Institutional Finance 08: Dynamic Arbitrage to Replicate Non-linear Payoffs

Binomial Option Pricing: Basics

(Chapter 10 of McDonald)

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Introduction to Binomial Option Pricing

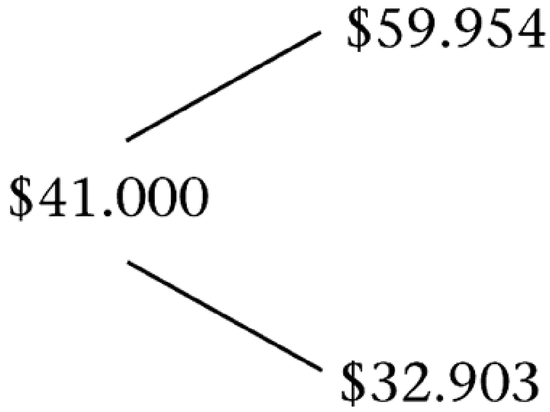
- Binomial option pricing enables us to determine the price of an option, given the characteristics of the stock or other underlying asset.

- The binomial option pricing model assumes that the price of the underlying asset follows a binomial distribution—that is, the asset price in each period can move only up or down by a specified amount.

- The binomial model is often referred to as the "Cox-Ross-Rubinstein pricing model."

A One-Period Binomial Tree

- Example:
  - Consider a European call option on the stock of XYZ, with a $40 strike and 1 year to expiration.
  - XYZ does not pay dividends, and its current price is $41.
  - The continuously compounded risk-free interest rate is 8%.
  - The following figure depicts possible stock prices over 1 year, i.e., a **binomial tree**.



Computing the option price

- Next, consider two portfolios:
  - *Portfolio A*: Buy one call option.
  - *Portfolio B*: Buy 0.7376 shares of XYZ and borrow \$22.405 at the risk-free rate.
- Costs:
  - *Portfolio A*: The call premium, which is unknown.
  - *Portfolio B*: 0.7376 × \$41 – \$22.405 = \$7.839.

Computing the option price

- Payoffs:
  - *Portfolio A*:

	Stock Price in 1 Year	
	$32.903	**$59.954**
Payoff	0	$19.954

  - *Portfolio B*:

	Stock Price in 1 Year	
	$32.903	**$59.954**
0.7376 purchased shares	$24.271	$44.225
Repay loan of $22.405	– $24.271	– $24.271
Total payoff	0	$19.954

Computing the option price

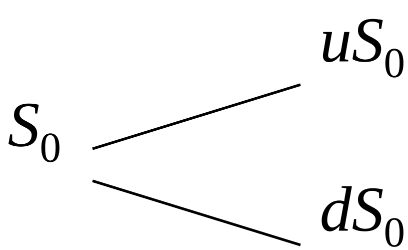
- Portfolios A and B have the same payoff. Therefore,
  - Portfolios A and B should have the same cost. Since Portfolio B costs $7.839, the price of one option must be $7.839.
  - There is a way to create the payoff to a call by buying shares and borrowing. Portfolio B is a *synthetic* call.
  - One option has the risk of 0.7376 shares. The value 0.7376 is the *delta* (Δ) of the option: The number of shares that replicates the option payoff.

The binomial solution

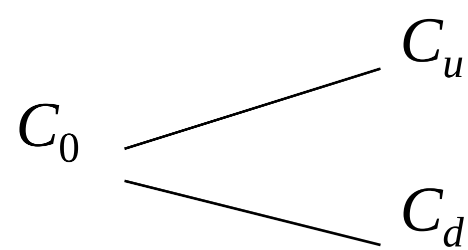
- How do we find a replicating portfolio consisting of Δ shares of stock and a dollar amount B in lending, such that the portfolio imitates the option whether the stock rises or falls?
  - Suppose that the stock has a continuous dividend yield of δ, which is reinvested in the stock. Thus, if you buy one share at time t, at time $t+h$ you will have $e^{\delta h}$ shares.
  - If the length of a period is h, the interest factor per period is e^{rh}.
  - uS_0 denotes the stock price when the price goes up, and dS_0 denotes the stock price when the price goes down.

The binomial solution

- Stock price tree:



- Corresponding tree for the value of the option:



  – Note that *u* (*d*) in the stock price tree is interpreted as one plus the rate of capital gain (loss) on the stock if it foes up (down).

