INSTITUTIONAL FINANCE

Lecture 05: Portfolio Choice, CAPM, Black-Litterman

OVERVIEW

1. Portfolio Theory in a Mean-Variance world
2. Capital Asset Pricing Model (CAPM)
3. Estimating Mean and CoVariance matrix
4. Black-Litterman Model
   - Taking a view
   - Bayesian Updating

EXPECTED RETURNS & VARIANCE

- Expected returns (linear)

$$\mu_p := E[r_p] = w_j \mu_j, \text{ where each } w_j = \frac{h^j}{\sum_j h^j}$$

- Variance

$$\begin{aligned}
\sigma_p^2 := Var[r_p] &= w'Vw = (w_1\ w_2)\begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\
&= (w_1\sigma_1^2 + w_2\sigma_{21} \quad w_1\sigma_{12} + w_2\sigma_2^2)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\
&= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} \geq 0 \\
& \quad since\ \sigma_{12} \leq -\sigma_1\sigma_2.
\end{aligned}$$

recall that correlation coefficient $\in$ [-1,1]

ILLUSTRATION OF 2 ASSET CASE

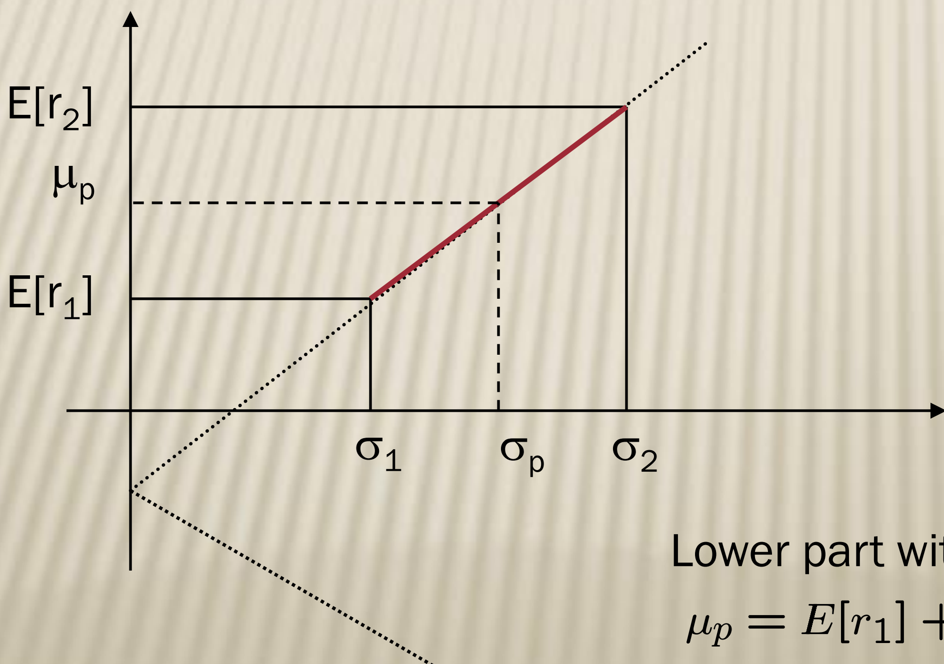
- For certain weights: w_1 and $(1\text{-}w_1)$
  $\mu_p = w_1\, E[r_1] + (1\text{-}w_1)\, E[r_2]$
  $\sigma^2{}_p = w_1{}^2\, \sigma_1{}^2 + (1\text{-}w_1)^2\, \sigma_2{}^2 + 2\, w_1(1\text{-}w_1)\sigma_1\, \sigma_2\, \rho_{1,2}$
  (Specify $\sigma^2{}_p$ and one gets weights and μ_p's)
- Special cases [w_1 to obtain certain σ_R]
  + $\rho_{1,2} = 1 \;\; \Rightarrow w_1 = (+/\text{-}\sigma_p - \sigma_2) / (\sigma_1 - \sigma_2)$
  + $\rho_{1,2} = \text{-}1 \;\; \Rightarrow w_1 = (+/\text{-}\ \sigma_p + \sigma_2) / (\sigma_1 + \sigma_2)$

2 ASSETS $\rho = 1$

$$\begin{aligned} \sigma_p &= |w_1\sigma_1 + (1 - w_1)\sigma_2| \\ \mu_p &= w_1\mu_1 + (1 - w_1)\mu_2 \end{aligned}$$

Hence, $w_1 = \frac{\pm\sigma_p - \sigma_2}{\sigma_1 - \sigma_2}$

$$\mu_p = \mu_1 + \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}(\S\,\sigma_p \text{ ¡ } \sigma_1)$$



Lower part with ... is irrelevant

$$\mu_p = E[r_1] + \frac{E[r_2] - E[r_1]}{\sigma_2 - \sigma_1}(-\sigma_R - \sigma_1)$$

