Lecture 06: Factor Pricing

Prof. Markus K. Brunnermeier

Overview

- Theory of Factor Pricing (APT)
  - ➢ Merits of Factor Pricing
  - ➢ Exact Factor Pricing and Factor Pricing Errors
  - ➢ Factor Structure and Pricing Error Bounds
  - ➢ Single Factor and Beta Pricing (and CAPM)
  - ➢ (Factor) Mimicking Portfolios
  - ➢ Unobserved Factor Models
  - ➢ Multi-period outlook
- Empirical Factor Pricing Models
  - ➢ Arbitrage Pricing Theory (APT) Factors
  - ➢ The Fama-French Factor Model + Momentum
  - ➢ Factor Models from the Street
    - Salomon Smith Barney's and Morgan Stanley's Model

The Merits of Factor Models

- Without any structure one has to estimate
  - J expected returns $E[R^j]$ (for each asset j)
  - J standard deviations
  - J(J-1)/2 co-variances
- Assume that the correlation between any two assets is explained by systematic components/factors, one can restrict attention to only K (non-diversifiable) factors
  - Advantages:
    - Drastically reduces number of input variables
    - Models expected returns (priced risk)
    - Allows to estimate systematic risk (even if it is not priced, i.e. uncorrelated with SDF)
    - Analysts can specialize along factors
  - Drawbacks:
    - Purely statistical model (no theory) (does not explain why factor deserves compensation: risk vs mispricing)
    - relies on past data and assumes stationarity

Factor Pricing Setup …

- K factors $f_1, f_2, \ldots, f_K$
  - $E[f_k]=0$
  - K is small relative to dimension of $\mathcal{M}$
  - f_k are not necessarily in $\mathcal{M}$
- $\mathcal{F}$ space spanned by $f_1,\ldots,f_K,e$
- in payoffs

$$x_j = E(x_j)1 + \sum_{k=1}^{K} b_{jk} f_k + \delta_j,$$

with $\delta_j \perp \mathcal{F}$, and in particular $E[\delta_j] = 0$.

  - $b_{j,k}$ factor loading of payoff x_j

…Factor Pricing Setup

- in returns

$$r_j = E[r_j] + \sum_{k=1}^{K} \beta_{jk} f_k + \epsilon_j, \qquad (1)$$

with $\beta_{jk} = \frac{b_{jk}}{p_j}$, the factor loading of return r_j, and $\epsilon_j = \frac{\delta_j}{p_j}$.

- Remarks:
  - One can always choose orthogonal factors $Cov[f_k, f_{k'}]=0$
  - Factors can be observable or unobservable

Factor Structure

- Definition of "factor structure:"

$$r_j = E[r_j] + \sum_{k=1}^{K} \beta_{jk} f_k + \epsilon_j \text{ (1), where}$$

$$\text{cov}(\epsilon_j, \epsilon_i) = 0 \text{ if } i \neq j, \ E[\epsilon_j] = 0 \text{ and}$$
$$\text{cov}(\epsilon_j, f_k) = 0 \text{ for each } (j, k).$$

- $\Rightarrow$ risk can be split in *systematic* risk and *idiosyncratic (diversifiable)* risk

Exact vs. Approximate Factor Pricing

- Multiplying (1) by k_q and taking expectations

$$1 = E[r_j]E[k_q] + \sum_{k=1}^{K} \beta_{jk} E[k_q f_k] + E[k_q \epsilon_j].$$

- Rearranging

$$E[r_j] = \underbrace{\frac{1}{E[k_q]}}_{=:\gamma_0} + \sum_{k=1}^{K} \beta_{jk} \underbrace{\frac{-E[k_q f_k]}{E[k_q]}}_{=:\gamma_k} \underbrace{-\frac{E[k_q \epsilon_j]}{E[k_q]}}_{=:\psi_j \text{[error]}}$$

- *Exact factor pricing:*
  - error: $\psi_j = 0$ (i.e. ε_j s orthogonal to k_q)
  - e.g. if $k_q \in \mathcal{F}$

Bound on Factor Pricing Error…

- Recall error $\psi_j = -\frac{E[k_q \epsilon_j]}{E[k_q]}$
  - Note, if ∃ risk-free asset and all $f_k \in \mathcal{M}$, then $\psi_j = -\bar{r}q(\epsilon_j)$.
- If $k_q \in \mathcal{F}$, then factor pricing is exact
- If $k_q \notin \mathcal{F}$, then $k_q = k_q^{\mathcal{F}} + \eta$, with $\eta \perp \mathcal{F}$, $E[k_q \epsilon_j] = E[\eta \epsilon_j]$.
  - Let's make use of the Cauchy-Schwarz inequality (which holds for any two random variables z_1 and z_2)

$$|E[z_1 z_2]| \leq \sqrt{E[z_1^2]}\sqrt{E[z_2^2]}.$$

$$|E[\eta \epsilon_j]| \leq \sqrt{E[\eta^2]}\sqrt{E[\epsilon_j^2]} = \sqrt{E[(k_q - k_q^{\mathcal{F}})^2]}\sigma(\epsilon_j).$$

  - Error-bound

$$|\psi_j| \leq \frac{1}{E[k_q]}\sigma(\epsilon_j)\sqrt{E[(k_q - k_q^{\mathcal{F}})^2]}.$$

