Lecture 02: One Period Model

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Overview

1. Securities Structure
   - Arrow-Debreu securities structure
   - Redundant securities
   - Market completeness
   - Completing markets with options

2. Pricing (no arbitrage, state prices, SDF, EMM …)
The Economy

- **State space (Evolution of states)**
  - Two dates: $t=0, 1$
  - $S$ states of the world at time $t=1$

- **Preferences**
  - $U(c_0, c_1, \ldots, c_S)$
  - $MRS^A_{s,0} = -\frac{\partial U^A}{\partial c_s^A} / \frac{\partial U^A}{\partial c_0^A}$ (slope of indifference curve)

- **Security structure**
  - Arrow-Debreu economy
  - General security structure
Security Structure

- Security $j$ is represented by a payoff vector $\left( x^j_1, x^j_2, \ldots, x^j_S \right)$

- Security structure is represented by payoff matrix $X = \begin{pmatrix}
  x^j_1 & x^j_2 & \cdots & x^j_{S-1} & x^j_S \\
  x^{j+1}_2 & x^{j+1}_2 & \cdots & x^{j+1}_{S-1} & x^{j+1}_S \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  x^{J-1}_1 & x^{J-1}_2 & \cdots & x^{J-1}_{S-1} & x^{J-1}_S \\
  x^J_1 & x^J_2 & \cdots & x^J_{S-1} & x^J_S
\end{pmatrix}$

- NB. Most other books use the transpose of $X$ as payoff matrix.
Arrow-Debreu Security Structure in $R^2$

One A-D asset $e_1 = (1,0)$

This payoff cannot be replicated!

Payoff Space $<X>$

⇒ Markets are incomplete
Arrow-Debreu Security Structure in $R^2$

Add second A-D asset $e_2 = (0,1)$ to $e_1 = (1,0)$
Arrow-Debreu Security Structure in $\mathbb{R}^2$

Add **second** A-D asset $e_2 = (0,1)$ to $e_1 = (1,0)$

Any payoff can be replicated with two A-D securities
Arrow-Debreu Security Structure in $R^2$

Add **second** asset $(1,2)$ to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

*New asset is **redundant** – it does not enlarge the payoff space*
Arrow-Debreu Security Structure

\[ X = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \]

- \( S \) Arrow-Debreu securities
- each state \( s \) can be insured individually
- All payoffs are linearly independent
- Rank of \( X = S \)
- Markets are complete
General Security Structure

Only bond $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Payoff space $\langle X \rangle$
General Security Structure

Only bond $x^{\text{bond}} = (1,1)$

Payoff space $\langle X \rangle$

can’t be reached

$(1, 1)$
General Security Structure

Add security \((2,1)\) to bond \((1,1)\)
General Security Structure

Add security (2,1) to bond (1,1)

- Portfolio of
  - buy 3 bonds
  - sell short 1 risky asset
General Security Structure

Payoff space \( \langle X \rangle \)

Two assets span the payoff space

Market are complete with security structure

Payoff space coincides with payoff space of

\[
\begin{pmatrix}
1 & 1 \\
2 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\]
General Security Structure

- Portfolio: vector $h \in \mathbb{R}^J$ (quantity for each asset)
- Payoff of Portfolio $h$ is $\sum_j h^j x^j = h'X$
- Asset span $\langle X \rangle = \{z \in \mathbb{R}^S : z = h'X \text{ for some } h \in \mathbb{R}^J\}$
  - $\langle X \rangle$ is a linear subspace of $\mathbb{R}^S$
  - Complete markets $\langle X \rangle = \mathbb{R}^S$
  - Complete markets if and only if $\text{rank}(X) = S$
  - Incomplete markets $\text{rank}(X) < S$
  - Security $j$ is redundant if $x^j = h'X$ with $h^j = 0$
Introducing derivatives

- Securities: property rights/contracts
- Payoffs of derivatives *derive* from payoff of underlying securities
- Examples: forwards, futures, call/put options

