Lecture 05: Mean-Variance Analysis & Capital Asset Pricing Model (CAPM)

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Overview

- **Simple CAPM with quadratic utility functions** (derived from state-price beta model)

- Mean-variance preferences
  - Portfolio Theory
  - CAPM (intuition)

- CAPM
  - Projections
  - Pricing Kernel and Expectation Kernel
Recall State-price Beta model

Recall:

$$E[R^h] - R^f = \beta^h E[R^*- R^f]$$

where $\beta^h := \frac{\text{Cov}[R^*, R^h]}{\text{Var}[R^*]}$

very general – but what is $R^*$ in reality?
Simple CAPM with Quadratic Expected Utility

1. All agents are identical
   - Expected utility \( U(x_0, x_1) = \sum_s \pi_s u(x_0, x_s) \) \( \Rightarrow \) \( m = \partial_1 u / E[\partial_0 u] \)
   - Quadratic \( u(x_0, x_1) = v_0(x_0) - (x_1 - \alpha)^2 \)
     \( \Rightarrow \partial_1 u = [-2(x_1, -\alpha), \ldots, -2(x_S, -\alpha)] \)
   - \( E[R^h] - R^f = -\text{Cov}[m, R^h] / E[m] \)
     \( = -R^f \text{Cov}[\partial_1 u, R^h] / E[\partial_0 u] \)
     \( = -R^f \text{Cov}[-2(x_1 - \alpha), R^h] / E[\partial_0 u] \)
     \( = R^f 2\text{Cov}[x_1, R^h] / E[\partial_0 u] \)
   - Also holds for market portfolio
     - \( E[R^m] - R^f = R^f 2\text{Cov}[x_1, R^m] / E[\partial_0 u] \)
     \[ \Rightarrow \frac{E[R^h] - R^f}{E[R^m] - R^f} = \frac{\text{Cov}[x_1, R^h]}{\text{Cov}[x_1, R^m]} \]
Simple CAPM with Quadratic Expected Utility

\[
\frac{E[R^h] - R^f}{E[R^m] - R^f} = \frac{Cov[x_1, R^h]}{Cov[x_1, R^m]}
\]

2. Homogenous agents + Exchange economy
   \[
   \Rightarrow x_1 = \text{agg. endowment and is perfectly correlated with } R^m
   \]
   \[
   \frac{E[R^h] - R^f}{E[R^m] - R^f} = \frac{Cov[R^m, R^h]}{Var[R^m]}
   \]
   since \( \beta^h = \frac{Cov[R^h, R^m]}{Var[R^m]} \)

\[
E[R^h] = R^f + \beta^h \{E[R^m] - R^f\} \quad \text{Market Security Line}
\]

N.B.: \( R^* = R^f (a + b_1 R^M)/(a + b_1 R^f) \) in this case (where \( b_1 < 0 \)! 

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Mean-Variance Analysis and CAPM 
Slide 05-5
Overview

• Simple CAPM with quadratic utility functions (derived from state-price beta model)
• Mean-variance analysis
  – Portfolio Theory (Portfolio frontier, efficient frontier, …)
  – CAPM (Intuition)
• CAPM
  – Projections
  – Pricing Kernel and Expectation Kernel
Definition: Mean-Variance Dominance & Efficient Frontier

- Asset (portfolio) A mean-variance dominates asset (portfolio) B if \( \mu_A \leq \mu_B \) and \( \sigma_A < \sigma_B \) or if \( \mu_A > \mu_B \) while \( \sigma_A \leq \sigma_B \).

- **Efficient frontier**: loci of all non-dominated portfolios in the mean-standard deviation space. By definition, no ("rational") mean-variance investor would choose to hold a portfolio not located on the efficient frontier.
Expected Portfolio Returns & Variance

- **Expected returns (linear)**
  \[ \mu_p := E[r_p] = w_j \mu_j, \text{ where each } \mu_j = \frac{h^j}{\sum_j h^j} \]

- **Variance**
  \[ \sigma_p^2 := Var[r_p] = w'Vw = (w_1 \ w_2) \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \]
  \[ = (w_1 \sigma_1^2 + w_2 \sigma_{21} \ w_1 \sigma_{12} + w_2 \sigma_2^2) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \]
  \[ = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_{12} \leq 0 \]
  since \( \sigma_{12} \leq -\sigma_1 \sigma_2 \).

recall that correlation coefficient \( \in [-1,1] \)
Illustration of 2 Asset Case

- For certain weights: $w_1$ and $(1-w_1)$
  
  $\mu_p = w_1 \ E[r_1] + (1-w_1) \ E[r_2]$  
  $\sigma_p^2 = w_1^2 \sigma_1^2 + (1-w_1)^2 \sigma_2^2 + 2 \ w_1(1-w_1)\sigma_1 \sigma_2 \rho_{1,2}$  
  (Specify $\sigma_p^2$ and one gets weights and $\mu_p$’s)