Error-Bound if Factor Structure Holds

- Factor structure ⇒ split idiosyncratic from systematic risk
- ⇒ all idiosyncratic risk ε_j are linearly independent and span space orthogonal to $\mathcal{F}$. Hence, $\eta = \sum_j^J a_j \epsilon_j$
- Note $E[k_q \epsilon_j] = E[\eta \epsilon_j] = a_j E[\epsilon_j^2] = a_j \sigma^2(\epsilon_j^2)$
- Error $\psi_j = -\frac{E[k_q \epsilon_j]}{E[k_q]} = -\frac{1}{E[k_q]} a_j \sigma^2(\epsilon_j)$
- *Pythagorean Thm:* If $\{z_1, \ldots, z_n\}$ is orthogonal system in Hilbert space, then $\|\sum_{i=1}^n z_i\|^2 = \sum_{i=1}^n \|z_i\|^2$
  - Follows from def. of inner product and orthogonality

Error-Bound if Factor Structure Holds

Applying Pythagorean Thm to $\eta = \sum_j^J a_j \epsilon_j$
implies $\sum_j^J a_j^2 E[\epsilon_j^2] = ||\eta||^2$

$\sum_j^J a_j^2 \sigma^2(\epsilon_j) = ||k_q - k_q^{\mathcal{F}}||^2$

Multiply by $(1/E[k_q]^2)\max_j\{\sigma^2(\epsilon_j)\}$ and making use of $\sigma^2(\epsilon_j) \leq \max_j\{\sigma^2(\epsilon_j)\}$

$$\sum_j^J \psi_j^2 \leq \frac{1}{E[k_q]^2} E[(k_q - k_q^{\mathcal{F}})^2] max_j\{\sigma^2(\epsilon_j)\}.$$

RHS is constant for constant $\max[\sigma^2(\varepsilon_j)]$.

⇒ For large J, most securities must have small pricing error

- Intuition for Approximate Factor Pricing:
  Idiosyncratic risk can be diversified away

One Factor Beta Model…

- Let r be a risky frontier return and set $f = r - E[r]$ (i.e. f has zero mean)
  - ➢ $q(f) = q(r) - q(E[r])$
- Risk free asset exists with gross return of $\bar{r}$
  - ➢ $q(f) = 1 - E[r]/\bar{r}$
- f and $\bar{r}$ span $\mathcal{E}$ and hence $k_q \in \mathcal{F}$

$\Rightarrow$ *Exact Factor Pricing*

…One Factor Beta Model

- Recall $E[r_j] = \underbrace{\frac{1}{E[k_q]}}_{=:\gamma_0=\bar{r}} + \sum_{k=1}^{K} \beta_{jk} \underbrace{\frac{-E[k_q f_k]}{E[k_q]}}_{=:\gamma_k} - \underbrace{\frac{E[k_q \epsilon_j]}{E[k_q]}}_{=:\psi_j=0}$
  - $E[r_j] = \bar{r} - \beta_j \bar{r}\, q(f)$
  - $E[r_j] = \bar{r} - \beta_i \{E[r] - \bar{r}\}$
- Recall $r_j = E[r_j] + \sum_{k=1}^{K} \beta_{jk} f_k + \epsilon_j$
  - β_j = Cov[r_j, f] / Var[f] = Cov[r_j, r] / Var[r]
- If $r_m \in \mathcal{E}$ then CAPM

Mimicking Portfolios…

- Regress on factor directly or on portfolio that mimics factor
  - ➢ Theoretical justification: project factor on $\mathcal{M}$
  - ➢ Advantage: portfolios have smaller measurement error
- Suppose portfolio contains shares $\alpha_1, \ldots, \alpha_J$ with $\sum_j^J \alpha_j = 1$.
- Sensitivity of portfolio w.r.t. to factor f_k is $\gamma_k = \sum_j \alpha_j \beta_{jk}$
- Idiosyncratic risk of portfolio is $\nu = \sum_j \alpha\ \varepsilon_j$
  - ➢ $\sigma^2(\nu) = \sum_j \alpha^2\ \sigma(\varepsilon_j)$
  - ➢ diversification

…Mimicking Portfolios

- Portfolio is *only* sensitive to factor k_0 (and idiosyncratic risks) if for each $k \neq k_0$ $\gamma_k = \sum \alpha_j \beta_{jk} = 0$, and $\gamma_{k0} = \sum \alpha_j \beta_{jk0} \neq 0$.
- The dimension of the space of portfolios sensitive to a particular factor is J-(K-1).
- A portfolio mimics factor k_0 if it is the portfolio with smallest idiosyncratic risk among portfolios that are sensitive only to k_0.

Observable vs. Unobservable Factors…

- Observable factors: GDP, inflation etc.
- Unobservable factors:
  - ➢ Let data determine "abstract" factors
  - ➢ Mimic these factors with "mimicking portfolios"
  - ➢ Can always choose factors such that
    - factors are orthogonal, $Cov[f_k, f_{k'}]=0$ for all $k \neq k'$
    - Factors satisfy "factor structure" (systemic & idiosyncratic risk)
    - Normalize variance of each factor to ONE

      ⇒ pins down factor sensitivity (but not sign, - one can always change sign of factor)

…Unobservable Factors…

- Empirical content of factors
  - ➢ $Cov[r_i,r_j] = \sum_k \beta_{ik}\beta_{jk}\sigma^2(f_k)$
  - ➢ $\sigma^2(r_j) = \sum_k \beta_{jk}\beta_{jk}\sigma^2(f_k)+\sigma^2(\varepsilon_j)$
  - ➢ $\sigma(f_k)=1$ for $k=1,\cdots,K$. (normalization)
  - ➢ In matrix notation
    - $Cov[r,r'] = \sum_k \beta_k{}'\beta_k\sigma^2(f_k) + D$,
      - where $\beta_k = (\beta_{1k},\ldots,\beta_{Jk})$.
    - $\Omega = B\,B' + D$,
      - where $B_{jk}=\beta_{jk}$, and D diagonal.
      - For PRINCIPAL COMPONENT ANALYSIS assume D=0 (if D contains the same value along the diagonal it does affect eigenvalues but not eigenvectors – which we are after)