The Efficient Frontier: Two Perfectly Correlated Risky Assets

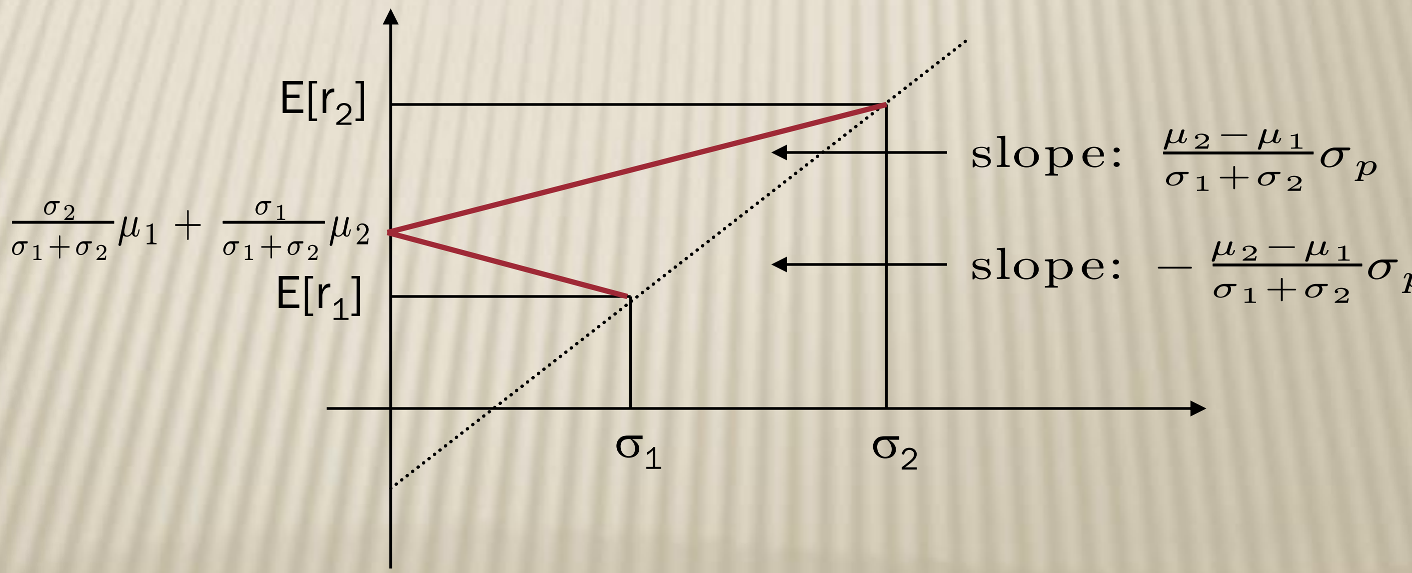
2 ASSETS ρ = -1

For $\rho_{1,2}$ = -1:

$$\sigma_p = |w_1\sigma_1 - (1-w_1)\sigma_2|$$
$$\mu_p = w_1\mu_1 + (1-w_1)\mu_2$$

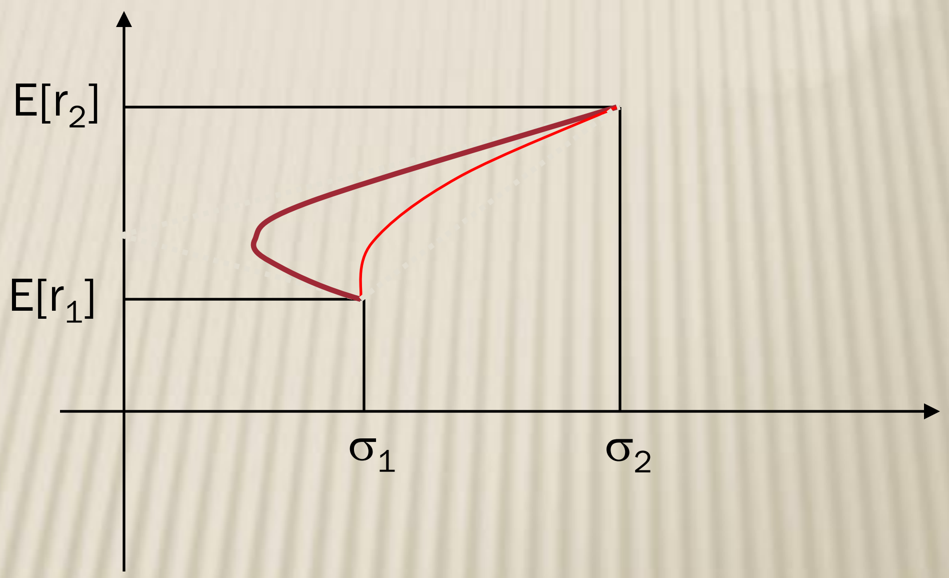
Hence, $w_1 = \frac{\pm\sigma_p+\sigma_2}{\sigma_1+\sigma_2}$

$$\mu_p = \frac{\sigma_2}{\sigma_1+\sigma_2}\mu_1 + \frac{\sigma_1}{\sigma_1+\sigma_2} \pm \frac{\mu_2-\mu_1}{\sigma_1+\sigma_2}\sigma_p$$



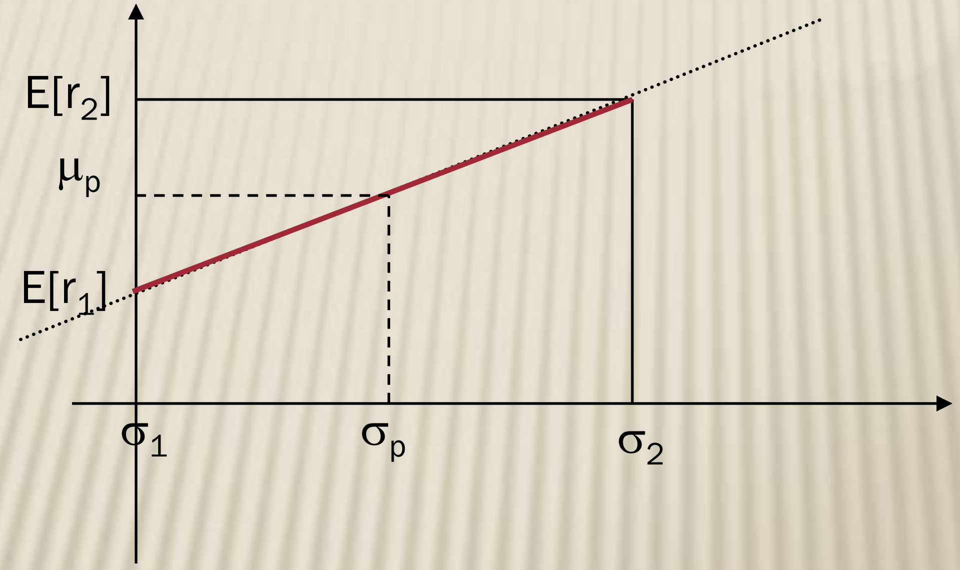
Efficient Frontier: Two Perfectly Negative Correlated Risky Assets

