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COMPETING MECHANISMS IN A COMMON VALUE ENVIRONMENT

BY BRUNO BIAIS, DAVID MARTIMORT, AND JEAN-CHARLES ROCHET[1]

Consider strategic risk-neutral traders competing in schedules to supply liquidity to a risk-averse agent who is privately informed about the value of the asset and his hedging needs. Imperfect competition in this *common value environment* is analyzed as a *multi-principal* game in which liquidity suppliers offer trading mechanisms in a decentralized way. Each liquidity supplier behaves as a monopolist facing a residual demand curve resulting from the maximizing behavior of the informed agent and the trading mechanisms offered by his competitors. There exists a unique equilibrium in convex schedules. It is symmetric and differentiable and exhibits typical features of market-power: Equilibrium trading volume is lower than ex ante efficiency would require. Liquidity suppliers charge positive mark-ups and make positive expected profits, but these profits decrease with the number of competitors. In the limit, as this number goes to infinity, ask (resp. bid) prices converge towards the upper (resp. lower) tail expectations obtained in Glosten (1994) and expected profits are zero.

KEYWORDS: Multi-principals, mechanism design, competing market-makers, financial markets microstructure.

1. INTRODUCTION

THIS PAPER ANALYZES IMPERFECT COMPETITION under adverse selection in financial markets as a multiprincipal game in which liquidity suppliers offer trading mechanisms in a decentralized way.

Financial markets are sometimes presented as textbook examples of perfect competition. In practice, however, the number of traders actively and regularly providing liquidity to the market is limited. In fact Christie and Schultz (1994a, 1994b) offer empirical evidence consistent with the hypothesis that NASDAQ market-makers post rather noncompetitive schedules, at which they earn positive profits.

Traders demanding liquidity in financial markets may do so to use their information on the underlying value of the asset or to share risk. Because informational signals and risk sharing needs are private information, adverse selection matters on these markets. Moreover, it is natural to assume that there is a *common value* element to the valuation of securities.

We capture both of these features of the market by modeling strategic liquidity suppliers posting nonlinear price schedules (such as limit order schedules) at which they stand ready to trade with a risk-averse agent who has private

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information on the fundamental value of the asset as well as on his hedging needs.

In this context, we address the following issues: How are prices formed and liquidity supplies determined in the presence of asymmetric information when market-makers are competing through liquidity supply? What sort of inefficiencies are driven by this asymmetric information? How are these inefficiencies affected by the number of competing market-makers? What are the consequences of the strategic behavior of the liquidity suppliers and of the informed agent on the distribution of informational rents in the economy?

That the trading mechanisms offered by the market-makers have both allocative and redistributive roles is a well-known lesson from the mechanism design literature. The analysis of optimal auctions (Myerson (1982)), optimal nonlinear pricing (Goldman, Leland, and Sibley (1984) and Maskin and Riley (1984)) and optimal bargaining procedures (Myerson and Sattherwaite (1983)) have all highlighted a fundamental trade-off between informational rents and efficiency.[2] We extend the analysis of this trade-off to a world of competing mechanism designers.

Contrary to standard mechanism design, we do not see the allocation of resources by financial markets as coming from the implementation of a single grand-mechanism. To fit more closely to the real behavior of market-makers, we see financial markets as a set of *competing incentive mechanisms* offered in a decentralized way by several mechanism designers: the market-makers. Taking this new perspective allows to understand better the consequences of strategic behavior on the supply side of the market.

While the standard mechanism design perspective assumes monopolistic behavior on the supply side of the market, we stress the multiprincipal nature of the competition among market-makers. In this case, equilibrium schedules reflect competition for market shares. This competition introduces an externality between market-makers. When posting his quotes, each liquidity supplier does not take into account the consequences of his mechanism offer on the rent-efficiency trade-offs achieved by his competitors.

We show that there exists a *unique* equilibrium with convex schedules.[3] Convexity of the oligopolistic price schedules points at the contrast with the monopoly case, whereby concave schedules, i.e., quantity discounts, could arise in equilibrium. This illustrates how the rent-efficiency trade-off arising in this mechanism design problem is radically altered once one steps away from the monopoly case to the oligopoly case.

[2] Wilson (1985) presents an insightful synthesis.

[3] Since transfer schedules are convex, they can be implemented by menus of limit orders. Therefore, they would also arise as an equilibrium if we constrained market-makers to post sequences of limit orders. Hence our analysis can also be viewed as a study of competition between agents placing limit orders. This points at its relevance for the analysis of order driven markets such as the Paris Bourse, the Tokyo Stock Exchange, and the New York Stock Exchange. It also points at the relation between our analysis and that of Glosten (1994) who studies competition in limit orders in the limiting case where there is an infinite number of market makers.

Certain features of the trading mechanisms, however, reflecting the desire by the competitors to extract surplus from the informed agent, or equivalently to minimize his informational rent, are qualitatively similar to those arising in the monopoly case. As in the monopoly case, equilibrium trading volume under oligopolistic screening is below its optimal level. This results from the endeavor by the oligopolists to reduce the informational rent of the agents. In particular, as a consequence of adverse selection, traders with relatively low willingness to trade are rationed and excluded from the market. Also, oligopolists quote marginal prices above the expectation of the asset value conditional on the size of the trade. These mark-ups are qualitatively similar to those quoted by the monopolist, although smaller. Corresponding to the pooling of some investors at zero trading, there exists a "small trade" spread, to use the term coined by Glosten (1989), whereby the ask price demanded for the sale of a very small positive quantity is strictly larger than the bid price quoted for a very small negative quantity.

Still, competition among market-makers leads to a deeper market, and a larger trading volume than in the monopoly case. In fact, the overall volume of trade increases and the spread decreases with the number of market-makers. Competition erodes the ability of each market-maker to reduce the supply of liquidity for rent-extraction reasons.

Moreover, and this is a central point in the paper, competition in this common value environment is limited. The market-makers do not bid the asset price to the expected value of this asset. Hence, as long as they are in finite number on this market, they earn strictly positive expected profits and the volume of trade remains far from the ex ante efficient one. The intuitive reason for this phenomenon is the following. To reflect the informational content of trades, unit prices quoted by each market-maker are increasing in trade size. This adjustment of prices to quantities, combined with the optimal response of the informed agent, implies that each market-maker faces a residual demand that is not infinitely elastic. In this context, the standard trade-off between price and quantity that arises under monopolistic screening is still present under oligopolistic screening. Hence each liquidity supplier quotes prices at which he earns positive expected profits. The mark-up between marginal costs and marginal prices reflects the elasticity of the above mentioned residual demand. In the special case where there is no asymmetric information on the value of the asset, but only on the hedging needs of the agent, prices no longer need to adjust to quantities to reflect information about the value of the asset. In this case, the residual demand curve faced by each competitor is infinitely elastic. Hence they have no choice but to quote constant marginal prices just equal to the expected value of the asset. Thus our analysis shows that adverse selection with common values plays a crucial role in driving a wedge between the outcome of competition in schedules and the Bertrand outcome.

In the limiting case where the number of market-makers goes to infinity, the expected profits earned by each individual market-maker as well as by market-makers as a whole go to zero. There still exists a strictly positive small trade spread, however, and the volume of trade remains bounded away from the ex ante efficient one. Moreover, ask (respectively bid) prices are equal to upper tail (respectively lower tail) expectations of the value of the asset, i.e., the expectation of the value conditional on its being above (respectively below) a certain threshold (as in Glosten (1994)).

Section 2 briefly surveys the literature. Section 3 presents the model. In Section 4, we present as a benchmark the case of monopolistic screening. Section 5 sets up the stage for the analysis of the oligopolistic screening case. We describe the best response mapping of each market-maker to his competitors' schedules in an appropriate strategy space. Then we derive the equilibrium. Section 6 discusses the properties of the equilibrium. Section 7 deals with the case of a large number of market-makers. Section 8 concludes. Proofs not presented in the text are in the Appendix.

2. REVIEW OF THE LITERATURE

Glosten (1989, 1994), and Bernhardt and Hughson (1997) analyze the screening game where, first, market-makers post price schedules, and second, one strategic informed trader optimally chooses the amount he desires to buy from or sell to each market-maker. Glosten (1989) analyzes the monopoly case where there is only one market-maker. Glosten (1994) analyzes the limiting case where the number of liquidity suppliers goes to infinity, so that they behave competitively. Bernhardt and Hughson (1997) focus on the duopoly case. They show that, in this case, equilibrium cannot be such that market-makers earn zero profit, but they do not solve for the equilibrium or prove its existence.[4] Their positive profits result is in contrast with Kyle (1985) who postulates a zero-profit condition but does not provide a game theoretic foundation for it. The present paper reexamines the monopoly case, and then offers a complete analysis of the equilibrium prevailing for a given number of oligopolists, and of its limit when this number goes to infinity.

Competition in schedules has also been analyzed in the theory of industrial organization. Wilson (1979) and Bernheim and Whinston (1987) study complete information environments. The uniqueness of our equilibrium contrasts with the multiplicity of subgame perfect equilibria they describe. Bernheim and Whinston (1987) select within this set by imposing an exogenous "truthfulness" criterion. In contrast, in our asymmetric information framework, the incentive compatibility constraints which arise endogenously pin down the marginal price schedule at any equilibrium point. Klemperer and Meyer (1989) analyze competition in supply functions under uncertainty and impose differentiability to reduce the set of equilibria. The difference with our approach is twofold: First, we assume adverse selection rather than ex ante uncertainty over the demand

[4] Dennert (1993) also offers an interesting analysis of competition between market-makers in a screening game. To simplify the analysis, however, Dennert (1993) considers the case where the uncertainty follows a two point distribution and where trades are equal to a constant or 0.

realizations; second, the structure of the market is different. In our analysis, market-makers are competing through schedules mapping output choices into monetary transfers while Klemperer and Meyer (1989) analyze a uniform price Walrasian auction. In the case of asymmetric information with private values, Oren, Smith, and Wilson (1982) study oligopolistic pricing under exogenous conjectures on how the competitors react to a deviation. McAfee (1993) and Peters (1997) discuss competition between mechanism designers offering auction-like mechanisms. As they do, we take a complete game theoretic approach to characterize the equilibrium. However, in their contexts, exclusivity is imposed and therefore competition creates contractual externalities among mechanism designers only through its effect on the participation constraints of the agents who are attracted or not by a given mechanism.[5] Instead, we allow agents to split their consumptions among several mechanism designers. Hence, the contractual externality we analyze thereafter also goes through the incentive constraints of the agents. This particular aspect of our analysis is similar to that highlighted by the multiprincipal literature developed in other contexts in Stole (1991), Ivaldi and Martimort (1994), and Martimort (1992) and (1996). The difference between the present paper and this literature is our focus on a common value environment. We comment on this important feature of the environment all along the paper.

3. THE MODEL

We analyze a financial market for a risky asset where n risk-neutral market-makers (the principals),[6] denoted by M_i for $i=1,\dots,n$, supply liquidity to a risk-averse and expected utility-maximizing informed agent. This agent, denoted by A, desires to buy or sell the risky asset before the realization of its final value v.

3.1. *Information Structure*

The final value of the asset is $v=s+\varepsilon$, with s and ε independently distributed. ε has zero mean and variance σ^2. s is privately observed by the informed trader.[7] The informed trader also observes his endowment in the risky asset: I (I can be positive in which case the trader holds a long position in the asset, or negative in which case the trader holds a short position). Unlike most of previous market microstructure papers (such as Kyle (1985) and Glosten

[5] See also the analysis of the common value environment in Rothschild and Stiglitz (1976) in the case of the insurance market. They also insist on exclusivity.

[6] Gould and Verrecchia (1985) offer a justification for the assumption that market-makers are risk-neutral.

[7] Note that we assume $v=s+\varepsilon$, with s independent of ε and not $s=v+\varepsilon$, with v independent of ε, as in other models in the literature (see, e.g., Grossman and Stiglitz (1980) and Glosten (1989)). As a result, the inference problem of the trader is very simple: $E[v|s]=s$ and $V(v|s)=\sigma^2$. Hence, the conditional variance of v is constant without assuming a normal distribution for s.

(1989)), we do not make parametric assumptions on the distribution of the adverse selection parameters s and I. We only assume that the random variables s and I have bounded supports, respectively $[\underline{s}, \bar{s}]$ and $[\underline{I}, \bar{I}]$, that I is independent of ε, and that certain technical conditions, stated below, are satisfied.

3.2. *Competition Among Market-Makers*

In financial markets, liquidity suppliers post quotes against which agents demanding liquidity decide how much to trade. In order driven markets, such as the NYSE, the Paris Bourse, or the Tokyo Stock Exchange, these quotes are sequences of limit orders. In quote driven markets, such as the NASDAQ or the London SEAQ, these quotes are bid and ask prices and associated quantities. To model this quote setting behavior, we assume, as in Glosten (1994), or Bernhardt and Hughson (1997), that liquidity suppliers post price schedules. For example market-maker M_i posts the schedule $T_i(\cdot)$, which states that he is willing to trade q_i shares, against transfer $T_i(q_i)$. To see the relationship between such transfer schedules and sequences of limit orders, rewrite $T_i(q_i)$ as

$$T_i(q_i) = \int_0^{q_i} t_i(z)\,dz$$

where $t_i(z)$ is the marginal price at which market-maker i trades the zth unit. If the transfer schedule is convex, i.e., if $t_i(z)$ is increasing in z, then the sequence of marginal prices $t_i(z)$ amounts to a sequence of limit orders. Convexity is required to ensure equivalence between transfer schedules and limit orders and to reflect the execution priority of the limit sell (resp. buy) orders placed at lower (resp. higher) prices.