- The value of the replicating portfolio at time *h*, with stock price S_h, is

$$\Delta S_h + e^{rh} B$$

The binomial solution

- At the prices $S_h = uS$ and $S_h = dS$, a replicating portfolio will satisfy

$$\left(\Delta \times uS \times e^{\delta h}\right) + (B \times e^{rh}) = C_u$$

$$\left(\Delta \times dS \times e^{\delta h}\right) + (B \times e^{rh}) = C_d$$

- Solving for Δ and B gives

(10.1)

$$\Delta = e^{-\delta h} \frac{C_u - C_d}{S(u-d)} \tag{10.2}$$

$$B = e^{-rh} \frac{uC_d - dC_u}{u-d}$$

The binomial solution

- The cost of creating the option is the cash flow required to buy the shares and bonds. Thus, the cost of the option is $\Delta S+B$.

$$\Delta S + B = e^{-rh}\left(C_u \frac{e^{(r-\delta)h} - d}{u-d} + C_d \frac{u - e^{(r-\delta)h}}{u-d} \right) \qquad (10.3)$$

- The no-arbitrage condition is

$$u > e^{(r-\delta)h} > d \qquad (10.4)$$

Arbitraging a mispriced option

- If the observed option price differs from its theoretical price, arbitrage is possible.
  - If an option is overpriced, we can sell the option. However, the risk is that the option will be in the money at expiration, and we will be required to deliver the stock. To hedge this risk, we can buy a synthetic option at the same time we sell the actual option.
  - If an option is underpriced, we buy the option. To hedge the risk associated with the possibility of the stock price falling at expiration, we sell a synthetic option at the same time.

A graphical interpretation of the binomial formula

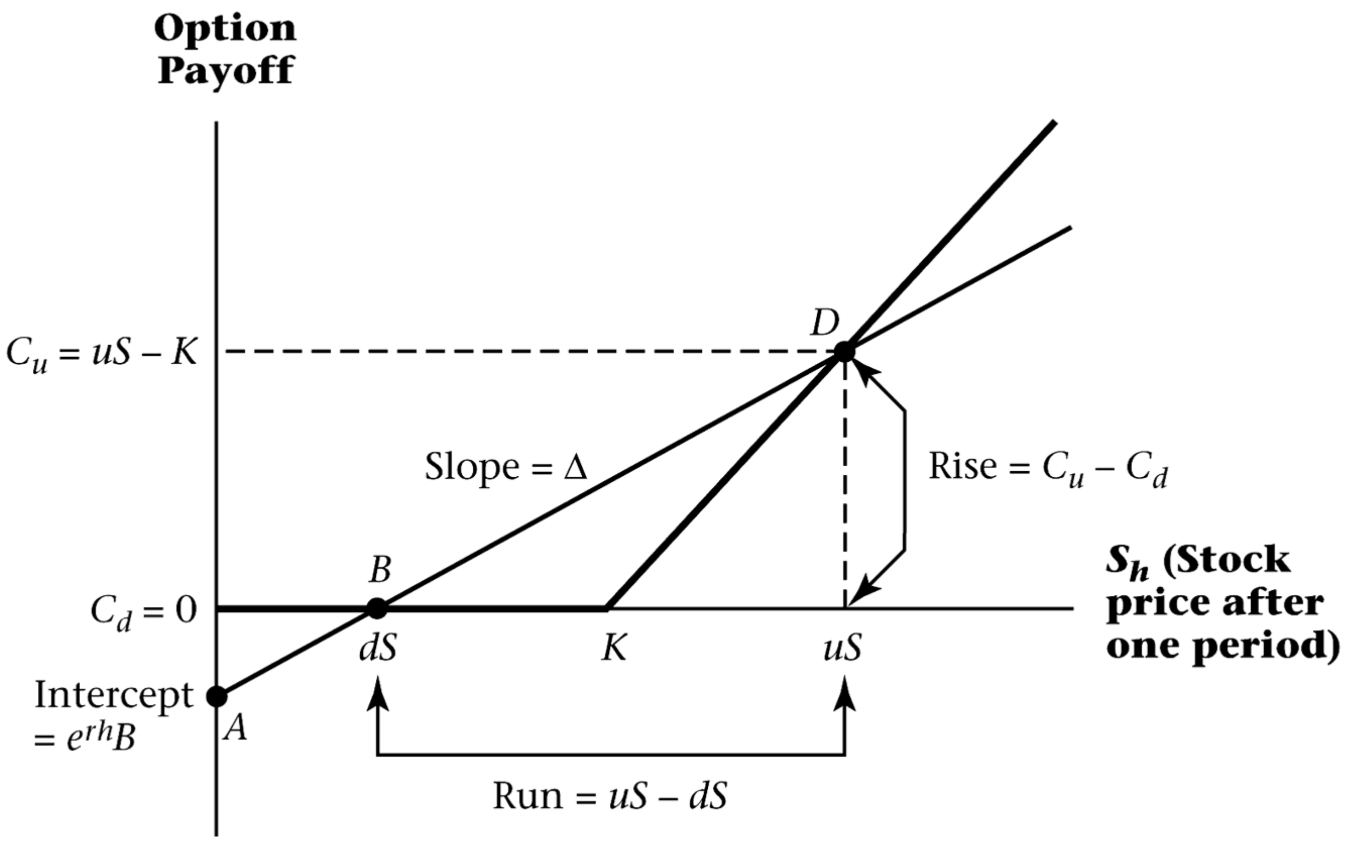
- The portfolio describes a line with the formula