2 ASSETS $-1 < \rho < 1$



Efficient Frontier: Two Imperfectly Correlated Risky Assets

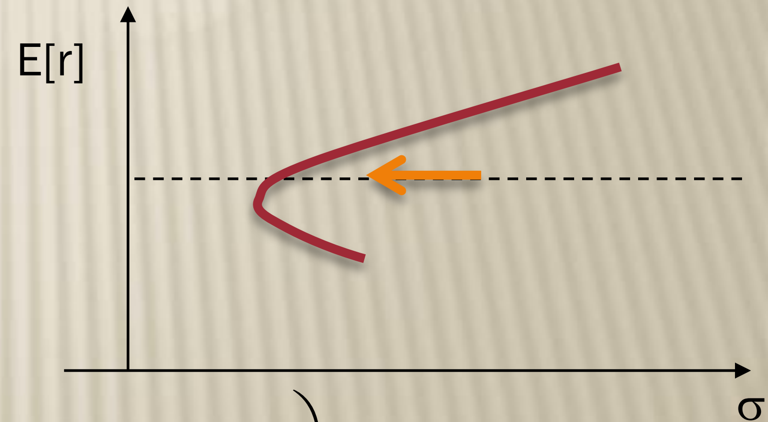
2 ASSETS $\sigma_1 = 0$



The Efficient Frontier: One Risky and One Risk Free Asset

EFFICIENT FRONTIER WITH N RISKY ASSETS

- *A frontier portfolio is one which displays minimum variance among all feasible portfolios with the same expected portfolio return.*



$$\min_{w} \frac{1}{2} w^T V w$$

$$(\lambda) \quad \text{s.t.} \quad w^T e = E \qquad \left(\sum_{i=1}^{N} w_i E(\tilde{r}_i) = E \right)$$

$$(\gamma) \qquad w^T 1 = 1 \qquad \left(\sum_{i=1}^{N} w_i = 1 \right)$$

$$\frac{\partial \mathcal{L}}{\partial w} = Vw - \lambda e - \gamma 1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = E - w^T e = 0$$
$$\frac{\partial \mathcal{L}}{\partial \gamma} = 1 - w^T 1 = 0$$

The first FOC can be written as:

$$Vw_p = \lambda e + \gamma 1 \text{ or}$$
$$w_p = \lambda V^{-1} e + \gamma V^{-1} 1$$
$$e^T w_p = \lambda (e^T V^{-1} e) + \gamma (e^T V^{-1} 1)$$

Noting that $e^T w_p = w^T{}_p e$, using the first foc, the second foc can be written as

$$E[\tilde{r}_p] = e^T w_p = \lambda \underbrace{(e^T V^{-1} e)}_{:=B} + \gamma \underbrace{(e^T V^{-1} 1)}_{=:A}$$

pre-multiplying first foc with 1 (instead of e^T) yields

$$1^T w_p = w_p^T 1 = \lambda(1^T V^{-1} e) + \gamma(1^T V^{-1} 1) = 1$$
$$1 = \lambda \underbrace{(1^T V^{-1} e)}_{=:A} + \gamma \underbrace{(1^T V^{-1} 1)}_{=:C}$$

Solving both equations for λ and γ

$$\lambda = \frac{CE-A}{D} \text{ and } \gamma = \frac{B-AE}{D}$$

where $D = BC - A^2$.

Hence, $w_p = \lambda V^{-1}e + \gamma V^{-1}1$ becomes

$$w_p = \frac{CE - A}{D} V^{-1}e + \frac{B - AE}{D} V^{-1}\mathbf{1}$$

(vector) (vector)

λ (scalar) γ (scalar)

$$= \frac{1}{D}\left[B\left(V^{-1}1\right) - A\left(V^{-1}e\right)\right] + \frac{1}{D}\left[C\left(V^{-1}e\right) - A\left(V^{-1}1\right)\right]E$$

$$w_p = g + h \quad E \qquad (6.15)$$

(vector) (vector) (scalar)

