representation for \( \succsim \).

Additionally, let \((u, C)\) be a state dependent Bewley representation for \( \succsim \), \( \tilde{u} = (\tilde{u}_s)_{s \in S} \)
be such that for all \( s \in S \), \( \tilde{u}_s : \Delta(X_s) \to \mathbb{R} \) is affine and \( \tilde{C} \subseteq \Delta(S) \) be closed and convex.
Then \((\tilde{u}, \tilde{C})\) is a state dependent Bewley representation for \( \succsim \) if and only if there exist
\( a \in \mathbb{R}^S_{++}, b \in \mathbb{R}^S \) such that
\[
\tilde{C} = \left\{ \left( \frac{a_s^{-1} p_s}{\sum_{s' \in S} a_{s'}^{-1} p_{s'}} \right)_{s \in S} \bigg| p \in C \right\},
\]
and for all \( s \in S^p \),
\[
\tilde{u}_s = a_s u_s + b_s.
\]

Next, we consider the relation to state independent Bewley representation. Assume a
state independent setting with for all \( s \in S \), \( X_s = X \). We show that the only difference between
state independent and state dependent Bewley representations is the following monotonicity
with between state comparisons.

Axiom 6B For all \( s, s' \in S, f, g \in H, x, y \in \Delta(X) \),
\[
(x_s, f_{s-s}) \succ (y_s, f_{s-s}) \implies (x_{s'}, g_{s'-s'}) \succeq (y_{s'}, g_{s'-s'}),
\]
The next result shows that this gives state independent Bewley representation with the
same utility in every state of the world. Additionally, we show that any statewise positive
affine transformation of utilities gives an alternative state dependent Bewley representation.

**Theorem 95 (State Independent Bewley Representation)** \(\succeq\) satisfies Axioms 1B-6B if and only if there exists \((u, C)\) that is a state independent Bewley representation for \(\succeq\).

Additionally, let \((u, C)\) be a state independent Bewley representation for \(\succeq\), \(\bar{u} = (\bar{u}_s)_{s \in S}\) be such that for all \(s \in S\), \(\bar{u}_s : \Delta(X_s) \rightarrow \mathbb{R}\) is affine and \(\bar{C} \subseteq \Delta(S)\) be closed and convex. Then \((\bar{u}, \bar{C})\) is a state dependent Bewley representation for \(\succeq\) if and only if there exist \(a \in \mathbb{R}_{++}^S, b \in \mathbb{R}^S\) such that

\[
\bar{C} = \left\{ \left( \frac{a_{s_1}^{-1}p_{s_1}}{\sum_{s' \in S} a_{s'}^{-1}p_{s'}} \right) \middle| p \in C \right\},
\]

and for all \(s \in S^p\),

\[
\bar{u}_s = a_s u + b_s.
\]

### F.1 Bewley Existence Proof

For all \(s \in S\), define \(\succeq_s\) and \(\succ_s\) as in Section 2.3. Our first lemma shows that statewise preferences have an expected utility representation.

**Lemma 96** Let \(\succeq\) satisfy Axioms 1B-5B. Then for each \(s \in S\), there exists \(u_s : \Delta(X_s) \rightarrow \mathbb{R}\) that is affine such that for all \(x_s, y_s \in \Delta(X_s)\),

\[
x_s \succeq_s y_s \iff u_s(x_s) \geq u_s(y_s).
\]

**Proof.** Let \(s \in S\). We will show that \(\succeq_s\) is complete, transitive, mixture continuous, and satisfy Independence. Let \(x_s, y_s, z_s \in \Delta(X_s)\), \(f, g \in H\).

1. **Complete:** By Axiom 3B, either \((x_s, f_{-s}) \succeq (y_s, f_{-s})\) or \((y_s, f_{-s}) \succeq (x_s, f_{-s})\). Hence, by Axiom 4B, respectively either for all \(g \in H\), \((x_s, g_{-s}) \succeq (y_s, g_{-s})\) or for all \(g \in H\), \((y_s, g_{-s}) \succeq (x_s, g_{-s})\). Thus by the definition, either \(x_s \succeq_s y_s\) or \(y_s \succeq_s x_s\) respectively.

2. **Transitive:** Let \(x_s \succeq_s y_s\) and \(y_s \succeq_s z_s\). By the definition, for all \(f \in H\), \((x_s, f_{-s}) \succeq (y_s, f_{-s})\) and \((y_s, f_{-s}) \succeq (z_s, f_{-s})\). Thus by Axiom 1B, for all \(f \in H\), \((x_s, f_{-s}) \succeq (y_s, f_{-s})\) and hence \(x_s \succeq_s z_s\).

3. **Mixture Continuous:** Let \(x_s \succeq_s y_s\) and \(y_s \succeq_s z_s\). By definition, there exist \(f, g \in H\) such that \((x_s, f_{-s}) \succ (y_s, f_{-s}) \succeq (z_s, f_{-s})\) and \((x_s, g_{-s}) \succeq (y_s, g_{-s}) \succ (z_s, g_{-s})\). By Axiom 2B, the
Define the extended preferences in the utility space $\hat{h}$ representation. We extend the preferences linearly starting from some interior act $s$; these induced preferences linearly to the set $H$. Similarly, they are nonempty since $1 \in \{ \alpha \in [0, 1] | (\alpha x_s + (1 - \alpha) z_s, f_{-s}) \succ (y_s, f_{-s}) \}$ and $0 \in \{ \alpha \in [0, 1] | (\alpha x_s + (1 - \alpha) z_s, g_{-s}) \succ (y_s, g_{-s}) \}$ are open as the complements of the previous sets. Additionally, they are nonempty since $1 \in \{ \alpha \in [0, 1] | (\alpha x_s + (1 - \alpha) z_s, f_{-s}) \succ (y_s, f_{-s}) \}$ and $0 \in \{ \alpha \in [0, 1] | (\alpha x_s + (1 - \alpha) z_s, g_{-s}) \succ (y_s, g_{-s}) \}$. Thus, there exist $\alpha, \beta \in (0, 1)$ such that

$$
(\alpha x_s + (1 - \alpha) z_s, f_{-s}) \succ (y_s, f_{-s}) \text{ and } (y_s, g_{-s}) \succ (\beta x_s + (1 - \beta) z_s, g_{-s}).
$$