…Unobservable Factors…

- For any symmetric JxJ matrix A (like BB'), which is semi-positive definite, i.e. $y'Ay \geq 0$, there exist numbers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \text{lambda}_J \geq 0$ and non-zero vectors $y_1, \ldots, y_J$ such that
  - y_j is an eigenvector of A assoc. w/ eigenvalue λ_j, that is $A y_j = \lambda_j y_j$
  - $\sum_j^J y^i_j y^i_{j'} = 0$ for $j \neq j'$
  - $\sum_j^J y^i_j y^i_j = 1$
  - rank (A) = number of non-zero λ 's
  - The y_j 's are unique (except for sign) if the λ_i 's are distinct
- Let Y be the matrix with columns $(y_1,\ldots,y_J)$, and let Λ the diagonal matrix with entries λ_i then

$$A = Y\sqrt{\Lambda}\sqrt{\Lambda}Y'$$

…Unobservable Factors

- If K-factor model is true, BB' is a symmetric positive semi-definite matrix of rank $K.$
  - Exactly K non-zero eigenvalues $\lambda_1,\ldots,\lambda_k$ and associated eigenvectors $y_1,\ldots,y_K$
  - Y_K the matrix with columns given by $y_1,\ldots,y_K$ Λ_K the diagonal matrix with entries λ_i, j=1,…, K.
  - $BB' = Y_K\sqrt{\Lambda_K}\sqrt{\Lambda_K}Y_K'$.

    Hence,

$$r_j = \sum_{k=1}^{K} (Y_K\sqrt{\Lambda_K})_{jk} f_k + \epsilon_j$$

- Factors are not identified but sensitivities are (except for sign.)
- In practice choose K so that λ_k is small for k>K.

Why more than ONE mimicking portfolio?

- Mimic (un)observable factors with portfolios
  [Projection of factor on asset span]
- Isn't a ***single*** portfolio which mimics pricing kernel sufficient $\Rightarrow$ ONE factor
- So why multiple factors?
  - Not all assets are included (real estate, human capital …)
  - Other factors capture dynamic effects
    [since e.g. conditional $\neq$ unconditional. CAPM]
    (more later to this topic)

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APT Factors of Chen, Roll and Ross (1986)

1. Industrial production
   (reflects changes in cash flow expectations)
2. Yield spread btw high risk and low risk corporate bonds
   (reflects changes in risk preferences)
3. Difference between short- and long-term interest rate
   (reflects shifts in time preferences)
4. Unanticipated inflation
5. Expected inflation (less important)

Note: The factors replicate market portfolio.

Fama-MacBeth 2 Stage Method

- Stage 1: Use *time series* data to obtain estimates for each individual stock's β^j

$$R_t^j - R^f = \alpha + \beta^j (R_t^m - R_t^f) + \epsilon_t^j$$

(e.g. use monthly data for last 5 years)

Note: $\widehat{\beta}^j$ is just an estimate [around true β^j]

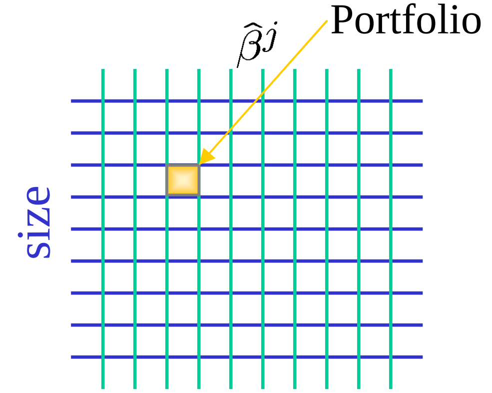
- Stage 2: Use *cross sectional* data and estimated β^js to estimate SML

$$R^j_{\text{next month}} = a + b\widehat{\beta}^j + e^j$$

b=market risk premium

CAPM β–Testing Fama French (1992)

