Lecture 03: Sharpe Ratio, Bounds and the Equity Premium Puzzle

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The long-term gains from the stock market have been astounding.

**TODAY’S VALUE OF $1 INVESTED IN 1972**
Including reinvestment of interests and dividends

- **Exponential growth of stock prices**
- **1y-Bonds earned less, but grew very smoothly**
- **Stock account dipping below $1**

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*Source: Mertens, Data from Ibbotson Associates*
Sharpe Ratios and Bounds

• Consider a one period security available at date $t$ with payoff $x_{t+1}$. We have

$$p_t = E_t[m_{t+1} \cdot x_{t+1}]$$

or

$$p_t = E_t[m_{t+1}] \cdot E_t[x_{t+1}] + Cov[m_{t+1}, x_{t+1}]$$

• For a given $m_{t+1}$ we let

$$R_{t+1}^f = 1 / E_t[m_{t+1}]$$

– Note that $R_{t}^f$ will depend on the choice of $m_{t+1}$ unless there exists a riskless portfolio
Sharpe Ratios and Bounds (ctd.)

– Hence

\[ p_t = \left(1/R_{t+1}^f\right) E_t[x_{t+1}] + \text{Cov}[m_{t+1}, x_{t+1}] \]

– price = expected PV + Risk adjustment

– positive correlation with the discount factor adds value
in Returns

\[ E_t [m_{t+1} x_{t+1}] = p_t \]

- divide both sides by \( p_t \) and note that \( x_{t+1} = R_{t+1} \)

\[ E_t [m_{t+1} R_{t+1}] = 1 \quad \text{(vector)} \]

- using \( R^f_{t+1} = 1/ E_t[m_{t+1}] \), we get

\[ E_t [m_{t+1} (R_{t+1} - R^f_{t+1})] = 0 \]

- \( m \)-discounted expected excess return for all assets is zero.
in Returns

– Since \( E_t [m_{t+1} (R_{t+1} - R_{t+1}^f)] = 0 \)

\[
\text{Cov}_t[m_{t+1},R_{t+1}-R_{t+1}^f] = E_t[m_{t+1}(R_{t+1} - R_{t+1}^f)] - E_t[m_{t+1}]E_t[R_{t+1} - R_{t+1}^f]
\]

\[
= -E_t[m_{t+1}]E_t[R_{t+1} - R_{t+1}^f]
\]

• That is, risk premium or expected excess return

\[
E_t [R_{t+1} - R_{t+1}^f] = -\text{Cov}_t[m_{t+1},R_{t+1}] / E[m_{t+1}]
\]

is determined by its covariance with the stochastic discount factor
Sharpe Ratio

- Multiply both sides with portfolio \( h \)
  
  \[
  E_t [(R_{t+1} - R_t^f)h] = - \frac{\text{Cov}_t[m_{t+1}, R_{t+1}h]}{E[m_{t+1}]}
  \]

- NB: All results also hold for unconditional expectations \( E[\cdot] \)
  
  \[
  E[(R_{t+1} - R_t^f)h] = - \frac{\rho(m_{t+1}, R_{t+1}h) \sigma(R_{t+1}h) \sigma(m_{t+1})}{E[m_{t+1}]}
  \]

- Rewritten in terms of Sharpe Ratio = ...

\[
\frac{E[(R_{t+1} - R_t^f)h]}{\sigma(R_{t+1}h)} = - \frac{\sigma(m_{t+1})}{E[m_{t+1}]} \rho(m_{t+1}, R_{t+1}h)
\]
Hansen-Jagannathan Bound

- Since $\rho \in [-1, 1]$ we have

$$\frac{\sigma(m_{t+1})}{E[m_{t+1}]} \geq \sup_h \left| \frac{E[(R_{t+1} - R_{t+1}^f)h]}{\sigma(R_{t+1}h)} \right|$$

- Theorem (Hansen-Jagannathan Bound): The ratio of the standard deviation of a stochastic discount factor to its mean exceeds the Sharpe Ratio attained by any portfolio.
Hansen-Jagannathan Bound