Additionally, by Axiom 4B, for all $h \in H$, $(\alpha x_s + (1 - \alpha) z_s, h_{-s}) \succ (y_s, h_{-s})$ and $(y_s, h_{-s}) \succ (\beta x_s + (1 - \beta) z_s, h_{-s})$. Thus by the definition of $\succ_s$, $\alpha x_s + (1 - \alpha) z_s \succ_s y_s$ and $y_s \succ_s \beta x_s + (1 - \beta) z_s$.

(4) Independence: Let $\alpha \in (0, 1)$. For all $f \in H$, by Axiom 5B,

$$
(x_s, f_{-s}) \succ (y_s, f_{-s}) \iff (\alpha x_s + (1 - \alpha) z_s, f_{-s}) \succ (\alpha y_s + (1 - \alpha) z_s, f_{-s}).
$$

Thus by the definition of $\succ_s$,

$$
x_s \succ_s y_s \iff \alpha x_s + (1 - \alpha) z_s \succ_s \alpha y_s + (1 - \alpha) z_s.
$$

By von Neumann–Morgenstern utility theorem (Fishburn, 1970), there exists affine function $u_s : \Delta(X_s) \rightarrow \mathbb{R}$ such that for all $x_s, y_s \in \Delta(X_s)$,

$$
x_s \succ_s y_s \iff u_s(x_s) \geq u_s(y_s).
$$

Next, we consider the induced preferences in the utility space. Additionally, we extend these induced preferences linearly to $\mathbb{R}^{SP}$. In this case, they have a state independent Bewley representation. We extend the preferences linearly starting from some interior act $h^*$. Let $h^* \in H$ be such that for all $s \in SP$, there exist $x_s, y_s \in \Delta(X_s)$ such that $x_s \succ_s h^* \succ_s y_s$. Define the extended preferences in the utility space $\hat{\succ}$ on $\mathbb{R}^{SP}$ by for all $\varphi, \psi \in \mathbb{R}^{SP}$,

$$
\varphi \hat{\succ} \psi \iff \exists \alpha \in (0, 1), f, g \in H, \alpha \varphi + (1 - \alpha) u(h^*) = u(f), \alpha \psi + (1 - \alpha) u(h^*) = u(g), f \succ g,
$$

where we embed $\varphi$ to $\mathbb{R}^S$ as $\varphi_s = u_s(h^*_s)$ for all $s \notin SP$.

The next lemma shows that in the definition of $\hat{\succ}$, we can use for all quantifier for $\alpha$ by linearity.
Lemma 97  Let $\succsim$ satisfy Axioms 1$^B$-5$^B$. For all $\varphi, \psi \in \mathbb{R}^{SP}$,

\[ \varphi \overset{\sim}{\succsim} \psi \iff \left[ \forall \alpha \in (0, 1], f, g \in H, \alpha \varphi + (1 - \alpha) u(h^*) = u(f), \alpha \psi + (1 - \alpha) u(h^*) = u(g) \Rightarrow f \succsim g \right]. \]

Proof. First, we show the if direction. Let $\varphi \overset{\sim}{\succsim} \psi, \alpha \in (0, 1], f, g \in H$ be such that $\alpha \varphi + (1 - \alpha) u(h^*) = u(f), \alpha \psi + (1 - \alpha) u(h^*) = u(g)$. We show that $f \succsim g$. Since $\varphi \overset{\sim}{\succsim} \psi$, there exist $\alpha^* \in (0, 1), f^*, g^* \in H$ such that $f^* \succsim g^*$ and $\alpha^* \varphi + (1 - \alpha^*) u(h^*) = u(f^*), \alpha^* \psi + (1 - \alpha^*) u(h^*) = u(g^*)$.

We consider two cases: 1) $\alpha \geq \alpha^*$: Now by the linearity of $u$, $u\left(\alpha^*/\alpha f + (1 - \alpha^*/\alpha) h^* \right) = u(f^*)$ and $\left(\alpha^*/\alpha g + (1 - \alpha^*/\alpha) h^* \right) = u(g^*)$. Since $u_s$ represents $\succsim_s$ and by Lemma 26,

\[ \alpha^*/\alpha f + (1 - \alpha^*/\alpha) h^* \sim f^* \succsim g^* \sim \alpha^*/\alpha g + (1 - \alpha^*/\alpha) h^*. \]

Thus by Axioms 1$^B$ and 5$^B$, $f \succsim g$.

2) $\alpha^* > \alpha$: Now by the linearity of $u$, $u\left(\alpha/\alpha^* f^* + (1 - \alpha/\alpha^*) h^* \right) = u(f)$ and $\left(\alpha/\alpha^* g^* + (1 - \alpha/\alpha^*) h^* \right) = u(g)$. Since $u_s$ represents $\succsim_s$ and by Lemma 26 and Axiom 5$^B$,

\[ f \sim \alpha/\alpha^* f^* + (1 - \alpha/\alpha^*) h^* \succsim \alpha/\alpha^* g^* + (1 - \alpha/\alpha^*) h^* \sim g. \]

Thus by Axiom 1$^B$, $f \succsim g$.

