Lecture 05: Mean-Variance Analysis & Capital Asset Pricing Model (CAPM)

Prof. Markus K. Brunnermeier

Overview

- Simple CAPM with quadratic utility functions (derived from state-price beta model)
- Mean-variance preferences
  - Portfolio Theory
  - CAPM (intuition)
- CAPM
  - Projections
  - Pricing Kernel and Expectation Kernel

Recall State-price Beta model

Recall:

$$E[R^h] - R^f = \beta^h E[R^* - R^f]$$

where $\beta^h := Cov[R^*, R^h] / Var[R^*]$

very general – but what is R^* in reality?

Simple CAPM with Quadratic Expected Utility

1. All agents are identical
   - Expected utility $U(x_0, x_1) = \sum_s \pi_s\, u(x_0, x_s) \Rightarrow m = \partial_1 u \,/\, E[\partial_0 u]$
   - Quadratic $u(x_0,x_1)=v_0(x_0) - (x_1 - \alpha)^2$

     $\Rightarrow \partial_1 u = [-2(x_{1,1} - \alpha),\ldots, -2(x_{S,1} - \alpha)]$
   - $E[R^h] - R^f = - Cov[m,R^h] \,/\, E[m]$

     $= -R^f\, Cov[\partial_1 u, R^h] \,/\, E[\partial_0 u]$

     $= -R^f\, Cov[-2(x_1 - \alpha), R^h] \,/\, E[\partial_0 u]$

     $= R^f\, 2Cov[x_1,R^h] \,/\, E[\partial_0 u]$
   - Also holds for market portfolio
   - $E[R^m] - R^f = R^f\, 2Cov[x_1,R^m]/E[\partial_0 u]$

$$\Rightarrow \frac{E[R^h]-R^f}{E[R^m]-R^f} = \frac{Cov[x_1,R^h]}{Cov[x_1,R^m]}$$

Simple CAPM with Quadratic Expected Utility

$$\frac{E[R^h]-R^f}{E[R^m]-R^f} = \frac{Cov[x_1,R^h]}{Cov[x_1,R^m]}$$

2. Homogenous agents + Exchange economy

$\Rightarrow x_1$ = agg. endowment and is perfectly correlated with R^m

$$\frac{E[R^h]-R^f}{E[R^m]-R^f} = \frac{Cov[R^m,R^h]}{Var[R^m]}$$

$$\text{since } \beta^h = \frac{Cov[R^h,R^m]}{Var[R^m]}$$

$E[R^h]=R^f + \beta^h \{E[R^m]-R^f\}$ Market Security Line

N.B.: $R^*=R^f (a+b_1R^M)/(a+b_1R^f)$ in this case (where $b_1<0$)!

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- Mean-variance analysis
  - Portfolio Theory (Portfolio frontier, efficient frontier, …)
  - CAPM (Intuition)
- CAPM
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Definition: Mean-Variance Dominance & Efficient Frontier

- Asset (portfolio) A **mean-variance dominates** asset (portfolio) B
if $\mu_A \leq \mu_B$ and $\sigma_A < \sigma_B$ or if $\mu_A > \mu_B$ while $\sigma_A \leq \sigma_B$.
- **Efficient frontier**: loci of all non-dominated portfolios in the mean-standard deviation space. By definition, no ("rational") mean-variance investor would choose to hold a portfolio not located on the efficient frontier.

Expected Portfolio Returns & Variance

- Expected returns (linear)

$$\mu_p := E[r_p] = w_j \mu_j, \text{ where each } \mu_j = \frac{h^j}{\sum_j h^j}$$

- Variance

$$\begin{aligned} \sigma_p^2 := Var[r_p] &= w'Vw = (w_1\, w_2) \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= (w_1\sigma_1^2 + w_2\sigma_{21} \quad w_1\sigma_{12} + w_2\sigma_2^2) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} \leq 0 \\ & since\ \sigma_{12} \leq -\sigma_1\sigma_2. \end{aligned}$$

recall that correlation coefficient $\in$ [-1,1]

Illustration of 2 Asset Case

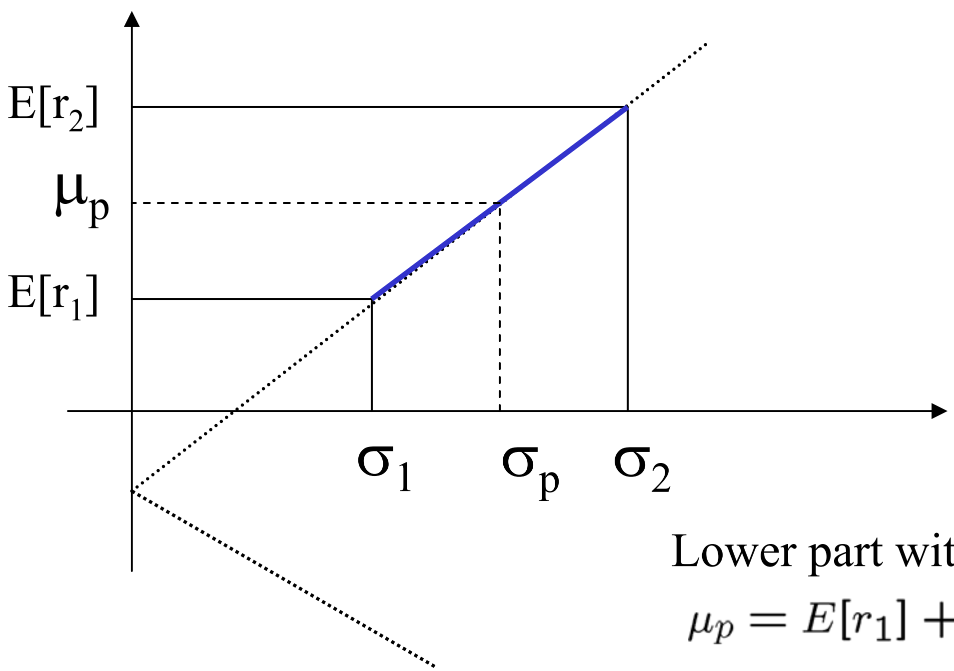
- For certain weights: w_1 and $(1-w_1)$
  $\mu_p = w_1 E[r_1] + (1-w_1) E[r_2]$
  $\sigma^2_p = w_1^2 \sigma_1^2 + (1-w_1)^2 \sigma_2^2 + 2 w_1(1-w_1)\sigma_1 \sigma_2 \rho_{1,2}$
  (Specify σ^2_p and one gets weights and μ_p's)
- Special cases [w_1 to obtain certain σ_R]
  - $\rho_{1,2} = 1 \Rightarrow w_1 = (+/-\sigma_p - \sigma_2) / (\sigma_1 - \sigma_2)$
  - $\rho_{1,2} = -1 \Rightarrow w_1 = (+/- \sigma_p + \sigma_2) / (\sigma_1 + \sigma_2)$

For $\rho_{1,2}$ = 1:

$$\sigma_p = |w_1\sigma_1 + (1-w_1)\sigma_2|$$
$$\mu_p = w_1\mu_1 + (1-w_1)\mu_2$$

Hence, $w_1 = \frac{\pm\sigma_p - \sigma_2}{\sigma_1 - \sigma_2}$

$$\mu_p = \mu_1 + \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}(\pm\sigma_p - \sigma_1)$$



Lower part with … is irrelevant

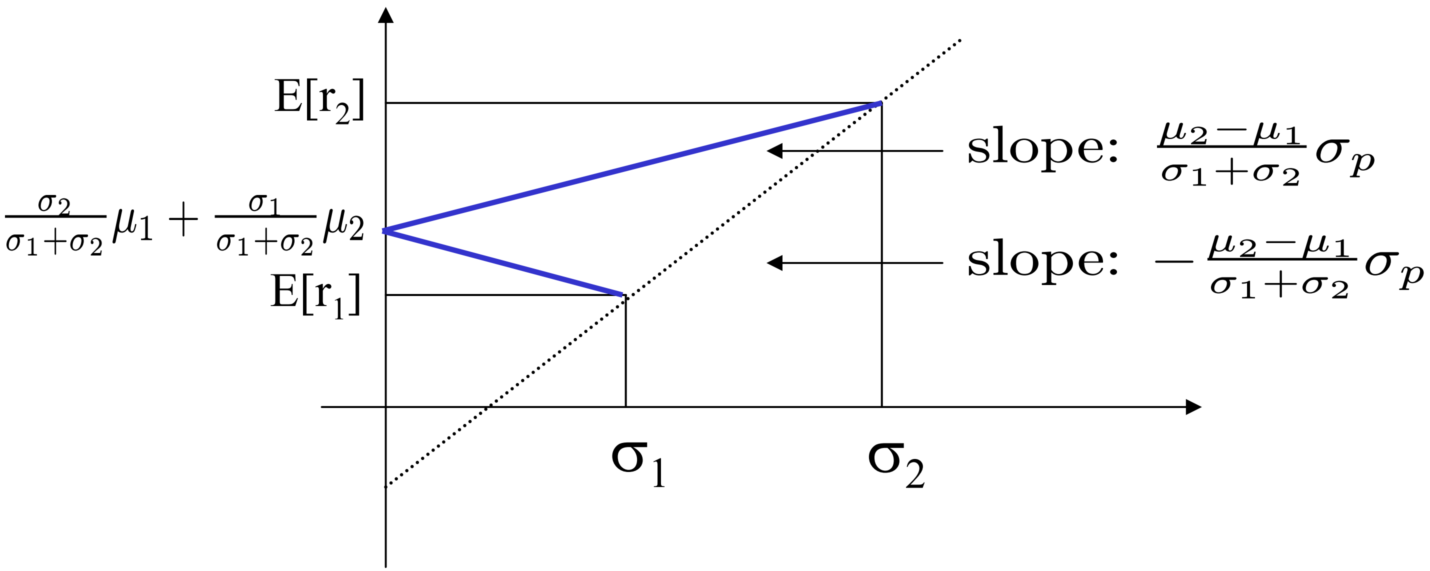
$$\mu_p = E[r_1] + \frac{E[r_2]-E[r_1]}{\sigma_2-\sigma_1}(-\sigma_R - \sigma_1)$$

The Efficient Frontier: Two Perfectly Correlated Risky Assets

For $\rho_{1,2}$ = -1:

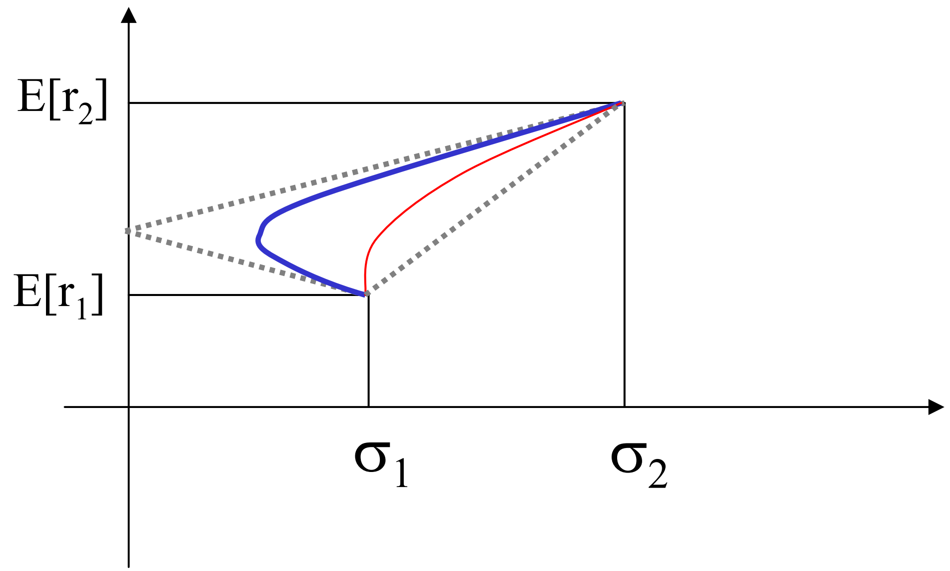
$$\sigma_p = |w_1\sigma_1 - (1-w_1)\sigma_2|$$
$$\mu_p = w_1\mu_1 + (1-w_1)\mu_2$$