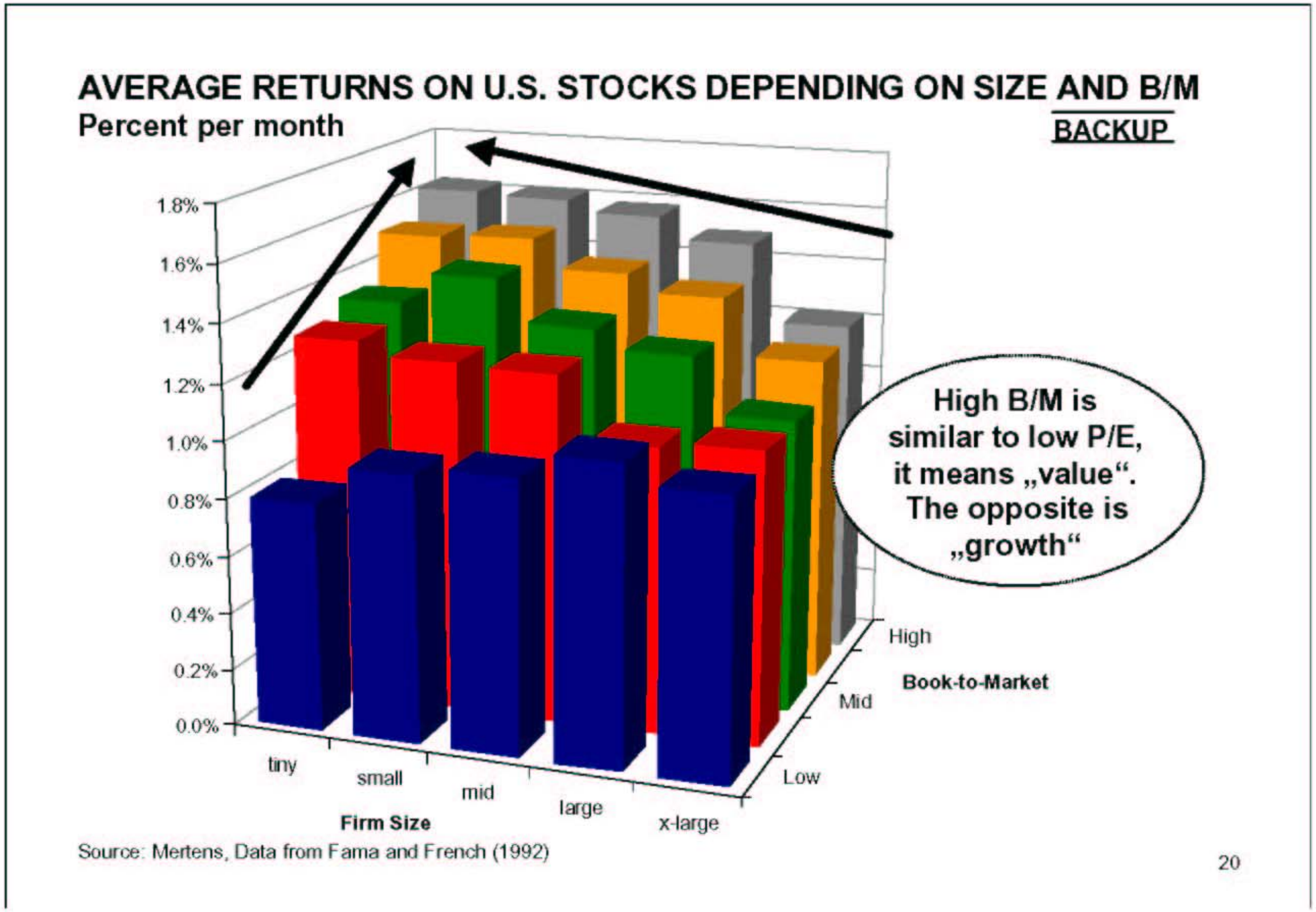
- Using newer data slope of SML b is not significant (adding size and B/M)
- Dealing with econometrics problem:
  - ➢ $\hat{\beta}^j$ s are only noisy estimates, hence estimate of b is biased
  - ➢ Solution:
    - Standard Answer: Find instrumental variable
    - Answer in Finance: Derive $\hat{\beta}$ estimates for portfolios
      - Group stocks in 10 x 10 groups sorted to size and estimated $\hat{\beta}^j$
      - Conduct Stage 1 of Fama-MacBeth for portfolios
      - Assign all stocks in same portfolio same β
      - Problem: Does not resolve insignificance



- *CAPM predictions:* b is significant, all other variables insignificant
- *Regressions:* size and B/M are significant, b becomes insignificant
  - ➢ Rejects CAPM

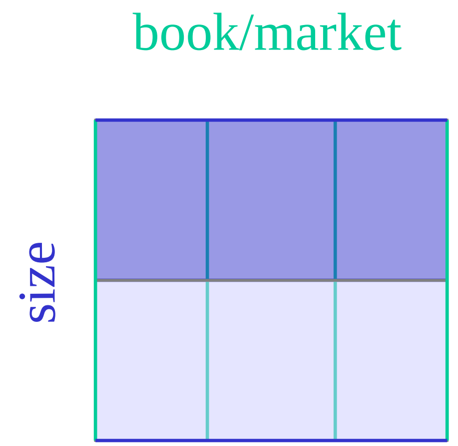
Book to Market and Size

Small „value“ companies have higher returns



Fama French Three Factor Model

- Form 2x3 portfolios
  - ➢Size factor (SMB)
    - Return of small minus big
  - ➢Book/Market factor (HML)
    - Return of high minus low



- For $R_t^j - R_t^f = \alpha^p + \beta^p (R_t^m - R_t^f)$

  αs are big and βs do not vary much

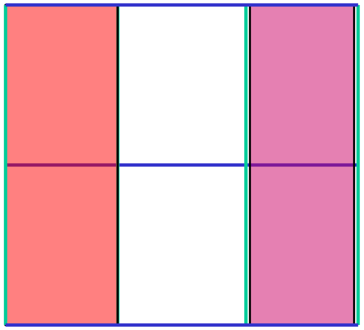
- For $R_t^p - R_t^f = \alpha^p + \beta^p (R_t^m - R_t^f) + \gamma^p \mathsf{SMB}_t^p + \delta^p \mathsf{HML}_t^p$

  (for each portfolio p using time series data)

  αs are zero, coefficients significant, high R^2.