- Special cases [$w_1$ to obtain certain $\sigma_R$]
  
  - $\rho_{1,2} = 1 \ \Rightarrow \ w_1 = (+/-\sigma_p - \sigma_2) / (\sigma_1 - \sigma_2)$
  - $\rho_{1,2} = -1 \ \Rightarrow \ w_1 = (+/-\sigma_p + \sigma_2) / (\sigma_1 + \sigma_2)$
For $\rho_{1,2} = 1$:

$$\sigma_p = |w_1 \sigma_1 + (1 - w_1) \sigma_2|$$

$$\mu_p = w_1 \mu_1 + (1 - w_1) \mu_2$$

Hence, $w_1 = \frac{\pm \sigma_p - \sigma_2}{\sigma_1 - \sigma_2}$

$$\mu_p = \mu_1 + \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} (\pm \sigma_p - \sigma_1)$$

The Efficient Frontier: Two Perfectly Correlated Risky Assets
For $\rho_{1,2} = -1$: 

\[ \sigma_p = |w_1 \sigma_1 - (1 - w_1) \sigma_2| \]
\[ \mu_p = w_1 \mu_1 + (1 - w_1) \mu_2 \]

Hence, 

\[ w_1 = \frac{\pm \sigma_p + \sigma_2}{\sigma_1 + \sigma_2} \]

Efficient Frontier: Two Perfectly Negative Correlated Risky Assets
For $-1 < \rho_{1,2} < 1$:

Efficient Frontier: Two Imperfectly Correlated Risky Assets
For $\sigma_1 = 0$

The Efficient Frontier: One Risky and One Risk Free Asset
Efficient Frontier with $n$ risky assets and one risk-free asset

The Efficient Frontier: One Risk Free and $n$ Risky Assets
Mean-Variance Preferences

- \( U(\mu_p, \sigma_p) \) with \( \frac{\partial U}{\partial \mu_p} > 0, \frac{\partial U}{\partial \sigma_p^2} < 0 \)
  - quadratic utility function (with portfolio return \( R \))
    \[
    U(R) = a + b R + c R^2
    \]
  - vNM: \( E[U(R)] = a + b E[R] + c E[R^2] \)
    \[
    = a + b \mu_p + c \mu_p^2 + c \sigma_p^2
    = g(\mu_p, \sigma_p)
    \]

- asset returns normally distributed \( \Rightarrow R=\sum_j w^j r^j \) normal
  - if \( U(.) \) is CARA \( \Rightarrow \) certainty equivalent = \( \mu_p - \rho_A/2\sigma_p^2 \)
    (Use moment generating function)
Optimal Portfolio: Two Fund Separation

Optimal Portfolios of Two Investors with Different Risk Aversion

Price of Risk = highest Sharpe ratio
Equilibrium leads to CAPM

• Portfolio theory: only analysis of demand
  – price/returns are taken as given
  – composition of risky portfolio is same for all investors

• Equilibrium Demand = Supply (market portfolio)

• CAPM allows to derive
  – equilibrium prices/returns.
  – risk-premium
The CAPM with a risk-free bond

• The market portfolio is efficient since it is on the efficient frontier.
• All individual optimal portfolios are located on the half-line originating at point (0, r_f).
• The slope of Capital Market Line (CML): \[
\frac{E[R_M] - R_f}{\sigma_M}
\]

\[
E[R_p] = R_f + \frac{E[R_M] - R_f}{\sigma_M} \sigma_p
\]
The Capital Market Line

$\sigma_p$

$\sigma_M$

$r_f$

$r_M$

$M$

$CML$

The Capital Market Line (CML) is a graphical representation that illustrates the relationship between risk and expected return. It is a straight line that represents the set of efficient portfolios that offer the highest expected return for a given level of risk. The CML is tangent to the efficient frontier, which represents the set of all possible portfolios. The point where the CML intersects the horizontal axis represents the risk-free rate ($r_f$). The CML is an upward-sloping line, indicating that as risk increases, so does the expected return. The point $M$ represents the market portfolio, which is a combination of all assets in the market, weighted by their market values.
**Proof of the CAPM relationship**  
[old traditional derivation]

- Refer to previous figure. Consider a portfolio with a fraction $1 - \alpha$ of wealth invested in an arbitrary security $j$ and a fraction $\alpha$ in the market portfolio

$$\mu_p = \alpha \mu_M + (1 - \alpha) \mu_j$$

$$\sigma_p^2 = \alpha^2 \sigma_M^2 + (1 - \alpha)^2 \sigma_j^2 + 2\alpha(1 - \alpha)\sigma_{jM}$$

As $\alpha$ varies we trace a locus which

- passes through $M$ (- and through $j$)
- cannot cross the CML (why?)
- hence must be tangent to the CML at $M$