Hence, $w_1 = \frac{\pm\sigma_p + \sigma_2}{\sigma_1+\sigma_2}$



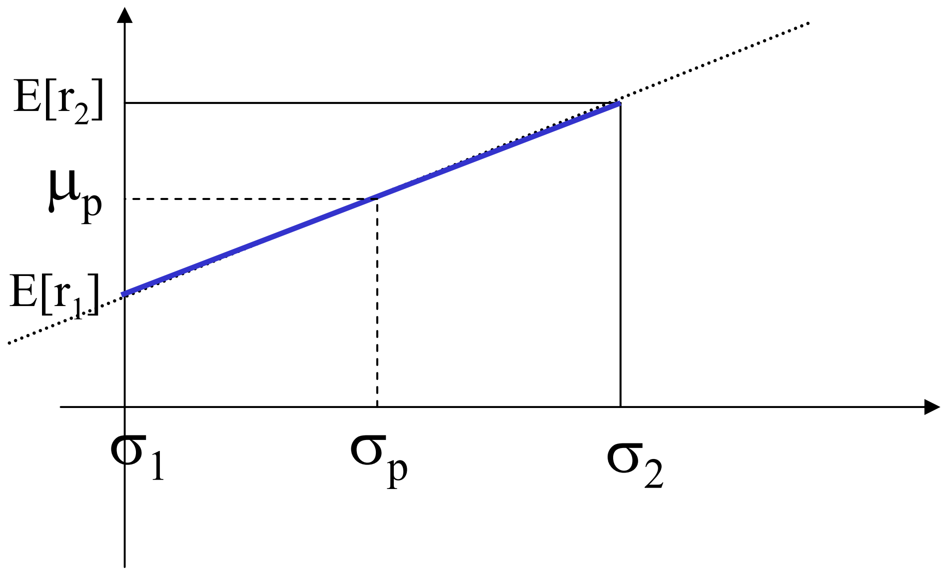
Efficient Frontier: Two Perfectly Negative Correlated Risky Assets

For $-1 < \rho_{1,2} < 1$:



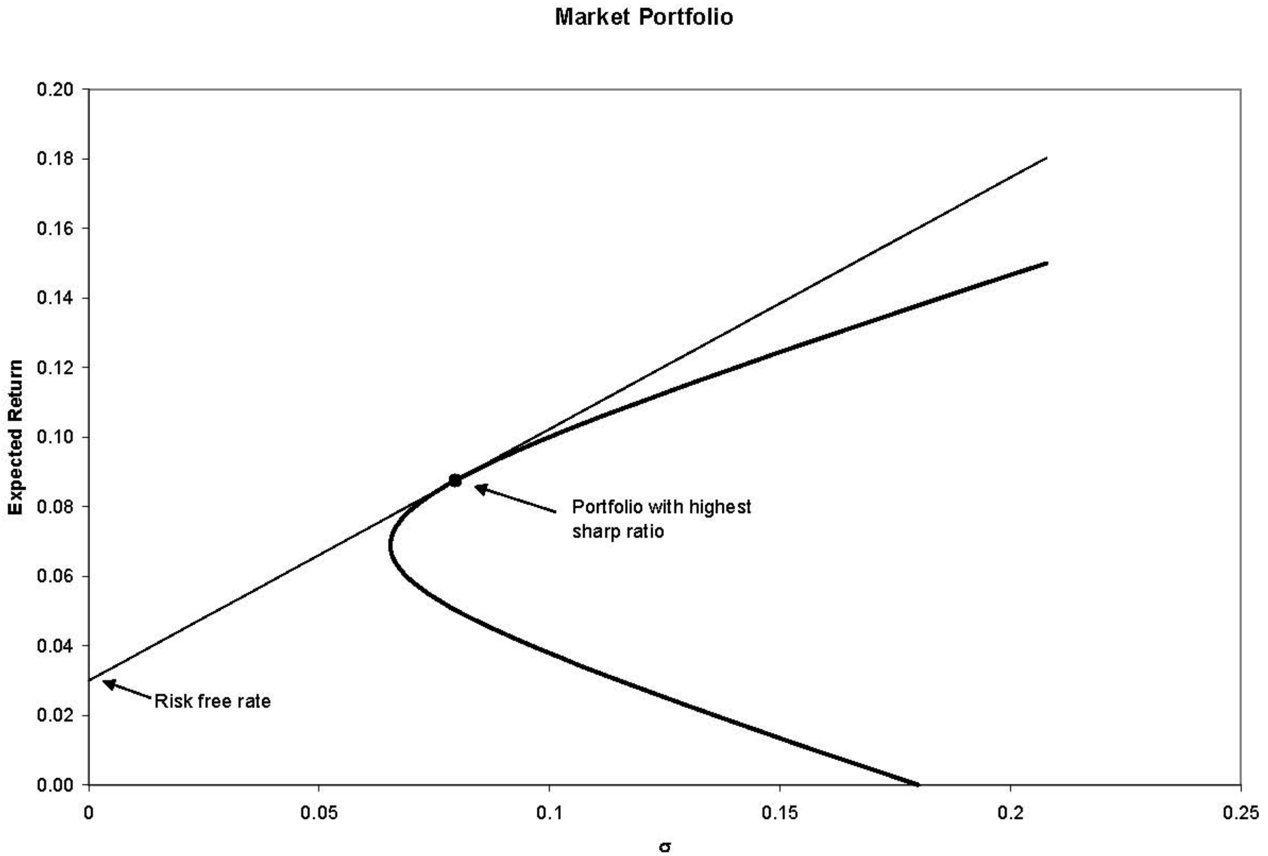
Efficient Frontier: Two Imperfectly Correlated Risky Assets

For $\sigma_1 = 0$



The Efficient Frontier: One Risky and One Risk Free Asset

Efficient Frontier with
n risky assets and one risk-free asset



The Efficient Frontier: One Risk Free and n Risky Assets

Mean-Variance Preferences

- $U(\mu_p, \sigma_p)$ with $\frac{\partial U}{\partial \mu_p} > 0, \frac{\partial U}{\partial \sigma_p^2} < 0$
  - quadratic utility function (with portfolio return R)

$$U(R) = a + b\,R + c\,R^2$$

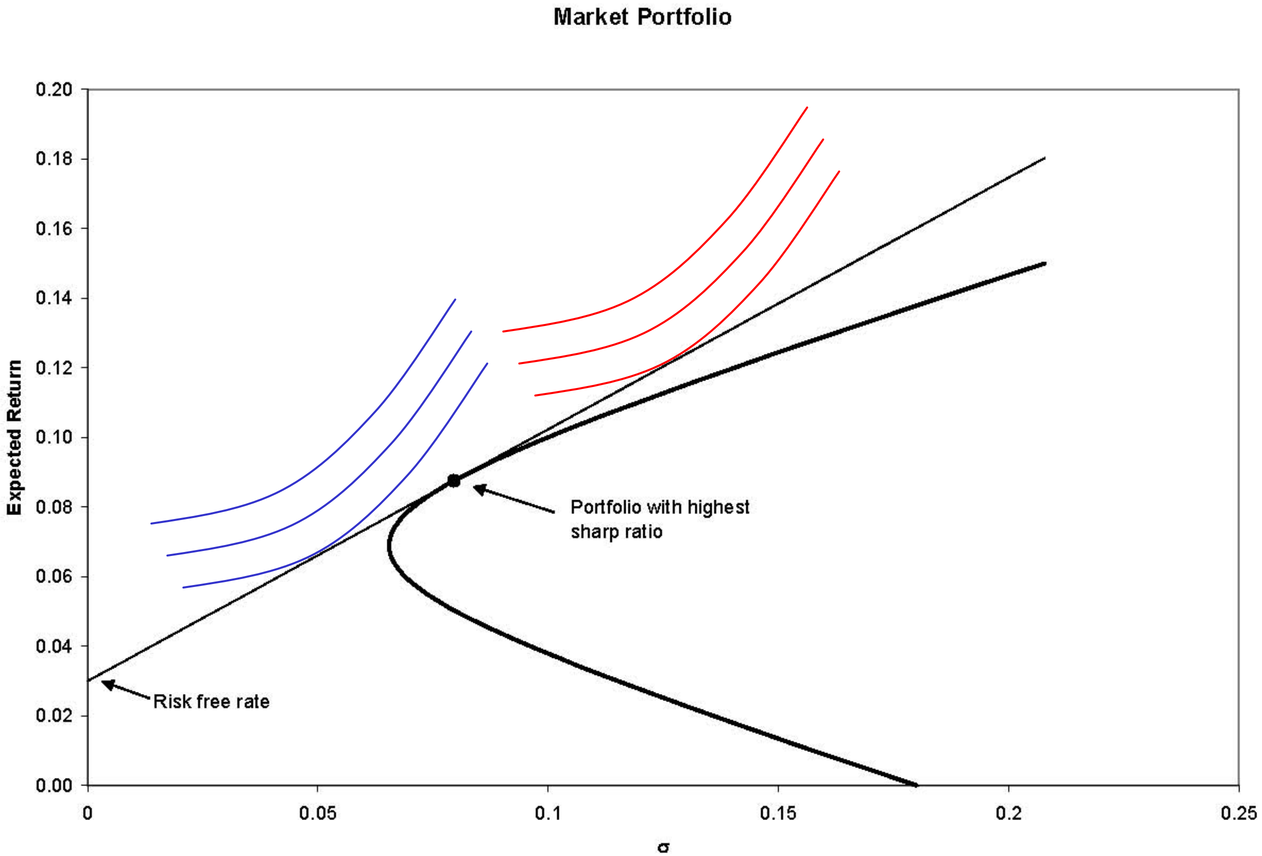
$$\text{vNM: } E[U(R)] = a + b\,E[R] + c\,E[R^2]$$

$$= a + b\,\mu_p + c\,\mu_p^2 + c\,\sigma_p^2$$

$$= g(\mu_p, \sigma_p)$$

  - asset returns normally distributed $\Rightarrow R=\sum_j w^j r^j$ normal
    - if U(.) is CARA $\Rightarrow$ certainty equivalent = $\mu_p - \rho_A/2\sigma^2_p$
      (Use moment generating function)

Optimal Portfolio: Two Fund Separation



Price of Risk =
= highest Sharpe ratio

Optimal Portfolios of Two Investors with Different Risk Aversion

Equilibrium leads to CAPM

- Portfolio theory: only analysis of demand
  - price/returns are taken as given
  - composition of risky portfolio is same for all investors
- Equilibrium Demand = Supply (market portfolio)
- CAPM allows to derive
  - equilibrium prices/ returns.
  - risk-premium

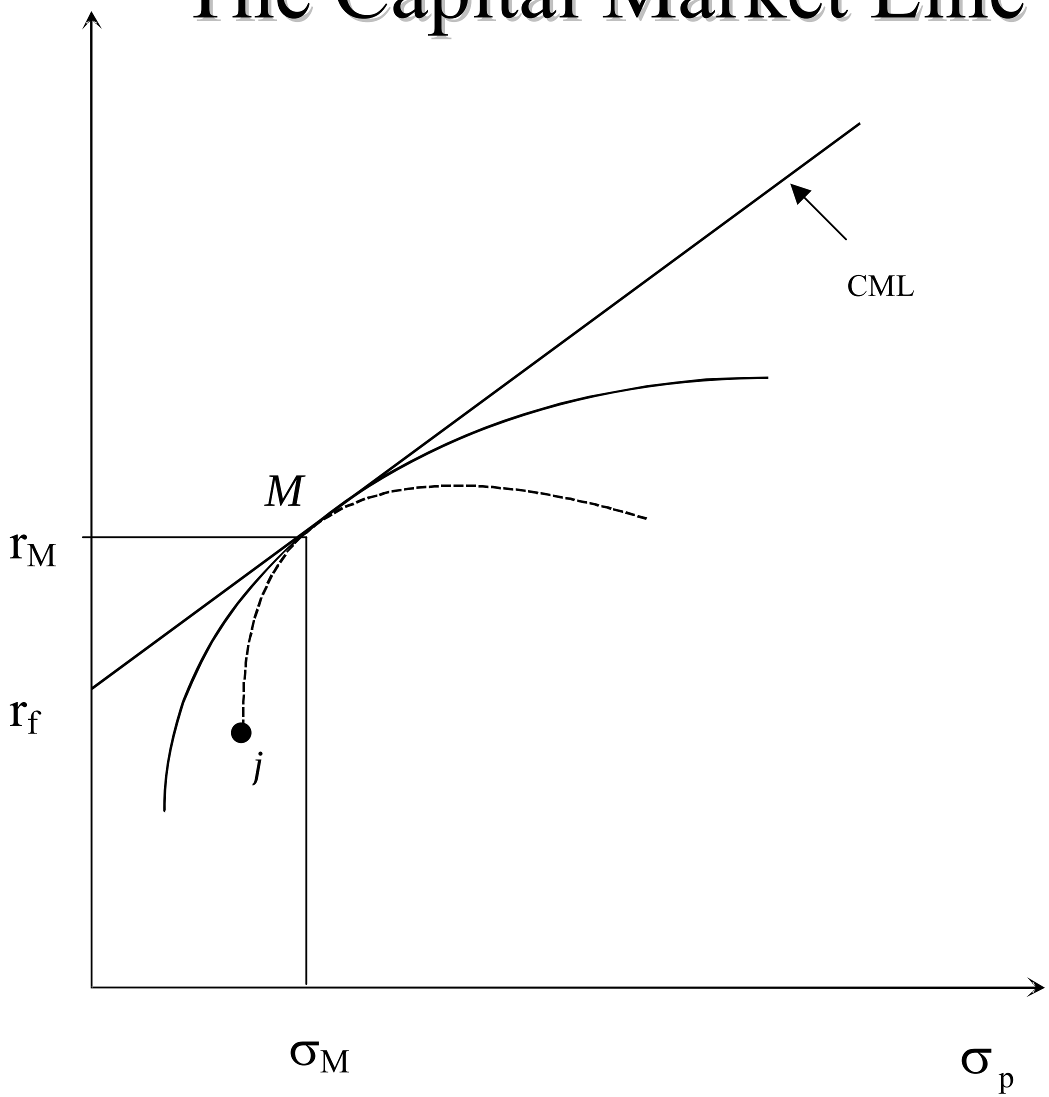
The CAPM with a risk-free bond

- The market portfolio is efficient since it is on the efficient frontier.
- All individual optimal portfolios are located on the half-line originating at point $(0,r_f)$.
- The slope of Capital Market Line (CML): $\frac{E[R_M]-R_f}{\sigma_M}$

.

$$E[R_p] = R_f + \frac{E[R_M]-R_f}{\sigma_M}\sigma_p$$

The Capital Market Line



Proof of the CAPM relationship
[old traditional derivation]

- Refer to previous figure. Consider a portfolio with a fraction 1 - α of wealth invested in an arbitrary security j and a fraction α in the market portfolio

$$\mu_p = \alpha \mu_M + (1-\alpha)\mu_j$$

$$\sigma_p^2 = \alpha^2 \sigma_M^2 + (1-\alpha)^2 \sigma_j^2 + 2\alpha(1-\alpha)\sigma_{jM}$$

As α varies we trace a locus which

- passes through M

(- and through j)

- cannot cross the CML (why?)