Note that competition in schedules between liquidity suppliers can be interpreted as competition among trading mechanisms. Each competitor is then viewed as a principal, facing the informed agent and offering a trading mechanism $T_i(\cdot)$. This mechanism only depends on the quantity that is sold by M_i.[8] We do not allow this trading mechanism to depend on the quantities sold by competing market-makers. Similarly and consistently with previous models of competition in mechanisms (Stole (1990) and Martimort (1992)) we assume that M_i cannot contract on the mechanisms offered by competing market-makers. These assumptions correspond to the institutions observed in practice in financial markets where quotes cannot be made contingent on the quotes or trades made by others.

Our model can also be viewed as describing the behavior of market-makers facing a continuous distribution of traders receiving different shocks on endowments and different signals. In this interpretation of the model, the set of mechanisms available is somewhat restricted since the principal cannot condi-

[8] These schemes are anonymous since they do not depend on the identity of the buyer.

tion the payment of any single agent on the whole volume of trade that the market-maker is doing with other agents.[9]

3.3. *The Trading Game*

The extensive form of the game is the following:

- First, nature chooses s and I. This information is learned by the agent A.
- Second, the n market-makers simultaneously post trading mechanisms $\{T_i(\cdot)\}_{i=1,\dots,n}$.
- Third, the informed agent selects the vector of his trades $\{q_i\}_{i=1,\dots,n}$ with the n market-makers and the corresponding set of transfers $\{T_i(q_i)\}_{i=1,\dots,n}$, to maximize his expected utility.
- Finally, ε and therefore v are realized and consumption takes place.

We study the perfect Bayesian equilibria[10] of this screening game. For the sake of simplicity we focus on pure strategy equilibria. In these equilibria, market-makers post transfer schedules that are best responses to the strategies of the other market-makers given the behavior of the informed agent in the subsequent stage of the game.

3.4. *Preferences*

When the agent trades $\{q_i\}_{i=1,\dots,n}$, his final wealth is[11]

$$W = (q + I)v - T(q) \tag{1}$$

where

$$q = \sum_{i=1}^{n} q_i, \tag{2}$$

and

$$T(q) = \sum_{i=1}^{n} T_i(q_i). \tag{3}$$

We assume that the utility of the informed agent is CARA with absolute risk-aversion parameter γ and that ε is normally distributed. Then the objective function of the informed agent is

$$\tilde{U} = E[W|I, s] - \frac{\gamma}{2} V(W|I, s), \tag{4}$$

[9] Since traders get signals that are correlated among each other, a principal unrestricted in the set of mechanisms could extract costlessly the common information of the traders by conditioning the payments of the agent on the ex post information contained in the total volume of trade (see Cremer and McLean (1988) for such mechanisms in the case of a single principal).

[10] For a formal definition, see for example Fudenberg and Tirole (1991).

[11] For simplicity the risk-free rate is normalized to zero.

where W is defined as in equation (1).[12] Equation (4) can be rewritten as

$$\tilde{U} = (q+I)s - \frac{\gamma\sigma^2}{2}(q+I)^2 - T(q)$$

or

$$\tilde{U} = \left(Is - \frac{\gamma\sigma^2}{2}I^2\right) + \left(\theta q - \frac{\gamma\sigma^2}{2}q^2 - T(q)\right) \quad (5)$$

where θ is defined as

$$\theta = s - \gamma\sigma^2 I. \quad (6)$$

The first term on the right-hand-side of (5) measures the reservation utility of the agent, which he would obtain if he did not participate to the market. The second term measures the gains from trades obtained by the agent, or to put it differently his informational rent.

θ reflects the blend of the agent's informational and risk sharing motivations to trade. Intuitively θ is the marginal valuation of the agent for the asset. On the one hand, this valuation is increasing in the signal on the asset value that is observed by the agent s. On the other hand, because of risk-aversion, this valuation is decreasing in the initial position of the agent in the asset, I.

Two remarks are in order regarding this information structure:

- Consider a slightly modified version of Grossman and Stiglitz (1980), whereby uncertainty stems from random endowments in the risky asset (instead of stemming from random supply noise in prices as in the original version). In such a model, the sufficient statistic, which is informationally equivalent to the price, is very similar to θ in our model. Indeed it is equal to a linear combination of the conditional expectation of the value of the asset given the private signal (in our notation s) and of the endowment shock (in our notation I).
- In general, the problem we address is a two-dimensional adverse selection problem, and therefore potentially mathematically extremely complex.[13] Because of the mean-variance structure of the objective function of the informed agent, however, this two-dimensional problem is made much simpler by the fact that one single variable (θ), which is a linear combination of the two adverse selection variables, captures entirely the dependence of the agent's utility on these adverse selection variables. Since θ reflects both the private signal s and the endowment shock (I), trades convey an ambiguous message to the market-makers. Large sales, for example, could well stem from very negative signals, or from very large endowments. Since the agent's utility depends on s and I only

[12] Note that, although to obtain this quadratic objective function we assume normality of ε (which is a random variable about which there is no informational asymmetry), we do not make any parametric assumption about the distribution of the adverse selection variables s and I. In fact, the objective in (4) could also have been obtained without imposing normality of ε, by directly assuming mean-variance preferences for the agent.

[13] See Rochet and Choné (1998).

through θ, only the latter can be revealed in equilibrium. Hence there will be pooling in equilibrium, in the sense that agents with different s and I but with equal θ will conduct the same trade.

θ being the important variable of the model, we directly make assumptions on the distribution of this random variable. We denote by $[\underline{\theta}, \bar{\theta}]$, $f(\theta)$, and $F(\theta)$ respectively the support, the density, and the cumulative of this distribution, which is absolutely continuous. Some technical restrictions on f and F will be made in the sequel.

We denote by $v(\theta) = E(v \mid \theta)$ the expectation of the value of the asset given θ. We make also the natural assumption that larger valuations θ tend to stem from larger signals s. More precisely, the expectation of the asset value v, conditional on θ, weakly increases with θ:[14]

$$0 \leq \dot{v}(\theta).$$

For technical reasons, we also assume that this conditional expectation increases at a rate less than one:[15]

$$1 > \dot{v}(\theta).$$

Below, we will interpret $v(\theta)$ as the opportunity cost of the asset for the market-makers. This cost depends on the agent's valuation for the good. This is the fundamental *common value* dimension of our analysis.

3.5. *Ex Ante Efficiency*

As a benchmark, we first consider the case where a benevolent social planner chooses a trading mechanism so as to maximize social welfare. Following Holmström and Myerson (1983), efficiency may be defined at an *ex ante stage*, i.e., before the agent learns any asymmetric information. An ex ante optimal trading mechanism is a pair $\{\tau(\theta), q(\theta)\}$ of transfer and trading volume (contingent on the future realization of θ) that solves the following problem:

$$\mathrm{Max}_{\{\tau(\cdot), q(\cdot)\}} \int_{\underline{\theta}}^{\bar{\theta}} \left(\theta q(\theta) - \frac{\gamma \sigma^2}{2} q(\theta)^2 - \tau(\theta) \right) f(\theta)\, d\theta$$

subject to

$$\int_{\underline{\theta}}^{\bar{\theta}} (\tau(\theta) - v(\theta) q(\theta)) f(\theta)\, d\theta \geq \underline{\pi}, \tag{7}$$

where the constraint is the ex ante participation constraint of the market-makers.

[14] The limit case $\dot{v}(\theta) \equiv 0$ corresponds to a private value environment, which possesses markedly different properties, as we discuss below.

[15] Using the definition of θ and v, this condition means also that $(dE(I \mid \theta)/d\theta) < 0$, i.e., higher values of θ correspond to conditional expectations of endowments that are lower. Rather intuitively, those agents who are eager to buy the asset are also likely to have low endowments in the asset.

Solving this problem is immediate. The participation constraint of the market-makers is binding.[16] The ex ante optimal trading volume is given by

$$q^*(\theta) = \frac{\theta - v(\theta)}{\gamma\sigma^2}. \tag{8}$$

Note that $q^*(\theta) = E(-I \mid \theta)$. Intuitively, the ex ante efficient risk-sharing is to trade an amount equal to minus the conditional expectation of the agent's endowment in the asset.

We assume thereafter that $q^*(\underline{\theta}) < 0 < q^*(\bar{\theta})$, i.e., it is efficient that the agent with the lowest (resp. highest) valuation for the good sells (resp. buys) it. Therefore, the two sides of the market will be active at equilibrium. More precisely, since we have assumed $\dot{v}(\cdot) < 1$, $q^*(\cdot)$ is increasing and there exists a unique θ_0 such that $q^*(\theta_0) = 0$. As we will see, traders with a valuation $\theta > \theta_0$ will always buy (possibly a zero amount), while traders with a valuation $\theta < \theta_0$ will always sell (possibly a zero amount).[17]

4. MONOPOLISTIC SCREENING

Glosten (1989) analyzes the monopoly case. He shows that monopoly power leads to a small trade spread, i.e., a noninfinitesimal difference between the marginal prices of infinitesimal purchases and sales. Our analysis in this section is qualitatively very similar. It serves as a benchmark to assess our original results, for the oligopoly case, presented in the next sections. There are some technical differences between our analysis of the monopoly case and that of Glosten (1989), however. As discussed in the previous section, our informational assumptions enable us to solve the problem without making any parametric assumptions about the adverse selection variables s and I.[18] Second, we use a somewhat different mathematical technique. By focusing on the dual problem where the rent of the agent is the control variable, rather than the transfer schedule, we can use the powerful tools of the calculus of variations.

The monopolistic market-maker posts a transfer schedule $T_m(q)$ to maximize his expected profit:

$$\int_{\underline{\theta}}^{\bar{\theta}} (T_m(q(\theta)) - v(\theta)q(\theta))f(\theta)\,d\theta.$$

Under adverse selection, the mechanism offered by the market-maker must be incentive compatible and satisfy the individual rationality condition of the informed agent in all states of nature.

[16] The ex ante participation constraint of the agent that we do not write above is then automatically satisfied when $\underline{\pi}$ is small enough.

[17] The limiting case $\dot{v}(\theta) \equiv 1$ corresponds to a situation where endowments are publicly observed. As a result, only one side of the market is active at equilibrium: for example a trader with a positive endowment will never find a market-maker willing to sell him more stocks.

[18] In particular, unlike Glosten (1989), we do not assume normality of s and I.

In the presence of adverse selection, the market-maker must elicit information revelation from the informed agent. As mentioned above, only θ, and not s and I separately, can be revealed.

In the case of a single mechanism designer, there is no loss of generality in applying the Revelation Principle.[19] Any implementable allocation achieved with a nonlinear schedule $T(q)$ can also be achieved with a truthful direct mechanism $\{\tau(\cdot); q(\cdot)\}$ that stipulates a transfer and a trading volume as a function of the agent's report on his type. Therefore, incentive compatibility requires that

$$(9) \qquad \theta \in \operatorname{Argmax}_{\hat{\theta}} \left(\theta q(\hat{\theta}) - \frac{\gamma\sigma^2}{2} q(\hat{\theta})^2 - \tau(\hat{\theta}) \right).$$

We denote by $U(\theta)$ the corresponding informational rent:

$$(10) \qquad U(\theta) = \max_{\hat{\theta}} \left(\theta q(\hat{\theta}) - \frac{\gamma\sigma^2}{2} q(\hat{\theta})^2 - \tau(\hat{\theta}) \right).$$

To study the mechanism design problem of the monopolistic market-maker, we can focus on the transfers and allocations $\{\tau(\cdot), q(\cdot)\}$, or alternatively we can take the dual approach and focus on the informational rent $U(\cdot)$ (defined in equation (10)) left to the agent. We can then characterize the set of informational rents corresponding to an incentive compatible mechanism. This set is characterized in the following lemma.

LEMMA 1: *A pair $\{U(\cdot), q(\cdot)\}$ is implementable if and only if*

$$(11) \qquad U(\cdot) \text{ is convex on } \left[\underline{\theta}, \bar{\theta}\right],$$

and for a.e. θ

$$(12) \qquad \dot{U}(\theta) = q(\theta).$$

The Lemma follows from the fact that $U(\theta)$ is the maximum of a family of affine functions of θ, as can be seen from its definition in (10). Convexity of $U(\cdot)$ and the fact that $q(\theta) = \dot{U}(\theta)$ imply that $q(\theta)$ must be weakly increasing in θ. The latter condition is rather intuitive. Independently of prices, the quantity bought by the agent increases with his valuation of the asset.

Because the market-maker is risk-neutral and the informed agent has CARA utility, the total gain from trade can be measured by the sum of the certainty equivalents. Hence the profit of the monopolist is equal to the total gain from trade minus the informational rent.[20] Therefore, the optimal allocation $\{U_m(\theta),$

[19] See Myerson (1979), among others.