• Question: Are derivatives necessarily redundant assets?
Forward contracts

• Definition: A binding agreement (obligation) to buy/sell an underlying asset in the future, at a price set today

• Futures contracts are same as forwards in principle except for some institutional and pricing differences

• A forward contract specifies:
  - The features and quantity of the asset to be delivered
  - The delivery logistics, such as time, date, and place
  - The price the buyer will pay at the time of delivery
Reading price quotes

Index futures

Settlement price (last transaction of the day)

Low of the day

High of the day

The open price

Expiration month

Daily change

Lifetime high

Lifetime low

Open interest

<table>
<thead>
<tr>
<th>INDEX</th>
</tr>
</thead>
<tbody>
<tr>
<td>DJ Industrial Average (CBOT)-$10 times average</td>
</tr>
<tr>
<td>Mar</td>
</tr>
<tr>
<td>June</td>
</tr>
<tr>
<td>Est vol: 21,000; vol Fri 17,070; open int: 28,254, +822.</td>
</tr>
<tr>
<td>Index: Hi 9905.06; Lo 9677.54; Close 9687.09; -220.17.</td>
</tr>
<tr>
<td>S&amp;P 500 Index (CME)-$250 times Index</td>
</tr>
<tr>
<td>Mar</td>
</tr>
<tr>
<td>June</td>
</tr>
<tr>
<td>Dec</td>
</tr>
<tr>
<td>Est vol: 75,914; vol Fri 65,230; open int: 502,626; -701.</td>
</tr>
<tr>
<td>Index: Hi 1122.20; Lo 1092.25; Close 1094.44; -27.76.</td>
</tr>
<tr>
<td>Mini S&amp;P 500 (CME)-$50 times index</td>
</tr>
<tr>
<td>Mar</td>
</tr>
<tr>
<td>Vol Fri 193,622; open int: 100,323; -4,791.</td>
</tr>
<tr>
<td>S&amp;P Midcap 400 (CME)-$500 times index</td>
</tr>
<tr>
<td>Mar</td>
</tr>
<tr>
<td>Est vol: 1,140; vol Fri 1,101; open int: 13,435; -207.</td>
</tr>
<tr>
<td>Index: Hi 504.26; Lo 492.74; Close 493.38; -10.88.</td>
</tr>
<tr>
<td>Nikkei 225 Stock Average (CBOT)-$5 times index</td>
</tr>
<tr>
<td>Mar</td>
</tr>
<tr>
<td>Est vol: 667; vol Fri 2,100; open int: 15,817; -17.</td>
</tr>
<tr>
<td>Index: Hi 9809.82; Lo 9523.99; Close: 9631.93; -159.50.</td>
</tr>
<tr>
<td>Nasdaq 100 (CME)-$100 times index</td>
</tr>
<tr>
<td>Mar</td>
</tr>
<tr>
<td>Est vol: 18,215; vol Fri 17,500; open int: 51,812; +763.</td>
</tr>
<tr>
<td>Index: Hi 1528.30; Lo 1471.52; Close: 1479.17; -48.98.</td>
</tr>
</tbody>
</table>
Payoff diagram for forwards

- Long and short forward positions on the S&R 500 index:
Forward vs. outright purchase

- Forward + bond = Spot price at expiration - $1,020 + $1,020
  = Spot price at expiration

Forward payoff
Bond payoff

Long forward
$1000 bond
Forward + bond
Additional considerations (ignored)

- Type of settlement
  - Cash settlement: less costly and more practical
  - Physical delivery: often avoided due to significant costs

- Credit risk of the counter party
  - Major issue for over-the-counter contracts
    - Credit check, collateral, bank letter of credit
  - Less severe for exchange-traded contracts
    - Exchange guarantees transactions, requires collateral
Call options

- A non-binding agreement (right but not an obligation) to buy an asset in the future, at a price set today
- Preserves the upside potential (😊), while at the same time eliminating the unpleasant (🙁) downside (for the buyer)
- The seller of a call option is obligated to deliver if asked
Definition and Terminology