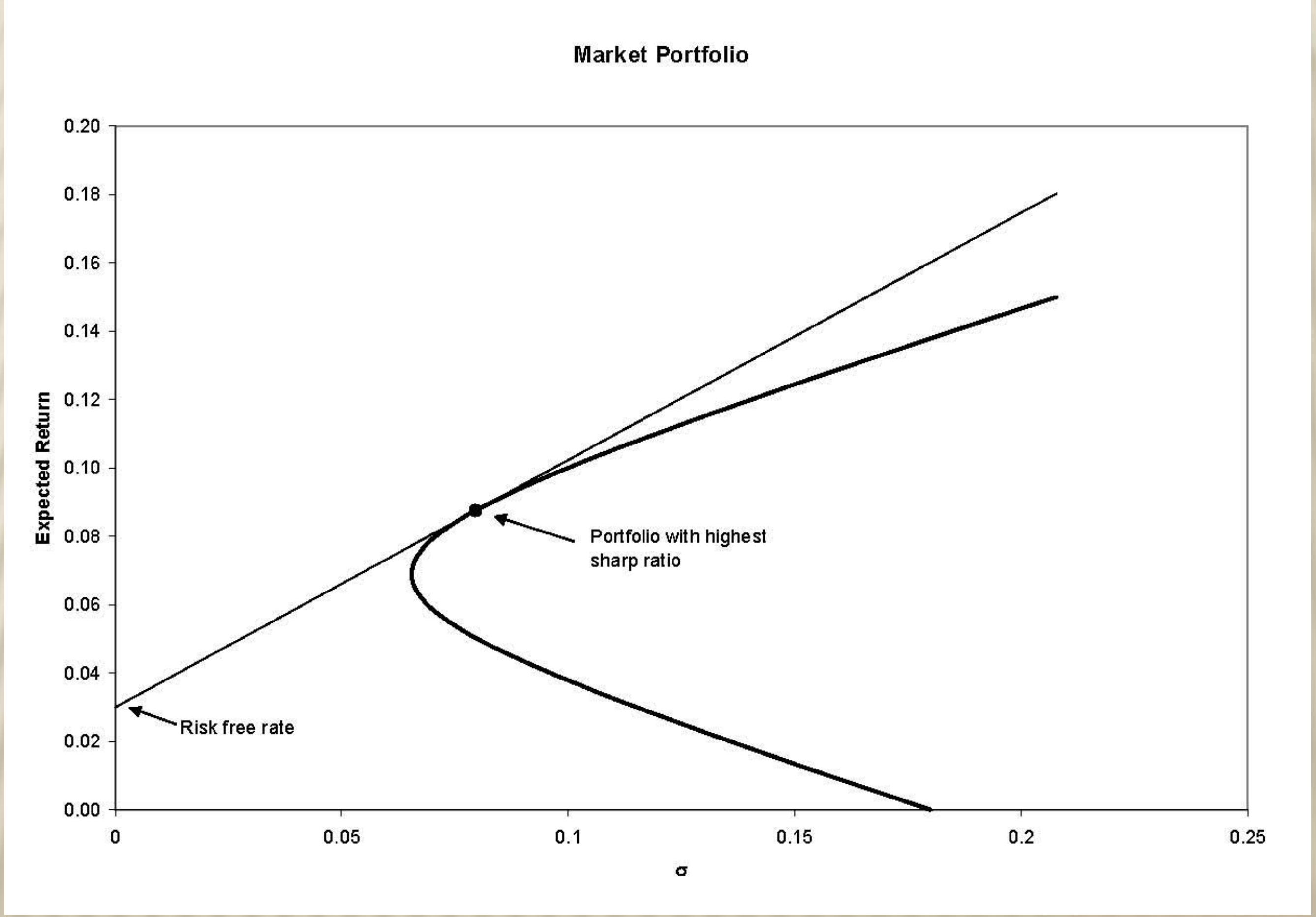
linear in expected return E!

If E = 0, $w_p = g$

If E = 1, $w_p = g + h$

Hence, g and g+h are portfolios on the frontier.

EFFICIENT FRONTIER WITH RISK-FREE ASSET



The Efficient Frontier: One Risk Free and n Risky Assets

EFFICIENT FRONTIER WITH RISK-FREE ASSET

$$\min_w \frac{1}{2} w^T V w$$
$$\text{s.t. } w^T e + (1 - w^T 1) r_f = E[r_p]$$

FOC: $$w_p = \lambda V^{-1}(e - r_f 1)$$

Multiplying by $(e - r_f\ 1)^T$ and solving for λ yields $$\lambda = \frac{E[r_p] - r_f}{(e - r_f 1)^T V^{-1} (e - r_f 1)}$$

$$w_p = \underbrace{V^{-1}(e - r_f \mathbf{1})}_{n \times 1} \frac{E[r_p] - r_f}{H^2}$$

where $H = \sqrt{B - 2Ar_f + Cr_f^2}$ is a number

EFFICIENT FRONTIER WITH RISK-FREE ASSET

- Result 1: Excess return in frontier excess return

$$
\begin{aligned}
Cov[r_q, r_p] &= w_q^T V w_p \\
&= \underbrace{w_q^T (e - r_f \mathbf{1})}_{E[r_q]-r_f} \frac{E[r_p] - r_f}{H^2} \\
&= \frac{(E[r_q] - r_f)([E[r_p] - r_f)}{H^2} \\
Var[r_p, r_p] &= \frac{(E[r_p] - r_f)^2}{H^2}
\end{aligned}
$$

$$
E[r_q] - r_f = \underbrace{\frac{Cov[r_q, r_p]}{Var[r_p]}}_{:=\beta_{q,p}} (E[r_p] - r_f)
$$

Holds for any frontier portfolio *p*, in particular the market portfolio!