[20] This is why, as discussed in the introduction, the market-maker is interested in generating a large gain from trade (this is the allocative role of the mechanism) as long as this does not give an excessive informational rent to the agent (this is the distributive role of the mechanism).

$q_m(\theta)\}$ for the monopolist solves:

$$\mathrm{Max}_{\{U(\cdot),\, q(\cdot)\}} \int_{\underline{\theta}}^{\bar{\theta}} \left((\theta - v(\theta)) q(\theta) - \frac{\gamma\sigma^2}{2} q(\theta)^2 - U(\theta) \right) f(\theta)\, d\theta$$

subject to (11), (12), and

(13) $\qquad U(\theta) \geq 0,$

where the latter constraint is the ex post participation constraint of the trader. Taking the dual approach, we replace the quantity traded by the derivative of the informational rent, using (12). We can rewrite the expected profit of the monopolist as

(14) $$B_m(U, \dot{U}) = \int_{\underline{\theta}}^{\bar{\theta}} \left((\theta - v(\theta)) \dot{U}(\theta) - \frac{\gamma\sigma^2}{2} \dot{U}(\theta)^2 - U(\theta) \right) f(\theta)\, d\theta.$$

The monopolist problem, thereafter denoted by (M), can be written as a calculus of variations problem:

(15) $\qquad \mathrm{Max}_{U(\cdot)}\, B_m(U, \dot{U})$

subject to (11) and (13).

The participation constraint (13) must be binding somewhere. If it was not, then the monopolist could raise prices uniformly by a small amount, and thus obtain larger revenues, while still inducing participation of all types of agents.

Note that in problem (M) the control variable is not the transfer but rather its dual: the informational rent. This dual approach simplifies the resolution techniques. In the oligopoly case, it will also facilitate the characterization of the agent's trades as a function of the set of nonlinear prices posted by the traders.

We adopt a standard strategy to solve this problem. First, we neglect the convexity condition (11) and solve the so-called "relaxed" problem (M'). Second, we check ex post that the solution of (M') satisfies the convexity constraint (11) and thus solves the initial problem (M). This will be true for example if the distribution of θ satisfies the following monotonicity properties:

(16) $$\forall \theta > \theta_0 \quad \frac{d}{d\theta}\left(\frac{1 - F(\theta)}{f(\theta)} \right) < 0,$$

(17) $$\forall \theta < \theta_0 \quad \frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)} \right) > 0.$$

Inequality (16) (resp. (17)) amounts to the property that $F(\cdot)$ (resp. $1 - F(\cdot)$) is log-concave.[21] It turns out that these properties are not very restrictive, since they are implied by the log concavity of *either* the density of s *or* that of I. This is because log-concavity is preserved under convolution as we show in the Appendix (Proposition 16).

[21] See Bagnoli and Bergstrom (1989) for some discussion of this property.

4.1. *Trading Volume and Bid-Ask Spread*

PROPOSITION 2: *Under assumptions* (16) *and* (17), *there exist* $\theta_a^m > 0$ *and* $\theta_b^m < 0$ *such that the trading volume offered by the monopolistic market-maker entails*:

- *for all* $\theta \in [\underline{\theta}, \theta_b^m)$

$$q_m(\theta) = q^*(\theta) + \frac{F(\theta)}{\gamma\sigma^2 f(\theta)}; \tag{18}$$

- *for all* $\theta \in [\theta_b^m, \theta_a^m]$

$$q_m(\theta) = 0; \tag{19}$$

- *for all* $\theta \in (\theta_a^m, \bar{\theta}]$

$$q_m(\theta) = q^*(\theta) - \frac{1 - F(\theta)}{\gamma\sigma^2 f(\theta)}. \tag{20}$$

Proposition 2 shows that trading in the monopoly case is lower than the ex ante efficient trading volume. As mentioned above, the mechanism designer faces a trade-off between the allocative and redistributive roles of the mechanism. Since intense trading increases the informational rent of the agent, the monopolist finds it optimal to lower trading in order to reduce this costly informational rent.

Inspection of equations (18) and (20) in Proposition 2 shows that conditions (16) and (17) (together with the fact that the ex-ante optimal trade is increasing in θ) imply that demand is increasing in the trader's valuation for the asset, which was required for incentive compatibility.

The values of θ_b^m and θ_a^m are given by the smooth pasting condition that requires $q_m(\cdot)$ to be continuous. Further, the monopolist transfer schedule $T_m(\cdot)$ is shown to be differentiable everywhere except at 0. The marginal price schedule optimally chosen by the monopolist is

$$t_m(q_m(\theta)) = \theta - \gamma\sigma^2 q_m(\theta) = v(\theta) - \frac{F(\theta) - \Lambda(\theta)}{f(\theta)},$$

where $\Lambda(\theta)$ equals 1 (resp. 0) when $\theta \in (\theta_a^m, \bar{\theta}]$ (resp. $\theta \in [\underline{\theta}, \theta_b^m)$). Hence we can state the following proposition:

PROPOSITION 3: *The nonlinear price schedule optimally quoted by the monopolist is differentiable everywhere except at* 0. *Its derivative is as follows*:

- *for all* $\theta \in [\underline{\theta}, \theta_b^m)$,

$$t_m(q_m(\theta)) = v(\theta) - \frac{F(\theta)}{f(\theta)};$$

- *for all* $\theta \in (\theta_a^m, \bar{\theta}]$,

$$t_m(q_m(\theta)) = v(\theta) + \frac{1 - F(\theta)}{f(\theta)}.$$

Proposition 3 shows that the monopolist market-maker quotes marginal prices $t_m(q_m(\theta))$ equal to the sum of the marginal cost $v(\theta)$ and a monopolist mark-up that depends on the distribution of types. Note that the optimal schedule is discontinuous at 0:

$$t_m(0^+) = \theta_a^m > \theta_b^m = t_m(0^-).$$

The strict inequality between θ_a^m and θ_b^m stems from the fact that they are defined by

$$q^*(\theta_a^m) = \frac{1-F(\theta_a^m)}{f(\theta_a^m)\gamma\sigma^2} > 0$$

and

$$q^*(\theta_b^m) = -\frac{F(\theta_b^m)}{f(\theta_b^m)\gamma\sigma^2} < 0,$$

respectively, which implies, since $q^*(\cdot)$ is increasing in θ, that $\theta_b^m < \theta_a^m$. Building on these remarks we can state the following corollary:

COROLLARY 4: *θ_a^m and θ_b^m can be interpreted as the ask and bid quotes for infinitesimal trades. Since $\theta_a^m > \theta_b^m$ there is a small trade spread.*

This result shows the extreme form of contraction in trading volume that takes place in this model. A positive measure of agents with intermediate types are rationed as a consequence of adverse selection.

4.2. *Concavity or Convexity of the Price Schedule*

Whether the price schedule is convex or concave is of economic interest. Concave transfers correspond to quantity discounts, whereby large trades obtain better prices than small trades. In contrast, convex transfers correspond to the case where, for purchases, the unit price is increasing in the quantity traded. Note that, in the monopoly case, there is no general result about the convexity or concavity of the optimal nonlinear schedule.[22] This is stated more precisely in the following corollary (which obtains from straightforward manipulations of the optimal schedule given in Proposition 3).

COROLLARY 5: *In the monopoly case, the optimal nonlinear schedule is concave (resp. convex) when:*

$$\text{(21)} \qquad \forall\theta \in \left(\theta_a^m, \bar{\theta}\right], \quad \dot{v}(\theta) < -\frac{d}{d\theta}\left(\frac{1-F(\theta)}{f(\theta)}\right) \qquad (\textit{resp.} >),$$

[22] This is not a peculiar aspect of our model but a constant characteristic of monopolistic screening models (see Maskin and Riley (1984) and Spulber (1989) in the context of nonlinear pricing).

and

$$(22) \qquad \forall \theta \in [\underline{\theta}, \theta_b^m), \quad \dot{v}(\theta) < \frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right) \qquad (resp. >).$$ [23]

4.3. *Comparison with the Private Value Case*

It is interesting to note that the formulas derived in Propositions 2 and 3 are compatible with the property derived by Oren, Smith, and Wilson (1983) and Goldman, Leland, and Sibley (1984) in a private value environment, namely that the seller charges the monopoly price for each separate marginal unit q. This can be seen simply as follows. Suppose trader θ wants to purchase the qth unit at price t_m. In other words, we assume $q_m(\theta) \geq q$. Using the first order condition for the agent's problem, this is equivalent to

$$\theta \geq \gamma\sigma^2 q + t_m.$$

Denoting by $\theta(t_m) = \gamma\sigma^2 q + t_m$ the type of the marginal buyer, the expected profit of the market-maker for the sale of the qth unit is

$$(23) \qquad t_m(1 - F(\theta(t_m))) - \int_{\theta(t_m)}^{\bar{\theta}} v(\theta) f(\theta)\, d\theta,$$

where $1 - F(\theta(t_m))$ is the mass of agents conducting this trade while

$$\frac{\int_{\theta(t_m)}^{\bar{\theta}} v(\theta) f(\theta)\, d\theta}{1 - F(\theta(t_m))}$$

is the expected value of the asset given that the trade has been conducted, i.e., the marginal cost of the qth unit for the monopolistic seller. The property discovered by Goldman, Leland, and Sibley (1984), which extends to our common value context, is that the optimal monopolistic schedule is obtained by maximizing (23) over t_m. The first order condition is

$$(24) \qquad 1 - F(\theta(t_m)) - t_m f(\theta(t_m)) + v(\theta(t_m)) f(\theta(t_m)) = 0.$$

Hence

$$(25) \qquad t_m = v(\theta(t_m)) + \frac{1 - F(\theta(t_m))}{f(\theta(t_m))},$$

which corresponds to the result stated in Proposition 3.[24]

[23] In particular, when $v(\theta) = \alpha\theta$, and the distribution of θ is uniform, we have $\alpha < 1$ by assumption and the monopoly price exhibits some discounts. More generally, the right-hand-side of (21) is equal to one when $\theta = \bar{\theta}$ as soon as $f(\bar{\theta}) > 0$ and $|f'(\bar{\theta})|$ is bounded. Since we assume also that $\dot{v}(\bar{\theta}) < 1$, the monopolist always offers some discounts for large trades.

[24] Similar remarks apply to the buy side of the market.

Note that equation (24) can be rewritten as

$$1-F(\theta(t_m))=f(\theta(t_m))(t_m-v(\theta(t_m))).$$

By raising its price by one unit, the monopolist increases its profit on all agents having a supramarginal valuation for the asset (revenue effect on the left-hand side). However, it loses the profit made on the agent with the marginal valuation for the asset (demand effect on the right-hand side).

5. OLIGOPOLISTIC SCREENING

We now turn to the case where n market-makers offer simultaneously and noncooperatively the price schedules $\{T_i(q_i)\}_{i=1,\dots,n}$. For a given set of $n-1$ nonlinear prices $T_i(\cdot)$ offered by market-makers M_i, $i=2,\dots,n$, the problem of finding the optimal best response $T_1^*(T_2,\dots,T_n)$ of the first market-maker M_1 exhibit some similarities with the problem faced by the monopolist. The main difference is that M_1 does not offer contracts where transfers are contingent on the total trading volume but only on his own trades. From the point of view of M_1, the trades conducted with his rivals play the role of nonverifiable moral hazard variables which therefore reduce his control over the agent.

5.1. *The Agent's Problem*

Confronted with the price schedules posted by the market-makers, the trader chooses his optimal bundle of trades $\{q_i\}_{i=1,\dots,n}$ by solving

$$U(\theta)=\max_{\{q_i\}_{i=1,\dots,n}}\left(\theta\left(\sum_i q_i\right)-\frac{\gamma\sigma^2}{2}\left(\sum_i q_i\right)^2-\sum_i T_i(q_i)\right). \tag{26}$$

Actually, the trader's utility only depends on the aggregate price schedule, defined for all q by

$$T(q)=\min\left(\sum_i T_i(q_i),\ \sum_i q_i=q\right). \tag{27}$$

At this point it is appropriate to introduce a notation that will be useful in the paper. For two given transfer functions $T_1(\cdot)$ and $T_2(\cdot)$, the infimal convolution of these functions $(T_1\ \square\ T_2)(\cdot)$ is defined as[25]

$$(T_1\ \square\ T_2)(q)=\min_{q_1}(T_1(q_1)+T_2(q-q_1))\quad\forall q.$$

By extension, if we denote $T_{-1}(\cdot)$ the aggregate schedule quoted by market-makers $i=2,\dots,n$,[26] the total aggregate price schedule is the infimal convolu-

[25] See Rockafellar (1970).

[26] By a slight abuse of notation and for notational simplicity, we denote by $T_{-1}(\cdot)$ both the vector $(T_2(\cdot),\dots,T_n(\cdot))$ of the other market-makers schedule and the aggregate schedule they offer $(T_2\ \square\ \cdots\ \square\ T_n)(\cdot)$.

tion of $T_1(\cdot)$ and $T_{-1}(\cdot)$:

$$T(q) = (T_1 \,\square\, T_{-1})(q) \quad \forall q.$$

5.2. *Technical Restrictions on the Market-Makers' Strategy Space*

The essential restriction we make on the price schedules quoted by the market-makers is that the marginal price $t_i(q_i)$ has at most a finite number of discontinuities, and that elsewhere it is differentiable.