Tangency = \[ \frac{d\mu_p}{d\sigma_p} \bigg|_{\alpha=1} = \text{slope of the locus at } M = \frac{\mu_M - r_f}{\sigma_M} \]

= \text{slope of CML} = \frac{\mu_M - r_f}{\sigma_M}
\[ \mu_p = \alpha \mu_M + (1 - \alpha) \mu_j \]
\[ \sigma_p^2 = \alpha^2 \sigma_M^2 + (1 - \alpha)^2 \sigma_j^2 + 2\alpha(1 - \alpha) \sigma_j \sigma_M \]
\[ \frac{d\mu_p}{d\sigma_p} = \frac{d\mu_p/d\alpha}{d\sigma_p/d\alpha} \]
\[ \frac{d\mu_p}{d\alpha} = \mu_M - \mu_j \]
\[ 2\sigma_p \frac{\sigma_p}{d\alpha} = 2\alpha \sigma_M^2 - 2(1 - \alpha) \sigma_j^2 + 2(1 - 2\alpha) \sigma_j \sigma_M \]

\[ \text{slope of } \frac{d\mu_p}{d\sigma_p} \bigg|_{\alpha=1} = \frac{(\mu_M - \mu_j) \sigma_M}{\sigma_M^2 - \sigma_j \sigma_M} = \frac{\mu_M - r_f}{\sigma_M} \]

\[ \mu_M - \mu_j = \frac{(\mu_M - r_f)(\sigma_M^2 - \sigma_j \sigma_M)}{\sigma_M^2} \]

\[ E[r_j] = \mu_j = r_f + \frac{\sigma_j \sigma_M}{\sigma_M^2} (\mu_M - r_f) \]

Do you see the connection to earlier state-price beta model?  \( R^* = \sigma_M \)
The Security Market Line

\[ \text{slope SML} = \frac{E(r_i) - r_f}{\beta_i} \]

\[ E(r) \]

\[ E(r_i) \]

\[ E(r_M) \]

\[ r_f \]

\[ \beta_M = 1 \]

\[ \beta_i \]
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• CAPM (modern derivation)
  – Projections
  – Pricing Kernel and Expectation Kernel
Projections

• States $s=1,\ldots,S$ with $\pi_s > 0$

• Probability inner product

$$[x, y]_\pi = (xy)_\pi = \sum_s \pi_s x_s y_s = \sum_s (\sqrt{\pi_s} x_s \sqrt{\pi_s} y_s)$$

• $\pi$-norm $||x|| = \sqrt{[x, x]_\pi}$ (measure of length)

  (i) $||x|| > 0 \ \forall x \neq 0$ and $||x|| = 0$ if $x = 0$

  (ii) $||\lambda x|| = |\lambda||x||$

  (iii) $||x + y|| \leq ||x|| + ||y|| \ \forall x, y \in IR^S$
x and y are $\pi$-orthogonal iff $[x, y]_\pi = 0$, I.e. $E[xy]=0$
...Projections...

- \( Z \) space of all linear combinations of vectors \( z_1, \ldots, z_n \)
- Given a vector \( y \in \mathbb{R}^s \) solve
  \[
  \min_{\alpha \in \mathbb{R}^n} E[ y - \sum_{j=1}^n \alpha_j z_j ]^2
  \]
  FOC: (for each \( j = 1, \ldots, n \))
  \[
  \sum_s \pi_s (y_s - \sum_j \alpha_j z^j_s) z^j = 0
  \]
  \( \Rightarrow \hat{\alpha} \) the solution
  \[
  y^Z = \sum_j \hat{\alpha}_j z^j, \quad \epsilon := y - y^Z
  \]
- [smallest distance between vector \( y \) and \( Z \) space]
E[ε zj] = 0 for each j = 1, ..., n (from FOC)
ε ⊥ z
y^Z is the (orthogonal) projection on Z
y = y^Z + ε', y^Z ∈ Z, ε ⊥ z
Expected Value and Co-Variance...

\[ x = \hat{x} + \tilde{x} \]

\[ [x,y] = E[xy] = \text{Cov}[x,y] + E[x]E[y] \]

\[ [x,x] = E[x^2] = \text{Var}[x] + E[x]^2 \]

\[ ||x|| = E[x^2]^{\frac{1}{2}} \]

Squeeze axis by \( \sqrt{\pi_s} \)
...Expected Value and Co-Variance

\[ x = \hat{x} + \tilde{x}, \text{ where} \]
\[ \hat{x} \text{ is projection of } x \text{ onto } <1> \]
\[ \tilde{x} \text{ is projection of } x \text{ onto } <1> \perp \]

\[ E[x] = [x, 1]_\pi = [\hat{x}, 1]_\pi = ||\hat{x}|| \]
\[ Var[x] = [\tilde{x}, \tilde{x}]_\pi = E[\tilde{x}^2] = Var[\tilde{x}] \]
\[ \sigma_x = ||\tilde{x}||_\pi = \text{standard deviation of } x \]
\[ Cov[x, y] = Cov[\tilde{x}, \tilde{y}] = [\tilde{y}, \tilde{x}] \]