EFFICIENT FRONTIER WITH RISK-FREE ASSET

- Result 2: Frontier is linear in (E[r], σ)-space

$$
\begin{aligned}
Var[r_p, r_p] &= \frac{(E[r_p] \text{ ¡ } r_f)^2}{H^2} \\
E[r_p] &= r_f + H\sigma_p
\end{aligned}
$$

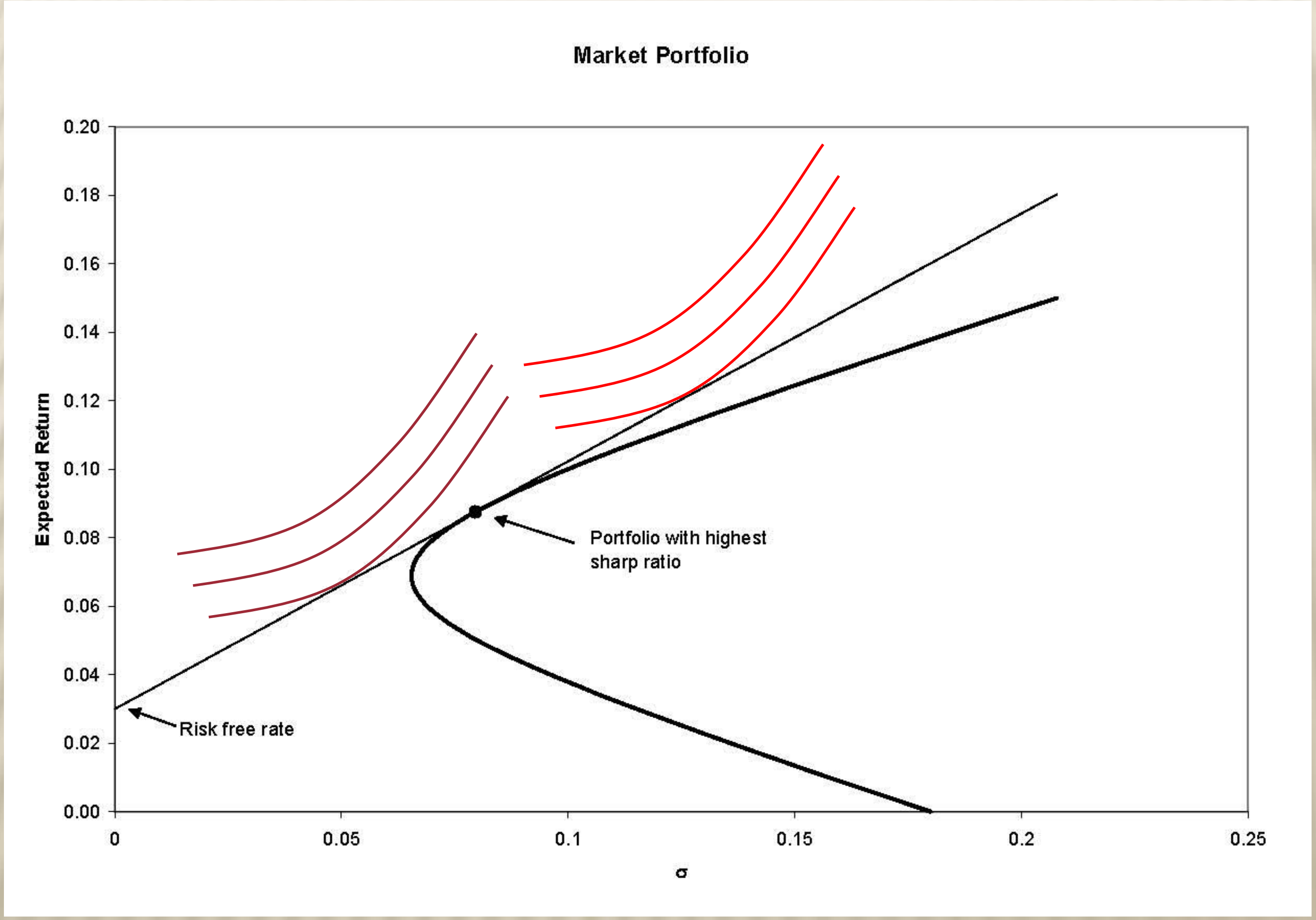
$$H = \frac{E[r_p] - r_f}{\sigma_p}$$

where H is the Sharpe ratio

TWO FUND SEPARATION

- Doing it in two steps:
  - First solve frontier for *n* risky asset
  - Then solve tangency point
- Advantage:
  - Same portfolio of n risky asset for different agents with different risk aversion
  - Useful for applying equilibrium argument (later)

TWO FUND SEPARATION



Price of Risk = = highest Sharpe ratio

Optimal Portfolios of Two Investors with Different Risk Aversion

MEAN-VARIANCE PREFERENCES

- $U(\mu_p, \sigma_p)$ with $\frac{\partial U}{\partial \mu_p} > 0, \ \frac{\partial U}{\partial \sigma_p^2} < 0$
  - Example: $E[W]$ ¡ $\frac{\gamma}{2} Var[W]$
- Also in expected utility framework
  - quadratic utility function (with portfolio return R)

    $U(R) = a + b\,R + c\,R^2$

    vNM: $E[U(R)] = a + b\,E[R] + c\,E[R^2]$

    $= a + b\,\mu_p + c\,\mu_p^2 + c\,\sigma_p^2$

    $= g(\mu_p, \sigma_p)$
  - asset returns normally distributed $\Rightarrow R = \sum_j w^j r^j$ normal
    - if U(.) is CARA $\Rightarrow$ certainty equivalent = $\mu_p - \rho_A/2\sigma^2_p$
      (Use moment generating function)

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2. EQUILIBRIUM LEADS TO CAPM