DEFINITION 1: The marginal price function $t_i(\cdot)$ is *admissible* if it has (at most) a finite number of jumps and is differentiable elsewhere. For all q_i we denote by $t_i^-(q_i)$ and $t_i^+(q_i)$ *the left- and right-limits* of $t_i(\cdot)$ at q_i.

For technical reasons, we restrict our attention to convex price schedules (i.e., nondecreasing marginal price functions). For this strategy space we prove the existence of a unique Nash equilibrium among the market-makers. Next, we prove that no market-maker can gain by unilaterally offering a nonconvex schedule. This establishes that the Nash equilibrium obtained when the strategy space is restricted to convex schedules is also an equilibrium of the game where the strategy space is not restricted. Interestingly, it can be observed that in practice, the transfer schedule faced by traders in limit order markets are convex by construction of the limit order book.

In the convex case, it is useful to define the supply correspondence of each market-maker.

DEFINITION 2: The *supply correspondence* $S_i(\cdot)$ of market-maker i is the inverse of the subdifferential of $T_i(\cdot)$[27]

$$q_i \in S_i(p) \Leftrightarrow p \in \partial T_i(q_i).$$

If q_i is a continuity point of $t_i(\cdot)$, then $\partial T_i(q_i) = \{t_i(q_i)\}$. Therefore, $S_i(\cdot)$ can be interpreted as the "inverse" mapping of $t_i(\cdot)$. The relationship is indeed one to one at every couple (q_i, $p = t_i(q_i)$) such that $\partial T_i(q_i)$ and $S_i(p)$ are singletons. In that case, q_i is called a "regularity point" of $t_i(\cdot)$.[28]

[27] The subdifferential of a convex function $T(\cdot)$ of one variable is defined as $\partial T(q) = [T'^-(q), T'^+(q)]$. In our notations, we have $\partial T_i(q_i) = [t_i^-(q_i), t_i^+(q_i)]$. See Rockafellar (1970, p. 215).

[28] Note that there is an important difference between our definition of the supply correspondence and the standard definition that can be found, for example, in Klemperer and Meyer (1989). In Definition 2, q_i is the quantity offered when the marginal price is p while in Klemperer and Meyer (1989) the supply function is determined in terms of the average unit price. This difference is related to the fact that we study a sequential screening game where the agent selects quantities in the schedules posted by the market-makers, while Klemperer and Meyer (1989) study a uniform price market clearing mechanism, where the average unit price is the relevant price variable.

5.3. *The Market-Makers' Problem*

Given the transfer schedules posted by his competitors, the problem of market-maker M_1 is to design his schedule $T_1(\cdot)$, so as to maximize his own expected profits defined as

$$(28) \qquad B_1(T_1,\dots,T_n)=\int_{\underline{\theta}}^{\bar{\theta}}(T_1(q_1(\theta))-v(\theta)q_1(\theta))f(\theta)\,d\theta.$$

As under monopolistic screening, this expected profit must be maximized under incentive and participation constraints.

However, both these constraints are now affected by competition:

• Under oligopolistic screening, the incentive constraint writes as (26). This constraint tells us that the quantity $q_1(\theta)$ in the maximand above must belong to the trades that maximize the agent's utility.

Note that we do not rely on direct truthfulness mechanisms to characterize the incentive compatibility constraints under oligopoly. In this competing mechanism framework, the Revelation Principle is of little help to characterize equilibrium outcomes since it does not apply in a simple manner to all mechanism designers simultaneously.[29,30]

• Under oligopolistic screening, the agent's participation constraint is

$$(29) \qquad \forall\theta \quad U(\theta)\geq U_{-1}(\theta)$$
$$=\max_{q_2,\dots,q_n}\left(\theta\left(\sum_{i=2}^{n}q_i\right)-\tfrac{1}{2}\gamma\sigma^2\left(\sum_{i=2}^{n}q_i\right)^2-\sum_{i=2}^{n}T_i(q_i)\right),$$

where the right-hand side of equation (29) corresponds to the best that the agent can obtain if she does not trade with market-maker M_1 and trades instead with all his rivals.

This program defines the best response of market-maker M_1 to the schedules of his competitors $T_2(\cdot),\dots,T_n(\cdot)$. We denote the best response mapping by $T_1^*(T_2,\dots,T_n)$. Our objective is to characterize the Nash equilibria of this game, which correspond to the fixed-points of the best response mappings obtained for each market-maker.

[29] The Revelation Principle nevertheless applies in a rather restrictive way for each principal separately. Given the pure strategy nonlinear schedules offered by his rivals, there is no loss of generality in restricting any principal to offer a direct revelation mechanism using reports from the agent on his type only in order to compute his best response mechanism. Peters (1999) and Martimort and Stole (1999) show that there is no loss of generality in restricting each principal to offer nonlinear prices of the form $T_i(q_i)$ to the common agent. This validates our focus on nonlinear prices.

[30] Epstein and Peters (1990) show that there is always a sense in which the Revelation Principle holds simultaneously for both principals if one defines the agent's "type" as belonging to a sufficiently large space constructed through an infinite regress procedure. This procedure amounts to having the agent report his valuation and his "market information" which summarizes his knowledge of competing mechanisms.

In fact, computations are again highly simplified by using the dual approach. Instead of taking the transfer schedule of M_1 as an instrument, we still use the informational rent $U(\cdot)$ of the informed agent as the control variable.

Let us come back now to the trader's choice, obtained as the solution to (26). By concavity of the objective, this solution is characterized by the first order condition:

$$[\theta - \gamma\sigma^2 q(\theta)] \in \partial T_i(q_i(\theta)) \quad \text{for all } i \text{ and } \theta \tag{30}$$

where $q(\theta) = \sum_{i=1}^{n} q_i(\theta)$. The left-hand side of equation (30) is the marginal benefit obtained by the agent from trading, while the right-hand side is the marginal price. Since the distribution of θ is absolutely continuous, we can neglect the (possible) nondifferentiability points of $U(\cdot)$, and write (30) as

$$\forall i, \forall \theta, \quad q_i(\theta) = S_i\big(\theta - \gamma\sigma^2 \dot{U}(\theta)\big). \tag{31}$$

Also, as in the monopoly case, the optimality of the agent's choice of trades implies that (11) and (12) still hold under oligopoly.

We are now in a position to write the expected profit of M_1 as a function of the rent $U(\theta)$, the total quantity traded $q(\theta)$ and the transfers and trading volumes of the other market-makers $(T_{-1}, S_{-1}) = (T_2, \dots, T_n, S_2, \dots, S_n)$. This is stated in the following lemma:

LEMMA 6: *Written as a function of the agent's information rent, the total trading volume, and the transfers of his competitors, the expected profit of market-maker M_1 is $B_1(U, \dot{U}, T_{-1})$ such that*

$$\begin{aligned} B_1\big(U, \dot{U}, T_{-1}\big) = \int_{\underline{\theta}}^{\bar{\theta}} \bigg[& (\theta - v(\theta))\dot{U}(\theta) - \frac{\gamma\sigma^2}{2}\dot{U}^2(\theta) - U(\theta) \\ & - \sum_{i=2}^{n} T_i \circ S_i\big(\theta - \gamma\sigma^2\dot{U}(\theta)\big) \\ & + v(\theta) \sum_{i=2}^{n} S_i\big(\theta - \gamma\sigma^2 \dot{U}(\theta)\big) \bigg] f(\theta)\, d\theta. \end{aligned} \tag{32}$$

Lemma 6 simply amounts to saying that the expected profit of market-maker M_1 is equal to the expected total surplus minus the sum of the expected informational rent of the agent and the expected profits of the other market-makers.

If the market-maker is to leave a given rent $U(\cdot)$ to the agent, he must ensure that the aggregate price schedule $T(\cdot)$ (resulting from his own transfer $T_1(\cdot)$ and those of his competitors $T_{-1}(\cdot)$) gives rise to this rent. That is, he must ensure that $T(\cdot)$ is such that

$$U(\theta) = \max_q \left(\theta q - \frac{\gamma\sigma^2}{2} q^2 - T(q) \right). \tag{33}$$

Hence, in its dual form, the program of the market-maker, thereafter denoted by (M_1) reduces to the following:

$$\text{(34)} \qquad \mathrm{Max}_{U(\cdot)}\, B_1\big(U, \dot{U}, T_{-1}\big)$$

subject to (11) and (29) and such that there exists $T_1(\cdot)$ solution to $T(\cdot) = T_1 \,\square\, T_{-1}(\cdot)$.

Following the standard approach in mechanism design, we will study the relaxed problem, whereby the convexity condition is not imposed, and then check ex post that the solution of the relaxed problem does satisfy the convexity constraint.

We will also relax the problem in a second and much less standard way, reflecting the fact that we consider a multiprincipal (rather than a single principal) environment. We first do not impose the constraint that there exists a solution $T_1(\cdot)$ to $T(\cdot) = T_1 \,\square\, T_{-1}(\cdot)$, and then we prove ex post that the solution we find does indeed satisfy this constraint. To summarize we solve the following relaxed dual problem (M_1'):

$$\text{(35)} \qquad \mathrm{Max}_{U}\, B_1\big(U, \dot{U}, T_{-1}\big)$$

subject to (29) where $B_1(\cdot)$ is defined as in Lemma 6.

5.4. *Derivation of the Best Response Mapping*

First, we define the Lagrangian associated to (M_1'):

$$L_1(U, \dot{U}) = B_1\big(U, \dot{U}, T_{-1}\big) + \int_{\underline{\theta}}^{\bar{\theta}} (U(\theta) - U_{-1}(\theta))\, d\Lambda_1(\theta);$$

$\Lambda_1(\theta) = \int_{\underline{\theta}}^{\theta} \mathrm{d}\Lambda_1(s)$ is the integral of the Lagrange multiplier associated to constraint (29).[31] Integrating by parts, we can simplify $U(\theta)$ out of the expression of $L_1(\cdot)$:

$$\begin{aligned} L_1(U, \dot{U}) = \int_{\underline{\theta}}^{\bar{\theta}} \Bigg[& (\theta - v(\theta))\dot{U}(\theta) - \frac{\gamma\sigma^2}{2}\dot{U}^2(\theta) + \frac{F(\theta) - \Lambda_1(\theta)}{f(\theta)}\dot{U}(\theta) \\ & - \sum_{i=2}^{n} T_i \circ S_i\big(\theta - \gamma\sigma^2 \dot{U}(\theta)\big) \\ & + v(\theta) \sum_{i=2}^{n} S_i\big(\theta - \gamma\sigma^2 \dot{U}(\theta)\big) \Bigg] f(\theta)\, d\theta, \end{aligned}$$

where we neglect to write an additive term involving $U_{-1}(\theta)$, in which the control variable $U(\theta)$ does not enter.

[31] In particular, $\Lambda_1(\bar{\theta}) = 1$ and $\Lambda_1(\cdot)$ is constant on all the intervals where $q_1(\theta) \neq 0$ since then the participation constraint (29) is slack.

Using the definition of $q^*(\theta)$ in (8) to simplify expressions, pointwise maximization with respect to $\dot{U}(\theta)$ gives, at all regularity points,

$$0 = \gamma\sigma^2\big(q^*(\theta) - \dot{U}(\theta)\big) + \frac{F(\theta) - \Lambda_1(\theta)}{f(\theta)} \tag{36}$$

$$+\gamma\sigma^2 \sum_{i=2}^{n} \Big((T_i \circ S_i)'\big(\theta - \gamma\sigma^2\dot{U}(\theta)\big) - v(\theta)S_i'\big(\theta - \gamma\sigma^2\dot{U}(\theta)\big)\Big). \tag{37}$$

Recall that, at points where the transfer function is differentiable,

$$\theta - \gamma\sigma^2 q(\theta) = t_i(q_i(\theta)). \tag{38}$$

Or equivalently,

$$S_i(\theta - \gamma\sigma^2 q(\theta)) = q_i(\theta).$$

This implies that

$$S_i'(\theta - \gamma\sigma^2 q(\theta)) = \frac{1}{t_i'(q_i(\theta))}.$$

Consequently,

$$(T_i \circ S_i)'\big(\theta - \gamma\sigma^2\dot{U}(\theta)\big) = \frac{t_i(q_i(\theta))}{t_i'(q_i(\theta))} = \frac{\theta - \gamma\sigma^2\dot{U}(\theta)}{t_i'(q_i(\theta))}.$$

Therefore, condition (36) can be written as

$$\forall i, \quad \gamma\sigma^2\left(\sum_{i=2}^{n} \frac{1}{t_i'(q_i(\theta))}\right)\big(q^*(\theta) - \dot{U}(\theta)\big) = \dot{U}(\theta) - \left(q^*(\theta) + \frac{F(\theta) - \Lambda_1(\theta)}{\gamma\sigma^2 f(\theta)}\right).\text{[32]} \tag{39}$$

Similarly, by differentiating (38) at a regularity point of $t_i(\cdot)$ we find

$$t_i'(q_i(\theta))\dot{q}_i(\theta) = 1 - \gamma\sigma^2\dot{q}(\theta).\text{[33]} \tag{40}$$

In fact, condition (40) shows that all the $\dot{q}_i(\theta)$ have the same sign as $1 - \gamma\sigma^2\dot{q}(\theta)$ (since $t_i'(\cdot)$ is positive). Since $\dot{q}(\theta)$ is nonnegative and $\dot{q}(\theta) = \sum_{i=1}^{n}\dot{q}_i(\theta)$, we conclude that all the $\dot{q}_i(\theta)$ are also nonnegative. Condition (40) also implies that

[32] In fact, nonregularity points correspond to situations where, at least for one market-maker i, $t_i'(q_i)$ equals 0 or $+\infty$. Modulo this slight abuse of notation, condition (39) extends to these cases.