Proof: \[ [x, y]_\pi = [\hat{x}, \tilde{y}]_\pi + [\tilde{x}, \tilde{y}]_\pi, \text{ since} \]
\[ [\tilde{y}, \tilde{x}]_\pi = [\tilde{y}, \tilde{x}]_\pi = 0, [x, y]_\pi = E[\tilde{y}]E[\tilde{x}] + Cov[\tilde{x}, \tilde{y}] \]
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• **CAPM** (modern derivation)
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New (LeRoy & Werner) Notation

- Main changes (new versus old)
  - gross return: \( r = R \)
  - SDF: \( \mu = m \)
  - pricing kernel: \( k_q = m^* \)
  - Asset span: \( \mathcal{M} = \langle X \rangle \)
  - income/endowment: \( w_t = e_t \)
Pricing Kernel $k_q$...

- $M$ space of feasible payoffs.
- If no arbitrage and $\pi >> 0$ there exists SDF $\mu \in \mathbb{R}^S$, $\mu >> 0$, such that $q(z) = \mathbb{E}(\mu z)$.
- $\mu \in M$ – SDF need not be in asset span.
- A pricing kernel is a $k_q \in M$ such that for each $z \in M$, $q(z) = \mathbb{E}(k_q z)$.
- $(k_q = m^*$ in our old notation.)
...Pricing Kernel - Examples...

- Example 1:
  - $S=3, \pi^s=1/3$ for $s=1,2,3$,
  - $x_1=(1,0,0), x_2=(0,1,1), p=(1/3,2/3)$.
  - Then $k=(1,1,1)$ is the unique pricing kernel.

- Example 2:
  - $S=3, \pi^s=1/3$ for $s=1,2,3$,
  - $x_1=(1,0,0), x_2=(0,1,0), p=(1/3,2/3)$.
  - Then $k=(1,2,0)$ is the unique pricing kernel.
Pricing Kernel – Uniqueness

- If a state price density exists, there exists a unique pricing kernel.
  - If \( \text{dim}(\mathcal{M}) = m \) (markets are complete), there are exactly \( m \) equations and \( m \) unknowns.
  - If \( \text{dim}(\mathcal{M}) \leq m \), (markets may be incomplete)
    For any state price density (=SDF) \( \mu \) and any \( z \in \mathcal{M} \)
    \[ E[\mu - k_q]z = 0 \]
    \[ \mu = (\mu - k_q) + k_q \Rightarrow k_q \text{ is the } \text{“projection” } \text{of } \mu \text{ on } \mathcal{M}. \]
- Complete markets \( \Rightarrow \), \( k_q = \mu \) (SDF=state price density)
Expectations Kernel $k_e$

- An expectations kernel is a vector $k_e \in \mathcal{M}$
  - Such that $E(z) = E(k_e z)$ for each $z \in \mathcal{M}$.

- Example
  - $S=3$, $\pi^s=1/3$, for $s=1,2,3$, $x_1=(1,0,0)$, $x_2=(0,1,0)$.
  - Then the unique $k_e=(1,1,0)$.

- If $\pi >> 0$, there exists a unique expectations kernel.
- Let $e=(1,\ldots, 1)$ then for any $z \in \mathcal{M}$
- $E[(e-k_e)z]=0$
- $k_e$ is the “projection” of $e$ on $\mathcal{M}$
- $k_e = e$ if bond can be replicated (e.g. if markets are complete)
Mean Variance Frontier

- **Definition 1:** \( z \in \mathcal{M} \) is in the mean variance frontier if there exists no \( z' \in \mathcal{M} \) such that \( E[z'] = E[z] \), \( q(z') = q(z) \) and \( \text{var}[z'] < \text{var}[z] \).

- **Definition 2:** Let \( \mathcal{E} \) the space generated by \( k_q \) and \( k_e \).
  - Decompose \( z = z^\mathcal{E} + \varepsilon \), with \( z^\mathcal{E} \in \mathcal{E} \) and \( \varepsilon \perp \mathcal{E} \).
  - Hence, \( E[\varepsilon] = E[\varepsilon k_e] = 0 \), \( q(\varepsilon) = E[\varepsilon k_q] = 0 \)
  - \( \text{Cov}[\varepsilon, z^\mathcal{E}] = E[\varepsilon z^\mathcal{E}] = 0 \), since \( \varepsilon \perp \mathcal{E} \).
  - \( \text{var}[z] = \text{var}[z^\mathcal{E}] + \text{var}[\varepsilon] \) (price of \( \varepsilon \) is zero, but positive variance)
  - If \( z \) in mean variance frontier \( \Rightarrow z \in \mathcal{E} \).
  - Every \( z \in \mathcal{E} \) is in mean variance frontier.
Frontier Returns...