Fama French Three Factor Model

- Form 2x3 portfolios
  - ➢Size factor (SMB)
    - Return of small minus big
  - ➢Book/Market factor (HML)
    - Return of high minus low



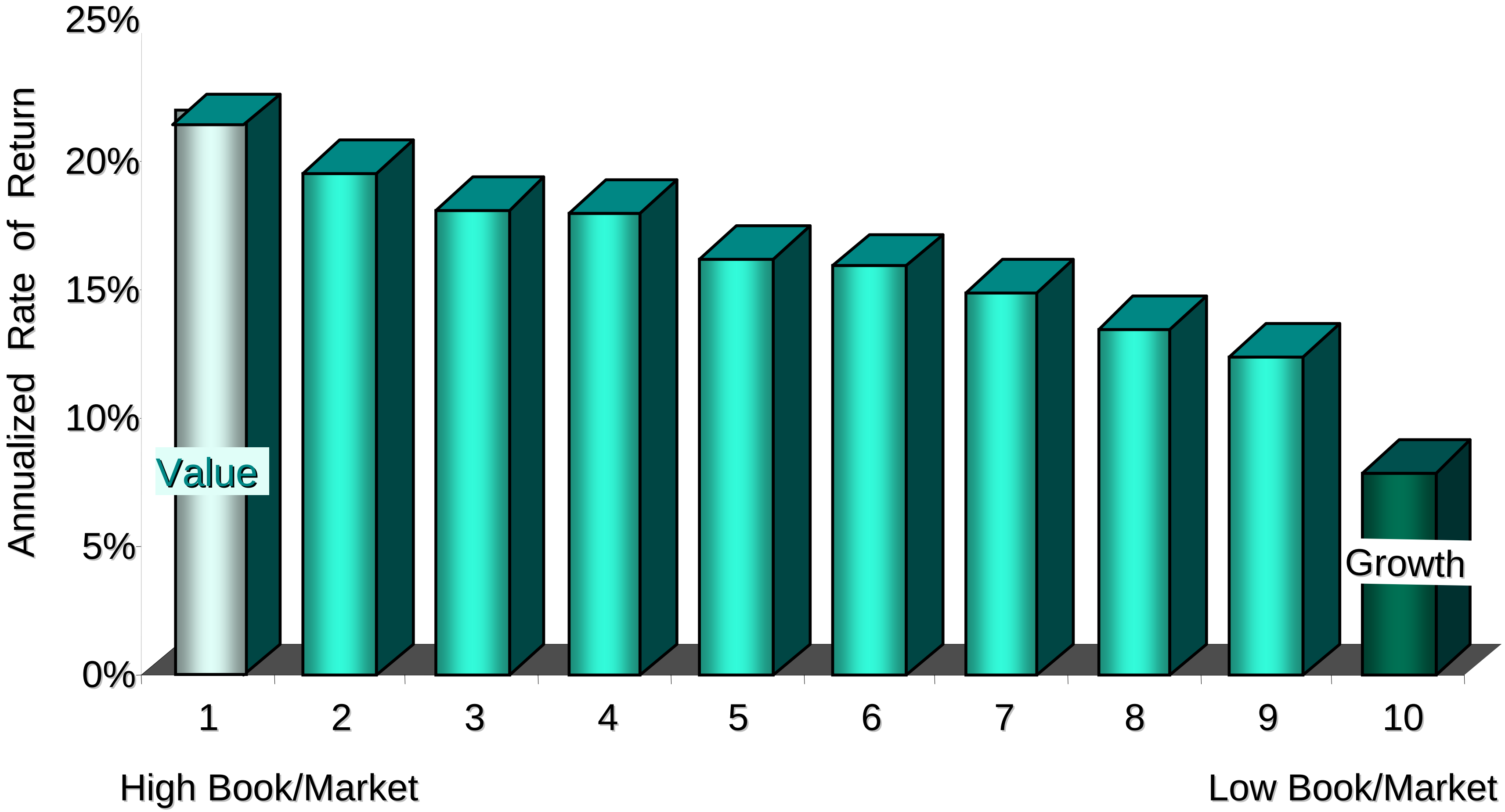
- For $R_t^j - R_t^f = \alpha^p + \beta^p (R_t^m - R_t^f)$

  αs are big and βs do not vary much
- For $R_t^p - R_t^f = \alpha^p + \beta^p (R_t^m - R_t^f) + \gamma^p \mathsf{SMB}_t^p + \delta^p \mathsf{HML}_t^p$