[33] This condition can be extended to nonregularity points by allowing $\dot{q}_i(\theta)$ to be equal to 0 or $+\infty$.

(39) can be rewritten as

$$
\begin{aligned}
(41)\qquad &\gamma\sigma^2\left(\frac{\dot q(\theta)-\dot q_1(\theta)}{1-\gamma\sigma^2\dot q(\theta)}\right)\left(q^*(\theta)-\dot U(\theta)\right)\\
&\quad=\dot U(\theta)-\left(q^*(\theta)+\frac{F(\theta)-\Lambda_1(\theta)}{\gamma\sigma^2 f(\theta)}\right).
\end{aligned}
$$

Equation (41) characterizes the best response mapping of market-maker M_1 expressed as a function of the dual variable $U(\theta)$. Doing the same for all market-makers M_i, $i=2,\dots,n$, would give similar expressions simply by permuting indices.

5.5. *Derivation of the Equilibrium*

From equation (41), an equilibrium of the game among market-makers is characterized by a system of differential equations:

$$
\begin{aligned}
(42)\qquad &\forall i,\forall\theta,\quad \gamma\sigma^2\left(\frac{\dot q(\theta)-\dot q_i(\theta)}{1-\gamma\sigma^2\dot q(\theta)}\right)(q^*(\theta)-q(\theta))\\
&\quad=q(\theta)-\left(q^*(\theta)+\frac{F(\theta)-\Lambda_i(\theta)}{\gamma\sigma^2 f(\theta)}\right),
\end{aligned}
$$

where $q(\theta)=\sum_i q_i(\theta)$, and for all i, $\dot q_i(\theta)\geq 0$, $d\Lambda_i(\theta)\geq 0$, and $\dot q_i(\theta)\,d\Lambda_i(\theta)=0$. In particular, for each market-maker M_i there exist two thresholds θ_b^i,θ_a^i such that

$$
\begin{cases}
\forall\theta\leq\theta_b^i \qquad q_i(\theta)\leq 0 \quad\text{and}\quad \Lambda_i(\theta)=0,\\
\forall\theta\geq\theta_a^i \qquad q_i(\theta)\geq 0 \quad\text{and}\quad \Lambda_i(\theta)=1.
\end{cases}
$$

Building on equation (42) the following proposition obtains.

PROPOSITION 7: *Any equilibrium in convex schedules must have the following properties*:

• *It is symmetric, i.e., market-makers share equally the market* $q_i(\theta)=q_j(\theta)$ $\forall(i,j)$.

• *The corresponding total trading volume* $q^n(\theta)$ *is the solution to the differential equation*

$$
(43)\qquad
\begin{aligned}
&\dot q^n(\theta)=\frac{1}{\gamma\sigma^2}\left(1+\frac{(n-1)(q^*(\theta)-q^n(\theta))}{n(q^n(\theta)-q_m(\theta))}\right)^{-1} \qquad \theta\notin[\theta_b^n,\theta_a^n],\\
&q^n(\theta)=0 \quad \textit{on} \quad [\theta_b^n,\theta_a^n],
\end{aligned}
$$

where θ_b^n *and* θ_a^n *are such that* $q^n(\cdot)$ *is continuous on* $[\underline{\theta},\overline{\theta}]$, *and where* $q_m(\cdot)$ *is defined in equations* (18) *and* (20), *except for the bounds, which are changed from* θ_a^m *and* θ_b^m *to* θ_a^n *and* θ_b^n.

• *The boundary conditions of this differential equation are* $q^n(\bar{\theta}) = q^*(\bar{\theta})$ *and* $q^n(\underline{\theta}) = q^*(\underline{\theta})$.

(43) is a differential equation with singularities at the extreme points $\bar{\theta}$ and $\underline{\theta}$. As in previous analysis of competing contracts (Stole (1990), Martimort (1992, 1996)) the exact behavior of the solution will be determined by a local analysis around these singularities. Having characterized the possible solutions to the system of equations (42) we can establish existence and unicity of the solution.

We need a last technical assumption[34] on the distribution of θ:

$$\lim_{\theta \to \bar{\theta}^-} \frac{d}{d\theta}\left(\frac{1 - F(\theta)}{f(\theta)}\right) = -1 \tag{44}$$

and

$$\lim_{\theta \to \underline{\theta}^+} \frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right) = 1. \tag{45}$$

PROPOSITION 8: *Under assumptions* (44) *and* (45), *there exists a unique convex equilibrium. It is symmetric and differentiable. The total volume of trade in this equilibrium* $q^n(\theta)$ *has the following properties*:

- $q^m(\theta) \le q^n(\theta) \le q^*(\theta)$ *for all* $\theta \in [\theta_0, \bar{\theta}]$ *with both equalities only for* $\theta = \bar{\theta}$.
- $q_m(\theta) \ge q^n(\theta) \ge q^*(\theta)$ *for all* $\theta \in [\underline{\theta}, \theta_0]$ *with both equalities only for* $\theta = \underline{\theta}$.
- $q^n(\theta)$ *is strictly increasing with* θ, *for* θ *outside* $[\theta_b^n, \theta_a^n]$.

5.6. *Ex Post Validity*

Having proved, in the proof of Proposition 7, that the condition that there exists a solution T_1 to $T = T_1 \,\square\, T_{-1}$ is satisfied, and in Proposition 8 that $q(\theta)$ is increasing (which corresponds to $U(\cdot)$ being convex), we have established that the solution of the relaxed problem also solves the original problem. This is stated in the next proposition.

PROPOSITION 9: *The solution of the relaxed problem* (35) *is also a solution to the full problem* (34).

[34] All our analysis would also hold (at the cost of more complicated algebra) when

$$\lim_{\theta \to \bar{\theta}^-} \frac{d}{d\theta}\left(\frac{1 - F(\theta)}{f(\theta)}\right) = -k_1 \quad \text{and}$$

$$\lim_{\theta \to \underline{\theta}^+} \frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right) = k_2$$

where k_1 and k_2 are two positive numbers. Notice also that conditions (44) and (45) are insured when $f(\bar{\theta})$ and $f(\underline{\theta})$ are strictly positive and $|f'(\theta)|$ is bounded, i.e., when the distribution of θ is sufficiently regular.

Having characterized the equilibrium arising when market-makers must post convex schedules, we now establish that this equilibrium also prevails when market-makers are not constrained to post convex schedules.

PROPOSITION 10: *The equilibrium obtained when strategies are constrained to be convex is also an equilibrium when this constraint is not imposed.*

6. PROPERTIES OF THE EQUILIBRIUM

6.1. *Monotonicity*

As requested by incentive compatibility, the larger the agent's valuation of the asset, the larger the trading volume. This property extends from the monopolistic to the oligopolistic cases.

6.2. *Trading Volume*

Trading volume in the oligopolistic market is lower than required by ex ante efficiency but larger than in the monopoly case. The heuristics of this property can be seen by manipulating the differential equation (43) into

$$q^n(\theta) = \frac{(n-1)\gamma\sigma^2\dot{q}^n(\theta)}{n-\gamma\sigma^2\dot{q}^n(\theta)}q^*(\theta) + \left(1 - \frac{(n-1)\gamma\sigma^2\dot{q}^n(\theta)}{n-\gamma\sigma^2\dot{q}^n(\theta)}\right)q_m(\theta), \tag{46}$$

which shows that, since $\dot{q}^n(\theta) > 0$ (by the convexity of U) and since $\gamma\sigma^2\dot{q}^n(\theta) < 1$ (by the convexity of the equilibrium schedule), $q^n(\theta)$ is a convex combination of $q^*(\theta)$ and $q_m(\theta)$. This illustrates the trade-off between the *allocative* and the *distributive* roles of the mechanism even under oligopolistic screening. Under monopoly, trading volume is low to reduce the costly informational rent of the trader. Competition among oligopolists leads to an increase in trading volume relative to this situation. Indeed suppose the oligopolists tried to achieve the fully collusive outcome where they would share equally the monopolistic trading volume $q_m(\theta)$ and charge the monopoly price $t_m(q_m(\theta))$. In this situation, taking as given the cooperative nonlinear schedule offered by his rivals, any individual market-maker, say M_1, would find it profitable to offer a side-deal to the agent. M_1 would offer a small reduction in prices to obtain an increase in his trades. The agent would accept this offer, and as a result purchase less from the remaining market-makers. In this situation, M_1 exerts a *positive externality* on his competitors. By reducing his own trading volume with the agent, M_1 shifts at the margin the choice of the agent towards buying more liquidity from his competitors.

In equilibrium, competition drives prices below the monopolistic level and thus increases trading volume. This discussion points at the fact that, in addition to the allocative and distributive roles played by the trading mechanism in the monopolistic case, in the oligopolistic case the competitors design the trading mechanisms they respectively offer with a view at a third objective: the gain of market shares.

6.3. *Mark-Ups*

To better understand the similarities and differences between mark-ups in the monopolistic and in the oligopolistic cases, and to shed some light on our results, we present them heuristically in a framework similar to that of Goldman, Leland, and Sibley (1984). Consider the choice by M_1 of the marginal transfer t_1 he will request for the sale of the q_1th unit. The informed agent's types purchasing this unit will be such that

$$\theta \geq \gamma\sigma^2 q + t_1 = \theta(t_1),$$

where q is the total quantity purchased from the market-makers. This total purchase can be decomposed into the trade of M_1, i.e., q_1, and the trades of each of his competitors $\phi^n(t_1)$:

$$q = q_1 + (n-1)\phi^n(t_1).$$

The trades of M_1's competitors depend on t_1 since when M_1 raises his prices, the agent alters the structure of her trades and buys more from M_1's competitors. More precisely, $\phi^n(\cdot)$ is related to the equilibrium marginal schedule offered by each market-maker $t^n(\cdot)$, by the following relation:

$$\phi^n(t^n(\tilde{q})) = \tilde{q}, \qquad \forall \tilde{q}.$$

M_1 optimally chooses t_1 to maximize his expected profits from the sale of the q_1th unit. As in the monopoly case, these expected profits can be written as

$$t_1(1 - F(\theta(t_1))) - \int_{\theta(t_1)}^{\bar{\theta}} v(\theta) f(\theta)\, d\theta, \tag{47}$$

where $1 - F(\theta(t_1))$ is the mass of agents conducting the trade with M_1 and

$$\frac{1}{1 - F(\theta(t_1))} \int_{\theta(t_1)}^{\bar{\theta}} v(\theta) f(\theta)\, d\theta$$

is the expected value of the asset given that the trade has been conducted with M_1, i.e., the marginal cost for M_1 of selling the unit. As in the monopoly case the optimal choice of t_1 is obtained by maximizing (47) with respect to t_1. This yields the following first order condition:

$$1 - F(\theta(t_1)) - t_1\theta'(t_1) f(\theta(t_1)) + v(\theta(t_1))\theta'(t_1) f(\theta(t_1)) = 0. \tag{48}$$

Hence, we get

$$t_1 = v(\theta(t_1)) + \frac{1 - F(\theta(t_1))}{\theta'(t_1) f(\theta(t_1))}. \tag{49}$$

Note that the difference between (49) and its monopolistic counterpart (25) stems from the presence of $\theta'(t_1)$. This term reflects that when M_1 raises his price, the agent alters the structure of his demand in favor of M_1's competitors. Indeed,

$$\theta'(t_1) = 1 + \gamma\sigma^2(n-1)(\phi^n)'(t_1) = 1 + \gamma\sigma^2\frac{(n-1)}{(t^n)'}, \tag{50}$$

which is larger than 1 since the equilibrium transfer schedule is convex. This highlights the role of the convexity of the schedules (of the competitors of M_1) in determining the residual demand faced by M_1, and as a result his own schedule. Because of the convexity of the transfers of his competitors, the oligopolistic mark-up of M_1 is lower than its monopolistic counterpart.

This heuristic discussion shows that, when choosing his optimal transfer schedules, M_1 trades-off price and quantity effects, similarly to the monopolist, except that the demand function he faces under oligopoly reflects not only the optimal behavior of the agent (as in the monopoly case) but also the schedules offered by competitors. Taking into account these combined effects, the oligopolist computes the elasticity of the residual demand and sets the mark-up of his marginal price over his marginal cost accordingly.[35]

6.4. *Common Values*

As shown above, although competition does increase trading and reduce mark-ups relative to the monopolistic case, it does not restore ex ante efficiency, nor does it bring prices to their break-even level. In fact, this is due to the *common value* aspect of the adverse selection problem studied here.