- A call option gives the owner the right but not the obligation to buy the underlying asset at a predetermined price during a predetermined time period.
- Strike (or exercise) price: The amount paid by the option buyer for the asset if he/she decides to exercise.
- Exercise: The act of paying the strike price to buy the asset.
- Expiration: The date by which the option must be exercised or become worthless.
- Exercise style: Specifies when the option can be exercised.
  - European-style: can be exercised only at expiration date.
  - American-style: can be exercised at any time before expiration.
  - Bermudan-style: can be exercised during specified periods.
Reading price quotes
S&P500 Index options

<table>
<thead>
<tr>
<th>Strike price</th>
</tr>
</thead>
<tbody>
<tr>
<td>S &amp; P 500(SPX)</td>
</tr>
<tr>
<td>STRIKE</td>
</tr>
<tr>
<td>Feb c 1100</td>
</tr>
<tr>
<td>Feb p 1100</td>
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<td>Mar c 1100</td>
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<td>Mar p 1100</td>
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<td>Apr c 1100</td>
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<td>Apr p 1100</td>
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<tr>
<td>Feb c 1110</td>
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<td>Feb p 1110</td>
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<tr>
<td>Feb c 1120</td>
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<tr>
<td>Feb p 1120</td>
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<td>Mar c 1120</td>
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<tr>
<td>Mar p 1120</td>
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<tr>
<td>Apr c 1120</td>
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</tbody>
</table>
Payoff/profit of a purchased call

- Payoff = $max [0, spot price at expiration – strike price]
- Profit = Payoff – future value of option premium
- Examples 2.5 & 2.6:
  - S&R Index 6-month Call Option
    - Strike price = $1,000, Premium = $93.81, 6-month risk-free rate = 2%
    - If index value in six months = $1100
      - Payoff = $max [0, $1,100 - $1,000] = $100
      - Profit = $100 – ($93.81 x 1.02) = $4.32
    - If index value in six months = $900
      - Payoff = $max [0, $900 - $1,000] = $0
      - Profit = $0 – ($93.81 x 1.02) = - $95.68
Diagrams for purchased call

- Payoff at expiration

![Payoff Diagram](image1)

- Profit at expiration

![Profit Diagram](image2)
Put options

- A put option gives the owner the right but not the obligation to sell the underlying asset at a predetermined price during a predetermined time period.
- The seller of a put option is obligated to buy if asked.
- Payoff/profit of a purchased (i.e., long) put:
  - Payoff = max [0, strike price – spot price at expiration]
  - Profit = Payoff – future value of option premium
- Payoff/profit of a written (i.e., short) put:
  - Payoff = - max [0, strike price – spot price at expiration]
  - Profit = Payoff + future value of option premium
A few items to note

- A call option becomes more profitable when the underlying asset appreciates in value
- A put option becomes more profitable when the underlying asset depreciates in value
- Moneyness:
  - In-the-money option: positive payoff if exercised immediately
  - At-the-money option: zero payoff if exercised immediately
  - Out-of-the-money option: negative payoff if exercised immediately
Options and insurance

• Homeowner’s insurance as a put option:

- Homeowner's insurance as a put option:
  - Gain on insurance due to damage: $160,000
  - Loss of $15,000 premium if house is undamaged: $175,000
  - Deductible
Equity linked CDs

- The 5.5-year CD promises to repay initial invested amount and 70% of the gain in S&P 500 index:

  - Assume $10,000 invested when S&P 500 = 1300
  - Final payoff = $10,000 \times \left( 1 + 0.7 \times \max \left[ 0, \frac{S_{\text{final}}}{1300} - 1 \right] \right)$
  - where $S_{\text{final}}$ = value of the S&P 500 after 5.5 years

![Payoff graph showing the payoff of the CD as a function of the S&P 500 at expiration.]
Option and forward positions

A summary

- **Long forward**
  - Profit vs. Stock Price

- **Short forward**
  - Profit vs. Stock Price

- **Long call**
  - Profit vs. Stock Price

- **Short call**
  - Profit vs. Stock Price

- **Long put**
  - Profit vs. Stock Price

- **Short put**
  - Profit vs. Stock Price
Options to Complete the Market

Stock’s payoff: \( x^j = (1, 2, \ldots, S) \) (= state space)