- Frontier returns are the returns of frontier payoffs with non-zero prices.
  \[ r_e = \frac{k_e}{q(k_e)} = \frac{k_e}{E(k_q)} \]
  \[ r_q = \frac{k_q}{q(k_q)} = \frac{k_q}{E(k_qk_q)} \]

- If \( z = \alpha k_q + \beta k_e \) then,
  \[ r_z = \frac{\alpha q(k_q)}{\alpha q(k_q) + \beta q(k_e)} r_q + \frac{\beta q(k_e)}{\alpha q(k_q) + \beta q(k_e)} r_e \]

- graphically: payoffs with price of p=1.
$\mathcal{M} = R^S = R^3$

Mean-Variance Return Frontier

$p=1$-line = return-line (orthogonal to $k_q$)

Mean-Variance Payoff Frontier
Mean-Variance (Payoff) Frontier

NB: graphical illustrated of expected returns and standard deviation changes if bond is not in payoff span.
Mean-Variance (Payoff) Frontier

- Efficient (return) frontier
- Inefficient (return) frontier
- Expected return
- Standard deviation

Point (1,1,1)
...Frontier Returns

If $k_e = \alpha k_q$, frontier returns $\equiv r_e$. (if agent is risk-neutral)

If $k_e \neq \alpha k_q$, frontier can be written as:

$$r_\lambda = r_e + \lambda(r_q - r_e)$$

Expectations and Variance are

$$E[r_\lambda] = E[r_e] + \lambda(E[r_q] - E[r_e])$$

$$\text{var}(r_\lambda) = \text{var}(r_e) + 2\lambda\text{cov}(r_e, r_q - r_e) + \lambda^2\text{var}(r_q - r_e)$$

(1)

If risk-free asset exists, they simplify to:

$$E[r_\lambda] = \bar{r} + \lambda(E[r_q] - \bar{r}).$$

$$\text{var}(r_\lambda) = \lambda^2\text{var}(r_q). \quad \sigma(r_\lambda) = |\lambda|\sigma(r_q).$$

$$E(r_\lambda) = \bar{r} \pm \sigma(r_\lambda) \frac{E(r_q) - \bar{r}}{\sigma(r_q)}$$
Minimum Variance Portfolio

• Take FOC w.r.t. $\lambda$ of

\[
\text{var}(r_\lambda) = \text{var}(r_e) + 2\lambda \text{cov}(r_e, r_q - r_e) + \lambda^2 \text{var}(r_q - r_e)
\]

(1)

• Hence, MVP has return of

\[r_e + \lambda_0 (r_q - r_e), \text{ with}\]

\[
\lambda_0 = -\frac{\text{cov}(r_e, r_q - r_e)}{\text{var}(r_q - r_e)}.
\]
Mean-Variance Efficient Returns

- **Definition**: A return is **mean-variance efficient** if there is no other return with same variance but greater expectation.

- Mean variance efficient returns are frontier returns with \( E[r_{\lambda}] \geq E[r_{\lambda 0}] \).

- If risk-free asset can be replicated
  - Mean variance efficient returns correspond to \( \lambda \leq 0 \).
  - Pricing kernel (portfolio) is not mean-variance efficient, since
    \[
    E[r_q] = \frac{E[k_q]}{E[k_q^2]} < \frac{1}{E[k_q]} = \bar{r}.
    \]
    Hint: \( E[k_q^2] > E[k_q]^2 \) since \( \text{Var}[k_q] > 0 \).
Zero-Covariance Frontier Returns

• Take two frontier portfolios with returns
  \( r_\lambda = r_e + \lambda(r_q - r_e) \) and \( r_\mu = r_e + \mu(r_q - r_e) \)

• \( \text{cov}(r_\mu, r_\lambda) = \text{var}(r_e) + (\lambda + \mu)\text{cov}(r_e, r_q - r_e) + \lambda\mu\text{var}(r_q - r_e) \)

• The portfolios have zero co-variance if
  \[ \mu = -\frac{\text{var}(r_e) + \lambda\text{cov}(r_e, r_q - r_e)}{\text{cov}(r_e, r_q - r_e) + \lambda\text{var}(r_q - r_e)} \]

• For all \( \lambda \neq \lambda_0 \) \( \mu \) exists

• \( \mu=0 \) if risk-free bond can be replicated
Illustration of MVP

\[ \mathcal{M} = R^2 \text{ and } S=3 \]

Expected return of MVP

Minimum standard deviation

(1,1,1)
Illustration of ZC Portfolio...

\[ \mathcal{M} = R^2 \text{ and } S=3 \]

Recall:

\[ \text{COV}[x, y] = [\tilde{x}, \tilde{y}] \pi \]
Illustration of ZC Portfolio

ZC\tilde{\rho}(p)

arbitrary portfolio p

(1,1,1)

\|\tilde{\rho}\|

ZC of p
Beta Pricing…

• Frontier Returns (are on linear subspace). Hence

\[ r_\beta = r_\mu + \beta (r_\lambda - r_\mu). \]

• Consider any asset with payoff \( x_j \)
  
  – It can be decomposed in \( x_j = x_j^\mathcal{E} + \varepsilon_j \)
  
  – \( q(x_j) = q(x_j^\mathcal{E}) \) and \( E[x_j] = E[x_j^\mathcal{E}] \), since \( \varepsilon \perp \mathcal{E} \).
  