Consider in contrast a private value environment, where there would not be any information asymmetry about the value of the asset v and where adverse selection would bear only on the risk sharing need of the agent (I). In this case, $v(\theta)$ (which represents the opportunity cost of trades for the competitors) would be a constant equal to the unconditional expectation of v. Inserting this expression into (43) and taking into account the boundary conditions at $\bar{\theta}$ and $\underline{\theta}$, we find that $q^n(\theta) = q^*(\theta)$ for all θ as soon as $n \geq 2$.

Hence, the ex ante efficient level of risk-sharing is achieved and the bid-ask spread disappears. The Bertrand result obtains (whereby each competitor sells at his constant marginal cost) thanks to competition between the (finite number of) liquidity suppliers. The intuition for the difference between the common and the private value environments can be grasped from inspection of the oligopolis-

[35] The intrinsic common agency literature (see Stole (1991) and Martimort (1992)) shows also that competing principals face an imperfectly elastic residual demand in the case where the activities they control are imperfect substitutes (case of Bertrand competition with differentiated goods). The results obtained in that context are similar to ours: equilibrium schedules entail positive mark-ups above marginal costs, but these mark-ups are eroded by competition.

tic mark-ups, as heuristically presented above:

$$\frac{1-F(\theta(t_1))}{\left(1+\dfrac{\gamma\sigma^2(n-1)}{(t^n)'}\right)f(\theta(t_1))}.$$

Suppose the competitors of M_1 offer constant marginal prices equal to v (note that they can do so without incurring losses since there is no asymmetric information on the value of the asset). In this case $(t^n)'(\cdot)=0$ and the mark-up of M_1 is also driven to 0. Indeed when his competitors quote constant marginal prices equal to v, the residual demand curve faced by M_1 is infinitely elastic. As soon as he raises his prices slightly above v, M_1 loses all his market share.

This discussion shows that adverse selection with common values reduces the aggressiveness of competition in schedules. It is possible to draw a parallel between this result and the analysis of the "*winner's curse*" in the literature on common-value auctions.[36] In these models, buyers bid for one unit of an indivisible good with an unknown value, after observing private signals on the latter. In this context, winning conveys some bad information since it means that the signal of the winner was too optimistic relative to the signals observed by his competitors. Because of this "winner's curse," buyers refrain from bidding too aggressively. Our setting is different in at least two senses: First, market-makers are bidding for a divisible good, the total trading volume; second, they do not directly observe exogenous private signals on the value of the asset v but instead can condition their estimate of this value on their own trading volume with the agent. In this context, similarly to the "winner's curse," the trade conducted by M_1 conveys bad news about the cost of selling liquidity. Selling one extra unit only provides the oligopolist with an estimate of a lower bound on the possible valuations of the agent for the asset. Realizing this, the oligopolist charges higher prices because he fears that selling liquidity is relatively costly for him. Oligopolists are reluctant to compete aggressively. This effect leads to relatively high mark-ups and reduces risk-sharing relative to the ex ante efficient outcome.

6.5. *Intensity of Competition*

We now study how the number n of market-makers affects the properties of the equilibrium that we have just derived. The best way to do so is to split (43) into two equations on $[\underline{\theta}, \theta_b^n]$ and $[\theta_a^n, \bar{\theta}]$ with respective solutions $q^{n-}(\cdot)$ and $q^{n+}(\cdot)$. The bid and ask prices at 0 are defined implicitly by

$$q^{n-}(\theta_b^n)=q^{n+}(\theta_a^n)=0.$$

Since $q_m(\theta)\le q^{n+}(\theta)\le q^*(\theta)$, the right-hand side of (43) is decreasing in n, which implies that, for all $\theta\in[\theta_a^n,\bar{\theta}]$, $q^{n+}(\theta)$ increases in n. Proceeding

[36] See Milgrom and Weber (1982).

similarly with $q^{n-}(\cdot)$, we obtain that for all $\theta \in [\underline{\theta}, \theta_b^n]$, $q^{n-}(\theta)$ decreases in n. This leads to the following corollaries:

COROLLARY 11: *The bid price, θ_b^n, is increasing and the ask price, θ_a^n, is decreasing in n, the number of market-makers; consequently the bid-ask spread is decreasing in n.*

COROLLARY 12: *For all θ the absolute value of the trade $|q^n(\theta)|$ is increasing in n, the number of market-makers.*

Intensifying competition on the market by increasing the number of market-makers (for instance by reducing entry requirements or merging existing markets) increases the trading volume and reduces the bid-ask spread. With more competitors, the residual demand faced by each single market-maker becomes more elastic and mark-ups diminish.

Moreover, as the number of market-makers increases the informational rent of the trader also increases. He can then better escape the control of the market-makers and the rent-efficiency trade-off is more and more tilted in his favor.

In this model of commitment to a price schedule before the agent's choice of consumption, the number of market-makers affects the equilibrium volume. This is similar to a result of Dennert (1993) in a different model and contrasts with models of no-commitment like that of Kyle (1985), where Bertrand competition among the market-makers leads to the same volume of trade whatever the number of market-makers.

7. THE CASE OF A LARGE NUMBER OF MARKET-MAKERS

We now turn to the limiting case where the number of market-makers goes to infinity. For all θ, the sequence $n \to q^n(\theta)$ is monotonic and bounded: It has a finite limit $q^\infty(\theta)$. By continuity, $q^\infty(\cdot)$ solves the differential equation obtained by letting $n \to \infty$ in (43):

$$\text{(51)} \qquad \dot{q}^\infty(\theta) = \frac{1}{\gamma\sigma^2}\left(1 + \frac{q^*(\theta) - q^\infty(\theta)}{q^\infty(\theta) - q_m(\theta)}\right)^{-1}.$$

Solving this equation, we obtain the following proposition.

PROPOSITION 13: *When the number of market-makers goes to infinity, the equilibrium volume of trade converges to a function $q^\infty(\cdot)$ such that there exist $\theta_a^\infty > 0$ and $\theta_b^\infty < 0$ with:*

- *for all $\theta \in (\theta_a^\infty, \overline{\theta}]$, $0 < q^\infty(\theta) \leq q^*(\theta)$ with the second inequality being an equality at $\overline{\theta}$;*
- *for all $\theta \in [\theta_b^\infty, \theta_a^\infty]$, $q^\infty(\theta) = 0$;*
- *for all $\theta \in [\underline{\theta}, \theta_b^\infty)$, $0 > q^\infty(\theta) \geq q^*(\theta)$ with the second inequality being an equality at $\underline{\theta}$. In particular the bid ask spread is bounded away from zero.*

Even when each market-maker has a negligible impact on the allocation of resources in the economy, the fundamental features of the equilibrium found with a finite number of market-makers remain. Competition under common value impedes the achievement of ex ante efficiency and a positive bid-ask spread remains. A positive measure of agents remains "rationed" in equilibrium even if a large number of market-makers allows trading volume to increase on this market.

Everything happens in the limit as if the agent were offered a nonlinear price schedule $T^{\infty}(q)$ that aggregates all individual market-maker's schedules. Denoting by $t^{\infty}(\cdot)$ the corresponding unit price and using the first order condition for the agent's problem (26), we have

$$t^{\infty}(q^{\infty}(\theta)) = \theta - \gamma\sigma^2 q^{\infty}(\theta).$$

PROPOSITION 14: *When the number of market-makers goes to infinity the equilibrium marginal price schedule converges to a limit characterized by*:

- *for* $\theta > \theta_a^{\infty}$,

$$t_+^{\infty}(q^{\infty}(\theta)) = E(v(s)|s \geq \theta) = \frac{1}{1-F(\theta)} \int_{\theta}^{\overline{\theta}} v(s) f(s)\, ds; \tag{52}$$

- *for* $\theta < \theta_b^{\infty}$,

$$t_-^{\infty}(q^{\infty}(\theta)) = E(v(s)|s \leq \theta) = \frac{1}{F(\theta)} \int_{\underline{\theta}}^{\theta} v(s) f(s)\, ds. \tag{53}$$

Noticeably, when the number of market-makers goes to infinity, the marginal prices equal the upper (resp. lower) tail conditional expectations which, when drawing an analogy between the monopoly case and the analysis conducted by Goldman, Leland, and Sibley (1984), we showed to be similar to marginal costs. Hence one interpretation of Proposition 14 is that the "*competitive outcome*" emerges when the number of market-makers goes to infinity. In the case of a private value environment (when there is no private signal on the value of the asset) this marginal cost is independent on θ. Competitive market-makers charge a constant unit price and end up making zero profit as a whole. In a common value environment, the competitive outcome is more delicate to define since the unit price requested changes for all different types of the agent.

Using Proposition 14, we can immediately establish that market-makers as a whole earn zero expected profit in equilibrium.

COROLLARY 15: *When the number of market-makers goes to infinity, the total profit of the market-makers at equilibrium goes to zero.*

The limit behavior of our economy has several features of a model of monopolistic competition among the market-makers. They charge a positive mark-up and make zero profit in equilibrium because they face an infinitesimal demand. In other words, the competitive schedule $T^{\infty}(\cdot)$ is such that no market-maker can deviate by offering an alternative schedule without losing money. The best that he can do is to match his own offer with this schedule. An interpretation of this result is that the schedule $T^{\infty}(\cdot)$ is *entry-proof*.[37] No market-maker can build an alternative institution or mechanism to enter and make positive profit on such a market.

8. CONCLUSION

Taking a mechanism design approach and using the powerful tools of the calculus of variations, this paper has analyzed financial markets as a nexus of competing trading mechanisms offered in a decentralized way. We derived a number of fundamental features of the equilibrium: Existence and uniqueness, symmetry, convexity of the price schedules, positive mark-ups even under oligopolistic screening, and finally trading volumes smaller than the ex ante efficient outcome but increasing with the intensity of competition.

A number of important extensions remain to be pursued. For example, how would the market-makers' knowledge of private signals on the underlying value of the asset reinforce the winner's curse illustrated in this paper? Addressing this question is presumably difficult, since it would require extending the informed-principal framework of Maskin and Tirole (1982) to the case of several principals. Similarly, what would happen if several risk-averse informed traders were simultaneously present? Finally, what happens when different market places or institutions are competing one with another, each of them with its own set of market-makers possibly restricted to use different mechanisms depending on the market-place?

IDEI-R and GREMAQ, Université de Toulouse I, Place Anatole-France, 31042 Toulouse Cedex, France,

IDEI-R and Université de Pau, Avenue du Doyen Poplawski, 6400 Pau, France,

and

IDEI-R and GREMAQ, Université de Toulouse I, Place Anatole-France, 31042 Toulouse Cedex, France.

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APPENDIX

LOG-CONCAVITY IS PRESERVED BY CONVOLUTION

PROPOSITION 16: *Let $f(\cdot)$ be a log-concave density and $H(\cdot)$ be the cumulative distribution associated to the convolution $h = f * g$ of f with an arbitrary density $g(\cdot)$ with a bounded support $[a, b]$; then $H(\cdot)$ and $1 - H(\cdot)$ are log-concave.*

[37] See also Glosten (1994) for some discussion of this issue.

Since f is log-concave f'/f is decreasing. Moreover, we know from Bagnoli and Bergstrom (1989) that F is also log-concave. Therefore f/F is decreasing, which implies that $f'F \leq f^2$. Let us define, for a given z, the function

$$\varphi(x,y) = f(z-x)f(z-y) - f'(z-x)F(z-y).$$

We are going to prove that $\varphi(x,y) \geq 0$ for all x, y. Indeed consider two cases:

- if $x \leq y$: then

$$\frac{f'(z-x)}{f(z-x)} \leq \frac{f'(z-y)}{f(z-y)}, \quad \text{so that}$$

$$\varphi(x,y) \geq \frac{f(z-x)}{f(z-y)}[f^2(z-y) - f'(z-y)F(z-y)] \geq 0;$$

- if $x \geq y$: then

$$\frac{f(z-x)}{F(z-x)} \leq \frac{f(z-y)}{F(z-y)} \quad \text{so that}$$

$$\varphi(x,y) \geq \frac{F(z-y)}{F(z-x)}[f^2(z-x) - f'(z-x)F(z-x)] \geq 0.$$

We are now in a position to prove that H is log-concave (the proof for $(1-H)$ is similar). Indeed, define $h(z) = \int_a^b f(z-y)g(y)\,dy$ and $H(z) = \int_a^b F(z-y)g(y)\,dy$. We have

$$\log(H(z))' = \frac{\int_a^b f(z-y)g(y)\,dy}{\int_a^b F(z-y)g(y)\,dy}$$

and

$$\log(H(z))'' = \frac{\int_a^b f'(z-y)g(y)\,dy}{\int_a^b F(z-y)g(y)\,dy} - \left(\frac{\int_a^b f(z-y)g(y)\,dy}{\int_a^b F(z-y)g(y)\,dy}\right)^2.$$

$H(\cdot)$ is log-concave if and only if

$$\left(\int_a^b f'(z-y)g(y)\,dy\right)\left(\int_a^b F(z-y)g(y)\,dy\right) \leq \left(\int_a^b f(z-y)g(y)\,dy\right)^2.$$