- hence must be tangent to the CML at M

Tangency = $\frac{d\mu_p}{d\sigma_p}\Big|_{\alpha=1}$ = slope of the locus at M =
= slope of CML = $\frac{\mu_M - r_f}{\sigma_M}$

$$\mu_p = \alpha\mu_M + (1-\alpha)\mu_j$$
$$\sigma_p^2 = \alpha^2\sigma_M^2 + (1-\alpha)^2\sigma_j^2 + 2\alpha(1-\alpha)\sigma_{jM}$$
$$\frac{d\mu_p}{d\sigma_p} = \frac{d\mu_p/d\alpha}{d\sigma_p/d\alpha}$$
$$\frac{d\mu_p}{d\alpha} = \mu_M - \mu_j$$
$$2\sigma_p\frac{\sigma_p}{d\alpha} = 2\alpha\sigma_M^2 - 2(1-\alpha)\sigma_j^2 + 2(1-2\alpha)\sigma_{jM}$$

at $\alpha = 1$
$\sigma_p = \sigma_M$

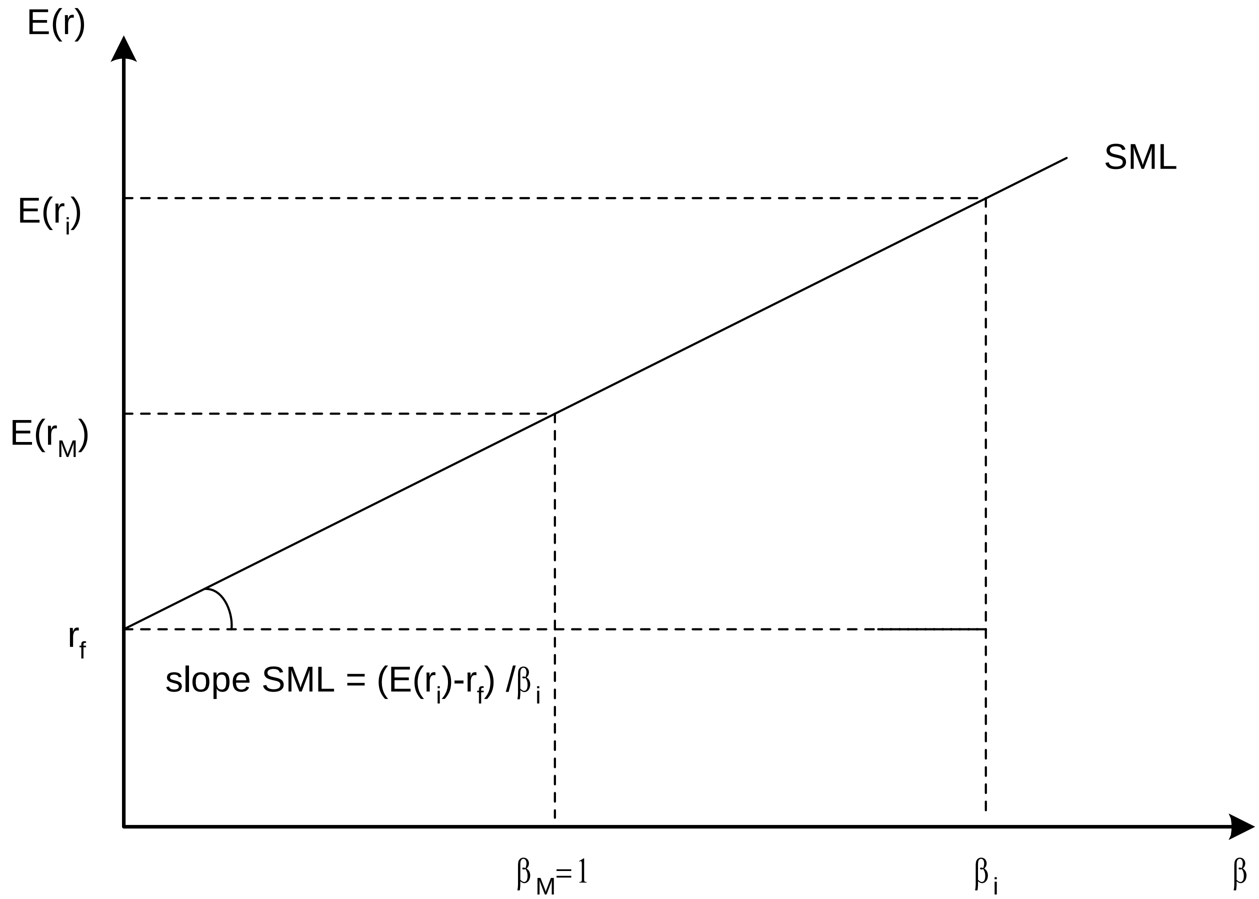
slope of locus $= \frac{d\mu_p}{d\sigma_p}\Big|_{\alpha=1} = \frac{(\mu_M-\mu_j)\sigma_M}{\sigma_M^2-\sigma_{jM}} = \frac{\mu_M - r_f}{\sigma_M} =$ slope of tangent!

$$(\mu_M - \mu_j) = \frac{(\mu_M - r_f)(\sigma_M^2 - \sigma_{jM})}{\sigma_M^2}$$

$$E[r_j] = \mu_j = r_f + \frac{\sigma_{jM}}{\sigma_M^2}(\mu_M - r_f)$$

Do you see the connection to earlier state-price beta model? $R^* = R_M$

The Security Market Line



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Projections

- States s=1,…,S with $\pi_s > 0$
- Probability inner product

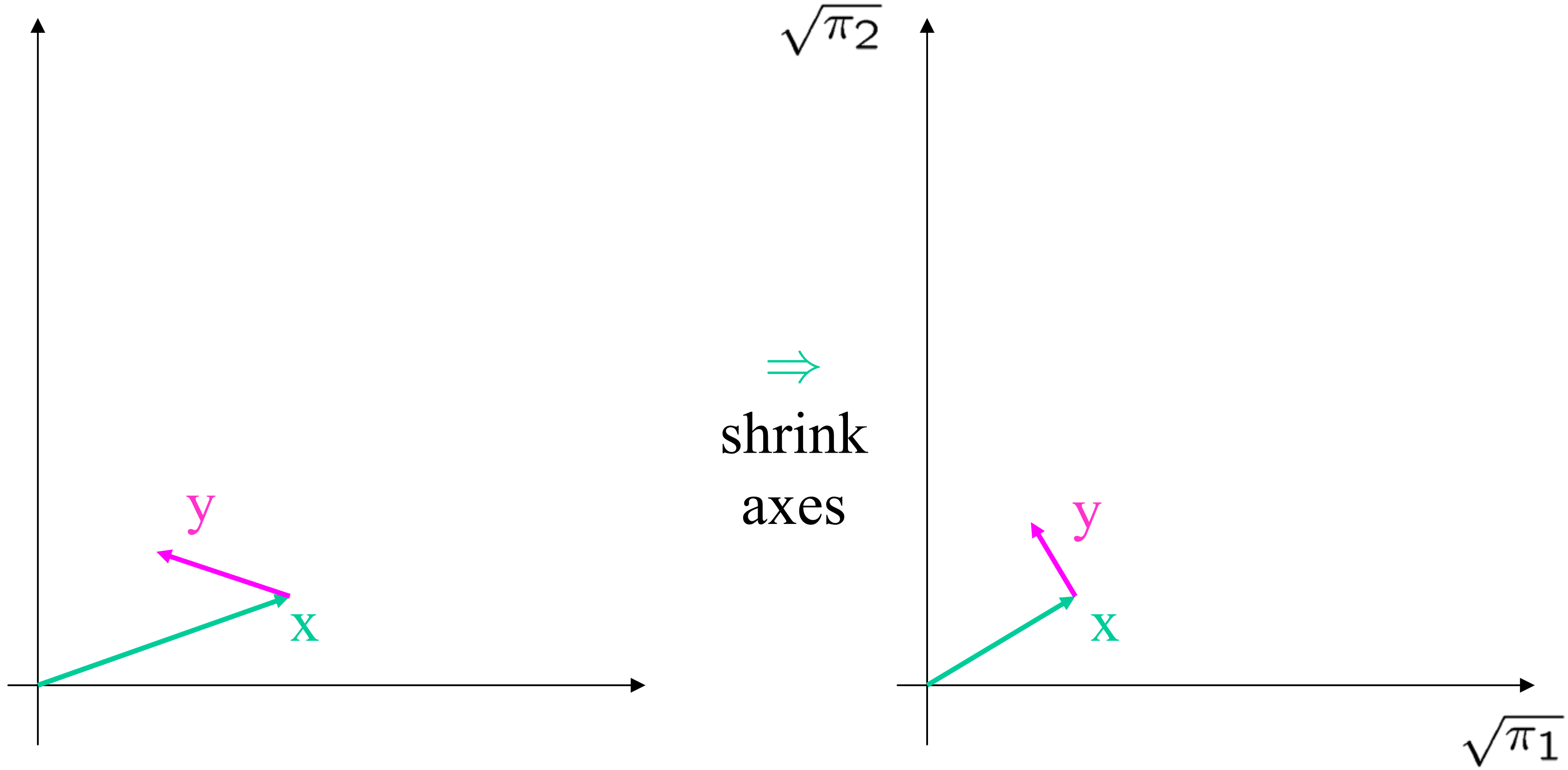
$$[x, y]_\pi = (xy)_\pi = \sum_s \pi_s x_s y_s = \sum_s (\sqrt{\pi_s} x_s \sqrt{\pi_s} y_s)$$

- π-norm $||x|| = \sqrt{[x, x]_\pi}$ (measure of length)

(i) $||x|| > 0 \ \forall x \neq 0$ and $||x|| = 0$ if $x = 0$

(ii) $||\lambda x|| = |\lambda| ||x||$

(iii) $||x + y|| \leq ||x|| + ||y|| \ \forall x, y \in \mathbb{R}^S$



x and y are π-orthogonal iff $[x,y]_\pi = 0$, I.e. E[xy]=0

…Projections…

- $\mathcal{Z}$ space of all linear combinations of vectors $z_1, \ldots, z_n$
- Given a vector $y \in R^S$ solve

$$min_{\alpha \in \mathbb{R}^n} E[y - \sum_{j=1,\ldots,n} \alpha^j z^j]^2$$

FOC: (for each $j = 1, \ldots, n$)