  – Let \( r_i^\mathcal{E} \) be the return of \( x_j^\mathcal{E} \)

\[
 r_j = r_j^\mathcal{E} + \frac{\varepsilon_j}{q(x_j)}. 
\]

– Using above and assuming \( \lambda \neq \lambda_0 \) and \( \mu \) is ZC-portfolio of \( \lambda \),

\[
 r_j = r_\mu + \beta_j (r_\lambda - r_\mu) + \frac{\varepsilon_j}{q(x_j)}. 
\]
...Beta Pricing

- Taking expectations and deriving covariance

\[ E[r_j] = E[r_{\mu}] + \beta_j (E[r_{\lambda}] - E[r_{\mu}]) \]

\[ \text{cov}(r_{\lambda}, r_j) = \beta_j \text{var}(r_{\lambda}) \Rightarrow \beta_j = \frac{\text{cov}(r_{\lambda}, r_j)}{\text{var}(r_{\lambda})} \]

- If risk-free asset can be replicated, beta-pricing equation simplifies to

\[ E[r_j] = \bar{r} + \beta_j (E[r_{\lambda}] - \bar{r}) \]

- Problem: How to identify frontier returns
Capital Asset Pricing Model...

- CAPM = market return is frontier return
  - Derive conditions under which market return is frontier return
  - Two periods: 0, 1,
  - Endowment: individual \( w_i \) at time 1, aggregate \( \bar{w}_1 = \bar{w}_1^M + \bar{w}_1^N \), where \( \bar{w}_1^M \) the orthogonal projection of \( \bar{w}_1 \) on \( M \) is.
  - The market payoff: \( m \equiv \bar{w}_1^M \)
  - Assume \( q(m) \neq 0 \), let \( r_m = m / q(m) \), and assume that \( r_m \) is not the minimum variance return.
Capital Asset Pricing Model

• If \( r_{m0} \) is the frontier return that has zero covariance with \( r_m \) then, for every security \( j \),

\[
E[r_j] = E[r_{m0}] + \beta_j (E[r_m] - E[r_{m0}]), \quad \text{with} \quad \beta_j = \frac{\text{cov}[r_j, r_m]}{\text{var}[r_m]}.
\]

• If a risk free asset exists, equation becomes,

\[
E[r_j] = r_f + \beta_j (E[r_m] - r_f)
\]

• N.B. first equation always hold if there are only two assets.
Outdated material follows

• Traditional derivation of CAPM is less elegant
• Not relevant for exams
Deriving the Frontier
n risky assets

• **Definition 6.1**: A frontier portfolio is one which displays minimum variance among all feasible portfolios with the same $E(\tilde{r}_p)$.

$$\min_w \frac{1}{2} w^T V w$$

$$(\lambda) \quad \text{s.t.} \quad w^T e = E \quad \left( \sum_{i=1}^{N} w_i E(\tilde{r}_i) = E \right)$$

$$(\gamma) \quad w^T 1 = 1 \quad \left( \sum_{i=1}^{N} w_i = 1 \right)$$
The first FOC can be written as:

\[
\frac{\partial L}{\partial w} = Vw - \lambda e - \gamma 1 = 0
\]

\[
\frac{\partial L}{\partial \lambda} = E - w^T e = 0
\]

\[
\frac{\partial L}{\partial \gamma} = 1 - w^T 1 = 0
\]

The first FOC can be written as:

\[
Vw_p = \lambda e + \gamma 1 \quad \text{or}
\]

\[
w_p = \lambda V^{-1}e + \gamma V^{-1}1
\]

\[
e^T w_p = \lambda (e^T V^{-1}e) + \gamma (e^T V^{-1}1)
\]
Noting that $e^T w_p = w_p^T e$, using the first foc, the second foc can be written as

$$E[\tilde{r}_p] = e^T w_p = \lambda \left( e^T V^{-1} e \right) + \gamma \left( e^T V^{-1} 1 \right)$$

pre-multiplying first foc with $1$ (instead of $e^T$) yields

$$1^T w_p = w_p^T 1 = \lambda \left( 1^T V^{-1} e \right) + \gamma \left( 1^T V^{-1} 1 \right) = 1$$

Solving both equations for $\lambda$ and $\gamma$

$$\lambda = \frac{CE - A}{D} \quad \text{and} \quad \gamma = \frac{B - AE}{D}$$

where $D = BC - A^2$. 
Hence, \( w_p = \lambda V^{-1}e + \gamma V^{-1}1 \) becomes

\[
wp = \frac{CE - A}{D} V^{-1}e + \frac{B - AE}{D} V^{-1}1
\]