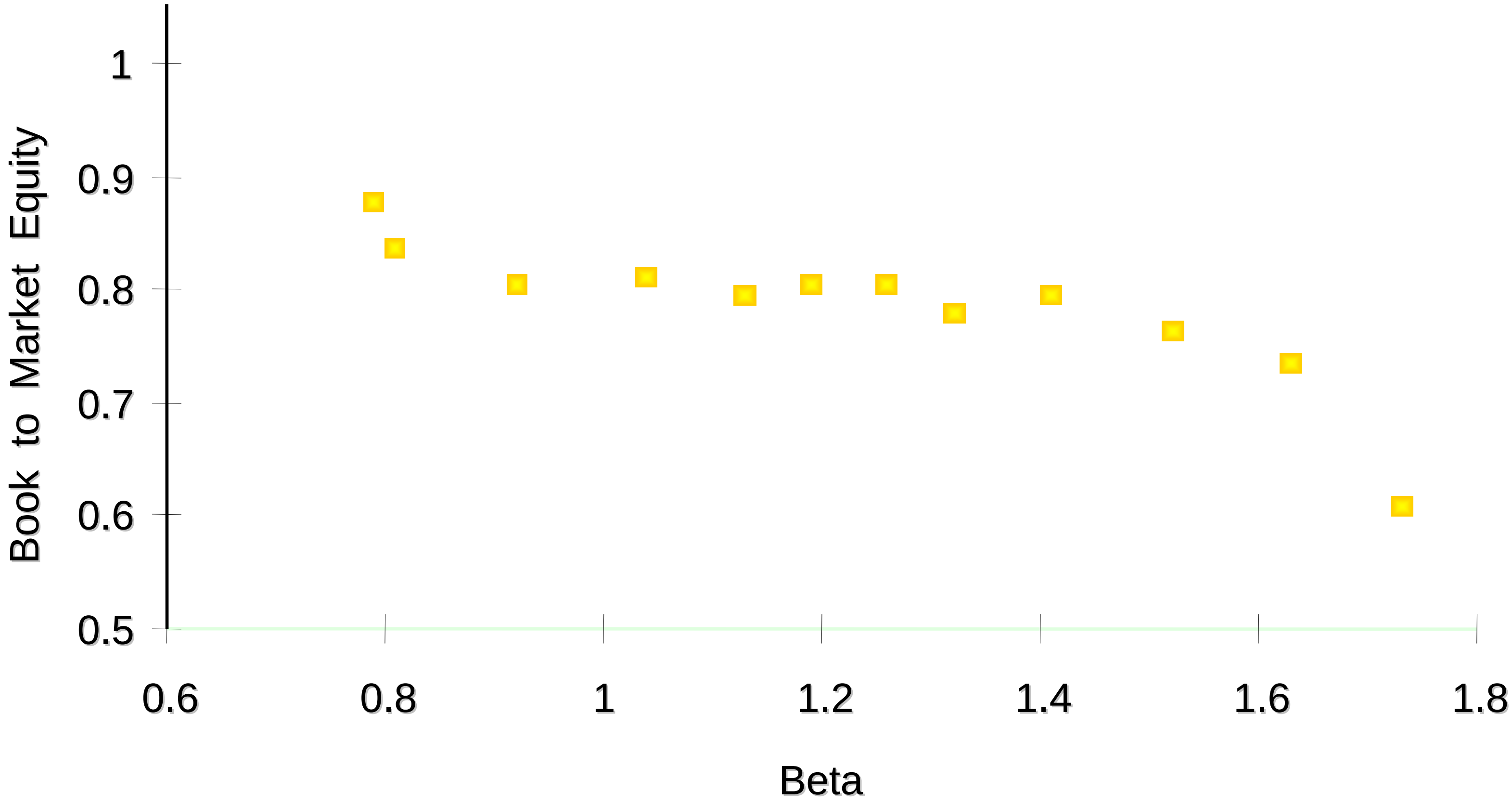
  (for each portfolio p using time series data)

  α^ps are zero, coefficients significant, high R^2.