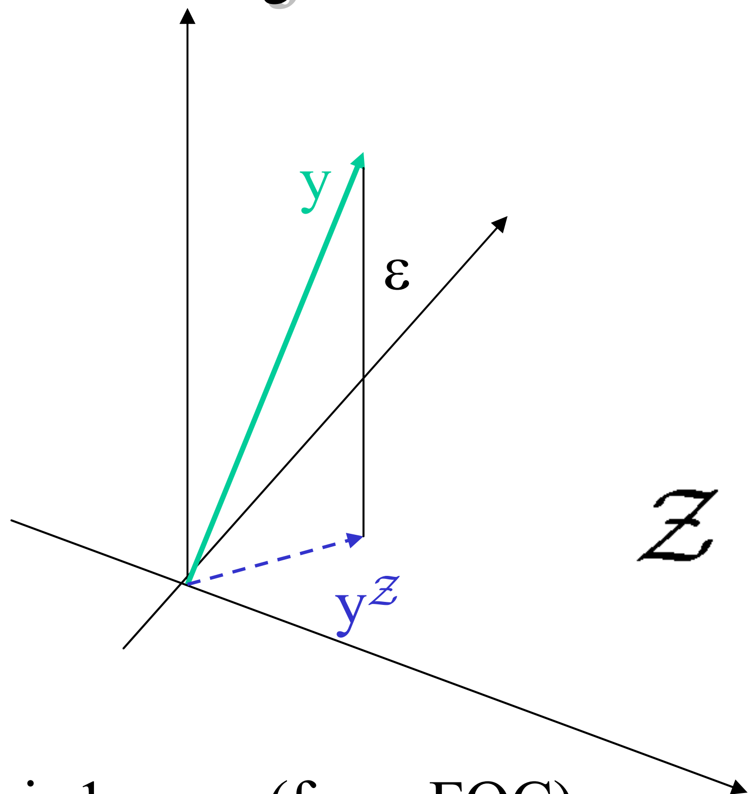
$$\sum_s \pi_s (y_s - \sum_j \alpha^j z_s^j) z^j = 0$$

$\Rightarrow \hat{\alpha}$ the solution

$$y^{\mathcal{Z}} = \sum_j \widehat{\alpha^j} z^j, \; \epsilon := y - y^{\mathcal{Z}}$$

- [smallest distance between vector y and $\mathcal{Z}$ space]

…Projections



$E[\varepsilon z^j]=0$ for each $j=1,\ldots,n$ (from FOC)

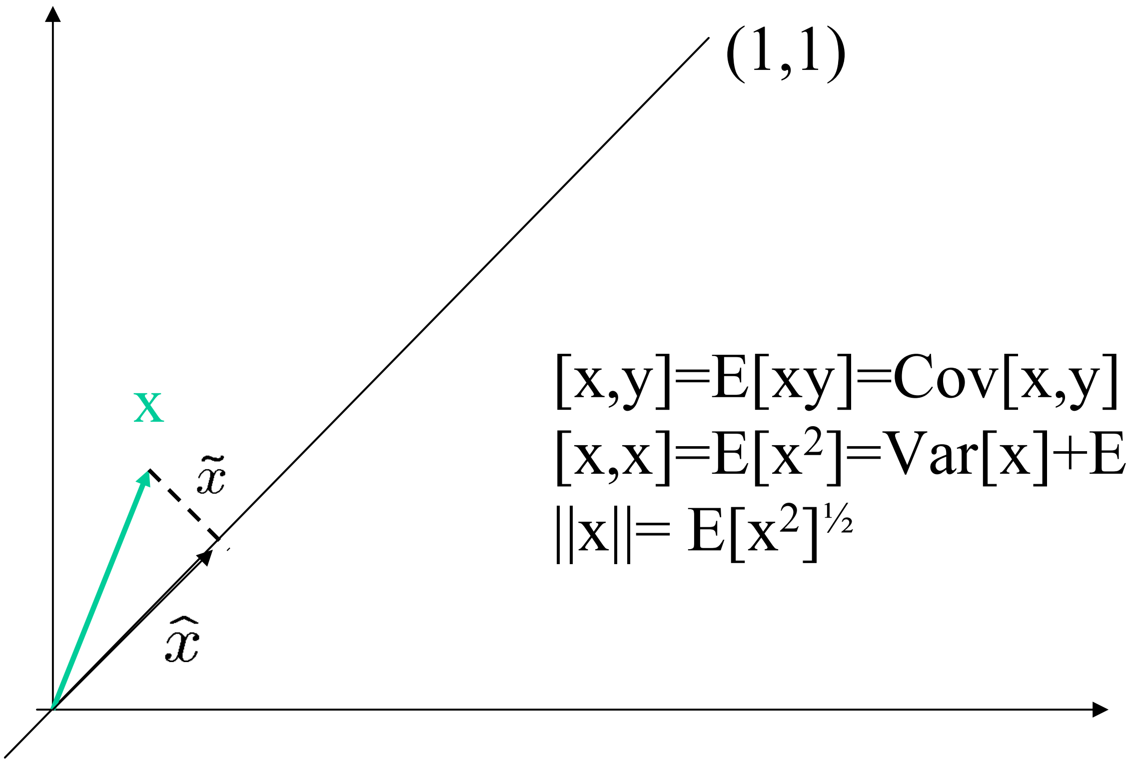
$\varepsilon \perp z$

y^Z is the (orthogonal) projection on $\mathcal{Z}$

$y = y^{\mathcal{Z}} + \varepsilon'$, $y^{\mathcal{Z}} \in \mathcal{Z}$, $\varepsilon \perp z$

Expected Value and Co-Variance…

squeeze axis by $\sqrt{\pi_s}$



[x,y]=E[xy]=Cov[x,y] + E[x]E[y]

[x,x]=E[x^2]=Var[x]+E[x]^2

||x||= E[x^2]^{½}

$$x = \hat{x} + \tilde{x}$$

…Expected Value and Co-Variance

$x = \hat{x} + \tilde{x}$, where

$\hat{x}$ is projection of x onto $< 1 >$

$\tilde{x}$ is projection of x onto $< 1 >^{\perp}$

$$E[x] = [x, 1]_\pi = [\hat{x}, 1]_\pi = ||\hat{x}||$$

$$Var[x] = [\tilde{x}, \tilde{x}]_\pi = E[\tilde{x}^2] = Var[\tilde{x}]$$

$$\sigma_x = ||\tilde{x}||_\pi = \text{standard deviation of } x$$

$$Cov[x, y] = Cov[\tilde{x}, \tilde{y}] = [\tilde{y}, \tilde{x}]$$

Proof: $[x, y]_\pi = [\hat{x}, \hat{y}]_\pi + [\tilde{x}, \tilde{y}]_\pi$, since

$[\hat{y}, \tilde{x}]_\pi = [\tilde{y}, \hat{x}]_\pi = 0$, $[x, y]_\pi = E[\hat{y}]E[\hat{x}] + Cov[\tilde{x}, \tilde{y}]$

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New (LeRoy & Werner) Notation

- Main changes (new versus old)
  - gross return: $r = R$
  - SDF: $\mu = m$
  - pricing kernel: $k_q = m^*$

  - Asset span: $\mathcal{M} = \langle X \rangle$
  - income/endowment: $w_t = e_t$

Pricing Kernel k_q…

- $\mathcal{M}$ space of feasible payoffs.
- If no arbitrage and $\pi >> 0$ there exists SDF $\mu \in R^S$, $\mu >> 0$, such that $q(z)=E(\mu z)$.
- $\mu \in \mathcal{M}$ – SDF need not be in asset span.
- A pricing kernel is a $k_q \in \mathcal{M}$ such that for each $z \in \mathcal{M}$, $q(z)=E(k_q z)$.
- ($k_q = m^*$ in our old notation.)

…Pricing Kernel - Examples…

- Example 1:
  - $S=3, \pi^s=1/3$ for $s=1,2,3$,
  - $x_1=(1,0,0)$, $x_2=(0,1,1)$, $p=(1/3,2/3)$.
  - Then $k=(1,1,1)$ is the unique pricing kernel.
- Example 2:
  - $S=3, \pi^s=1/3$ for $s=1,2,3$,
  - $x_1=(1,0,0)$, $x_2=(0,1,0)$, $p=(1/3,2/3)$.
  - Then $k=(1,2,0)$ is the unique pricing kernel.

…Pricing Kernel – Uniqueness

- If a state price density exists, there exists a ***unique*** pricing kernel.
  - If $\dim(\mathcal{M}) = m$ (markets are complete), there are exactly m equations and m unknowns
  - If $\dim(\mathcal{M}) \leq m$, (markets may be incomplete) For any state price density (=SDF) μ and any $z \in \mathcal{M}$

    $\mathbf{E[(\mu-k_q)z]=0}$

    $\mu=(\mu-k_q)+k_q \Rightarrow k_q$ is the "**projection**" of μ on $\mathcal{M}$.
    - Complete markets $\Rightarrow$, $k_q=\mu$ (SDF=state price density)

Expectations Kernel k_e

- An expectations kernel is a vector $k_e \in \mathcal{M}$
  - Such that $E(z)=E(k_e\, z)$ for each $z \in \mathcal{M}$.
- Example
  - $S=3$, $\pi^s=1/3$, for $s=1,2,3$, $x_1=(1,0,0)$, $x_2=(0,1,0)$.
  - Then the unique $k_e=(1,1,0)$.
- If $\pi >> 0$, there exists a unique expectations kernel.
- Let $e=(1,\ldots, 1)$ then for any $z \in \mathcal{M}$
- **$E[(e-k_e)z]=0$**
- k_e is the **"projection"** of e on $\mathcal{M}$
- $k_e = e$ if bond can be replicated (e.g. if markets are complete)

Mean Variance Frontier

- *Definition 1:* $z \in \mathcal{M}$ is in the mean variance frontier if there exists no $z' \in \mathcal{M}$ such that $E[z'] = E[z]$, $q(z') = q(z)$ and $var[z'] < var[z]$.

- *Definition 2:* Let $\mathcal{E}$ the space generated by k_q and k_e.
- Decompose $z = z^{\mathcal{E}} + \varepsilon$, with $z^{\mathcal{E}} \in \mathcal{E}$ and $\varepsilon \perp \mathcal{E}$.
- Hence, $E[\varepsilon] = E[\varepsilon\, k_e] = 0$, $q(\varepsilon) = E[\varepsilon\, k_q] = 0$

  $Cov[\varepsilon, z^{\mathcal{E}}] = E[\varepsilon\, z^{\mathcal{E}}] = 0$, since $\varepsilon \perp \mathcal{E}$.
- $var[z] = var[z^{\mathcal{E}}] + var[\varepsilon]$ (price of ε is zero, but positive variance)
- If z in mean variance frontier $\Rightarrow z \in \mathcal{E}$.
- Every $z \in \mathcal{E}$ is in mean variance frontier.

Frontier Returns…

- Frontier returns are the returns of frontier payoffs with non-zero prices.

$$r_e = \frac{k_e}{q(k_e)} = \frac{k_e}{E(k_q)}$$

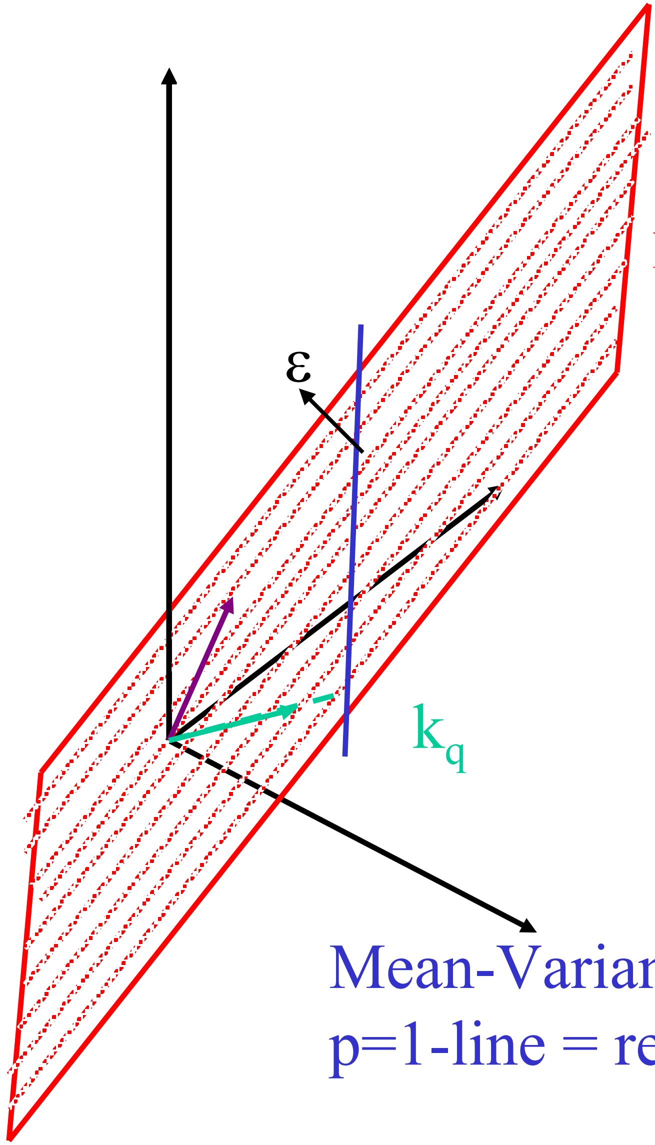
$$r_q = \frac{k_q}{q(k_q)} = \frac{k_q}{E(k_q k_q)}$$

- If $z = \alpha k_q + \beta k_e$ then,

$$r_z = \underbrace{\frac{\alpha q(k_q)}{\alpha q(k_q) + \beta q(k_e)}}_{\lambda} r_q + \underbrace{\frac{\beta q(k_e)}{\alpha q(k_q) + \beta q(k_e)}}_{1-\lambda} r_e$$

- graphically: payoffs with price of p=1.