This is equivalent to

$$\left(\int_a^b f'(z-x)g(x)\,dx\right)\left(\int_a^b F(z-y)g(y)\,dy\right)$$
$$\leq \left(\int_a^b f(z-x)g(x)\,dx\right)\cdot\left(\int_a^b f(z-y)g(y)\,dy\right)$$

or

$$\int_a^b \int_a^b \varphi(x,y)\,dx\,dy \geq 0,$$

which is implied by the positivity of φ. *Q.E.D.*

Proof of Proposition 2: A straightforward application of Theorem 1 in Luenberger (1969, Chapter 1), guarantees the existence of a positive measure Λ (the Lagrange multiplier associated with the agent's participation constraint (13)) such that the solution $U_m(\cdot)$ to the monopoly problem (M) maximizes the Lagrangian:

(54) $$L(U,\dot{U}) = B_m(U,\dot{U}) + \int_{\underline{\theta}}^{\overline{\theta}} U(\theta)\,d\Lambda(\theta).$$

Moreover, the complementarity slackness condition requires that the support of Λ be contained in the set $(U_m)^{-1}(0)$. Since $U_m(\cdot)$ is convex and nonnegative, this set is an interval, which we will denote by $[\theta_b^m, \theta_a^m]$. By a slight abuse of notation let us denote $\Lambda(\theta)$ the cumulative distribution function associated with the measure Λ:

$$\Lambda(\theta) = \int_{\underline{\theta}}^{\theta} d\Lambda(s). \tag{55}$$

Integrating by parts, the expression of $L(U, \dot{U})$ can be simplified as

$$\int_{\underline{\theta}}^{\bar{\theta}} U(\theta)\, d(\Lambda(\theta) - F(\theta)) = \int_{\underline{\theta}}^{\bar{\theta}} \dot{U}(\theta)\, d(\Lambda(\theta) - F(\theta)) + U(\bar{\theta})(\Lambda(\bar{\theta}) - 1).$$

Consequently,

$$L(U, \dot{U}) = \int_{\underline{\theta}}^{\bar{\theta}} \left(\left(\theta - v(\theta) + \frac{F(\theta) - \Lambda(\theta)}{f(\theta)} \right) \dot{U}(\theta) - \frac{\gamma\sigma^2}{2} \dot{U}(\theta)^2 \right) f(\theta)\, d\theta$$
$$+ U(\bar{\theta})(\Lambda(\bar{\theta}) - 1).$$

Since $U(\bar{\theta})$ is arbitrary, $L(\cdot)$ has a maximum only when $\Lambda(\bar{\theta}) = 1$. Moreover, pointwise maximization over $\dot{U}(\theta)$ in the expectation yields

$$\forall \theta \in [\underline{\theta}, \bar{\theta}], \quad q_m(\theta) = q^*(\theta) + \frac{F(\theta) - \Lambda(\theta)}{f(\theta)\gamma\sigma^2}.$$

Further, the complementary slackness condition gives

$$\forall \theta \in [\underline{\theta}, \theta_b^m), \quad \Lambda(\theta) = 0,$$

and

$$\forall \theta \in (\theta_a^m, \bar{\theta}], \quad \Lambda(\theta) = 1.$$

The volume of trade thus characterized is indeed increasing in θ, thanks to the technical conditions (16) and (17). Hence the solution of the relaxed problem (M') is also a solution of the complete problem (M). *Q.E.D.*

PROOF OF LEMMA 6: As in the monopoly case, the aggregate expected profit of all market-makers can be written as the difference between total surplus and the trader's utility:

$$B_1 + \cdots + B_n = \int_{\underline{\theta}}^{\bar{\theta}} \left((\theta - v(\theta))\dot{U}(\theta) - \frac{\gamma\sigma^2}{2} (\dot{U}(\theta))^2 - U(\theta) \right) f(\theta)\, d\theta, \tag{56}$$

where we have used (12) to replace $q(\theta)$ by $\dot{U}(\theta)$. Using condition (31), we have also

$$B_i = \int_{\underline{\theta}}^{\bar{\theta}} (T_i \circ S_i(\theta - \gamma\sigma^2 \dot{U}(\theta)) - v(\theta) S_i(\theta - \gamma\sigma^2 \dot{U}(\theta))) f(\theta)\, d\theta. \tag{57}$$

Combining equations (56) and (57), the desired result obtains. *Q.E.D.*

PROOF OF PROPOSITION 7

The proof of this proposition is divided in two parts. In the first part we show how the system of equations (42) gives rise to the symmetric solution characterized in equation (43). In the second part we show that for the equilibrium strategies of the market-maker the condition that there exists a solution T_1 to the convolution equation $T = T_1 \,\square\, T_{-1}$ is satisfied.

Symmetric Equilibrium:
Step 1: Conditions on q(θ): By summing up the n equations (42) we get

$$\gamma\sigma^2 \frac{(n-1)\dot{q}(\theta)(q^*(\theta)-q(\theta))}{1-\gamma\sigma^2\dot{q}(\theta)} = n\left(q(\theta)-q^*(\theta)-\frac{F(\theta)-\overline{\Lambda}(\theta)}{\gamma\sigma^2 f(\theta)}\right)$$

where $\overline{\Lambda}(\theta) = (1/n)\Sigma_{i=1}^n \Lambda_i(\theta)$. Solving for $\dot{q}(\theta)$ we obtain

$$\dot{q}(\theta) = \frac{1}{\gamma\sigma^2}\left(1+\frac{(n-1)(q^*(\theta)-q(\theta))}{n\left(q(\theta)-q^*(\theta)-\frac{F(\theta)-\overline{\Lambda}(\theta)}{\gamma\sigma^2 f(\theta)}\right)}\right)^{-1}.$$

Since $1-\gamma\sigma^2\dot{q}(\theta)$ is greater than 0 we deduce that

$$(q^*(\theta)-q(\theta))\left(q(\theta)-q^*(\theta)-\frac{F(\theta)-\overline{\Lambda}(\theta)}{\gamma\sigma^2 f(\theta)}\right) \geq 0.$$

Therefore, $\overline{\Lambda}(\theta)-F(\theta)$ has the same sign as $q^*(\theta)-q(\theta)$: positive for $\theta > \theta_0$ and negative for $\theta < \theta_0$.

Step 2: Positiveness of all $q_i(\theta)$: A priori, for a given θ, there may exist values of i such that $q_i(\theta) > 0$, resp. $q_i(\theta) = 0$ and < 0. Equation (42) shows that

$$0 = (q^*(\theta)-q(\theta))\left[1+\frac{\gamma\sigma^2}{1-\gamma\sigma^2\dot{q}(\theta)}(\dot{q}(\theta)-\dot{q}_i(\theta))\right]+\frac{F(\theta)-\Lambda_i(\theta)}{\gamma\sigma^2 f(\theta)}. \tag{58}$$

We know that all the $\dot{q}_i(\theta)$ are ≥ 0, and that $1-\gamma\sigma^2\dot{q}(\theta) \geq 0$. Consequently, $\dot{q}(\theta)-\dot{q}_i(\theta) = \Sigma_{j\neq i}\dot{q}_j(\theta) \geq 0$ and the bracketed term in equation (58) is positive. This means that for all i, $\Lambda_i(\theta)-F(\theta)$ has the same sign as $q^*(\theta)-q(\theta)$. If $\Lambda_i(\theta)-F(\theta) \geq 0$ then $q_i(\theta) \geq 0$ for all i (this is so because $q_i(\theta) = 0$ when $\Lambda_i(\theta) > 0$) and $q(\theta) \leq q^*(\theta)$.

Can we have $q_i(\theta) > 0$, say for $i = 1, \ldots, m$, and $q_i(\theta) = 0$ for the other market-makers? In that case, $\Lambda_i(\theta) = 1$ for $i = 1, \ldots, m$. Hence $q_i(\theta) = (1/m)q(\theta)$ for the active market-makers, and $q(\theta) < q^*(\theta)$. Therefore (58) has different expressions for the different market-makers:

$$\forall i \leq m, \quad 0 = (q^*(\theta)-q(\theta))\left(1+\frac{\gamma\sigma^2}{1-\gamma\sigma^2\dot{q}(\theta)}\frac{m-1}{m}\dot{q}(\theta)\right)-\frac{1-F(\theta)}{\gamma\sigma^2 f(\theta)},$$

$$\forall i > m, \quad 0 = (q^*(\theta)-q(\theta))\left(1+\frac{\gamma\sigma^2}{1-\gamma\sigma^2\dot{q}(\theta)}\dot{q}(\theta)\right)+\frac{F(\theta)-\Lambda_i(\theta)}{\gamma\sigma^2 f(\theta)}.$$

Subtracting, we get

$$0 = (q^*(\theta)-q(\theta))\frac{\gamma\sigma^2}{1-\gamma\sigma^2\dot{q}(\theta)}\left(-\frac{\dot{q}(\theta)}{m}\right)-\frac{1-\Lambda_i(\theta)}{\gamma\sigma^2 f(\theta)}. \tag{59}$$

Now since

$$q^*(\theta)-q(\theta) > 0,$$

$$\frac{\gamma\sigma^2\dot{q}(\theta)}{1-\gamma\sigma^2\dot{q}(\theta)} > 0 \quad \text{and}$$

$$-\frac{1-\Lambda_i(\theta)}{\gamma\sigma^2 f(\theta)} < 0,$$

equality (59) cannot hold. Hence we have proved by contradiction that when $q_i(\theta) > 0$ for i, then $q_j(\theta) > 0$ for all $j \neq i$. The case $q_i(\theta) \leq 0$ is handled symmetrically. Therefore the bid and ask prices are the same across market-makers, and $q_i(\theta) = q(\theta)/n$ for all i, θ.

Finally using that $\overline{\Lambda}(\overline{\theta}) = 1$ (resp. $\overline{\Lambda}(\underline{\theta}) = 0$), we obtain $q^*(\overline{\theta}) = q(\overline{\theta})$ (resp. $q^*(\underline{\theta}) = q(\underline{\theta})$).

Solution to the Equation $T = T_1 \,\square\, T_2$: For simplicity, but still without any loss of generality, this part of the proof is cast when there are only two market-makers. Using the dual approach that we have adopted in the text, we need to know, given $T_2(\cdot)$, how the trader's strategy $(q_1(\theta), q_2(\theta))$ depends on her rent $U(\cdot)$, which we use as the instrument of market-maker M_1. Recall that $U(\cdot)$ is defined by

$$(60) \qquad U(\theta) = \max_{q_1, q_2}\left(\theta(q_1+q_2) - \frac{\gamma\sigma^2}{2}(q_1+q_2)^2 - T_1(q_1) - T_2(q_2)\right)$$

where the maximum is attained for $q_1 = q_1(\theta)$ and $q_2 = q_2(\theta)$. The first question is implementability: Given $T_2(\cdot)$, is it possible to find $T_1(\cdot)$ such that $U(\cdot)$ satisfies (60)?

We answer it in two steps:

- We already know that, given $U(\cdot)$, it is possible to find an aggregate price schedule $T(\cdot)$ that implements $U(\cdot)$ if and only if

$$U(\cdot) \text{ is convex and for a.e. } \theta,\ q(\theta) = \dot{U}(\theta).$$

(where $q(\theta) = q_1(\theta) + q_2(\theta)$). In this case, $T(\cdot)$ is defined implicitly by

$$t(q(\theta)) = \theta - \gamma\sigma^2\dot{U}(\theta), \quad \text{a.e. } \theta,$$

$$T(q) = \int_0^q t(s)\,ds.$$

- We now have to find $T_1(q) = \int_0^q t_1(s)\,ds$ such that

$$(61) \qquad T = T_1 \,\square\, T_2.$$

We define the following function:

$$(62) \qquad T^*(\theta) = \max_q\{\theta q - T(q)\}.$$

T^* is the Fenchel dual of T.[38] It is a convex function (with values in $\mathbb{R} \cup \{+\infty\}$). The main property of this duality operator is that the bidual of T, defined as

$$T^{**} = (T^*)^*,$$

is the convex envelope of T, i.e., the supremum of all convex functions that are dominated by T. In particular $T^{**} = T$ if and only if T is convex. Moreover we have the following lemma.

LEMMA 17: $(T_1 \,\square\, T_2)^* = T_1^* + T_2^*$.[39]

PROOF: We have by definition

$$T^*(\theta) = \max_q\{\theta q - T(q)\}, \quad \text{and}$$

$$T(q) = \min_{q_1+q_2=q}\{T_1(q_1) + T_2(q_2)\}.$$

[38] See Rockafellar (1970).
[39] See Rockafellar (1970).

Therefore,

$$T^*(\theta) = \max_{q_1, q_2} (\theta(q_1 + q_2) - T_1(q_1) - T_2(q_2)),$$
$$= \max_{q_1} (\theta q_1 - T_1(\theta_1)) + \max_{q_2} (\theta q_2 - T_2(q_2)).$$

Thus $T^*(\theta) = T_1^*(\theta) + T_2^*(\theta)$. *Q.E.D.*

Therefore if T_1 solves equation (61) it must be that $T^* = T_1^* + T_2^*$ where $T_1^* = T^* - T_2^*$ is convex. Conversely if $(T^* - T_2^*)$ is convex, then a natural candidate to be a solution is

$$T_1 = (T^* - T_2^*)^*.$$

By the biduality theorem, it satisfies indeed $T_1^* = T^* - T_2^*$.