Introduce call options with final payoff at T:

\[
C_T = \max\{S_T - E, 0\} = [S_T - E]^+
\]

\[
c_{E=1} = (0, 1, 2, \ldots, S - 2, S - 1)
\]

\[
c_{E=2} = (0, 0, 1, \ldots, S - 3, S - 2)
\]

\[
\vdots
\]

\[
c_{E=S-1} = (0, 0, 0, \ldots, 0, 1)
\]
Options to Complete the Market

Together with the primitive asset we obtain

\[
\begin{pmatrix}
1 & 2 & 3 & \cdots & S-1 & S \\
0 & 1 & 2 & \cdots & S-2 & S-1 \\
0 & 0 & 1 & \cdots & S-3 & S-2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 2 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

Homework: check whether this markets are complete.
General Security Structure

• Price vector \( p \in R^J \) of asset prices
• Cost of portfolio \( h \),

\[
p \cdot h := \sum_j p^j h^j
\]

• If \( p^j \neq 0 \) the (gross) return vector of asset \( j \) is the vector

\[
R^j = \frac{x^j}{p^j}
\]
Overview

1. Securities Structure
   (AD securities, Redundant securities, completeness, …)

2. Pricing
   - LOOP, No arbitrage and existence of state prices
   - Market completeness and uniqueness of state prices
   - Pricing kernel $q^*$
   - Three pricing formulas (state prices, SDF, EMM)
   - Recovering state prices from options
Pricing

- State space (evolution of states)
- (Risk) preferences
- Aggregation over different agents
- Security structure – prices of traded securities

**Problem:**

- Difficult to observe risk preferences
- What can we say about existence of state prices without assuming specific utility functions/constraints for all agents in the economy
Vector Notation

- Notation: $y, x \in \mathbb{R}^n$
  - $y \geq x \iff y^i \geq x^i$ for each $i=1, \ldots, n$.  
  - $y > x \iff y \geq x$ and $y \neq x$.  
  - $y >> x \iff y^i > x^i$ for each $i=1, \ldots, n$.  

- Inner product
  - $y \cdot x = \sum_i yx$

- Matrix multiplication
Three Forms of No-ARBITRAGE

1. Law of one price (LOOP)
   If $h'X = k'X$ then $p \cdot h = p \cdot k$.

2. No strong arbitrage
   There exists no portfolio $h$ which is a strong arbitrage, that is $h'X \geq 0$ and $p \cdot h < 0$.

3. No arbitrage
   There exists no strong arbitrage nor portfolio $k$ with $k'X > 0$ and $p \cdot k \leq 0$. 
Three Forms of No-ARBITRAGE

• Law of one price is equivalent to every portfolio with zero payoff has zero price.
• No arbitrage ⇒ no strong arbitrage
  No strong arbitrage ⇒ law of one price
Pricing

- Define for each \( z \in \langle X \rangle \),

\[
q(z) := \{ p \cdot h : z = h'X \}
\]

- If LOOP holds \( q(z) \) is a single-valued and linear functional. (i.e. if \( h' \) and \( h' \) lead to same \( z \), then price has to be the same)

- Conversely, if \( q \) is a linear functional defined in \( \langle X \rangle \) then the law of one price holds.
Pricing

• LOOP ⇒ \( q(h'X) = p \cdot h \)

• A linear functional \( Q \) in \( R^S \) is a valuation function if \( Q(z) = q(z) \) for each \( z \in \langle X \rangle \).