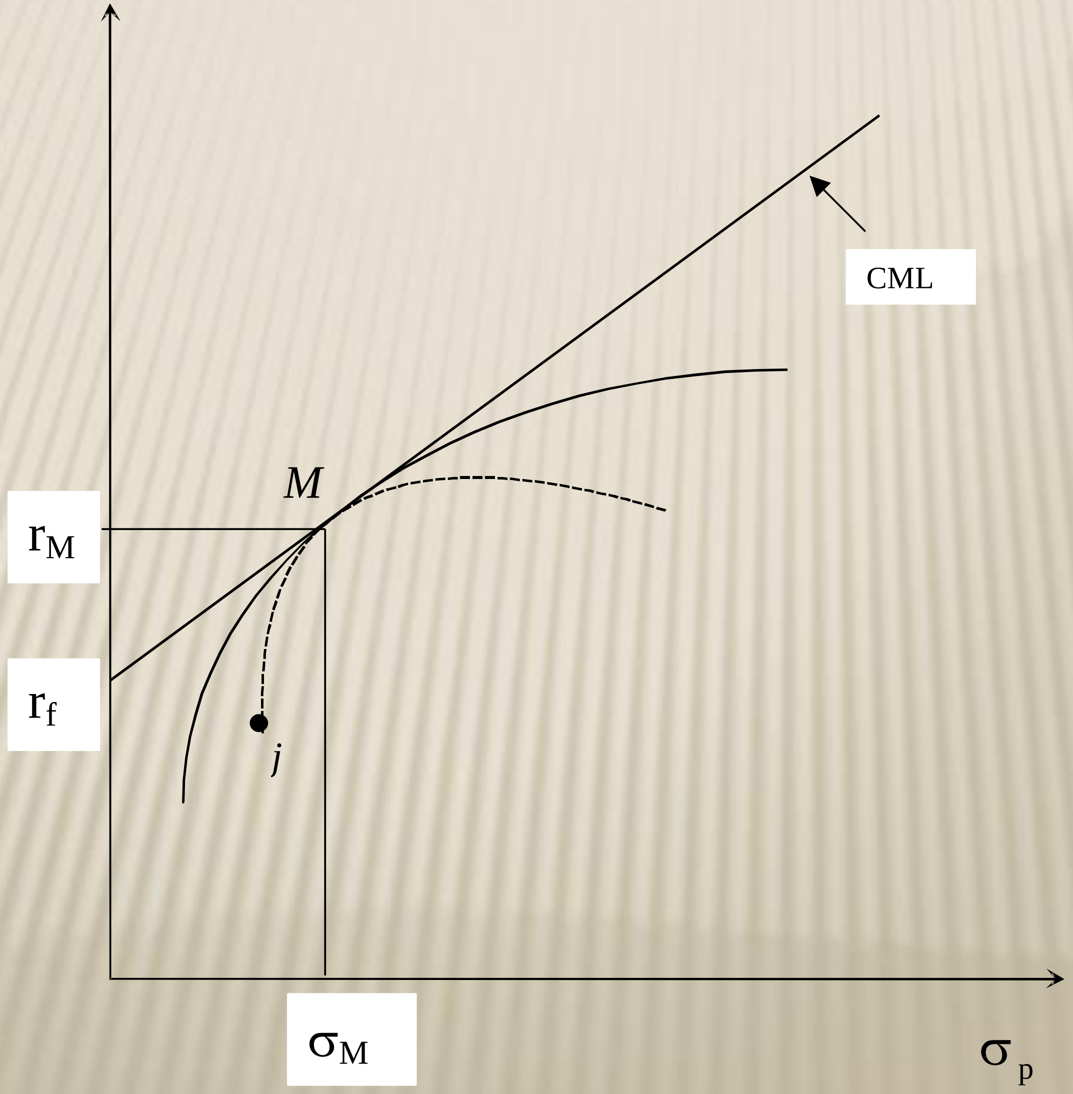
- Portfolio theory: only analysis of demand
  - price/returns are taken as given
  - composition of risky portfolio is same for all investors
- Equilibrium Demand = Supply (market portfolio)
- CAPM allows to derive
  - equilibrium prices/ returns.
  - risk-premium