Does this imply the desired result that $T = T_1 \ \square \ T_2$? Not necessarily, except if T_2 and T are convex. In this case we have

$$T = T_1 \ \square \ T_2 = (T_1 \ \square \ T_2)^{**} = (T_1^* + T_2^*)^* = T^{**} = T.$$

Thus, we have established the following lemma.

LEMMA 18: *Given T and T_2, there exists T_1 such that $T = T_1 \ \square \ T_2$ only when $T^* - T_2^*$ is convex. The converse is true when T and T_2 are convex, and we can take $T_1 = (T^* - T_2^*)^*$, which is itself convex.*

Of course there is an additional restriction, which is that $T_1(0) = 0$. It is easy to see that it is equivalent to say that $T_1^* \geq 0$. This condition is easier to formulate in terms of the function $U(\cdot)$: It is equivalent to

$$\forall \theta \quad U(\theta) \geq \sup_{q_2} \{\theta q_2 - \tfrac{1}{2}\gamma\sigma^2 q_2^2 - T_2(q_2)\} = U_2(\theta).$$

This justifies the method we have adopted in the text for finding the best response mapping of market-maker M_1: Find $U(\cdot)$ that maximizes the expected profit $B_1(U, \dot{U}, T_2)$ of M_1 under the constraints that $U(\cdot)$ is convex and $U(\theta) \geq U_2(\theta)$. Since the other restriction, implied by Lemma 17 that $(T^* - T_2^*)$ is convex, is very difficult to characterize in terms of U, we have checked it only ex post. *Q.E.D.*

PROOF OF PROPOSITION 8

This proof is divided in two parts. In the first part we prove that there exists a solution to the differential equation (43). In the second part we prove that the first order approach we have taken is valid. In particular, we exhibit conditions ensuring the concavity of each principal's problem.

Solution to the Differential Equation: We focus on the positive part $(\theta > \theta_0)$. Everything can be similarly done for $\theta < \theta_0$. The behavior of the solution to (43) around $\theta = \bar{\theta}$ is the same as that of the linearized differential equation that we obtain with simple Taylor expansions of the numerator and the denominator in the neighborhood of $\theta = \bar{\theta}$. We obtain

$$\dot{q}^n(\theta) = \frac{n}{\gamma\sigma^2}\left(\frac{(2 - \dot{v})(\theta - \bar{\theta}) - \gamma\sigma^2(q^n(\theta) - q^*(\bar{\theta}))}{(n + 1 - \dot{v})(\theta - \bar{\theta}) - \gamma\sigma^2(q^n(\theta) - q^*(\bar{\theta}))}\right) \tag{63}$$

where $\dot{v} = \dot{v}(\bar{\theta})$ and where we have used (43).

First of all it is easy to derive from this last equation the value k_n of the derivative $\dot{q}^n(\bar{\theta})$. We get that k_n must solve

$$P(k_n) = \gamma^2\sigma^4 k_n^2 - (2n + 1 - \dot{v})\gamma\sigma^2 k_n + n(2 - \dot{v}) = 0.$$

This second degree polynomial admits two roots

$$k'_n = \frac{2n+1-\dot{v}+\sqrt{4n(n-1)+(\dot{v}-1)^2}}{2\gamma\sigma^2},$$

$$k_n = \frac{2n+1-\dot{v}-\sqrt{4n(n-1)+(\dot{v}-1)^2}}{2\gamma\sigma^2}.$$

We note that both solutions are positive. Hence, they both correspond to schedules that are locally increasing around $\theta=\bar{\theta}$. We select nevertheless only k_n, the smallest of these solutions, since in fact $1-\gamma\sigma^2 k_n>0$ and $1-\gamma\sigma^2 k'_n<0$. As we will see later, this condition guarantees that price schedules are convex in equilibrium.

We define

$$\hat{q}^n(\theta) = q^*(\theta) - \frac{n}{\gamma\sigma^2}\frac{1-F(\theta)}{f(\theta)}.$$

We have also:

$$\dot{q}_m(\bar{\theta}) = \frac{2-\dot{v}}{\gamma\sigma^2}, \qquad \dot{q}^*(\bar{\theta}) = \frac{1-\dot{v}}{\gamma\sigma^2}, \qquad \dot{\hat{q}}^n(\bar{\theta}) = \frac{n+1-\dot{v}}{\gamma\sigma^2}.$$

Hence, $P(\dot{q}^*(\bar{\theta})) = n\dot{v}>0$, $P(\dot{q}_m(\bar{\theta})) = (2-\dot{v})(1-n)<0$, and $P(\dot{\hat{q}}^n(\bar{\theta})) = n(1-n)<0$ for $n\geq 2$. We deduce the following inequalities:

$$\dot{q}^*(\bar{\theta}) < k_n < \dot{q}_m(\bar{\theta}) < \dot{\hat{q}}^n(\bar{\theta}) < k'_n.$$

Locally, around $\theta=\bar{\theta}$, we have therefore $q^n(\theta)\in(q_m(\theta), q^*(\theta))$ (with equality only at $\theta=\bar{\theta}$).[40]

• Moreover, any solution to (43) must be such that $q^n(\theta)\in(q_m(\theta),q^*(\theta))$ for all $\theta<\bar{\theta}$. Suppose indeed that $q^n(\theta)$ crosses $q_m(\theta)$ for some $\theta'<\bar{\theta}$ and consider θ' as being the last of these crossing points before $\bar{\theta}$. Then $\dot{q}^n(\theta')=0$ and $\dot{q}_m(\theta')<0$. This implies that $q_m(\theta)<q^n(\theta)$ for $\theta\in[\theta',\theta'+\varepsilon)$ (for ε small enough). Since $q^n(\theta)>q_m(\theta)$ in the neighborhood of $\bar{\theta}$, we have a contradiction.

Similarly, denote θ'', the last point before $\bar{\theta}$ where $q^n(\theta)$ crosses $q^*(\theta)$. We have

$$\dot{q}^*(\theta'') = \frac{1-\dot{v}(\theta'')}{\gamma\sigma^2} < \dot{q}^n(\theta'') = \frac{1}{\gamma\sigma^2}.$$

This implies that $q^*(\theta)<q^n(\theta)$ for $\theta\in[\theta'',\theta''+\varepsilon)$ (for ε small enough). But around $\theta=\bar{\theta}$, know that $q^n(\theta)<q^*(\theta)$, and again we have a contradiction.

• All the solutions to (43) have the same derivative at $\theta=\bar{\theta}$; moreover all these possible solutions always remain in the interval $(q_m(\theta),q^*(\theta))$. If we prove that all these solutions are locally unique on some interval $(\bar{\theta}-\varepsilon,\bar{\theta})$, we will have in fact the global uniqueness result. At $\bar{\theta}-(\varepsilon/2)$, the right-hand side of (43) is Lipschitz continuous. The theorem of Cauchy-Lipschitz applies to prove the global uniqueness of the solution to (43). Changing notations $X^n=q^*(\bar{\theta})-q^n(\theta)$ and $Y=\bar{\theta}-\theta$, (63) can be solved by parameterizing X^n and Y as functions of a parameter t such that $X^n(t)=tY(t)$. Equation (63) becomes

$$t+\frac{Y}{\frac{dY}{dt}} = \frac{n}{\gamma\sigma^2}\left(\frac{(2-\dot{v})-\gamma\sigma^2 t}{(n+1-\dot{v})-\gamma\sigma^2 t}\right).$$

Solving for $Y(t)$ yields

$$Y(t) = C\frac{|t-k_n|^{1/2+(\dot{v}-1)/2\sqrt{4n(n-1)+(\dot{v}-1)^2}}}{|t-k'_n|^{1/2+(\dot{v}-1)/2\sqrt{4n(n-1)+(\dot{v}-1)^2}}}$$

[40] Note also that $\hat{q}^n(\theta)\leq q_m(\theta)\ \forall\theta\in[\theta_0,\bar{\theta}]$ with equality only for $\theta=\bar{\theta}$.

where C is an arbitrary constant. However the only possibility for having $X_n(t) = Y(t) = 0$ for some t as it is requested by the initial conditions of (63) is to have $X_n = k_n Y$. Hence, in a neighborhood of $\theta = \overline{\theta}$, the solution to (63) is unique.

• Finally, note that using (43) and the conditions $q^n(\theta) \in (q_m(\theta), q^*(\theta))$ we obtain that $\dot{q}^n(\theta) > 0$ and $1 - \gamma\sigma^2 \dot{q}^n(\theta) > 0$ for all $\theta \in [0, \overline{\theta})$. Hence

$$t'(q(\theta)) = \frac{1 - \gamma\sigma^2 \dot{q}(\theta)}{\dot{q}(\theta)} > 0$$

for any equilibrium trade and $T(\cdot)$ is convex.

Concavity of the Principal's Problem: First, we note that $B_1(U, \dot{U}, T_{-1})$ is linear in U. Moreover, we have

$$\frac{\partial^2 B_1}{\partial \dot{U}^2} = -f(\theta)\gamma\sigma^2 \left(1 + \frac{(n-1)\gamma\sigma^2}{T^{n''}\left(\dfrac{\dot{U}}{n}\right)} + \frac{(n-1)}{n} \frac{T^{n'''}\left(\dfrac{\dot{U}}{n}\right)}{\left(T^{n''}\left(\dfrac{\dot{U}}{n}\right)\right)^2} (\theta - v(\theta) - \gamma\sigma^2 \dot{U}) \right)$$

where $T^n(\cdot)$ is the equilibrium nonlinear price offered by other market-makers. When M_1 induces a total trading volume $q(\theta) = \dot{U}(\theta)$ such that $q(\theta) \le q^*(\theta)$, the previous expression will be concave if $T^{n'''}(q(\theta)/n)$ is positive. Easy computations show that

$$T^{*'''}\left(\frac{q^n(\theta)}{n}\right) = -n^2 \frac{\ddot{q}^n(\theta)}{\dot{q}^{n^3}(\theta)}.$$

After manipulations, we find that

$$\frac{\ddot{q}^n(\theta)}{\dot{q}^n(\theta)} = \frac{(n-1)\left(-\dot{v}(\theta)\dfrac{1-F(\theta)}{f(\theta)} + \gamma\sigma^2 (q^*(\theta) - q^n(\theta)) \left(\dfrac{d}{d\theta}\left(\dfrac{1-F(\theta)}{f(\theta)}\right) - \dfrac{1-F(\theta)}{f(\theta)} \right)\right)}{\gamma^2\sigma^4 (q_m(\theta) - q^n(\theta))(q^*(\theta) - q^n(\theta))}.$$

Using $\dot{v}(\theta) \ge 0$, (16), and $q^*(\theta) \ge q(\theta)$ yields that the numerator above is negative; hence $\ddot{q}^n(\theta) \le 0$ and $T^{n'''}(q)$ is positive for all $q \in [0, (q^n(\overline{\theta})/n)]$. For $q > (q^n(\overline{\theta})/n)$, we can extend the nonlinear schedule $T^n(\cdot)$ out of the equilibrium path in a continuous and linear way (it has therefore slope $v(\overline{\theta})$ for $q \ge (q(\overline{\theta})/n)$ so that $T^{n'''}(q) = 0$. This ensures the concavity of the principal's objective function. *Q.E.D.*

Proof of Proposition 10

This proof directly stems from Lemma 18. *Q.E.D.*

Proof of Proposition 13

It obtains by passing to the limit in the results of Proposition 7. The only thing to prove is that $q^\infty(\theta)$ never crosses $q^*(\theta)$ away from $\overline{\theta}$ and $\underline{\theta}$ and this can be proved as in the Proof of Proposition 7. *Q.E.D.*

PROOF OF PROPOSITION 14

Rearranging terms we obtain

$$\gamma\sigma^2\dot{q}^\infty(\theta)(q^\infty(\theta)-q_m(\theta)+q^*(\theta)-q^\infty(\theta))=q^\infty(\theta)-q_m(\theta). \tag{64}$$

Now

$$q^*(\theta)-q_m(\theta)=-\frac{F(\theta)-\Lambda(\theta)}{f(\theta)},$$

where $\Lambda(\theta)=1$ (resp. $F(\theta)$ and 0) when $\theta>\theta_a^m$ (resp. $\theta_a^m\geq\theta\geq\theta_b^m$ and $\theta_b^m>\theta$), so that (64) can be rewritten as

$$0=(\theta-v(\theta)-\gamma\sigma^2q^\infty(\theta))f(\theta)+(1-\gamma\sigma^2\dot{q}^\infty(\theta))(F(\theta)-\Lambda^\infty(\theta)),$$

where $\Lambda^\infty(\theta)=\lim_n\Lambda^n(\theta)$. Consider, for example, the region where $q^\infty(\theta)>0$ and $\Lambda^\infty(\theta)=1$. Integrating, we obtain,

$$(\theta-\gamma\sigma^2q^\infty(\theta))(1-F(\theta))=\int_\theta^{\bar\theta}v(s)f(s)\,ds.$$

Finally $t_a(q^\infty(\theta))=\theta-\gamma\sigma^2q^\infty(\theta)=E[v(s)|s\geq\theta]$. *Q.E.D.*

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