• \( Q(z) = q \cdot z \) for some \( q \in R^S \), where \( q^s = Q(e_s) \), and \( e_s \) is the vector with \( e_s^s = 1 \) and \( e_s^i = 0 \) if \( i \neq s \)

  \( e_s \) is an Arrow-Debreu security

• \( q \) is a vector of state prices
State prices $q$

- $q$ is a vector of state prices if $p = X q$, that is $p^j = x^j \cdot q$ for each $j = 1, ..., J$

- If $Q(z) = q \cdot z$ is a valuation functional then $q$ is a vector of state prices

- Suppose $q$ is a vector of state prices and LOOP holds. Then if $z = h'X$ LOOP implies that

$$q(z) = \sum_j h^j p^j = \sum_j (\sum_s x^j_s q_s) h^j =$$

$$= \sum_s (\sum_j x^j_s h^j) q_s = q \cdot z$$

- $Q(z) = q \cdot z$ is a valuation functional

$\iff q$ is a vector of state prices and LOOP holds
State prices $q$

$p(1,1) = q_1 + q_2$
$p(2,1) = 2q_1 + q_2$

*Value of portfolio $(1,2)$*

$3p(1,1) - p(2,1) = 3q_1 + 3q_2 - 2q_1 - q_2$

$= q_1 + 2q_2$
The Fundamental Theorem of Finance

• Proposition 1. Security prices exclude arbitrage if and only if there exists a valuation functional with \( q \gg 0 \).

• Proposition 1’. Let \( X \) be an \( J \times S \) matrix, and \( p \in R^J \). There is no \( h \) in \( R^J \) satisfying \( h \cdot p \leq 0 \), \( h' X \geq 0 \) and at least one strict inequality if, and only if, there exists a vector \( q \in R^S \) with \( q \gg 0 \) and \( p = X q \).

No arbitrage \( \Leftrightarrow \) positive state prices
Multiple State Prices $q$ & Incomplete Markets

What state prices are consistent with $p(1,1)$?

$$p(1,1) = q_1 + q_2$$

One equation – two unknowns $q_1, q_2$

There are (infinitely) many.

e.g. if $p(1,1) = .9$

$q_1 = .45, q_2 = .45$

or $q_1 = .35, q_2 = .55$
One Period Model

complete markets

Q(x)

x_1

x_2

\langle X \rangle

q
One Period Model

\[ Q(x) \]

\[ p = Xq \]

\[ \langle X \rangle \]

incomplete markets
One Period Model

\[ p = Xq^o \]

\[ Q(x) \]

\[ x_1 \]

\[ x_2 \]

\[ \langle X \rangle \]

incomplete markets
Multiple q in incomplete markets

Many possible state price vectors s.t. \( p = X'q \).

One is special: \( q^* \) - it can be replicated as a portfolio.
Uniqueness and Completeness

• Proposition 2. If markets are complete, under no arbitrage there exists a **unique** valuation functional.

• If markets are not complete, then there exists \( \nu \in \mathbb{R}^S \) with \( 0 = X\nu \).

Suppose there is no arbitrage and let \( q \gg 0 \) be a vector of state prices. Then \( q + \alpha \nu \gg 0 \) provided \( \alpha \) is small enough, and \( p = X (q + \alpha \nu) \). Hence, there are an infinite number of strictly positive state prices.
Four Asset Pricing Formulas

1. **State prices**
   \[ p^j = \sum_s q_s \, x^j_s \]

2. **Stochastic discount factor**
   \[ p^j = \mathbb{E}[mx^j] \]

3. **Martingale measure**
   \[ p^j = 1/(1+r^f) \, \mathbb{E}_\pi [x^j] \]
   (reflect risk aversion by over(under)weighing the “bad(good)” states!)

4. **State-price beta model**
   \[ \mathbb{E}[R^j] - R^f = \beta^j \, \mathbb{E}[R^* - R^f] \]
   (in returns \( R^j := x^j / p^j \))
1. State Price Model

• ... so far price in terms of Arrow-Debreu (state) prices

\[ p^j = \sum_s q_s x_s^j \]
2. Stochastic Discount Factor

\[ p^j = \sum_s q_s x_s^j = \sum_s \pi_s \frac{q_s}{\pi_s} x_s^j \]

- That is, stochastic discount factor \( m_s = q_s / \pi_s \) for all \( s \).