$\mathcal{M} = R^S = R^3$

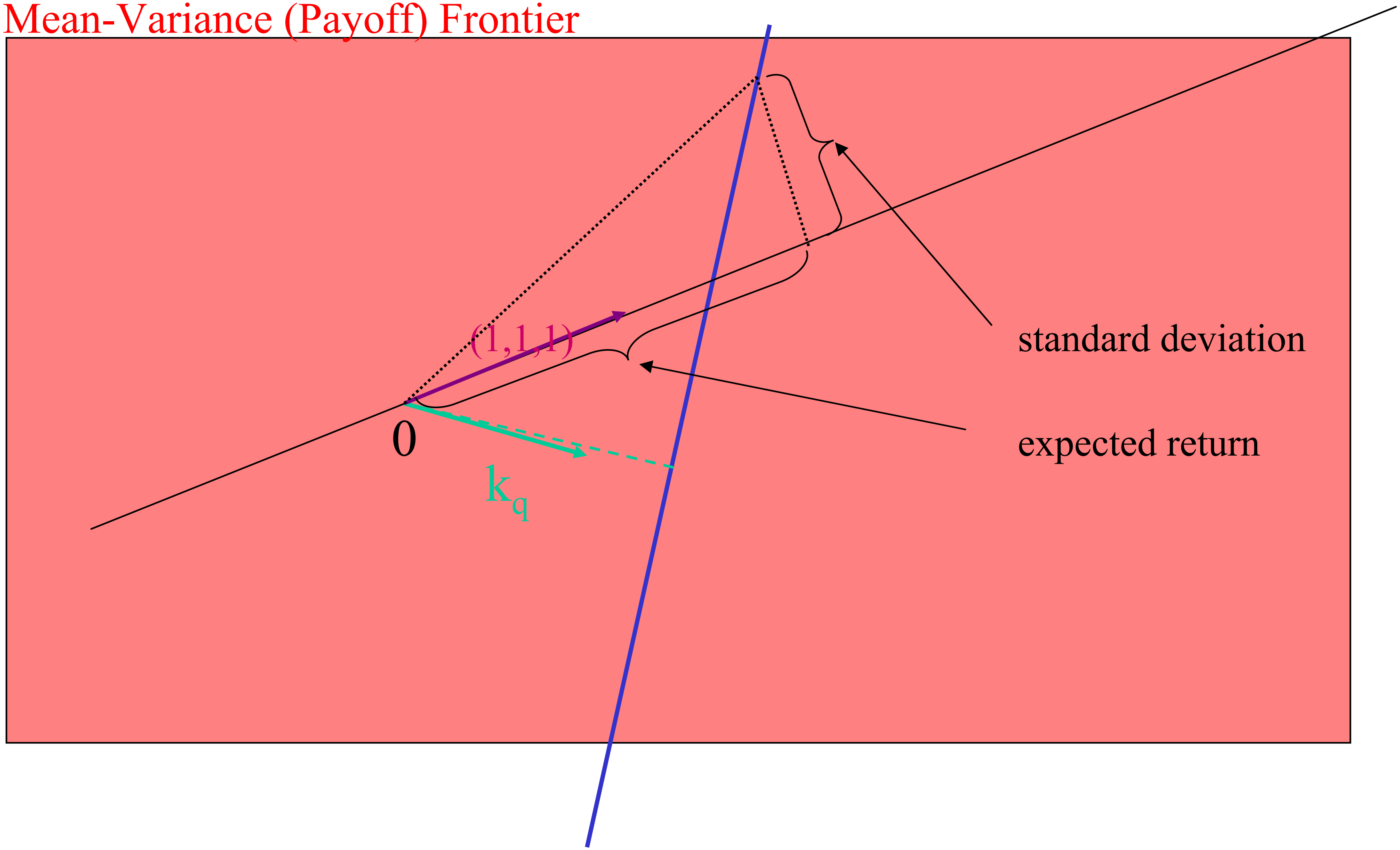


Mean-Variance Payoff Frontier

Mean-Variance Return Frontier

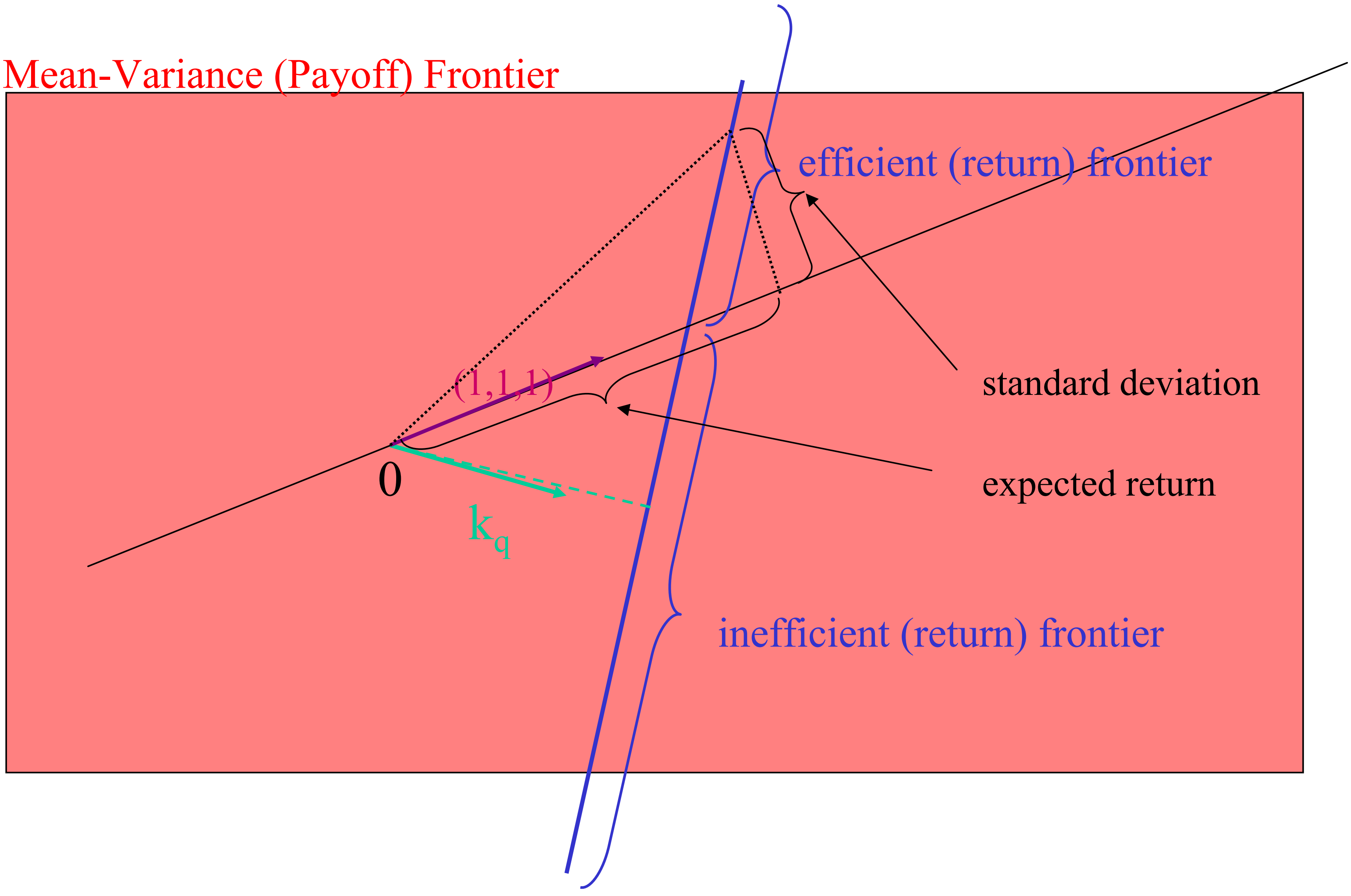
p=1-line = return-line (orthogonal to k_q)

Mean-Variance (Payoff) Frontier



NB: graphical illustrated of expected returns and standard deviation changes if bond is not in payoff span.

Mean-Variance (Payoff) Frontier



…Frontier Returns

If $k_e = \alpha k_q$, frontier returns $\equiv r_e$. (if agent is risk-neutral)

If $k_e \neq \alpha k_q$, frontier can be written as:

$$r_\lambda = r_e + \lambda(r_q - r_e)$$

Expectations and Variance are

$$E[r_\lambda] = E[r_e] + \lambda(E[r_q] - E[r_e])$$

$$\begin{aligned} \text{var}(r_\lambda) &= \text{var}(r_e) + 2\lambda \text{cov}(r_e, r_q - r_e) \\ &+ \lambda^2 \text{var}(r_q - r_e) \end{aligned} \qquad (1)$$

If risk-free asset exists, they simplify to:

$E[r_\lambda] = \bar{r} + \lambda(E[r_q] - \bar{r})$.

$\text{var}(r_\lambda) = \lambda^2 \text{var}(r_q)$. $\sigma(r_\lambda) = |\lambda| \sigma(r_q)$.

$E(r_\lambda) = \bar{r} \pm \sigma(r_\lambda) \frac{E(r_q) - \bar{r}}{\sigma(r_q)}$

Minimum Variance Portfolio

- Take FOC w.r.t. λ of

$$\begin{aligned} \mathrm{var}(r_\lambda) &= \mathrm{var}(r_e) + 2\lambda \mathrm{cov}(r_e, r_q - r_e) \\ &+ \lambda^2 \mathrm{var}(r_q - r_e) \end{aligned} \quad (1)$$

- Hence, MVP has return of

$r_e + \lambda_0(r_q - r_e)$, with

$$\lambda_0 = -\frac{\mathrm{cov}(r_e, r_q - r_e)}{\mathrm{var}(r_q - r_e)}.$$

Mean-Variance Efficient Returns

- *Definition:* A return is **mean-variance efficient** if there is no other return with same variance but greater expectation.
- Mean variance efficient returns are frontier returns with $E[r_\lambda] \geq E[r_{\lambda 0}]$.
- If risk-free asset can be replicated
  - Mean variance efficient returns correspond to $\lambda \leq 0$.
  - Pricing kernel (portfolio) is not mean-variance efficient, since

$$E[r_q] = \frac{E[k_q]}{E[k_q^2]} < \frac{1}{E[k_q]} = \bar{r}.$$

Hint: $E[k_q^2] > E[k_q]^2$ since $Var[k_q] > 0$

Zero-Covariance Frontier Returns

- Take two frontier portfolios with returns $r_\lambda = r_e + \lambda(r_q - r_e)$ and $r_\mu = r_e + \mu(r_q - r_e)$
- $$\text{cov}(r_\mu, r_\lambda) = \text{var}(r_e) + (\lambda + \mu)\text{cov}(r_e, r_q - r_e) + \lambda\mu\text{var}(r_q - r_e).$$
- The portfolios have zero co-variance if
$$\mu = -\frac{\text{var}(r_e) + \lambda\text{cov}(r_e, r_q - r_e)}{\text{cov}(r_e, r_q - r_e) + \lambda\text{var}(r_q - r_e)}$$
- For all $\lambda \neq \lambda_0$ μ exists
- $\mu=0$ if risk-free bond can be replicated