Next, we show the only if direction. Now $\emptyset \neq \{\alpha' \in (0, 1]|\alpha' \varphi + (1 - \alpha') u(h^*), \alpha' \psi + (1 - \alpha') u(h^*) \in H\}$ since $u(h^*) \in \text{int pr}_{SP} u(H)$, hence there exist $\alpha \in (0, 1), f, g \in H$ such that $\alpha \varphi + (1 - \alpha) u(h^*) = u(f), \alpha \psi + (1 - \alpha) u(h^*) = u(g)$. By assumption, $f \succsim g$ and hence by the definition, $\varphi \overset{\sim}{\succsim} \psi$.

The next lemma shows that $\overset{\sim}{\succsim}$ satisfies the axioms for state independent Bewley representation.

Lemma 98  Let $\succsim$ satisfy Axioms 1$^B$-5$^B$. $\overset{\sim}{\succsim}$ is reflexive, nontrivial, transitive, continuous, Independence, and monotonic.

Proof.  (1) Nontrivial: By Axiom 1$^B$, there exist $f, g \in H$ such that $f \succ g$ and hence by Lemma 97, $\text{pr}_{SP} u(f) \overset{\sim}{\succsim} \text{pr}_{SP} u(g)$.

(2) Reflexive: Let $\varphi \in \mathbb{R}^{SP}$. Since $u(h^*) \in \text{int u}(H)$, there exist $\alpha \in (0, 1), f \in H$ such that $\alpha \varphi + (1 - \alpha) u(h^*) = u(f)$. Since $\succsim$ is reflexive, $f \succsim f$ and hence $\varphi \overset{\sim}{\succsim} \varphi.$
(3) Transitive: Let \( \varphi \gtrless \psi \) and \( \psi \gtrless \theta \). Let \( \alpha^1, \alpha^2 \in (0, 1], f, g, g', h \in H \) be such that
\[
\alpha^1 \varphi + (1 - \alpha^1)u(h^*) = u(f), \alpha^1 \psi + (1 - \alpha^1)u(h^*) = u(g), \alpha^2 \psi + (1 - \alpha^2)u(h^*) = u(g'), \alpha^2 \theta + (1 - \alpha^2)u(h^*) = u(h) \text{ and } f \gtrsim g, g' \gtrsim h. \text{ Assume without loss of generality that } \alpha^1 \geq \alpha^2. \text{ By Axiom 5B, } \alpha^2/\alpha^1 f + (1 - \alpha^2/\alpha^1)h^* \gtrsim \alpha^2/\alpha^1 g + (1 - \alpha^2/\alpha^1)h^*. \text{ Additionally by the linearity of } u, u(\alpha^2/\alpha^1 g + (1 - \alpha^2/\alpha^1)h^*) = u(g'). \text{ Since } u_s \text{ represents } \gtrsim_s \text{ and by Lemma 26, } \alpha^2/\alpha^1 g + (1 - \alpha^2/\alpha^1)h^* \sim g'. \text{ By Axioms 1B and 5B, }
\[
\alpha^2/\alpha^1 f + (1 - \alpha^2/\alpha^1)h^* \gtrsim \alpha^2/\alpha^1 g + (1 - \alpha^2/\alpha^1)h^* \gtrsim g' \gtrsim h
\]
and \( \alpha^2 \varphi + (1 - \alpha^2)u(h^*) = u(\alpha^2/\alpha^1 f + (1 - \alpha^2/\alpha^1)h^*) \) and \( \alpha^2 \theta + (1 - \alpha^2)u(h^*) = u(h) \). Hence \( \varphi \gtrless \theta \).

(4) Mixture Continuous: We show that upper contour set is closed. The lower contour set follows symmetrically. Let \( \varphi, \psi, \theta \in \mathbb{R}^{S^p} \). Let \( \alpha^* \in (0, 1), f, g, h \in H \) be such that
\[
\alpha^* \varphi + (1 - \alpha^*)u(h^*) = u(f), \alpha^* \psi + (1 - \alpha^*)u(h^*) = u(g), \alpha^* \theta + (1 - \alpha^*)u(h^*) = u(h).
\]
Now for all \( \alpha \in [0, 1], \)
\[
\alpha^* \left( \alpha \varphi + (1 - \alpha) \psi \right) + (1 - \alpha^*)u(h^*) = u \left( \alpha f + (1 - \alpha) g \right).
\]
By Lemma 97, for all \( \alpha \in [0, 1], \)
\[
\alpha \varphi + (1 - \alpha) \psi \gtrsim \theta \iff \alpha f + (1 - \alpha) g \gtrsim h.
\]
Thus
\[
\{ \alpha \in [0, 1]| \alpha \varphi + (1 - \alpha) \psi \gtrsim \theta \} = \{ \alpha \in [0, 1]| \alpha f + (1 - \alpha) g \gtrsim h \}.
\]
By Axiom 2B, \( \{ \alpha \in [0, 1]| h \gtrsim \alpha f + (1 - \alpha) g \} \) is closed and hence \( \{ \alpha \in [0, 1]| \alpha \varphi + (1 - \alpha) \psi \gtrsim \theta \} \) is closed.