$$C_h = \Delta S_h + e^{rh} B \,,$$

where C_h and S_h are the option and stock value after one binomial period, and supposing $\delta = 0$.

  - We can control the slope of a payoff diagram by varying the number of shares, Δ, and its height by varying the number of bonds, B.
  - Any line replicating a call will have a positive slope ($\Delta > 0$) and negative intercept ($B < 0$). (For a put, $\Delta < 0$ and $B > 0$.)

A graphical interpretation of the binomial formula



Risk-neutral pricing

- We can interpret the terms $(e^{(r-\delta)h} - d)/(u - d)$ and $(u - e^{(r-\delta)h})/(u - d)$ as probabilities.
  - In equation (10.3), they sum to 1 and are both positive.

- Let

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} \qquad (10.5)$$

- Then equation (10.3) can then be written as

$$C = e^{-rh} [p^* C_u + (1 - p^*) C_d] , \qquad (10.6)$$

where p^* is the **risk-neutral probability** of an increase in the stock price.

Where does the tree come from?

- *In the absence of uncertainty*, a stock must appreciate at the risk-free rate less the dividend yield. Thus, from time t to time $t+h$, we have

$$S_{t+h} = S_t e^{(r-\delta)h} = F_{t,t+h}$$

The price next period equals the forward price.

Where does the tree come from?

- *With uncertainty*, the stock price evolution is

$$uS_t = F_{t,t+h}\, e^{+\sigma\sqrt{h}} \qquad (10.8)$$
$$dS_t = F_{t,t+h}\, e^{-\sigma\sqrt{h}} \,,$$

where σ is the annualized standard deviation of the continuously compounded return, and $\sigma\sqrt{h}$ is standard deviation over a period of length h.

- We can also rewrite (10.8) as

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} \qquad (10.9)$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}}$$