\[ p^j = E[mx^j] \]
2. Stochastic Discount Factor

shrink axes by factor $\sqrt{\pi_s}$

$c_2 \sqrt{\pi_2}$

$c_1 \sqrt{\pi_1}$

$m^*$

$m$
Risk-adjustment in payoffs

\[ p = E[mx^j] = E[m]E[x] + \text{Cov}[m,x] \]

Since \( 1 = E[mR] \), the risk free rate is \( R^f = 1/E[m] \)

\[ p = E[x]/R^f + \text{Cov}[m,x] \]

Remarks:

(i) If risk-free rate does not exist, \( R^f \) is the shadow risk free rate

(ii) In general \( \text{Cov}[m,x] < 0 \), which lowers price and increases return
3. Equivalent Martingale Measure

- Price of any asset
  \[ p^j = \sum_s q_s x_s^j \]

- Price of a bond
  \[ p_{\text{bond}} = \sum_s q_s = \frac{1}{1+r^f} \]

\[
p^j = \sum_{s'} q_{s'} \sum_s \frac{q_s}{\sum_{s'} q_{s'}} x_s^j \]

\[
p^j = \frac{1}{1+r^f} \sum_s \frac{q_s}{\sum_{s'} q_{s'}} x_s^j \]

\[
\sum_{s'} q_{s'} := \hat{\pi}_s
\]

\[
p^j = \frac{1}{1+r^f} E_{\hat{\pi}} [x^j]
\]
... in Returns: \( R^j = x^j/p^j \)

\[
E[mR^j] = 1 \quad \quad \quad \quad \quad \quad R^f E[m] = 1 \\
\Rightarrow E[m(R^j - R^f)] = 0 \\
E[m]\{E[R^j] - R^f\} + \text{Cov}[m,R^j] = 0
\]

\[
E[R^j] - R^f = -\frac{\text{Cov}[m,R^j]}{E[m]} \quad \quad \quad (2)
\]

also holds for portfolios \( h \)

Note:

- risk correction depends only on Cov of payoff/return with discount factor.
- Only compensated for taking on systematic risk not idiosyncratic risk.
4. State-price BETA Model

shrink axes by factor \( \sqrt{\pi S} \)

let underlying asset be \( x=(1.2,1) \)

\( p=1 \) (priced with \( m^* \))
4. State-price BETA Model

\[ E[R^j] - R^f = - \frac{\text{Cov}[m,R^j]}{E[m]} \]  \hspace{1cm} (2)

also holds for all portfolios \( h \) and we can replace \( m \) with \( m^* \)

Suppose (i) \( \text{Var}[m^*] > 0 \) and (ii) \( R^* = \alpha m^* \) with \( \alpha > 0 \)

\[ E[R^h] - R^f = - \frac{\text{Cov}[R^*,R^h]}{E[R^*]} \]  \hspace{1cm} (2’)

Define \( \beta^h := \frac{\text{Cov}[R^*,R^h]}{\text{Var}[R^*]} \) for any portfolio \( h \)
4. State-price BETA Model

(2) for $R^*$: $E[R^*] - R^f = -\text{Cov}[R^*, R^*]/E[R^*]$

$= -\text{Var}[R^*]/E[R^*]$

(2) for $R^h$: $E[R^h] - R^f = -\text{Cov}[R^*, R^h]/E[R^*]$

$= -\beta^h \text{Var}[R^*]/E[R^*]$

$$E[R^h] - R^f = \beta^h E[R^*- R^f]$$

where $\beta^h := \text{Cov}[R^*, R^h]/\text{Var}[R^*]$

very general – but what is $R^*$ in reality?

Regression $R^h_s = \alpha^h + \beta^h (R^*)_s + \varepsilon_s$ with $\text{Cov}[R^*, \varepsilon] = E[\varepsilon] = 0$
Four Asset Pricing Formulas

1. State prices
   \[ 1 = \sum_s q_s R^j_s \]

2. Stochastic discount factor
   \[ 1 = E[mR^j] \]

3. Martingale measure
   \[ 1 = 1/(1+r^f) \mathbb{E}_{\hat{\pi}} [R^j] \]
   (reflect risk aversion by over(under)weighing the “bad(good)” states!)