(5) Independence: Let \( \varphi, \psi, \theta \in \mathbb{R}^{S^p}, \alpha \in (0, 1) \). Let \( \alpha^* \in (0, 1), f, g, h \in H \) be such that
\[
\alpha^* \varphi + (1 - \alpha^*)u(h^*) = u(f), \alpha^* \psi + (1 - \alpha^*)u(h^*) = u(g), \alpha^* \theta + (1 - \alpha^*)u(h^*) = u(h).
\]
By Lemma 97 and Axiom 5B,
\[
\varphi \gtrsim \psi \iff f \gtrsim g \iff \alpha f + (1 - \alpha) h \gtrsim \alpha g + (1 - \alpha) h \iff \alpha \varphi + (1 - \alpha) \theta \gtrsim \alpha \psi + (1 - \alpha) \theta.
\]

(6) Monotonic: Let \( \varphi, \psi \in \mathbb{R}^{S^p} \) be such that for all \( s \in S^p, \varphi_s \geq \psi_s \). Let \( \alpha^* \in (0, 1), f, g \in H \) be such that \( \alpha^* \varphi + (1 - \alpha^*)u(h^*) = u(f), \alpha^* \psi + (1 - \alpha^*)u(h^*) = u(g) \). Now, for all \( s \in S^p, \)
\[
u_s(f) = \alpha^* \varphi_s + (1 - \alpha^*)u_s(h^*) \geq \alpha^* \psi_s + (1 - \alpha^*)u_s(h^*) = u_s(g).\]
Since for all \( s \in S \), \( u_s \) represents \( \succsim_s \), we have for all \( s \in S \),
\[
    f_s \succsim_s g_s.
\]
Thus by Lemma 26, \( f \succsim g \) and hence \( \varphi \succsim \psi \).

\[\Box\]

**Proposition 99** \( \succsim \) satisfies Axioms \( 1^B-5^B \) iff there exists \((u, C)\) that is a state dependent Bewley representation for \( \succsim \).

**Proof.** By Ghirardato et al.’s (2004) Theorem A.2, Gilboa et al.’s (2010) Lemma 3, and Lemma 98, there exist closed and convex set \( C \subseteq \Delta(S^P) \) and affine \( v : \mathbb{R} \to \mathbb{R} \) such that for all \( \varphi, \psi \in \mathbb{R}^{S^P} \),
\[
    \varphi \succsim \psi \iff \forall p \in C, v(\varphi) \cdot p \geq v(\psi) \cdot p.
\]

Since \( v \) is affine and increasing on \( \mathbb{R} \), there exist \( a > 0, b \in \mathbb{R} \) such that \( v = a \text{Id} + b \), where \( \text{Id} \) is the identity function. Thus especially by a positive affine transformation of utility, we have for all \( \varphi, \psi \in \mathbb{R}^{S^P} \),
\[
    \varphi \succsim \psi \iff \forall p \in C, \varphi \cdot p \geq \psi \cdot p.
\]

Finally, we can embed \( C \) to \( \Delta(S) \) with \( \tilde{C} \) by defining for all \( p \in C, s \notin S^P, p_s = 0 \). Thus \((u, \tilde{C})\) gives a Bewley representation for \( \succsim \) since for all \( f, g \in H \),
\[
    f \succsim g \overset{\text{Lem. 26}}{\iff} (f_{S^P}, h^*_{S^P}) \succsim (g_{S^P}, h^*_{S^P}) \overset{\text{Lem. 97}}{\iff} \text{pr}_{S^P} u(f) \succsim \text{pr}_{S^P} u(g) \iff \forall p \in C, \text{pr}_{S^P} u(f) \cdot p \geq \text{pr}_{S^P} u(g) \cdot p \iff \forall p \in \tilde{C}, u(f) \cdot p \geq u(g) \cdot p.
\]

Additionally, by non-triviality, \( \tilde{C} \neq \emptyset \).

Next, we show the only if direction.

(1) Nontrivial: Assume, per contra, for all \( f, g \in H, f \sim h \). Now by the definition of null-states, \( S^P = \emptyset \) and hence for all \( s \in S, p \in C, p_s = 0 \) which is a contradiction since \( C \neq \emptyset \).

(2) Reflexive: Let \( f \in H \). Now for all \( p \in C, p \cdot u(f) = p \cdot u(f) \) and hence \( f \succsim f \).

(3) Transitive: Let \( f, g, h \in H, f \succsim g, \) and \( g \succsim h \). Now by the definition of \( \succsim \), for all \( p \in C \),
\[
p \cdot u(f) \geq p \cdot u(g) \text{ and } p \cdot u(g) \geq p \cdot u(h)
\]
and hence by the transitivity of \( \geq \),
\[
p \cdot u(f) \geq p \cdot u(h).
\]

Thus \( f \succcurlyeq h \).

(4) Axiom 2B. Let \( f, g, h \in H \). Define for all \( p \in C \), an operator \( I_p : [0, 1] \to \mathbb{R} \),
\[
\alpha \xrightarrow{I_p} \alpha \left( p \cdot (u(f) - u(g)) \right) + p \cdot u(g) - p \cdot u(h).
\]

As a linear function for all \( p \in C \), \( I_p \) is continuous. Now
\[
\{ \alpha \in [0, 1] | \alpha f + (1 - \alpha) g \succcurlyeq h \} = \{ \alpha \in [0, 1] | \forall p \in C, p \cdot u(\alpha f + (1 - \alpha) g) \geq p \cdot u(h) \}
\]
\[
= \{ \alpha \in [0, 1] | \forall p \in C, \alpha p \cdot (u(f) - u(g)) + p \cdot u(g) - p \cdot u(h) \geq 0 \}
\]
\[
= \{ \alpha \in [0, 1] | \forall p \in C, I_p(\alpha) \geq 0 \} = \bigcap_{p \in C} I_p^{-1}[0, \infty)
\]

and symmetrically
\[
\{ \alpha \in [0, 1] | h \succcurlyeq \alpha f + (1 - \alpha) g \} = \bigcap_{p \in C} I_p^{-1}(-\infty, 0]
\]

are closed as the intersections of preimages of a continuous function over closed sets.