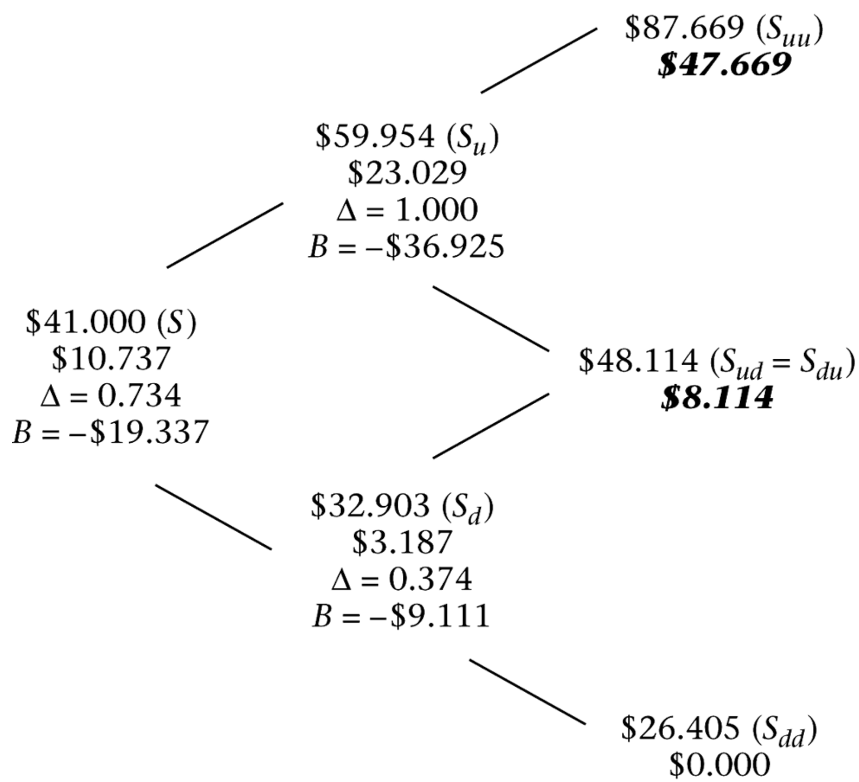
We refer to a tree constructed using equation (10.9) as a “forward tree.”

Summary

- In order to price an option, we need to know
  - stock price,
  - strike price,
  - standard deviation of returns on the stock,
  - dividend yield,
  - risk-free rate.
- Using the risk-free rate and σ, we can approximate the future distribution of the stock by creating a binomial tree using equation (10.9).
- Once we have the binomial tree, it is possible to price the option using equation (10.3).

A Two-Period European Call

- We can extend the previous example to price a 2-year option, assuming all inputs are the same as before.



A Two-Period European Call

- Note that an up move by the stock followed by a down move (S_{ud}) generates the same stock price as a down move followed by an up move (S_{du}). This is called a **recombining tree**. (Otherwise, we would have a **nonrecombining tree**).

$$S_{ud} = S_{du} = u \times d \times \$41 = e^{(0.08+0.3)} \times e^{(0.08-0.3)} \times \$41 = \$48.114$$

Pricing the call option

- To price an option with two binomial periods, we work *backward* through the tree.
  - *Year 2, Stock Price=$87.669*: Since we are at expiration, the option value is *max* (0, *S* – *K*) = $47.669.
  - *Year 2, Stock Price=$48.114*: Similarly, the option value is $8.114.
  - *Year 2, Stock Price=$26.405*: Since the option is out of the money, the value is 0.

Pricing the call option

– *Year 1, Stock Price=$59.954:* At this node, we compute the option value using equation (10.3), where *uS* is $87.669 and *dS* is $48.114.

$$e^{-0.08}\left(\$47.669\times\frac{e^{0.08}-0.803}{1.462-0.803}+\$8.114\times\frac{1.462-e^{0.08}}{1.462-0.803}\right)=\$23.029$$

– *Year 1, Stock Price=$32.903:* Again using equation (10.3), the option value is $3.187.

– *Year 0, Stock Price = $41:* Similarly, the option value is computed to be $10.737.