\( \lambda \) (scalar) \quad \gamma \) (scalar)

\[
= \frac{1}{D} [B(V^{-1}1) - A(V^{-1}e)] + \frac{1}{D} [C(V^{-1}e) - A(V^{-1}1)]E
\]

\[
w_p = g + h \quad E \quad \text{linear in expected return } E!
\]

If \( E = 0 \), \( w_p = g \)
If \( E = 1 \), \( w_p = g + h \)

Hence, \( g \) and \( g+h \) are portfolios on the frontier.
Characterization of Frontier Portfolios

- **Proposition 6.1**: The entire set of frontier portfolios can be generated by ("are convex combinations" of) $g$ and $g+h$.

- **Proposition 6.2**: The portfolio frontier can be described as convex combinations of any two frontier portfolios, not just the frontier portfolios $g$ and $g+h$.

- **Proposition 6.3**: Any convex combination of frontier portfolios is also a frontier portfolio.
Characterization of Frontier Portfolios...

- For any portfolio on the frontier, 
  \[
  \sigma^2(E[\tilde{r}_p]) = \left[g + hE(\tilde{r}_p)\right]^T V \left[g + hE(\tilde{r}_p)\right]
  \]
  with \(g\) and \(h\) as defined earlier.

Multiplying all this out yields:

\[
\sigma^2(E[\tilde{r}_p]) = \frac{C}{D} \left[E[\tilde{r}_p] - \frac{A}{C}\right]^2 + \frac{1}{C}
\]
Characterization of Frontier Portfolios...

- (i) the expected return of the minimum variance portfolio is $A/C$;

- (ii) the variance of the minimum variance portfolio is given by $1/C$;

- (iii) equation (6.17) is the equation of a parabola with vertex $(1/C, A/C)$ in the expected return/variance space and of a hyperbola in the expected return/standard deviation space. See Figures 6.3 and 6.4.
\[ E[\tilde{r}_p] = \frac{A}{C} \pm \sqrt{\frac{D}{C}}(\sigma^2 - \frac{1}{C}) \]

Figure 6-3    The Set of Frontier Portfolios: Mean/Variance Space
Figure 6-4  The Set of Frontier Portfolios: Mean/SD Space
Figure 6-5  The Set of Frontier Portfolios: Short Selling Allowed
Characterization of Efficient Portfolios
(No Risk-Free Assets)

• **Definition 6.2**: Efficient portfolios are those frontier portfolios which are not mean-variance dominated.

• **Lemma**: Efficient portfolios are those frontier portfolios for which the expected return exceeds A/C, the expected return of the minimum variance portfolio.
Zero Covariance Portfolio

• Zero-Cov Portfolio is useful for Zero-Beta CAPM

• Proposition 6.5: For any frontier portfolio \( p \), except the minimum variance portfolio, there exists a unique frontier portfolio with which \( p \) has zero covariance. We will call this portfolio the "zero covariance portfolio relative to \( p \)" and denote its vector of portfolio weights by \( ZC(p) \).

• Proof: by construction.
$Cov[r_p, r_q] := w_p^T V w_q$

$Cov[r_p, r_q] = [\lambda V^{-1} e + \gamma V^{-1} 1]^T V w_q$

$Cov[r_p, r_q] = \lambda e^T V^{-1} V w_q + \gamma 1^T V^{-1} V w_q$

$Cov[r_p, r_q] = \lambda e^T w_q + \gamma$

$Cov[r_p, r_q] = \lambda E[r_q] + \gamma$

where $\lambda = (CE[r_p] - A)/D$ and $\gamma = (B - AE[r_p])/D$

Hence,

$Cov[r_p, r_q] = \frac{CE[r_p] - A}{D} E[r_q] + \frac{B - AE[r_p]}{D}$

collect all expected returns terms, add and subtract $A^2 C/DC^2$

and note that the remaining term $(1/C)[(BC/D)-(A^2/D)]=1/C$, since $D=BC-A^2$

$Cov[r_p, r_q] = \frac{C}{D} [E[r_p] - \frac{A}{C}] [E[r_q] - \frac{A}{C}] + \frac{1}{C}$
\[\text{Cov}[r_p, r_q] = \frac{C}{D}[E[r_p] - \frac{A}{C}][E[r_q] - \frac{A}{C}] + \frac{1}{C}\]

For zero co-variance portfolio ZC(p)

\[\text{Cov}[r_p, r_{ZC(p)}] = 0\]

\[0 = \frac{C}{D}[E[r_p] - \frac{A}{C}][E[r_{ZC(p)}] - \frac{A}{C}] + \frac{1}{C}\]

\[E[r_{ZC(p)}] = \frac{A}{C} - \frac{D/C^2}{E[r_p] - A/C}\]