(5) Axiom 3B: Let \( s^* \in S, f \in H, x_{s^*}, y_{s^*} \in \Delta(X_{s^*}) \). Assume w.l.o.g. \( u_{s^*}(x_{s^*}) \geq u_{s^*}(y_{s^*}) \).

Now for all \( p \in C \),
\[
p \cdot (x_{s^*}, f_{-s^*}) = \sum_{s \in S \setminus \{s^*\}} p_s u_s(f_s) + p_{s^*} u_{s^*}(x_{s^*}) \geq \sum_{s \in S \setminus \{s^*\}} p_s u_s(f_s) + p_{s^*} u_{s^*}(y_{s^*}) = p \cdot (y_{s^*}, f_{-s^*}).
\]

Thus \( (x_{s^*}, f_{-s^*}) \succcurlyeq (y_{s^*}, f_{-s^*}) \).

(6) Axiom 4B: Let \( s^* \in S, f, g \in H, x_{s^*}, y_{s^*} \in \Delta(X_{s^*}) \) and \( (x_{s^*}, f_{-s^*}) \succcurlyeq (y_{s^*}, f_{-s^*}) \). If \( s^* \notin S^P \), then by definition, \( (x_{s^*}, g_{-s^*}) \succcurlyeq (y_{s^*}, g_{-s^*}) \). Thus assume \( s^* \in S^P \). Thus since \( C \neq \emptyset \), there exists \( p^* \in C \) such that \( p^*_{s^*} \). Since \( (x_{s^*}, f_{-s^*}) \succcurlyeq (y_{s^*}, f_{-s^*}) \), especially
\[
p^* \cdot (x_{s^*}, f_{-s^*}) \geq p^* \cdot (y_{s^*}, f_{-s^*}).
\]

Hence
\[
\sum_{s \in S \setminus \{s^*\}} p^*_{s} u_s(f_s) + p^*_{s^*} u_{s^*}(x_{s^*}) \geq \sum_{s \in S \setminus \{s^*\}} p^*_{s} u_s(f_s) + p^*_{s^*} u_{s^*}(y_{s^*})
\]
\[
\iff p^*_{s^*} u_{s^*}(x_{s^*}) \geq p^*_{s^*} u_{s^*}(y_{s^*}) \overset{p^*_{s^*} \geq 0}{\iff} u_{s^*}(x_{s^*}) \geq u_{s^*}(y_{s^*}).
\]

Thus by the above proof for Axiom 3B,
\[
(x_{s^*}, g_{-s^*}) \succcurlyeq (y_{s^*}, g_{-s^*}).
\]

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(7) Axiom 5B: Let \( f, g, h \in H, \alpha \in (0, 1) \). By the linearity of \( u \), we have

\[
f \succeq g \iff \forall p \in C, \sum_{s \in S} p_s u_s(f_s) \geq \sum_{s \in S} p_s u_s(g_s)
\]

\[
\overset{\alpha > 0}{\iff} \forall p \in C, \sum_{s \in S} p_s \alpha u_s(f_s) + \sum_{s \in S} p_s (1-\alpha) u_s(h_s) \geq \sum_{s \in S} p_s \alpha u_s(g_s) + \sum_{s \in S} p_s (1-\alpha) u_s(h_s)
\]

\[
\overset{u \text{ linear}}{\iff} \forall p \in C, \sum_{s \in S} p_s u_s(\alpha f_s + (1-\alpha) h_s) \geq \sum_{s \in S} p_s u_s(\alpha g_s + (1-\alpha) h_s)
\]

\[
\iff \alpha f_s + (1-\alpha) h_s \succeq \alpha g_s + (1-\alpha) h_s
\]

\[\square\]

### F.2 Bewley Uniqueness Proof

Next we show the uniqueness of the representation. First some notation. Let \( f^*, g^* \in H \) be such that for all \( s \in S^P \), \( f^*_s \succ_s g^*_s \) which exist by Axiom 1B and denote \( h^* := \frac{1}{2}f^* + \frac{1}{2}g^* \).

The next simple lemma shows that states are not proper iff they have zero probability.

**Lemma 100** Let \( \succ \) be an order on \( H \), \((u, C)\) be a state dependent Bewley representation for \( \succ \). \( s^* \notin S^P \) if and only if for all \( p \in C, p_{s^*} = 0 \).

**Proof.** First, we show the if direction. Let \( s^* \in S \) be such that for all \( p \in C, p_{s^*} = 0 \). Let \( x_{s^*}, y_{s^*} \in \Delta(X_{s^*}), f \in H \). Now for all \( p \in C \),

\[
p \cdot u\left(x_{s^*}, f_{-s^*}\right) = \sum_{s \in S \backslash \{s^*\}} p_s u_s(f_s) + p_{s^*} u_{s^*}(x_{s^*})
\]

\[
\overset{p_{s^*} = 0}{=} \sum_{s \in S \backslash \{s^*\}} p_s u_s(f_s) + p_{s^*} u_{s^*}(y_{s^*}) = p \cdot u\left(y_{s^*}, f_{-s^*}\right).
\]

and thus by the representation,

\[
(x_{s^*}, f_{-s^*}) \sim (y_{s^*}, f_{-s^*}).
\]

Hence by definition, \( s^* \) is null.

The only if direction follows directly from the definition of a Bewley representation. \[\square\]

The next three lemmas establish that \( u_s \) represents \( \succ_s \). First, we show a weak representation for the strict part.