THE CAPM WITH A RISK-FREE BOND

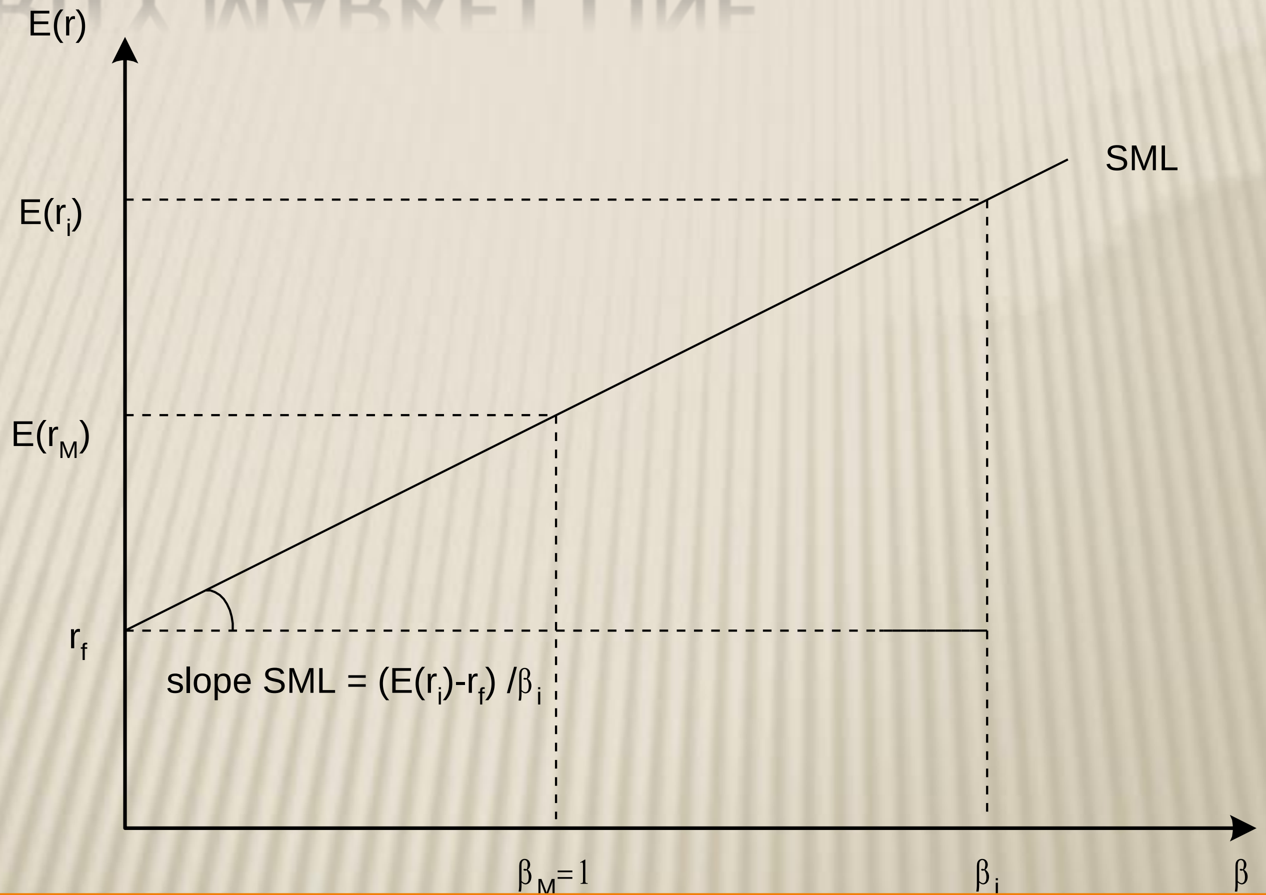
- The market portfolio is efficient since it is on the efficient frontier.
- All individual optimal portfolios are located on the half-line originating at point $(0, r_f)$.
- The slope of Capital Market Line (CML): $\frac{E[R_M] - R_f}{\sigma_M}$.

$$E[R_p] = R_f + \frac{E[R_M] - R_f}{\sigma_M} \sigma_p$$

CAPITAL MARKET LINE



SECURITY MARKET LINE



$$E[r_j] = \mu_j = r_f + \underbrace{\frac{Cov[r_j, r_M]}{Var[r_M]}}_{\beta_j}(E[r_M] - r_f)$$

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3. ESTIMATING MEAN AND CO-VARIANCE

- Consider returns as stochastic process (e.g. GBM)
- Mean return (drift)
  - For any partition of [0,T] with N points ($\Delta t = T/N$), $N * E[r] = \sum_{i=1}^{N} \mathbf{r}_{i\Delta t} = \mathbf{p}_T - \mathbf{p}_0$ (in log prices)
  - Knowing first $\mathbf{p}_0$ and last price $\mathbf{p}_T$ is sufficient
  - Estimation is very imprecise!
- Variance
  - $Var[r] = 1/N \sum_{i=1}^{N} (\mathbf{r}_{i\Delta t} - E[r])^2 \rightarrow \sigma^2$ as $N \rightarrow \infty$
  - Theory: Intermediate points help to estimate co-variance
  - Real world:
    - time-varying
    - Market microstructure noise

3. 1000 ASSETS

- Invert a 1000x1000 matrix
- Estimate 1000 expected returns
- Estimate 1000 variances
- Estimate 1000*1001/2 – 1000 co-variances

⇨ Reduce to fewer factors

… so far we used past data

(and assumed future will behave the same)

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4. BLACK-LITTERMAN MODEL

- So far we estimated expected returns using historical data.
- We ignored statistical priors:
  - A sector with an unusually high (or low) past return was assumed to earn (on average) the same high (or low) return going forward.
  - We should have attributed some of this past return to luck, and only some to the sector being unusual relative to the population.

EXPECTED RETURNS

- We also ignored economic priors:
  - A sector with a negative past return should not be expected to have negative expected returns going forward.
  - A sector that is highly correlated with another sector should probably have similar expected returns.
  - A "good deal" in the past (i.e. good realized return relative to risk) should not persist if everyone is applying mean-variance optimization.
- What is a good starting point from which to update based on our analysis?

BAYESIAN UPDATING

- Bayes' Rule allows one to update distribution after observing some signal/data
  - from prior to posterior distribution
- Recall if all variables are normally distributed with can use the projection theorem
  - E.g. prior: θ = N (μ,τ^2); signal/view x = $\theta + \epsilon$, where ϵ = N(0, σ^2)
  - Weights depend on relative precision/confidence of prior vs. signal/view (on portfolio)
- .

$$E(\theta \mid x) = \left(\frac{\sigma^2}{\tau^2 + \sigma^2}\right)\mu + \left(\frac{\tau^2}{\tau^2 + \sigma^2}\right)x$$

BLACK LITTERMAN PRIOR

- All expected returns are in proportion to their risk.
  → Expected returns are distributed *around*
  $\beta_i \, (E[R_m] - R_f)$

PROPERTIES OF A CAPM PRIOR

- All expected returns are in proportion to their risk.
  → Expected returns are distributed *around*
  $\beta_i\,(E[R_m] - R_f)$