Pricing the call option

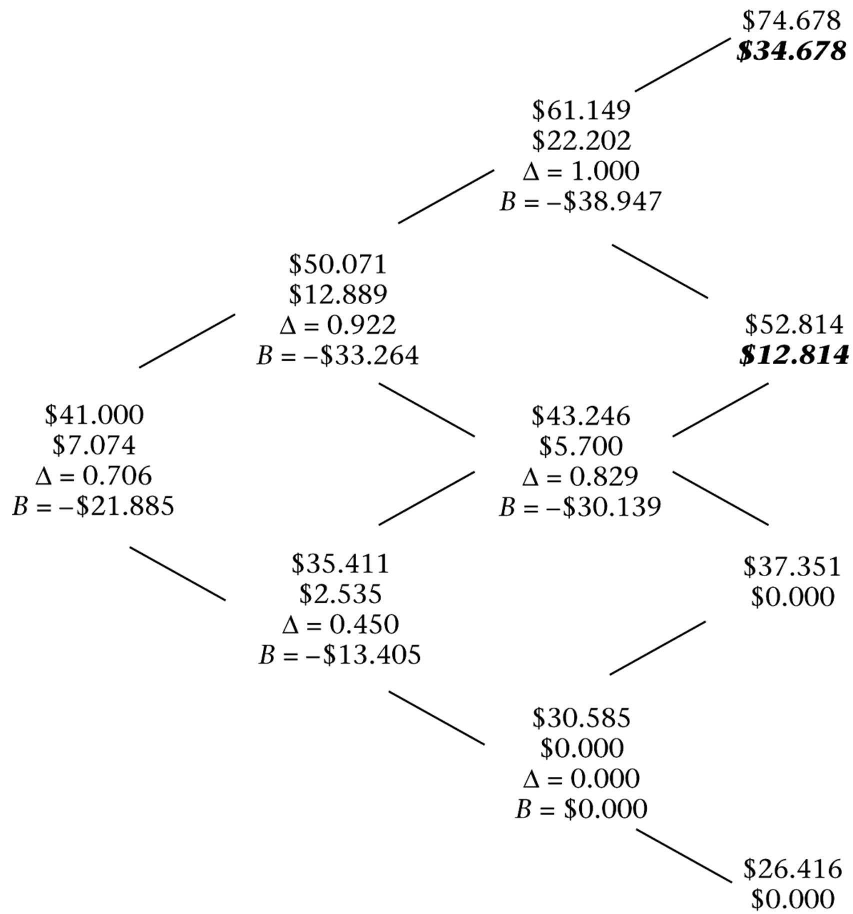
- Notice that:
  - The option was priced by working backward through the binomial tree.
  - The option price is greater for the 2-year than for the 1-year option.
  - The option's Δ and B are different at different nodes. At a given point in time, Δ increases to 1 as we go further into the money.
  - Permitting early exercise would make no difference. At every node prior to expiration, the option price is greater than $S - K$; thus, we would not exercise even if the option was American.

Many binomial periods

- Dividing the time to expiration into more periods allows us to generate a more realistic tree with a larger number of different values at expiration.
  - Consider the previous example of the 1-year European call option.
  - Let there be three binomial periods. Since it is a 1-year call, this means that the length of a period is $h = 1/3$.
  - Assume that other inputs are the same as before (so, $r = 0.08$ and $\sigma = 0.3$).

Many binomial periods

- The stock price and option price tree for this option:



Many binomial periods

- Note that since the length of the binomial period is shorter, u and d are smaller than before: u = 1.2212 and d = 0.8637 (as opposed to 1.462 and 0.803 with h = 1).
  - The second-period nodes are computed as follows:

$$S_u = \$41e^{0.08\times1/3+0.3\sqrt{1/3}} = \$50.071$$

$$S_d = \$41e^{0.08\times1/3-0.3\sqrt{1/3}} = \$35.411$$

The remaining nodes are computed similarly.

- Analogous to the procedure for pricing the 2-year option, the price of the three-period option is computed by working backward using equation (10.3).
  - The option price is $7.074.

Black-Scholes Formula

- Call Options:

$$C(S,K,\sigma,r,T,\delta) = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2)$$

- Put Options:

$$P(S,K,\sigma,r,T,\delta) = Ke^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1)$$

where

$$d_1 = \frac{ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

Black-Scholes (BS) assumptions

- Assumptions about stock return distribution
  - Continuously compounded returns on the stock are normally distributed and independent over time (no “jumps”)
  - The volatility of continuously compounded returns is known and constant
  - Future dividends are known, either as dollar amount or as a fixed dividend yield
- Assumptions about the economic environment
  - The risk-free rate is known and constant
  - There are no transaction costs or taxes
  - It is possible to short-sell costlessly and to borrow at the risk-free rate