**Lemma 101** Let \((u, C)\) be a state dependent Bewley representation for \( \succ \), \( s^* \in S^P, x_{s^*}, y_{s^*} \in \Delta(X_{s^*}) \). If \( x_{s^*} \succ_s y_{s^*} \), then \( u_{s^*}(x_{s^*}) > u_{s^*}(y_{s^*}) \).
Proof. Assume, per contra, \( u_{s^*}(y_{s^*}) \geq u_{s^*}(x_{s^*}) \). Now for all \( p \in C, f \in H \)

\[
\sum_{s \in S \setminus \{s^*\}} p_s u_s(f_s) + p_{s^*} u_{s^*}(y_{s^*}) \geq \sum_{s \in S \setminus \{s^*\}} p_s u_s(f_s) + p_{s^*} u_{s^*}(x_{s^*}).
\]

Thus by the representation, for all \( f \in H \),

\[
(y_{s^*}, f_{-s^*}) \succsim (x_{s^*}, f_{-s^*}).
\]

On the other hand, by the definition of \( \succsim_{s^*} \), since \( x_{s^*} \succsim_{s^*} y_{s^*} \) i.e. \( x_{s^*} \succsim_{s^*} y_{s^*} \) and \( y_{s^*} \succsim_{s^*} x_{s^*} \), for all \( f \in H \), \( (x_{s^*}, f_{-s^*}) \succsim (y_{s^*}, f_{-s^*}) \) and there exists \( f^0 \in H \) such that \( (y_{s^*}, f^0_{-s^*}) \not\succsim (x_{s^*}, f^0_{-s^*}) \) which is a contradiction with the above.

Second, we show that \( u(h^*) \) is in the interior of utility space.

**Corollary 102** Let \( (u, C) \) be a state dependent Bewley representation for \( \succsim \). Then \( \text{pr}_{s^*} u(h^*) \in \text{int} \text{pr}_{s^*} u(H) \).

**Proof.** By Proposition 99, \( \succsim \) satisfies Axioms 1B-5B. Let \( s^* \in S^P \). By Lemma 96, \( f^*_s \succsim_{s^*} h^*_s \succsim_{s^*} g^*_s \). Hence by Lemma 101, \( u_{s^*}(f^*_s) > u_{s^*}(h^*_s) > u_{s^*}(g^*_s) \). Thus by the linearity of \( u_{s^*} \), \( u_{s^*}(h^*_s) \in \text{int} \Delta(X_{s^*}) \). Since \( s^* \in S^P \) was arbitrary, \( \text{pr}_{s^*} u(h^*) \in \text{int} \text{pr}_{s^*} u(H) \).

Third, we show that \( u_s \) represents indifferences.

**Lemma 103** Let \( (u, C) \) be a state dependent Bewley representation for \( \succsim \), \( s^* \in S^P, x_{s^*}, y_{s^*} \in \Delta(X_{s^*}) \). If \( x_{s^*} \succsim_{s^*} y_{s^*} \), then \( u_{s^*}(x_{s^*}) = u_{s^*}(y_{s^*}) \).

**Proof.** Assume, per contra without loss of generality, \( u_{s^*}(y_{s^*}) > u_{s^*}(x_{s^*}) \). Since \( s^* \) is not null, by Lemma 100, there exists \( p^0 \in C \) such that \( p^0_{s^*} > 0 \). Let \( f^0 \in H \). Now

\[
p^0 \cdot u((x_{s^*}, f^0_{-s^*})) = \sum_{s \in S \setminus \{s^*\}} p^0_s u_s(f^0_s) + p^0_{s^*} u_{s^*}(x_{s^*})
\]

\[
\sum_{s \in S \setminus \{s^*\}} p^0_s u_s(f^0_s) + p^0_{s^*} u_{s^*}(y_{s^*}) = p^0 \cdot u((y_{s^*}, f^0_{-s^*})).
\]

Thus by the representation,

\[
(x_{s^*}, f^0_{-s^*}) \not\succsim (y_{s^*}, f^0_{-s^*})
\]

which is a contradiction since \( x_{s^*} \sim_{s^*} y_{s^*} \) and hence especially for all \( f \in H \),

\[
(x_{s^*}, f_{-s^*}) \succsim (y_{s^*}, f_{-s^*}).
\]

Thus \( u_{s^*}(x_{s^*}) = u_{s^*}(y_{s^*}) \).
The last result shows the uniqueness of the state dependent Bewley representation.

**Proposition 104**  If \((u, C)\) and \((\tilde{u}, \tilde{C})\) are state dependent Bewley representations for \(\succsim\), then there exist \(a \in \mathbb{R}^{S_{+}}\), \(b \in \mathbb{R}^{S}\) such that

\[
\tilde{C} = \left\{ \tilde{p} \in \Delta(S) \mid \exists p \in C, \forall s \in S, \tilde{p}_{s} = \frac{a_{s}^{-1}p_{s}}{\sum_{s \in S} a_{s}^{-1}p_{s}} \right\}
\]

and for all \(s \in S^{P}\),

\[
\tilde{u}_{s} = a_{s}u_{s} + b_{s}.
\]

**Proof.** Denote

\[
u_{S} := \text{pr}_{S^{P}} u \quad \text{and} \quad \tilde{u}_{S} := \text{pr}_{S^{P}} \tilde{u}
\]

and for all \(D \subseteq \Delta(S)\),

\[
D_{S} := \text{pr}_{S^{P}} D.
\]