– Theorem (Hansen-Jagannathan Bound): The ratio of the standard deviation of a stochastic discount factor to its mean exceeds the Sharpe Ratio attained by any portfolio.
– Can be used to easily check the “viability” of a proposed discount factor
– Given a discount factor, this inequality bounds the available risk-return possibilities
– The result also holds conditional on date $t$ info
Hansen-Jagannathan Bound

expected return

slope $\sigma (m) / E[m]$

available portfolios

$\sigma$

$R$
Assuming Expected Utility

- \( c_0 \in \mathbb{R} \), \( c_1 \in \mathbb{R}^S \)
- \( U(c_0, c_1) = \sum_s \pi_s \ u(c_0, c_{1,s}) \)
- \( \partial_0 u = (\frac{\partial u(c_{0,s}^*, c_{1,s}^*)}{\partial c_0}, \ldots, \frac{\partial u(c_{0,s}^*, c_{1,s}^*)}{\partial c_0}) \)
- \( \partial_1 u = (\frac{\partial u(c_{0,s}^*, c_{1,s}^*)}{\partial c_{1,s}^*}, \ldots, \frac{\partial u(c_{0,s}^*, c_{1,s}^*)}{\partial c_{1,s}^*}) \)
- Stochastic discount factor

\[
m = \left( \frac{MRS}{\pi} \right) = \left( \frac{\partial_1 u}{E[\partial_0 u]} \right) \in \mathbb{IR}^S
\]
• *Digression:* if utility is in addition time-separable
  \[ u(c_0, c_1) = v(c_0) + v(c_1) \]

• Then
  \[
  \begin{align*}
  \partial_0 u &= \left( \frac{\partial v(c_0^*)}{\partial c_0}, \ldots, \frac{\partial v(c_0^*)}{\partial c_0} \right) \\
  \partial_1 u &= \left( \frac{\partial v(c_1^*, 1)}{\partial c_1, 1}, \ldots, \frac{\partial v(v_1^*, s)}{\partial c_1, s} \right)
  \end{align*}
  \]

• and
  \[
  m_s = \frac{1}{\pi_s} \frac{\pi_s v'(c_1, s)}{v'(c_0)} = \frac{v'(c_1, s)}{v'(c_0)}
  \]
Equity Premium Puzzle

- Recall $E[R^j] - R^f = -R^f \text{Cov}[m, R^j]$
- Now: $E[R^j] - R^f = -R^f \text{Cov}[\partial_1 u, R^j] / E[\partial_0 u]$
- Recall Hansen-Jaganathan bound

\[
\frac{\sigma(m)}{E[m]} \geq \left| \frac{E[(R - R^f)]}{\sigma(R)} \right|; \quad E[m] = \frac{1}{R^f}
\]

\[
\sigma(m) \geq \frac{1}{R^f} \left| \frac{E[(R - R^f)]}{\sigma(R)} \right|
\]
Equity Premium Puzzle (ctd.)

\[ \sigma \left( \frac{\partial_1 u}{E[\partial_0 u]} \right) \geq \frac{1}{R^f} \left| \frac{E[(R - R^f)]}{\sigma(R)} \right| \]

Equity Premium Puzzle:
- high observed Sharpe ratio of stock market indices
- low volatility of consumption
  \( \Rightarrow \) (unrealistically) high level of risk aversion
A simple example

- $S=2$, $\pi_1 = \frac{1}{2}$, 
- 3 securities with $x^1 = (1,0)$, $x^2 = (0,1)$, $x^3 = (1,1)$
- Let $m = \left(\frac{1}{2}, 1\right)$, $\sigma = \frac{1}{4} = \sqrt{\frac{1}{2} \left(\frac{1}{2} - \frac{3}{4}\right)^2 + \frac{1}{2} \left(1 - \frac{3}{4}\right)^2}$
- Hence, $p^1 = \frac{1}{4}$, $p^2 = \frac{1}{2}$, $p^3 = \frac{3}{4}$ and
- $R^1 = (4,0)$, $R^2 = (0,2)$, $R^3 = \left(\frac{4}{3}, \frac{4}{3}\right)$
- $E[R^1] = 2$, $E[R^2] = 1$, $E[R^3] = \frac{4}{3}$
Example: Where does SDF come from?

• “representative agent” with
  – endowment: 1 in date 0, (2,1) in date 1
  – utility \( EU(c_0, c_1, c_2) = \sum_s \pi_s (\ln c_0 + \ln c_{1,s}) \)
  – i.e. \( u(c_0, c_{1,s}) = \ln c_0 + \ln c_{1,s} \) (additive) time separable u-function

• \( m = \frac{\partial_1 u (1,2,1)}{E[\partial_0 u(1,2,1)]} = (c_0/c_{1,1}, c_0/c_{1,2}) = (1/2, 1/1) \)
• \( m=(\frac{1}{2},1) \) since endowment=consumption
• Low consumption states are high “m-states”
• Risk-neutral probabilities combine true probabilities and marginal utilities.