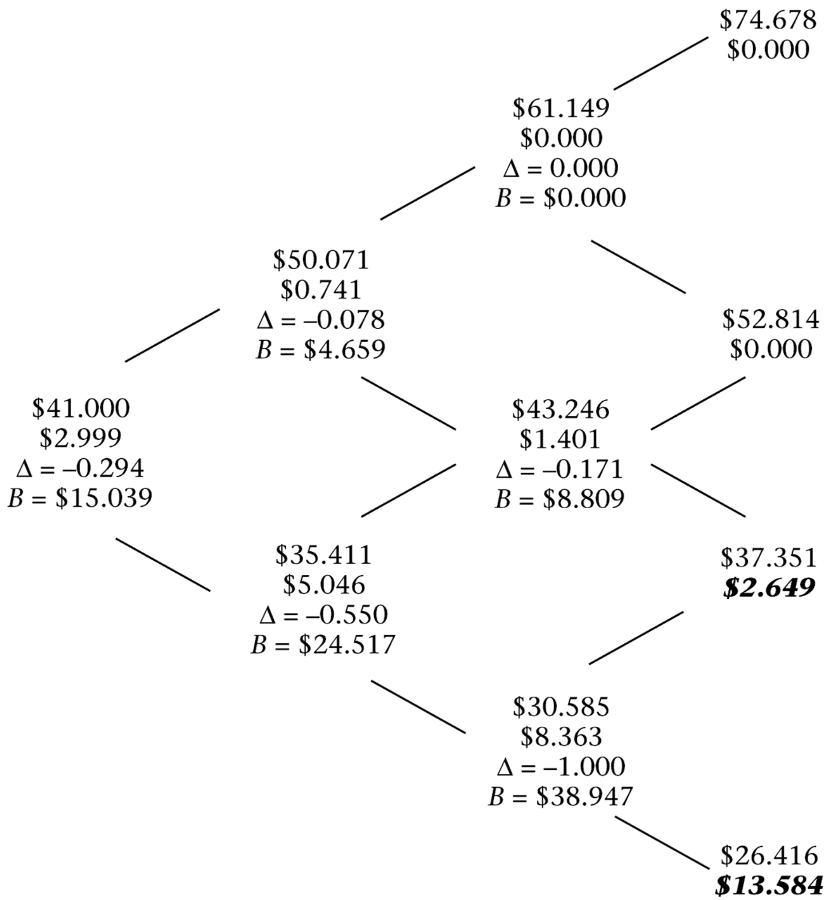
Put Options

- We compute put option prices using the same stock price tree and in the same way as call option prices.

- The only difference with a European put option occurs at expiration.
  - Instead of computing the price as $max\ (0, S - K)$, we use $max\ (0, K - S)$.

Put Options

- A binomial tree for a European put option with 1-year to expiration:



Put-Call Parity

- For European options with the same strike price and time to expiration the parity relationship is:

$$\text{Call} - \text{put} = PV\ (\text{forward price} - \text{strike price})$$

or

$$C(K, T) - P(K, T) = PV_{0,T}(F_{0,T} - K) = e^{-rT}(F_{0,T} - K)$$

- Intuition:
  - Buying a call and selling a put with the strike equal to the forward price ($F_{0,T} = K$) creates a synthetic forward contract and hence must have a zero price.

Parity for Options on Stocks

- If underlying asset is a stock and *Div* is the dividend stream, then $e^{-rT} F_{0,T} = S_0 - PV_{0,T}(Div)$, therefore

$$C(K, T) = P(K, T) + [S_0 - PV_{0,T}(Div)] - e^{-rT}(K)$$

- Rewriting above,

$$S_0 = C(K, T) - P(K, T) + PV_{0,T}(Div) + e^{-rT}(K)$$

- For index options, $S_0 - PV_{0,T}(Div) = S_0 e^{-\delta T}$, therefore

$$C(K, T) = P(K, T) + S_0 e^{-\delta T} - PV_{0,T}(K)$$

Summary of parity relationships

TABLE 9.9 Versions of put-call parity. Notation in the table includes the spot currency exchange rate, x_0; the risk-free interest rate in the foreign currency, r_f; and the current bond price, B_0.

Underlying Asset	Parity Relationship
Futures Contract	$e^{-rT} F_{0,T} = C(K,T) - P(K,T) + e^{-rT} K$
Stock, No-Dividend	$S_0 = C(K,T) - P(K,T) + e^{-rT} K$
Stock, Discrete Dividend	$S_0 - PV_{0,T}(Div) = C(K,T) - P(K,T) + e^{-rT} K$
Stock, Continuous Dividend	$e^{-\delta T} S_0 = C(K,T) - P(K,T) + e^{-rT} K$
Currency	$e^{-r_f T} x_0 = C(K,T) - P(K,T) + e^{-rT} K$
Bond	$B_0 - PV_{0,T}(Coupons) = C(K,T) - P(K,T) + e^{-rT} K$