Illustration of MVP

$\mathcal{M} = R^2$ and S=3

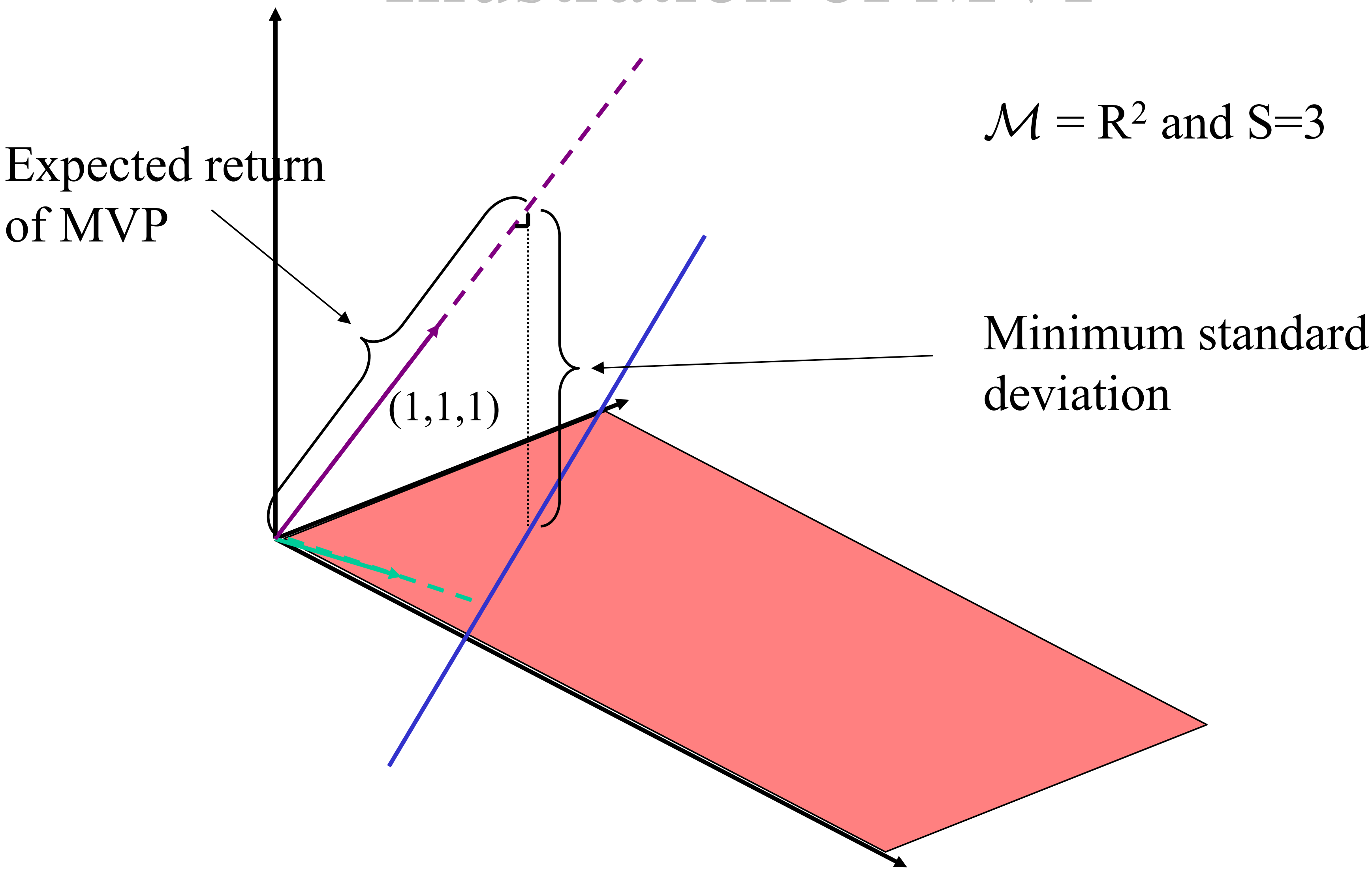
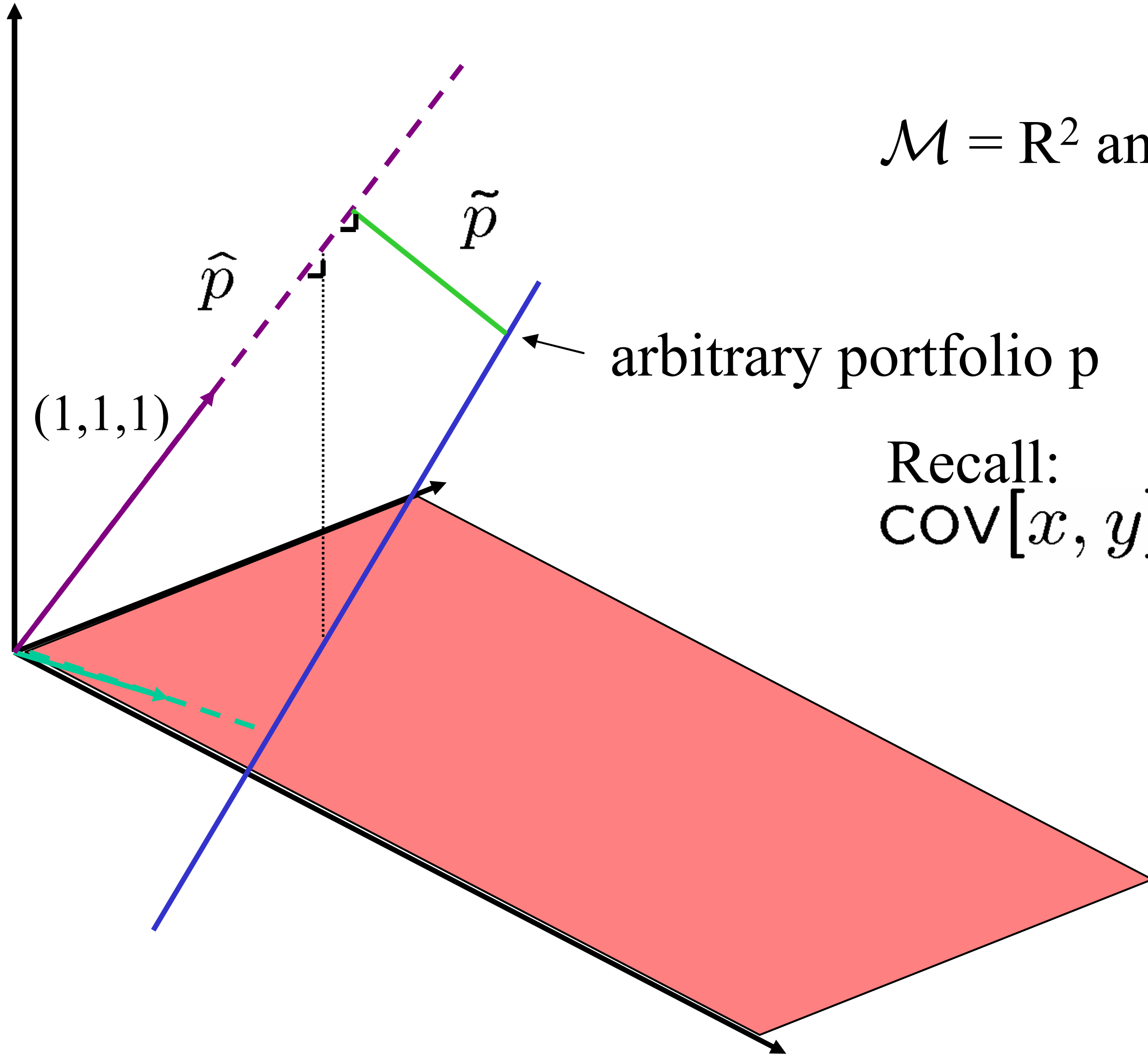


Illustration of ZC Portfolio…

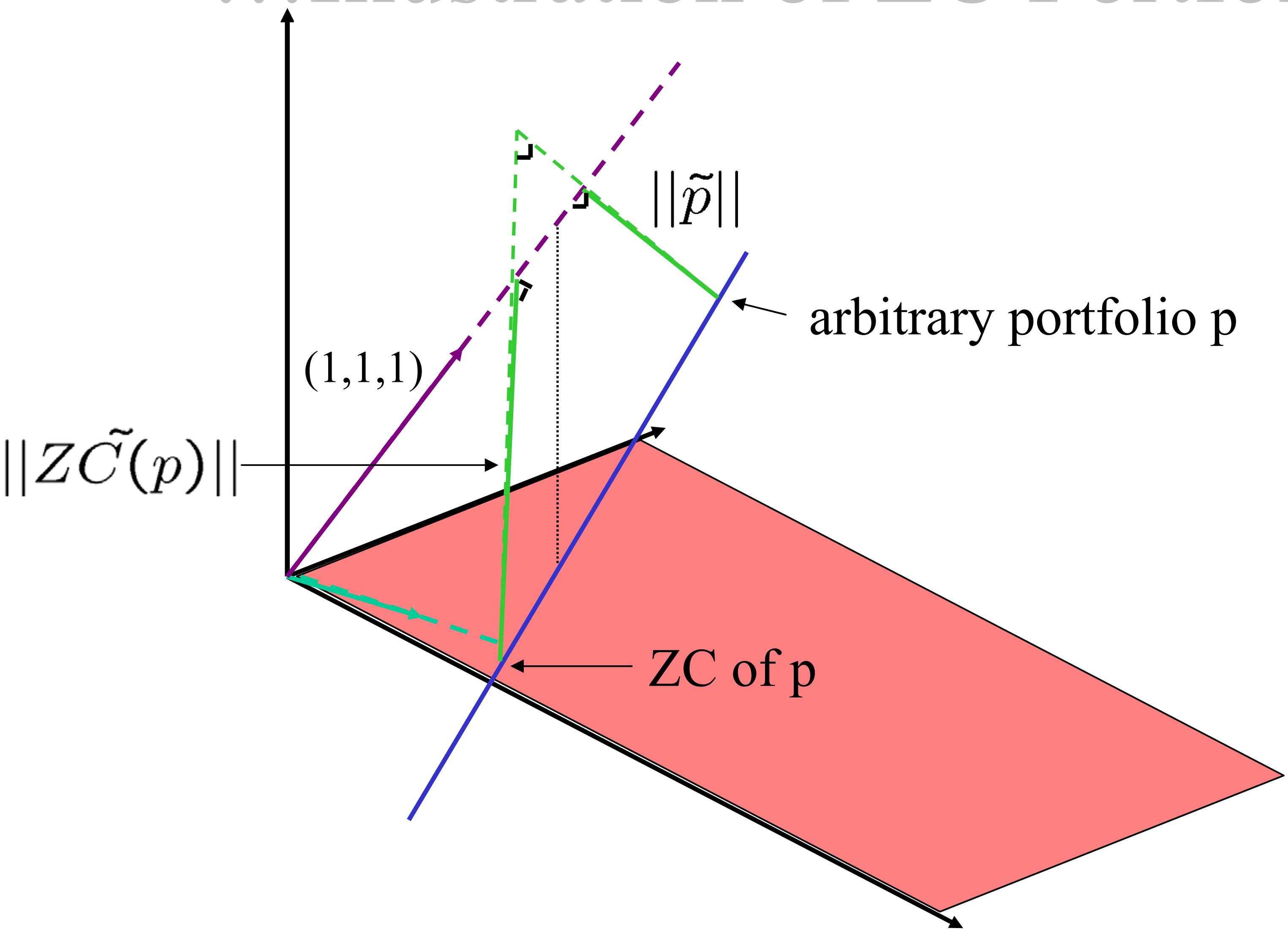
$\mathcal{M}$ = R^2 and S=3



Recall:

$$\text{cov}[x,y] = [\tilde{x},\tilde{y}]_\pi$$

…Illustration of ZC Portfolio



Beta Pricing…

- Frontier Returns (are on linear subspace). Hence
$$r_\beta = r_\mu + \beta(r_\lambda - r_\mu).$$
- Consider any asset with payoff x_j
  - It can be decomposed in $x_j = x_j^{\mathcal{E}} + \varepsilon_j$
  - $q(x_j)=q(x_j^{\mathcal{E}})$ and $E[x_j]=E[x_j^{\mathcal{E}}]$, since $\varepsilon \perp \mathcal{E}$.
  - Let $r_i^{\mathcal{E}}$ be the return of $x_j^{\mathcal{E}}$
  - $r_j = r_j^{\mathcal{E}} + \frac{\epsilon_j}{q(x_j)}.$
  - Using above and assuming $\lambda \neq \lambda_0$ and μ is ZC-portfolio of λ, $$r_j = r_\mu + \beta_j(r_\lambda - r_\mu) + \frac{\epsilon_j}{q(x_j)}$$

…Beta Pricing

- Taking expectations and deriving covariance
- $E[r_j] = E[r_\mu] + \beta_j(E[r_\lambda] - E[r_\mu])$ since $r_\lambda \perp \frac{\epsilon_j}{q(x_j)}$
  $\text{cov}(r_\lambda, r_j) = \beta_j \text{var}(r_\lambda) \Rightarrow \beta_j = \frac{\text{cov}(r_\lambda, r_j)}{\text{var}(r_\lambda)}.$
- If risk-free asset can be replicated, beta-pricing equation simplifies to
  $$E[r_j] = \bar{r} + \beta_j(E[r_\lambda] - \bar{r})$$
- Problem: How to identify frontier returns

Capital Asset Pricing Model…

- CAPM = market return is frontier return
  - Derive conditions under which market return is frontier return
  - Two periods: 0,1,
  - Endowment: individual w^i_1 at time 1, aggregate $\bar{w}_1 = \bar{w}_1^{\mathcal{M}} + \bar{w}_1^{\mathcal{N}}$, where $\bar{w}_1^{\mathcal{M}}$ the orthogonal projection of $\bar{w}_1$ on $\mathcal{M}$ is.
  - The market payoff: $m \equiv \bar{w}_1^{\mathcal{M}}$
  - Assume $q(m) \neq 0$, let $r_m = m / q(m)$, and assume that r_m is not the minimum variance return.