Book to Market as a Predictor of Return

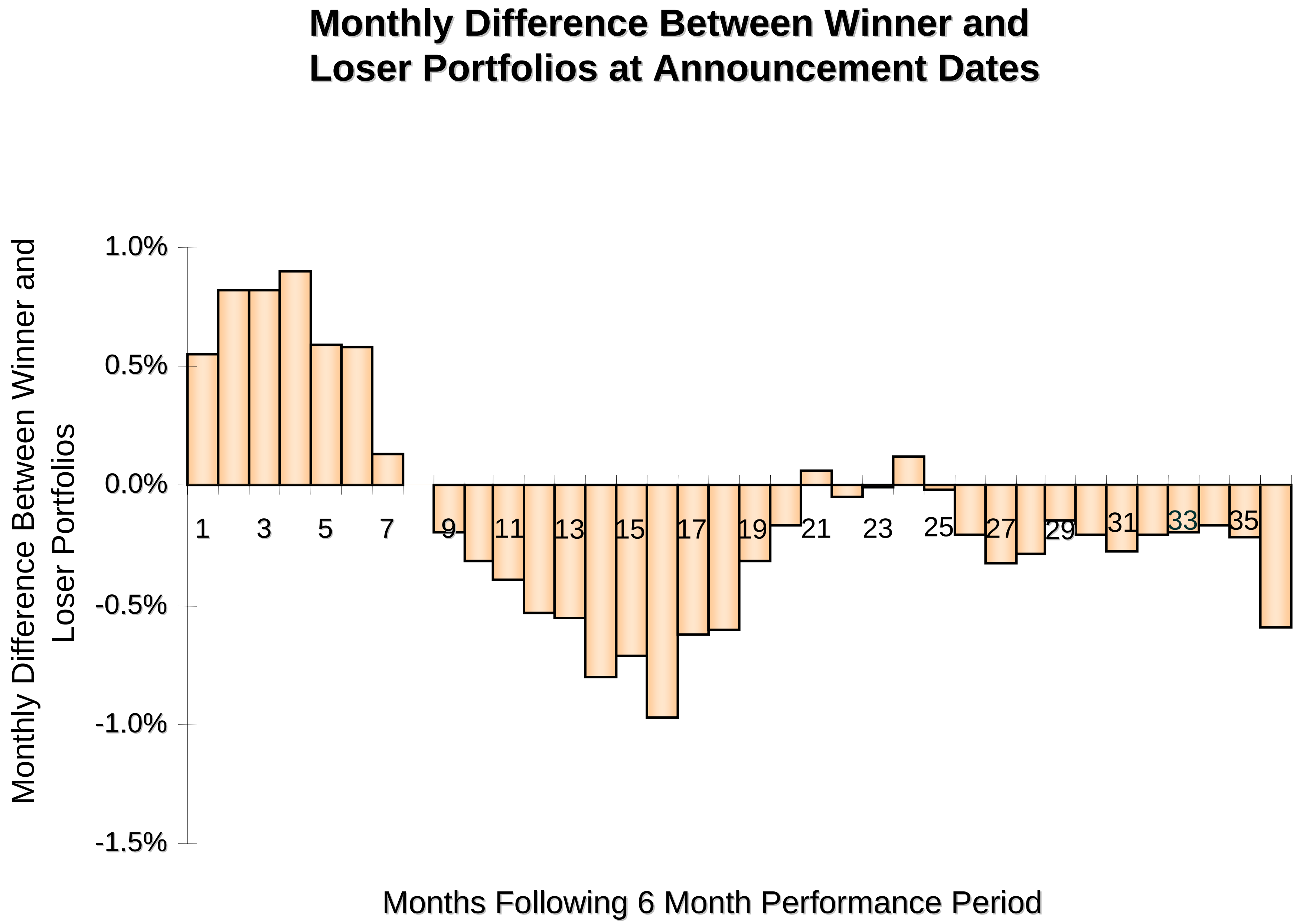


Book to Market Equity of Portfolios Ranked by Beta



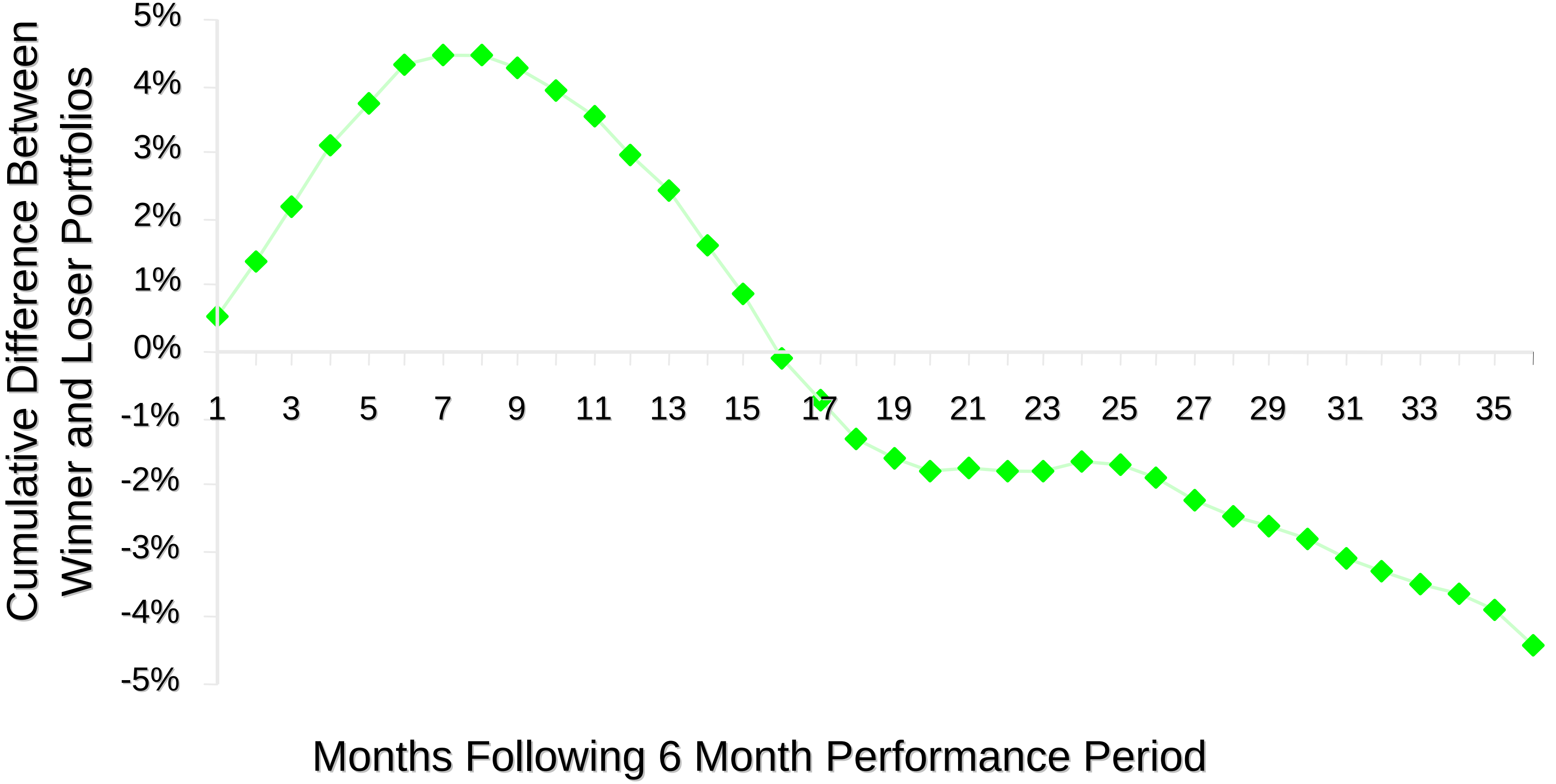
Adding Momentum Factor

- 5x5x5 portfolios
- Jegadeesh & Titman 1993 JF rank stocks according to performance to past 6 months
  - Momentum Factor
    Top Winner minus Bottom Losers Portfolios



Monthly Difference Between Winner and
Loser Portfolios at Announcement Dates
Monthly Difference Between Winner and
Loser Portfolios
1.0%
0.5%
0.0%
-0.5%
-1.0%
-1.5%
1
3
5
7
9
11
13
15
17
19
21
23
25
27
29
31
33
35
Months Following 6 Month Performance Period