By Lemmas 101 and 103, for all \(s \in S^{P}\), \(u_{s}\) and \(\tilde{u}_{s}\) represents \(\succsim_{s}\). By Lemma 96, \(\succsim_{s}\) satisfies, completeness, transitivity, mixture continuity, and independence and \(u_{s}\) and \(\tilde{u}_{s}\) are affine representations, hence by von Neumann–Morgenstern utility theorem (Fishburn, 1970), there exist \(a \in \mathbb{R}^{S_{+}}, b \in \mathbb{R}^{S}\) such that for all \(s \in S^{P}\),

\[
\tilde{u}_{s} = a_{s}u_{s} + b_{s}.
\]

Denote for all \(s \in S \setminus S^{P}\, a_{s} = 1\). Define

\[
\tilde{C} := \left\{ \left( \frac{a_{s}^{-1}p_{s}}{\sum_{s' \in S} a_{s'}^{-1}p_{s'}} \right)_{s \in S} \mid p \in C \right\}.
\]

Now \(\tilde{C} \subseteq \Delta(S)\) since for all \(s \in S, \, a_{s} > 0\). Additionally, for all \(\tilde{p} \in \tilde{C}, \tilde{p} \in \tilde{C}, s \notin S^{P}\),

\[
\tilde{p}_{s} = 0 = \tilde{p}_{s}.
\]

Thus \(\tilde{C}_{S-}, \tilde{C}_{S-} \subseteq \Delta(S^{P})\).

Define \(\succsim\) and \(\succsim\) on \(\mathbb{R}^{S^{P}}\) by for all \(\varphi, \psi \in \mathbb{R}^{S^{P}}\),

\[
\varphi \succsim \psi \iff \forall \tilde{p} \in \tilde{C}_{S-}, \, p \cdot \varphi \geq p \cdot \psi \quad \text{and} \quad \varphi \succsim \psi \iff \forall \tilde{p} \in \tilde{C}_{S-}, \, p \cdot \varphi \geq p \cdot \psi.
\]
Let \( \varphi, \psi \in \mathbb{R}^{S^P} \). We show that \( \varphi \succsim^C \psi \iff \varphi \succsim^C \psi \). Denote \( a^{-1} \varphi = (a_s^{-1} \varphi_s)_{s \in S^P} \), and similarly for \( \psi \). Let \( \alpha^* \in (0, 1] \) be such that \( \alpha^* a^{-1} \varphi + (1 - \alpha^*) u_{S^-}(h^*) \), \( \alpha^* a^{-1} \psi + (1 - \alpha^*) u_{S^-}(h^*) \) is \( u_{S^-}(H) \), which exists by Corollary 102 and let \( f, g \in H \) be such that

\[
\alpha^* A^{-1} \varphi + (1 - \alpha^*) u_{S^-}(h^*) = u_{S^-}(f) \quad \text{and} \quad \alpha^* A^{-1} \psi + (1 - \alpha^*) u_{S^-}(h^*) = u_{S^-}(g).
\]

Now we have when multiplications with \( a \) are define statewise

\[
\varphi \succsim^C \psi \iff \forall p \in \hat{C}_{S^-}, p \cdot \varphi \geq p \cdot \psi
\]

\[
\iff \forall \hat{p} \in \hat{C}_{S^-}, a^{-1} p \cdot \varphi + (1 - \alpha^*) \hat{p} \cdot a u_{S^-}(h^*) \geq a^{-1} \alpha^* \hat{p} \cdot \psi + (1 - \alpha^*) \hat{p} \cdot a u_{S^-}(h^*)
\]

\[
\iff \forall \hat{p} \in \hat{C}_{S^-}, a \left( a^{-1} p \cdot \varphi + (1 - \alpha^*) u_{S^-}(h^*) \right) \geq a \left( a^{-1} \alpha^* \hat{p} \cdot \psi + (1 - \alpha^*) u_{S^-}(h^*) \right)
\]

\[
\iff \forall p \in C_{S^-}, \sum_{s \in S} p_s a_s^{-1} \cdot a \left( a^{-1} p \cdot \varphi + (1 - \alpha^*) u_{S^-}(h^*) \right) \geq \sum_{s \in S} p_s a_s^{-1} \cdot a \left( a^{-1} \alpha^* \hat{p} \cdot \psi + (1 - \alpha^*) u_{S^-}(h^*) \right)
\]

\[
\iff \forall p \in C_{S^-}, p \cdot \left( a^{-1} p \cdot \varphi + (1 - \alpha^*) u_{S^-}(h^*) \right) \geq p \cdot \left( a^{-1} \alpha^* \hat{p} \cdot \psi + (1 - \alpha^*) u_{S^-}(h^*) \right)
\]

\[
\iff \forall p \in C_{S^-}, p \cdot u_{S^-}(f) \geq p \cdot u_{S^-}(g) \iff \forall p \in C, p \cdot u(f) \geq p \cdot u(g) \iff f \succsim g
\]

\[
\iff \forall \hat{p} \in \hat{C}, \hat{p} \cdot \tilde{u}(f) \geq \hat{p} \cdot \tilde{u}(g) \iff \varphi \succsim^C \psi
\]

where the last equivalence follows symmetrically to the first part.

Thus \( \succsim^C \) has state independent Bewley representations with \( (\text{Id}, \hat{C}_{S^-}) \) and \( (\text{Id}, \hat{C}_{S^-}) \). By Ghirardato et al.’s (2004) Theorem A.2,

\[
\hat{C}_{S^-} = \hat{C}_{S^-}.
\]

Finally, by Lemma 100 and (94),

\[
\hat{C} = \hat{C}.
\]

This proves the uniqueness. \( \Box \)

Putting all together, we have the following theorem.