- Is this a good starting point?
- We can still use optimization
- We don't throw out data (e.g. still can estimate covariance structure accurately)
- It is internally consistent – if we don't have an edge, the prior will lead us to holding the market

BLACK-LITTERMAN

- The Black-Litterman model simply takes the starting point that there are no good deals…
- And then adjusts returns according to any "views" that the investor has from:
  - Seeing abnormal returns in the past that expected to persist (or reverse)
  - Fundamental analysis
  - Alphas of active trading strategies
  - "views" concern portfolios and not necessarily individual assets

BLACK LITTERMAN PRIORS – MORE SPECIFIC

See He and Litterman

- Suppose returns of N-assets (in vector/matrix notation)

$$r \text{ » } N(\mu, \Sigma)$$

- Equilibrium risk premium,

$$\Pi = \gamma \Sigma w^{eq}$$

where γ risk aversion, **w**^{eq} market portfolio weights

- Bayesian prior (with imprecision)

$$\mu = \Pi + \varepsilon_0, \text{ where } \varepsilon_0 \text{ » } N(0, \tau\Sigma)$$

VIEWS

- View on a single asset affects many weights
- "Portfolios views"
  - views on K portfolios
  - P: K x N-matrix with portfolio weights
  - Q: K-vector of expected returns on these portfolios
- Investor's views

$$P\mu = Q + \varepsilon_v, \text{ where } \varepsilon_v \text{ » } N(0, \square)$$

  - Ω is a off-diagonal values are all zero
  - ε_v and ε_0 are all orthogonal

BAYESIAN POSTERIOR - REWRITTEN

$$E(\theta \mid x) = \left(\frac{\sigma^2}{\tau^2 + \sigma^2} \right) \mu + \left(\frac{\tau^2}{\tau^2 + \sigma^2} \right) x$$

$$= \left(\frac{1/\tau^2}{1/\tau^2 + 1/\sigma^2} \right) \mu + \left(\frac{1/\sigma^2}{1/\tau^2 + 1/\sigma^2} \right) x$$

$$= \frac{1}{1/\tau^2 + 1/\sigma^2} \left(1/\tau^2 \cdot \mu + 1/\sigma^2 \cdot x \right)$$

BAYESIAN UPDATING

- Black Litterman updates returns to reflect views using Bayes' Rule.
- The updating formula is just the multi-variate (matrix) version of

$$E(\theta \mid x) = \frac{1}{1/\tau^2 + 1/\sigma^2}\left(1/\tau^2 \cdot \mu + 1/\sigma^2 \cdot x\right)$$

$$E[R \mid Q] = \left[(\tau\Sigma)^{-1} + P^T \Omega^{-1} P\right]^{-1}\left[(\tau\Sigma)^{-1}\Pi + P^T \Omega^{-1} Q\right]$$

BAYESIAN UPDATING

$$E(\theta \mid x) = \boxed{\frac{1}{1/\tau^2 + 1/\sigma^2}} \left(1/\tau^2 \cdot \mu + 1/\sigma^2 \cdot x\right)$$

$$E[R \mid Q] = \boxed{\left[(\tau\Sigma)^{-1} + P^T \Omega^{-1} P\right]^{-1}} \left[(\tau\Sigma)^{-1}\Pi + P^T \Omega^{-1} Q\right]$$

Scaling term – Total precision

BAYESIAN UPDATING

$$E(\theta \mid x) = \frac{1}{1/\tau^2 + 1/\sigma^2}\left(1/\tau^2 \cdot \boxed{\mu} + 1/\sigma^2 \cdot x\right)$$

$$E[R \mid Q] = \left[(\tau\Sigma)^{-1} + P^T\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\boxed{\Pi} + P^T\Omega^{-1}Q\right]$$

CAPM Prior
expected returns

BAYESIAN UPDATING IN BLACK LITTERMAN

$$E(\theta \mid x) = \frac{1}{1/\tau^2 + 1/\sigma^2}\left(\boxed{1/\tau^2} \cdot \mu + 1/\sigma^2 \cdot x\right)$$

$$E[R \mid Q] = \left[(\tau\Sigma)^{-1} + P^T\Omega^{-1}P\right]^{-1}\left[\boxed{(\tau\Sigma)^{-1}}\Pi + P^T\Omega^{-1}Q\right]$$

Weighted by precision of CAPM Prior

BAYESIAN UPDATING

$$E(\theta \mid x) = \frac{1}{1/\tau^2 + 1/\sigma^2}\left(1/\tau^2 \cdot \mu + 1/\sigma^2 \cdot \boxed{x}\right)$$

$$E[R \mid Q] = \left[(\tau\Sigma)^{-1} + P^T\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}\boxed{Q}\right]$$

Vector of expected return views

BAYESIAN UPDATING

$$E(\theta \mid x) = \frac{1}{1/\tau^2 + 1/\sigma^2}\left(1/\tau^2 \cdot \mu + \boxed{1/\sigma^2} \cdot x\right)$$

$$E[R \mid Q] = \left[(\tau\Sigma)^{-1} + P^T\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\Pi + \boxed{P^T\Omega^{-1}}Q\right]$$

Weighted by
precision of views

ADVANTAGES OF BLACK-LITTERMAN

- Returns are only adjusted partially towards the investor's views using Bayesian updating
  - Recognizes that views may be due to estimation error
  - Only highly precise/confident views are weighted heavily
- Returns are modified in a way that is consistent with economic priors
  - highly correlated sectors have returns modified in the same way
- Returns can be modified to reflect absolute or relative views
- The resulting weights are reasonable and do not load up on estimation error

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