4. State-price beta model
   \[ E[R^j] - R^f = \beta_j E[R^* - R^f] \]
   (in returns \( R^j := x^j/p^j \))
What do we know about $q$, $m$, $\hat{\pi}$, $R^*$?

- Main results so far
  - Existence iff no arbitrage
    - Hence, single factor only
  - but doesn’t famos Fama-French factor model has 3 factors?
    - multiple factor is due to time-variation
      (wait for multi-period model)
  - Uniqueness if markets are complete
Different Asset Pricing Models

\[ p_t = E[m_{t+1} \times_{t+1}] \quad \Rightarrow \quad E[R^h] - R^f = \beta^h E[R^* - R^f] \]

where \( \beta^h := \frac{\text{Cov}[R^*, R^h]}{\text{Var}[R^*]} \)

General Equilibrium
\( f(\cdot) = \frac{\text{MRS}}{\pi} \)

Factor Pricing Model
\[ a + b_1 \ f_{1,t+1} + b_2 \ f_{2,t+1} \]

CAPM
\[ a + b_1 \ f_{1,t+1} = a + b_1 \ R^M \]

CAPM
\[ R^* = \frac{R^f (a + b_1 R^M)}{(a + b_1 R^f)} \]

where \( R^M = \) return of market portfolio
Is \( b_1 < 0? \)
Different Asset Pricing Models

• **Theory**
  - All economics and modeling is determined by
    \[ m_{t+1} = a + b' f \]
  - Entire content of model lies in restriction of SDF

• **Empirics**
  - \( m^* \) (which is a portfolio payoff) prices as well as \( m \)
    (which is e.g. a function of income, investment etc.)
  - Measurement error of \( m^* \) is smaller than for any \( m \)
  - Run regression on *returns* (portfolio payoffs)!
    (e.g. Fama-French three factor model)
One Period Model

- Specify Preferences & Technology
- Observe/specify existing Asset Prices
- Derive Asset Prices
- Derive Price for (new) asset

State Prices $q$
(or stochastic discount factor/Martingale measure)

Absolute asset pricing

- Evolution of states
- Risk preferences
- Aggregation

Relative asset pricing

Only works as long as market completeness doesn’t change
Recovering State Prices from Option Prices

• Suppose that $S_T$, the price of the underlying portfolio (we may think of it as a proxy for price of “market portfolio”), assumes a "continuum" of possible values.

• Suppose there are a “continuum” of call options with different strike/exercise prices $\Rightarrow$ markets are complete.

• Let us construct the following portfolio:
  
  for some small positive number $\varepsilon > 0$,

  $\Box$ Buy one call with $E = \hat{S}_T - \frac{\delta}{2} - \varepsilon$

  $\Box$ Sell one call with $E = \hat{S}_T - \frac{\delta}{2}$

  $\Box$ Sell one call with $E = \hat{S}_T + \frac{\delta}{2}$

  $\Box$ Buy one call with $E = \hat{S}_T + \frac{\delta}{2} + \varepsilon$
Recovering State Prices … (ctd.)

Value of the portfolio at expiration

Figure 8-2  Payoff Diagram: Portfolio of Options
Recovering State Prices … (ctd.)

• Let us thus consider buying \( \frac{1}{\varepsilon} \) units of the portfolio. The total payment, when \( \hat{S}_T - \frac{\delta}{2} \leq S_T \leq \hat{S}_T + \frac{\delta}{2} \), is \( \varepsilon \cdot \frac{1}{\varepsilon} = 1 \), for any choice of \( \varepsilon \). We want to let \( \varepsilon \mapsto 0 \), so as to eliminate the payments in the ranges \( S_T \in (\hat{S}_T - \frac{\delta}{2} - \varepsilon, \hat{S}_T - \frac{\delta}{2}) \) and \( S_T \in (\hat{S}_T + \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2} + \varepsilon) \). The value of \( \frac{1}{\varepsilon} \) units of this portfolio is:

\[
\frac{1}{\varepsilon} \left\{ C(S, E = \hat{S}_T - \frac{\delta}{2} - \varepsilon) - C(S, E = \hat{S}_T - \frac{\delta}{2}) - \left[ C(S, E = \hat{S}_T + \frac{\delta}{2}) - C(S, E = \hat{S}_T + \frac{\delta}{2} + \varepsilon) \right] \right\}
\]
Taking the limit $\varepsilon \to 0$