…Capital Asset Pricing Model

- If r_{m0} is the frontier return that has zero covariance with r_m then, for every security j,
- $E[r_j]=E[r_{m0}] + \beta_j\,(E[r_m]-E[r_{m0}])$, with $\beta_j=cov[r_j,r_m] / var[r_m]$.
- If a risk free asset exists, equation becomes,
- $E[r_j]= r_f + \beta_j\,(E[r_m]- r_f)$

- N.B. first equation always hold if there are only two assets.

Outdated material follows

- Traditional derivation of CAPM is less elegant
- Not relevant for exams

Deriving the Frontier
n risky assets

- Definition 6.1: *A frontier portfolio is one which displays minimum variance among all feasible portfolios with the same* $E(\tilde{r}_p)$.

$$\min_{w} \frac{1}{2} w^T V w$$

$$(\lambda) \quad \text{s.t.} \quad w^T e = E \qquad \left(\sum_{i=1}^{N} w_i E(\tilde{r}_i) = E \right)$$

$$(\gamma) \quad w^T 1 = 1 \qquad \left(\sum_{i=1}^{N} w_i = 1 \right)$$

$$\frac{\partial \mathcal{L}}{\partial w} = Vw - \lambda e - \gamma 1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = E - w^T e = 0$$
$$\frac{\partial \mathcal{L}}{\partial \gamma} = 1 - w^T 1 = 0$$

The first FOC can be written as:

$$Vw_p = \lambda e + \gamma 1 \text{ or}$$
$$w_p = \lambda V^{-1} e + \gamma V^{-1} 1$$
$$e^T w_p = \lambda (e^T V^{-1} e) + \gamma (e^T V^{-1} 1)$$

Noting that $e^T w_p = w^T{}_p e$, using the first foc, the second foc can be written as

$$E[\tilde{r}_p] = e^T w_p = \lambda \underbrace{(e^T V^{-1} e)}_{:=B} + \gamma \underbrace{(e^T V^{-1} 1)}_{=:A}$$

pre-multiplying first foc with 1 (instead of e^T) yields

$$1^T w_p = w_p^T 1 = \lambda(1^T V^{-1} e) + \gamma(1^T V^{-1} 1) = 1$$
$$1 = \lambda \underbrace{(1^T V^{-1} e)}_{=:A} + \gamma \underbrace{(1^T V^{-1} 1)}_{=:C}$$

Solving both equations for λ and γ

$$\lambda = \frac{CE - A}{D} \text{ and } \gamma = \frac{B - AE}{D}$$
$$\text{where } D = BC - A^2.$$

Hence, $w_p = \lambda V^{-1}e + \gamma V^{-1}1$ becomes

$$w_p = \frac{CE - A}{D} V^{-1}e + \frac{B - AE}{D} V^{-1}\mathbf{1}$$

(vector) (vector)

λ (scalar) γ (scalar)

$$= \frac{1}{D}\left[B\left(V^{-1}1\right) - A\left(V^{-1}e\right)\right] + \frac{1}{D}\left[C\left(V^{-1}e\right) - A\left(V^{-1}1\right)\right]E$$

$$w_p = g + h \quad E \qquad (6.15)$$

(vector) (vector) (scalar)

linear in expected return E!

If $E = 0$, $w_p = g$

If $E = 1$, $w_p = g + h$

Hence, g and g+h are portfolios on the frontier.

Characterization of Frontier Portfolios

- Proposition 6.1: *The entire set of frontier portfolios can be generated by ("are convex combinations" of)* g *and* g+h.

- Proposition 6.2. *The portfolio frontier can be described as convex combinations of any two frontier portfolios, not just the frontier portfolios* g *and* g+h.

- Proposition 6.3 : *Any convex combination of frontier portfolios is also a frontier portfolio.*

…Characterization of Frontier Portfolios…

- For any portfolio on the frontier, $\sigma^2\left(E[\tilde{r}_p]\right) = \left[g + hE\left(\tilde{r}_p\right)\right]^T V \left[g + hE\left(\tilde{r}_p\right)\right]$ with g and h as defined earlier.

  Multiplying all this out yields:

$$\sigma^2(E[\tilde{r}_p]) = \frac{C}{D}[E[\tilde{r}_p] - \frac{A}{C}]^2 + \frac{1}{C}$$

…Characterization of Frontier Portfolios…

- (i) the expected return of the minimum variance portfolio is A/C;

- (ii) the variance of the minimum variance portfolio is given by 1/C;

- (iii) equation (6.17) is the equation of a parabola with vertex (1/C, A/C) in the expected return/variance space and of a hyperbola in the expected return/standard deviation space. See Figures 6.3 and 6.4.

$$E[\tilde{r}_p] = \frac{A}{C} \pm \sqrt{\frac{D}{C}\left(\sigma^2 - \frac{1}{C}\right)}$$