Cumulative Difference Between Winner and Loser Portfolios at Announcement Dates



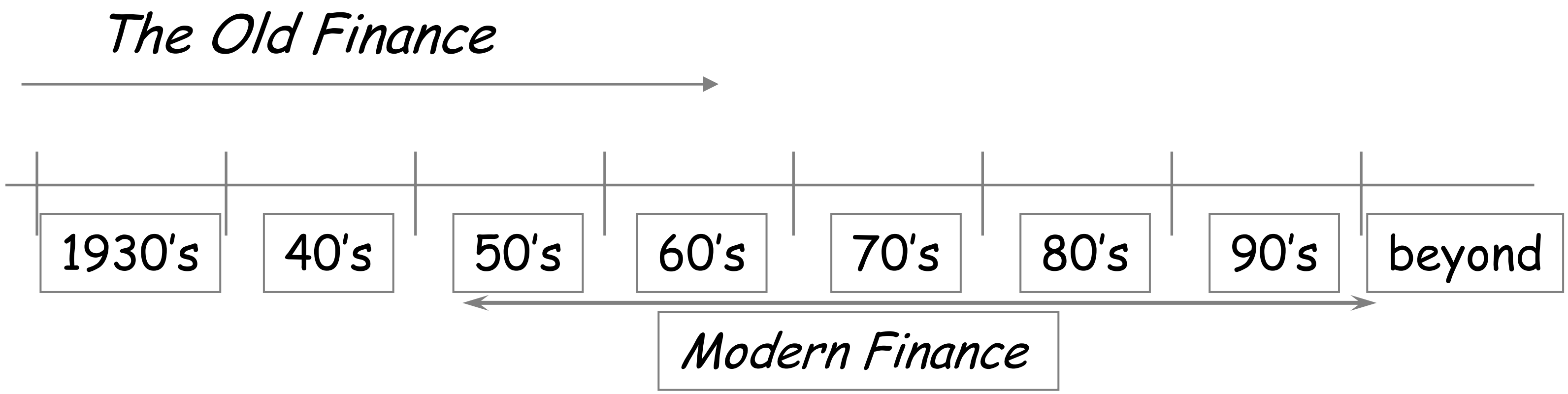
Morgan Stanley's Macro Proxy Model

- Factors
  - GDP growth
  - Long-term interest rates
  - Foreign exchange (Yen, Euro, Pound basket)
  - Market Factor
  - Commodities or oil price index
- Factor-mimicking portfolios ("Macro Proxy")
  - Stage 1: Regress individual stocks on macro factors
  - Stage 2: Create long-short portfolios of most and least sensitive stocks [5 quintiles]
    - Macro Proxy return predicts macro factor

Salomon Smith Barney Factor Model

- Factors
  - Market trend (drift)
  - Economic growth
  - Credit quality
  - Interest rates
  - Inflation shocks
  - Small cap premium

Haugen’s view: The Evolution of Academic Finance



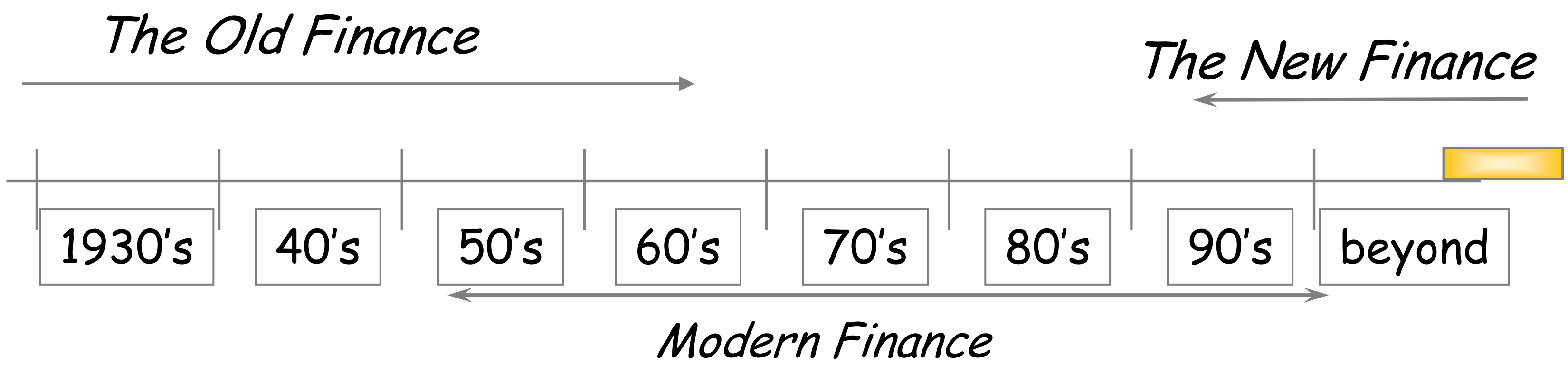
Modern Finance

Theme: Valuation Based on Rational Economic Behavior

Paradigms: Optimization *(Markowitz)* Irrelevance *(Modigliani & Miller)* CAPM *(Sharpe, Lintner & Mossen)* EMH *(Fama)*

Foundation: Financial Economics

Haugen's view: The Evolution of Academic Finance



The New Finance

Theme: Inefficient Markets

Paradigms: Inductive *ad hoc* Factor Models — Behavioral Models

Expected Return — Risk

Foundation: Statistics, Econometrics, and Psychology