For graphical illustration, let’s draw this line:

\[E[r] = \frac{A}{C} - \frac{D/C^2}{E[r_p] - A/C} + \frac{E[r_p] - A/C}{\sigma^2[r_p] - 1/C}\sigma^2[r]\]
Graphical Representation:

\[ E[r] = \frac{A}{C} - \frac{D/C^2}{E[r_p] - A/C} + \frac{E[r_p] - A/C}{\sigma^2[r_p] - 1/C} \sigma^2[r] \]

Line through
\[ (\text{Var}[r_p], E[r_p]) \quad \text{AND} \quad (1/C, A/C) \]

for \( \sigma^2(r) = 0 \) you get \( E[r_{ZC(p)}] \)

(line through
\[ (\text{Var}[r_p], E[r_p]) \quad \text{AND} \quad (1/C, A/C) \]

(use \( \sigma^2(\tilde{r}_p) = \frac{C}{D} \left( \frac{E(\tilde{r}_p) - A}{C} \right)^2 + \frac{1}{C} \))
Figure 6-6    The Set of Frontier Portfolios: Location of the Zero-Covariance Portfolio

16:14 Lecture 05      Mean-Variance Analysis and CAPM
Zero-Beta CAPM
(no risk-free asset)

(i) agents maximize expected utility with increasing and strictly concave utility of money functions and asset returns are multivariate normally distributed, or

(ii) each agent chooses a portfolio with the objective of maximizing a derived utility function of the form

\[ U(e, \sigma^2), U_1 > 0, U_2 < 0, U \text{ concave}. \]

(iii) common time horizon,

(iv) homogeneous beliefs about \( e \) and \( \sigma^2 \)
– All investors hold mean-variance efficient portfolios

– the market portfolio is convex combination of efficient portfolios

⇒ is efficient.

– Cov\[r_p,r_q]\] = \lambda \cdot E[r_q] + \gamma \quad (q \text{ need not be on the frontier}) (6.22)

– Cov\[r_p,r_{ZC(p)}\] = \lambda \cdot E[r_{ZC(p)}] + \gamma = 0

\rightarrow

– Cov\[r_p,r_q]\] = \lambda \cdot \{E[r_q] - E[r_{ZC(p)}]\}

– Var\[r_p\] = \lambda \cdot \{E[r_p] - E[r_{ZC(p)}]\}

Divide third by fourth equation:

\[ E(\tilde{r}_q) = E(\tilde{r}_{ZC(M)}) + \beta_{Mq} \left[ E(\tilde{r}_M) - E(\tilde{r}_{ZC(M)}) \right] \] (6.28)

\[ E(\tilde{r}_j) = E(\tilde{r}_{ZC(M)}) + \beta_{Mj} \left[ E(\tilde{r}_M) - E(\tilde{r}_{ZC(M)}) \right] \] (6.29)
Zero-Beta CAPM

- mean variance framework (quadratic utility or normal returns)
- In equilibrium, market portfolio, which is a convex combination of individual portfolios

\[
E[r_q] = E[r_{ZC(M)}] + \beta_{Mq}[E[r_M] - E[r_{ZC(M)}]]
\]

\[
E[r_j] = E[r_{ZC(M)}] + \beta_{Mj}[E[r_M] - E[r_{ZC(M)}]]
\]
The Standard CAPM
(with risk-free asset)

\[
\min_w \frac{1}{2} w^T V w \\
\text{s.t. } w^T e + (1 - w^T 1) r_f = E[r_p]
\]

FOC: \[ w_p = \lambda V^{-1} (e - r_f 1) \]

Multiplying by \((e - r_f 1)^T\) and solving for \(\lambda\) yields

\[
\lambda = \frac{E[r_p] - r_f}{(e - r_f 1)^T V^{-1} (e - r_f 1)}
\]

\[
w_p = V^{-1} (e - r_f 1) \frac{E(\tilde{r}_p) - r_f}{H}
\]

where \(H = B - 2A r_f + C r_f^2\)
\[ Cov[r_q, r_p] = (E[\tilde{r}_q - r_f]) \]

\[
\frac{1}{H}(e - r_f \mathbf{1})^T (V^{-1})^T (e - r_f \mathbf{1}) \frac{1}{H}(E[\tilde{r}_p - r_f])
\]

\[ \equiv G \]

\[ Var[r_p] = (E[\tilde{r}_p - r_f])^2 G \]

Rewrite first equation and replace \( G \) using second equation.

\[ E[r_q] - r_f = \frac{Cov[r_q, r_p]}{Var[r_p]} \frac{1}{E[r_p] - r_f} \]

\[ = \frac{Cov[r_q, r_p]}{Var[r_p]} (E[r_p] - r_f) \]

\[ \equiv \beta_{q,p} \]

Holds for any frontier portfolio, in particular the market portfolio.

NB: Derivation in DD is not correct.