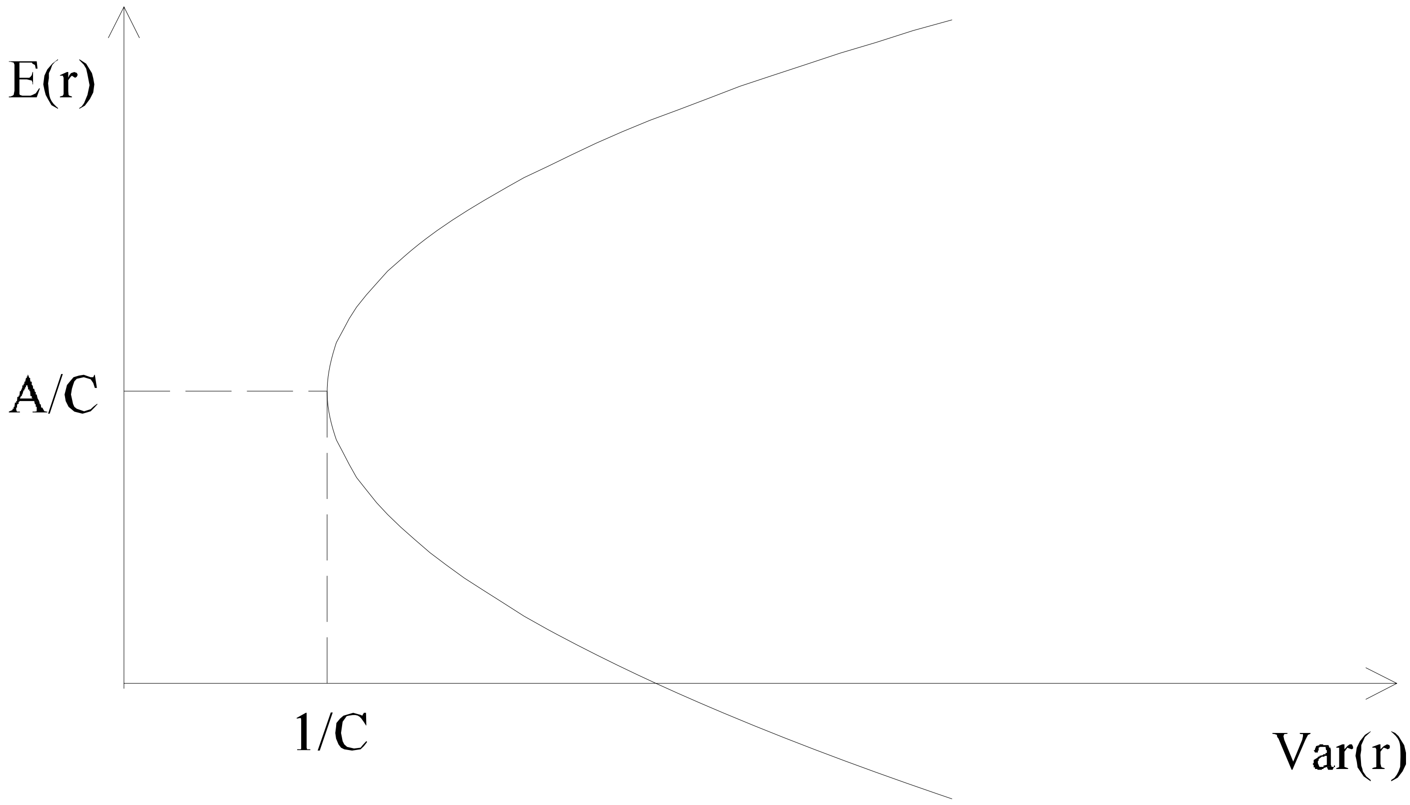


Figure 6-3 The Set of Frontier Portfolios: Mean/Variance Space

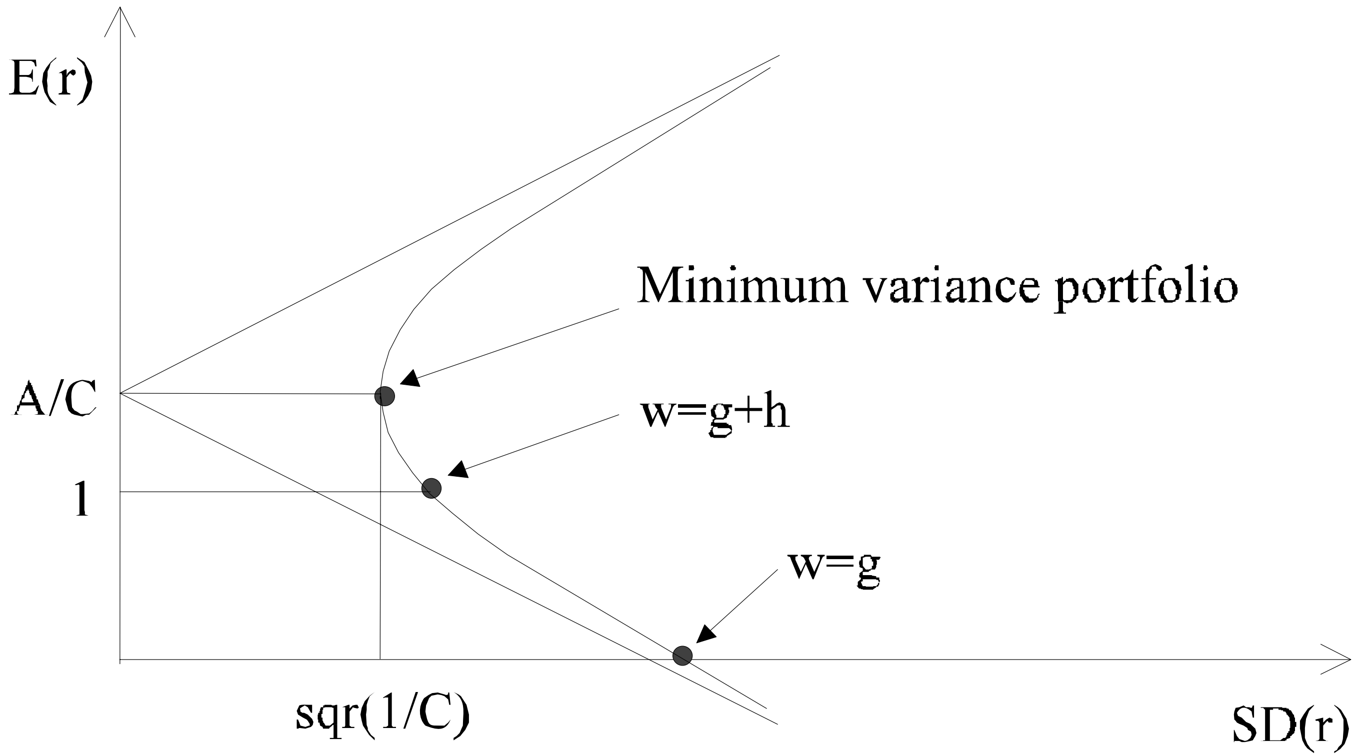


Figure 6-4 The Set of Frontier Portfolios: Mean/SD Space

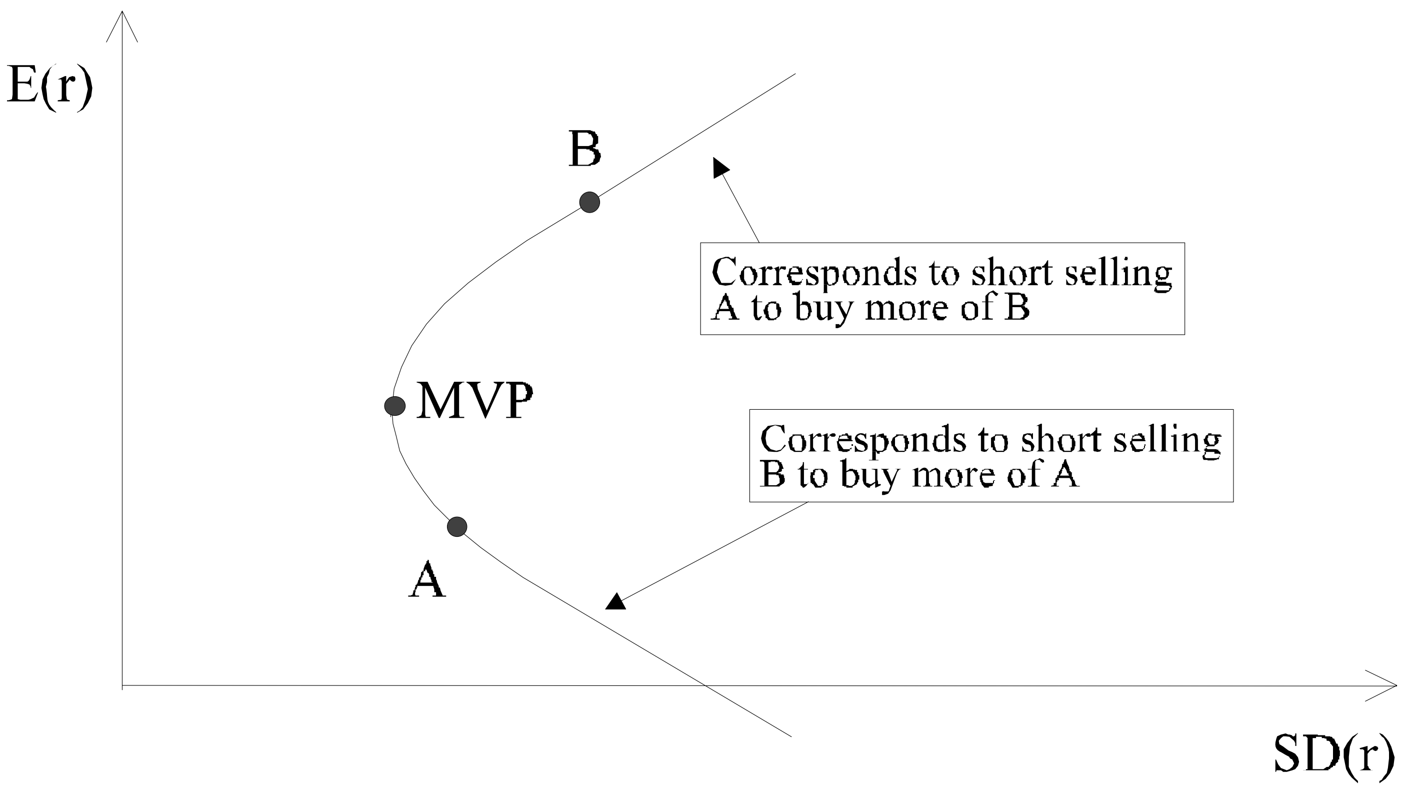


Figure 6-5 The Set of Frontier Portfolios: Short Selling Allowed

Characterization of Efficient Portfolios
(No Risk-Free Assets)

- Definition 6.2: *Efficient portfolios are those frontier portfolios which are not mean-variance dominated.*
- *Lemma: Efficient portfolios are those frontier portfolios for which the expected return exceeds* A/C, *the expected return of the minimum variance portfolio.*

Zero Covariance Portfolio

- Zero-Cov Portfolio is useful for Zero-Beta CAPM

- Proposition 6.5: *For any frontier portfolio* p, *except the minimum variance portfolio, there exists a unique frontier portfolio with which* p *has zero covariance.*

  We will call this portfolio the "*zero covariance portfolio relative to* p", and denote its vector of portfolio weights by

  $ZC(p)$.

- Proof: by construction.

$$Cov[r_p, r_q] := w_p^T V w_q$$
$$Cov[r_p, r_q] = [\lambda V^{-1} e + \gamma V^{-1} 1]^T V w_q$$
$$Cov[r_p, r_q] = \lambda e^T V^{-1} V w_q + \gamma 1^T V^{-1} V w_q$$
$$Cov[r_p, r_q] = \lambda e^T w_q + \gamma$$
$$Cov[r_p, r_q] = \lambda E[r_q] + \gamma$$

where $\lambda = (CE[r_p] - A)/D$ and $\gamma = (B - AE[r_p])/D$

Hence,

$$Cov[r_p, r_q] = \frac{CE[r_p]-A}{D} E[r_q] + \frac{B-AE[r_p]}{D}$$

collect all expected returns terms, add and subtract A^2C/DC^2 and note that the remaining term $(1/C)[(BC/D)-(A^2/D)]=1/C$, since $D=BC-A^2$

$$Cov[r_p, r_q] = \frac{C}{D}[E[r_p] - \frac{A}{C}][E[r_q] - \frac{A}{C}] + \frac{1}{C}$$

$$Cov[r_p, r_q] = \frac{C}{D}[E[r_p] - \frac{A}{C}][E[r_q] - \frac{A}{C}] + \frac{1}{C}$$

For zero co-variance portfolio ZC(p)

$$Cov[r_p, r_{ZC(p)}] = 0$$

$$0 = \frac{C}{D}[E[r_p] - \frac{A}{C}][E[r_{ZC(p)}] - \frac{A}{C}] + \frac{1}{C}$$

$$E[r_{ZC(p)}] = \frac{A}{C} - \frac{D/C^2}{E[r_p]-A/C}$$

For graphical illustration, let's draw this line:

$$E[r] = \frac{A}{C} - \frac{D/C^2}{E[r_p]-A/C} + \frac{E[r_p]-A/C}{\sigma^2[r_p]-1/C}\sigma^2[r]$$

Graphical Representation:

$$E[r] = \frac{A}{C} - \frac{D/C^2}{E[r_p]-A/C} + \frac{E[r_p]-A/C}{\sigma^2[r_p]-1/C}\sigma^2[r]$$

line through
p (Var[r_p], E[r_p]) AND
MVP (1/C, A/C) (use $\sigma^2(\tilde{r}_p) = \frac{C}{D}\left(E(\tilde{r}_p) - \frac{A}{C}\right)^2 + \frac{1}{C}$)

for $\sigma^2(r) = 0$ you get $E[r_{ZC(p)}]$

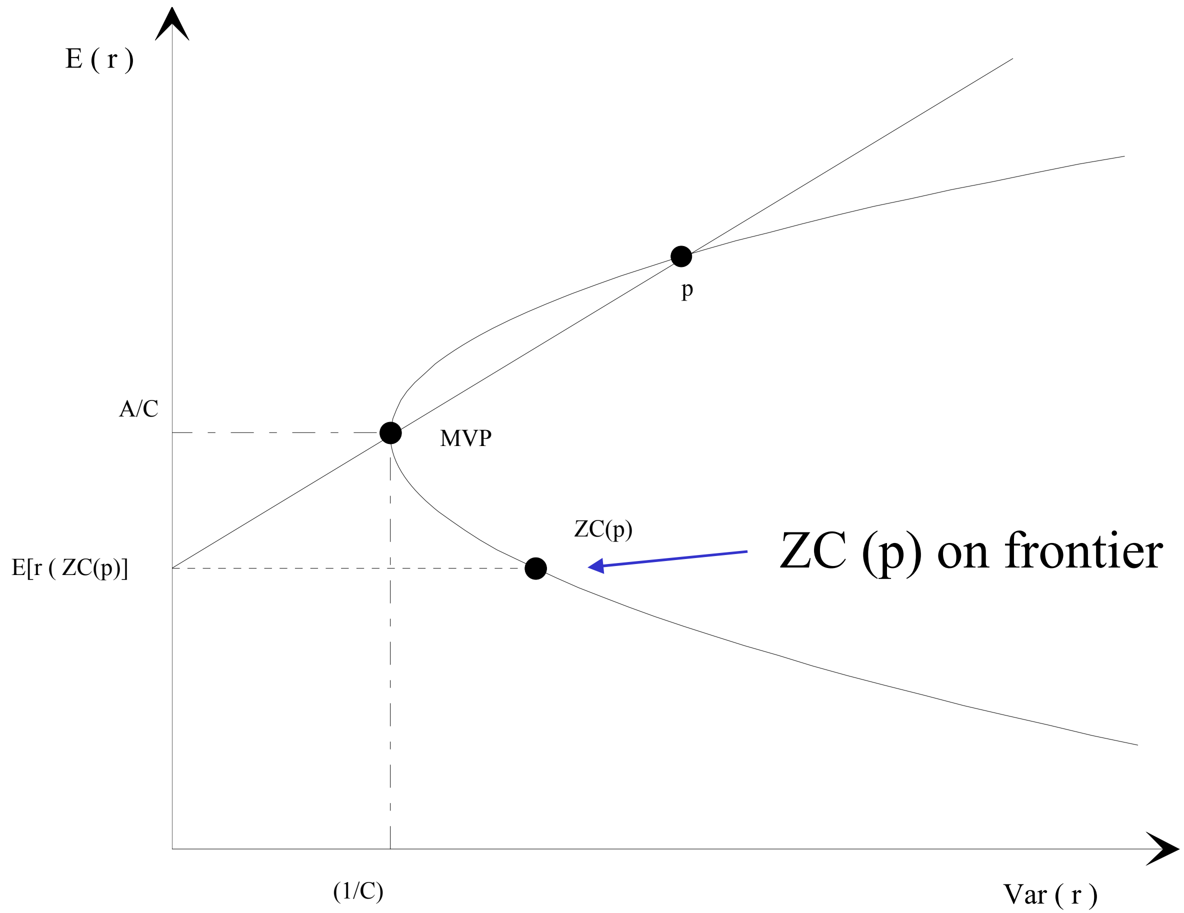


Figure 6-6 The Set of Frontier Portfolios: Location of the Zero-Covariance Portfolio

Zero-Beta CAPM

(no risk-free asset)

(i) agents maximize expected utility with increasing and strictly concave utility of money functions and asset returns are multivariate normally distributed, or

(ii) each agent chooses a portfolio with the objective of maximizing a derived utility function of the form $U(e, \sigma^2), U_1 > 0, U_2 < 0$, U concave.

(iii) common time horizon,

(iv) homogeneous beliefs about *e* and σ^2

– All investors hold mean-variance efficient portfolios

– the market portfolio is convex combination of efficient portfolios $\Rightarrow$ is efficient.

– $Cov[r_p, r_q] = \lambda\, E[r_q] + \gamma$ (*q* need not be on the frontier) (6.22)

– $Cov[r_p, r_{ZC(p)}] = \lambda\, E[r_{ZC(p)}] + \gamma = 0$

– $Cov[r_p, r_q] = \lambda\, \{E[r_q] - E[r_{ZC(p)}]\}$

– $Var[r_p] = \lambda\, \{E[r_p] - E[r_{ZC(p)}]\}$

Divide third by fourth equation:

$$E(\tilde{r}_q) = E(\tilde{r}_{ZC(M)}) + \beta_{Mq}\left[E(\tilde{r}_M) - E(\tilde{r}_{ZC(M)})\right] \quad (6.28)$$

$$E(\tilde{r}_j) = E(\tilde{r}_{ZC(M)}) + \beta_{Mj}\left[E(\tilde{r}_M) - E(\tilde{r}_{ZC(M)})\right] \quad (6.29)$$

Zero-Beta CAPM

- mean variance framework (quadratic utility or normal returns)
- In equilibrium, market portfolio, which is a convex combination of individual portfolios

$$E[r_q] = E[r_{ZC(M)}] + \beta_{Mq}[E[r_M] - E[r_{ZC(M)}]]$$

$$E[r_j] = E[r_{ZC(M)}] + \beta_{Mj}[E[r_M] - E[r_{ZC(M)}]]$$

The Standard CAPM
(with risk-free asset)

$$\min_w \tfrac{1}{2} w^T V w$$
$$\text{s.t. } w^T e + (1 - w^T 1) r_f = E[r_p]$$

FOC: $$w_p = \lambda V^{-1}(e - r_f 1)$$

Multiplying by $(e - r_f\, 1)^T$ and solving for λ yields $\lambda = \dfrac{E[r_p] - r_f}{(e - r_f 1)^T V^{-1} (e - r_f 1)}$

$$w_p = V^{-1}(e - r_f \mathbf{1}) \frac{E(\tilde{r}_p) - r_f}{H} \qquad (6.30)$$

V^{-1}: n x n; $(e - r_f \mathbf{1})$: n x 1; together: n x 1; $\frac{E(\tilde{r}_p) - r_f}{H}$: a number

where $H = B - 2Ar_f + Cr_f^2$

$$Cov[r_q, r_p] = (E[\tilde{r}_q - r_f]) \underbrace{\frac{1}{H}(e - r_f 1)^T (V^{-1})^T (e - r_f 1) \frac{1}{H}}_{:=G} (E[\tilde{r}_p - r_f])$$

$$Var[r_p] = (E[\tilde{r}_p - r_f])^2 G$$

NB:Derivation in DD is not correct.

Rewrite first equation and replace G using second equation.

$$E[r_q] - r_f = \frac{Cov[r_q, r_p]}{G} \frac{1}{E[r_p] - r_f}$$

$$= \underbrace{\frac{Cov[r_q, r_p]}{Var[r_p]}}_{:=\beta_{q,p}} (E[r_p] - r_f)$$

Holds for any frontier portfolio, in particular the market portfolio.