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ C(S, E = \hat{S}_T - \frac{\delta}{2} - \varepsilon) - C(S, E = \hat{S}_T - \frac{\delta}{2}) - \left[ C(S, E = \hat{S}_T + \frac{\delta}{2} + \varepsilon) - C(S, E = \hat{S}_T + \frac{\delta}{2}) \right] \right\}
$$

$$
= -\lim_{\varepsilon \to 0} \left\{ \frac{C(S, E = \hat{S}_T - \frac{\delta}{2} - \varepsilon) - C(S, E = \hat{S}_T - \frac{\delta}{2})}{-\varepsilon} \left\{ \frac{C(S, E = \hat{S}_T + \frac{\delta}{2} + \varepsilon) - C(S, E = \hat{S}_T + \frac{\delta}{2})}{\varepsilon} \right\} \right\}
$$

$$
\leq 0
$$

$$
= -\frac{\partial C}{\partial E}(S, E = \hat{S}_T - \frac{\delta}{2}) + \frac{\partial C}{\partial E}(S, E = \hat{S}_T + \frac{\delta}{2})
$$

Payoff

Divide by $\delta$ and let $\delta \to 0$ to obtain state price density as $\frac{\partial^2 C}{\partial E^2}$. 
Recovering State Prices … (ctd.)

Evaluating following cash flow

\[
\hat{C}_T = \begin{cases} 
0 \text{ if } S_T \notin \left[ \hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2} \right] \\
50000 \text{ if } S_T \in \left[ \hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2} \right].
\end{cases}
\]

The value today of this cash flow is:

\[
50000 \left[ \frac{\partial C}{\partial E}(S, E = \hat{S}_T + \frac{\delta}{2}) - \frac{\partial C}{\partial E}(S, E = \hat{S}_T - \frac{\delta}{2}) \right]
\]

\[
q(S_T^1, S_T^2) = \frac{\partial C}{\partial E}(S, E = S_T^2) - \frac{\partial C}{\partial E}(S, E = S_T^1)
\]
### Table 8.1 Pricing an Arrow-Debreu State Claim

<table>
<thead>
<tr>
<th>E</th>
<th>C(S,E)</th>
<th>Cost of position</th>
<th>Payoff if $S_T = \ E$</th>
<th>$\Delta C$</th>
<th>$\Delta(\Delta C) = q_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>3.354</td>
<td></td>
<td></td>
<td></td>
<td>-0.895</td>
</tr>
<tr>
<td>8</td>
<td>2.459</td>
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<tr>
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<td>1.670</td>
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<td>-0.789</td>
<td>0.164</td>
</tr>
<tr>
<td>10</td>
<td>1.045</td>
<td>-2.090</td>
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<td>0.184</td>
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<tr>
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<tr>
<td>13</td>
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<td></td>
<td>0 0 0 1 0 0 0</td>
<td>-0.161</td>
<td>0.118</td>
</tr>
</tbody>
</table>

Note: The table shows the pricing of Arrow-Debreu state claims for different values of $E$. The cost of holding the position, the payoff if $S_T = E$, and the change in the cost ($\Delta C$) and the change in the change ($\Delta(\Delta C) = q_s$) are calculated for each scenario.
**One Period Model**

- **specify Preferences & Technology**
  - evolution of states
  - risk preferences
  - aggregation

- **observe/specify existing Asset Prices**

**State Prices** $q$

- (or stochastic discount factor/Martingale measure)

**derive Asset Prices**

**derive Price for (new) asset**

Only works as long as market completeness doesn’t change
End of Lecture 02