**Theorem 94 (Bewley Representation)** \( \succsim \) satisfies Axioms 1B-5B if and only if there exists \( (u, C) \) that is a state dependent Bewley representation for \( \succsim \).
Additionally, let \((u, C)\) be a state dependent Bewley representation for \(\succsim\), \(\tilde{u} = (\tilde{u}_s)_{s \in S}\) be such that for all \(s \in S\), \(\tilde{u}_s : \Delta(X_s) \to \mathbb{R}\) is affine and \(\tilde{C} \subseteq \Delta(S)\) be closed and convex. Then \((\tilde{u}, \tilde{C})\) is a state dependent Bewley representation for \(\succsim\) if and only if there exist \(a \in \mathbb{R}_{++}^S, b \in \mathbb{R}^S\) such that
\[
\tilde{C} = \left\{ \left( \frac{a_s^{-1}p_s}{\sum_{s' \in S} a_{s'} p_{s'}} \right)_{s \in S} \left| p \in C \right. \right\}.
\]
and for all \(s \in S^P\),
\[
\tilde{u}_s = a_s u_s + b_s.
\]
\[\text{Proof.}\] Follows from Propositions 99 and 104.

\[\square\]

### F.3 State Independent Bewley Proof

Assume state independent setting with for all \(s \in S\), \(X_s = X\). The next lemma shows that under Axiom 6B, all state dependent utilities are positive affine transformations of each other.

**Lemma 105** Assume that \(\succsim\) satisfies Axioms 1B-6B and has a state dependent Bewley expected utility representation with \((u, C)\).

Then there exist an affine \(u^* : \Delta(X) \to \mathbb{R}\) and \(a \in \mathbb{R}_{++}^S, b \in \mathbb{R}^S\) such that for all \(s \in S^P\),
\[
u_s = a_s u^* + b_s.
\]
\[\text{Proof.}\] By Lemmas 101 and 103, for all \(s \in S^P\), \(u_s\) represents \(\succsim_s\). By the affinity of \(u\), for all \(s \in S, x, y, c \in \Delta(X), \alpha \in (0, 1),\)
\[
x \succsim_s y \Rightarrow \alpha x + (1 - \alpha)c \succsim_s \alpha y + (1 - \alpha)c. \tag{95}
\]
Additionally, by Axiom 6B, for all \(s, s' \in S, x, y \in \Delta(X),\)
\[
x \succsim_s y \Rightarrow x \succsim_{s'} y. \tag{96}
\]
Let \(s \in S^P\). Now there exist \(x^*, y^* \in \Delta(X)\) such that \(u_s(x^*) > u_s(y^*)\). Since \(u_s\) represents \(\succsim_s\), we have
\[
x^* \succsim_s \frac{1}{2} x^* + \frac{1}{2} y^* \succsim_s y^*. \tag{97}
\]
Let \( s' \in S^P \). By (95,96,97) and Lemma 49, \( u_{s'} \) gives a weak affine representation for \( \succcurlyeq_s \). By the affine representation, \( \succcurlyeq_s \) satisfies Axioms 1, 2, 4'. Additionally, \( u_s \) gives a weak affine representation for \( \succcurlyeq_s \). By Proposition 25, there exist \( a_{s'} > 0, b_{s'} \in \mathbb{R} \) such that

\[
u_{s'} = a_{s'} u_s + b_{s'}.
\]

Since \( s' \in S^P \) was arbitrary, this shows the claim. \(\square\)

This gives us state independent Bewley expected utility.

**Theorem 95 (State Independent Bewley Representation)** \( \succcurlyeq \) satisfies Axioms 1\textsuperscript{B}-6\textsuperscript{B} if and only if there exists \((u, C)\) that is a state independent Bewley representation for \( \succcurlyeq \).

Additionally, let \((u, C)\) be a state independent Bewley representation for \( \succcurlyeq \), \( \tilde{u} = (\tilde{u}_s)_{s \in S} \) be such that for all \( s \in S \), \( \tilde{u}_s : \Delta(X_s) \to \mathbb{R} \) is affine and \( \tilde{C} \subseteq \Delta(S) \) be closed and convex. Then \((\tilde{u}, \tilde{C})\) is a state dependent Bewley representation for \( \succcurlyeq \) if and only if there exist \( a \in \mathbb{R}^S_{++}, b \in \mathbb{R}^S \) such that

\[
\tilde{C} = \left\{ \left( \frac{a_s^{-1}p_s}{\sum_{s' \in S} a_{s'}^{-1}p_{s'}} \right)_{s \in S} \mid p \in C \right\}.
\]

and for all \( s \in S^P \),

\[
\tilde{u}_s = a_s u + b_s.
\]

**Proof.** We first show the sufficiency of the axioms. By Theorem 94, there exists \((u, C)\) that is a state dependent Bewley expected utility for \( \succcurlyeq \). By Lemma 105, there exist \( u^* : \Delta(X) \to \mathbb{R} \) and \( a \in \mathbb{R}^S_{++}, b \in \mathbb{R}^S \) such that for all \( s \in S^P \),

\[
u_s = a_s u^* + b_s.
\]

By the uniqueness in Theorem 94, there exists \( C^* \subseteq \Delta(S) \) such that \((u^*, C^*)\) is a state independent Bewley expected utility for \( \succcurlyeq \) since changing the utility to \( u^* \) for \( s \in S \setminus S^P \) does not affect the representation.

Next we show the necessity of the axioms. By Theorem 94, we only show Axiom 6\textsuperscript{B}. This follows from Lemmas 101 and 103 and the state independent utility.

Finally, the uniqueness follows from Theorem 94 and the existence of state independent Bewley representation.
References


Chandrasekher, Madhav; Frick, Mira; Iijima, Ryota, and Le Yaouanc, Yves (2020). Dual-self representations of ambiguity preferences.


