Lecture 03: One Period Model: Pricing

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Overview: Pricing

1. LOOP, No arbitrage
2. Parity relationship between options
3. No arbitrage and existence of state prices
4. Market completeness and uniqueness of state prices
5. Pricing kernel q^*
6. Four pricing formulas (state prices, SDF, EMM, state pricing)
7. Recovering state prices from options

Vector Notation

- Notation: $y,x \in R^n$
  - $y \geq x \Leftrightarrow y^i \geq x^i$ for each i=1,…,n.
  - $y > x \Leftrightarrow y \geq x$ and $y \neq x$.
  - $y >> x \Leftrightarrow y^i > x^i$ for each i=1,…,n.
- Inner product
  - $y \cdot x = \sum_i yx$
- Matrix multiplication

Three Forms of No-ARBITRAGE

1. Law of one price (LOOP)
   If $h'X = k'X$ then $p \cdot h = p \cdot k$.
2. No strong arbitrage
   There exists no portfolio h which is a strong arbitrage, that is $h'X \geq 0$ and $p \cdot h < 0$.
3. No arbitrage
   There exists no strong arbitrage
   nor portfolio k with $k' X > 0$ and $p \cdot k \leq 0$.

Three Forms of No-ARBITRAGE

- Law of one price is equivalent to every portfolio with zero payoff has zero price.
- No arbitrage $\Rightarrow$ no strong arbitrage
  No strong arbitrage $\Rightarrow$ law of one price

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Alternative ways to buy a stock

- Four different payment and receipt timing combinations:
  - Outright purchase: ordinary transaction
  - Fully leveraged purchase: investor borrows the full amount
  - Prepaid forward contract: pay today, receive the share later
  - Forward contract: agree on price now, pay/receive later
- Payments, receipts, and their timing:

TABLE 5.1 Four different ways to buy a share of stock that has price S_0 at time 0. At time 0 you agree to a price, which is paid either today or at time T. The shares are received either at 0 or T. The interest rate is r.

Description	Pay at Time:	Receive Security at Time:	Payment
Outright Purchase	0	0	S_0 at time 0
Fully Leveraged Purchase	T	0	$S_0 e^{rT}$ at time T
Prepaid Forward Contract	0	T	?
Forward Contract	T	T	$? \times e^{rT}$

Pricing prepaid forwards

- If we can price the *prepaid* forward (F^P), then we can calculate the price for a forward contract:

$$F = \text{Future value of } F^P$$

- Pricing by analogy
  - In the absence of dividends, the timing of delivery is irrelevant
  - Price of the prepaid forward contract same as current stock price
  - $F^P_{0,T} = S_0$ (where the asset is bought at $t = 0$, delivered at $t = T$)

- Pricing by discounted preset value (α: risk-adjusted discount rate)
  - If expected $t=T$ stock price at $t=0$ is $E_0(S_T)$, then $F^P_{0,T} = E_0(S_T)\, e^{-\alpha T}$
  - Since $t=0$ expected value of price at $t=T$ is $E_0(S_T) = S_0\, e^{\alpha T}$
  - Combining the two, $F^P_{0,T} = S_0\, e^{\alpha T}\, e^{-\alpha T} = S_0$

Pricing prepaid forwards (cont.)

- Pricing by arbitrage
  - If at time t=0, the prepaid forward price somehow exceeded the stock price, i.e., $F^P_{0,T} > S_0$, an arbitrageur could do the following:

TABLE 5.2 Cash flows and transactions to undertake arbitrage when the prepaid forward price, $F^P_{0,T}$, exceeds the stock price, S_0.

	Cash Flows	
Transaction	**Time 0**	**Time *T* (expiration)**
Buy Stock @ S_0	$-S_0$	$+S_T$
Sell Prepaid Forward @ $F^P_{0,T}$	$+F^P_{0,T}$	$-S_T$
Total	$F^P_{0,T} - S_0$	0

Pricing prepaid forwards (cont.)

- What if there are deterministic* dividends? Is $F^P_{0,T} = S_0$ still valid?
  - No, because the holder of the forward will not receive dividends that will be paid to the holder of the stock → $F^P_{0,T} < S_0$

$$F^P_{0,T} = S_0 - \text{PV}(\text{all dividends paid from t=0 to t=T})$$

  - For discrete dividends D_{t_i} at times t_i, $i = 1,\ldots, n$
    - The prepaid forward price: $F^P_{0,T} = S_0 - \sum^n_{i=1} PV_{0,t_i}(D_{t_i})$
  - For continuous dividends with an annualized yield δ
    - The prepaid forward price: $F^P_{0,T} = S_0 e^{-\delta T}$

*NB: if dividends are stochatistic, we cannot apply the one period model

Pricing prepaid forwards (cont.)

- Example 5.1
  - XYZ stock costs $100 today and will pay a quarterly dividend of $1.25. If the risk-free rate is 10% compounded continuously, how much does a 1-year prepaid forward cost?
  - $F^P_{0,1} = \$100 - \sum_{i=1}^{4} \$1.25 e^{-0.025i} = \$95.30$
- Example 5.2
  - The index is $125 and the dividend yield is 3% continuously compounded. How much does a 1-year prepaid forward cost?
  - $F^P_{0,1} = \$125 e^{-0.03} = \$121.31$

Pricing forwards on stock

- Forward price is the future value of the *prepaid* forward
  - No dividends: $F_{0,T} = FV(F^P_{0,T}) = FV(S_0) = S_0\, e^{rT}$
  - Discrete dividends: $F_{0,T} = S_0\, e^{rT} - \sum_{i=1}^{n} e^{r(T-t_i)} D_{t_i}$
  - Continuous dividends: $F_{0,T} = S_0\, e^{(r-\delta)T}$
- Forward premium
  - The difference between current forward price and stock price
  - Can be used to infer the current stock price from forward price
  - Definition:
    - Forward premium = $F_{0,T} / S_0$
    - Annualized forward premium =: $\pi^a = (1/T)\, ln\, (F_{0,T} / S_0)$ (from $e^{\pi T} = F_{0,T} / S_0$)

Creating a *synthetic* forward

- One can offset the risk of a forward by creating a *synthetic* forward to offset a position in the actual forward contract
- How can one do this? (assume continuous dividends at rate δ)
  - ❑ Recall the long forward payoff at expiration: = $S_T - F_{0,T}$
  - ❑ Borrow and purchase shares as follows:

TABLE 5.3 Demonstration that borrowing $S_0 e^{-\delta T}$ to buy $e^{-\delta T}$ shares of the index replicates the payoff to a forward contract, $S_T - F_{0,T}$.

	Cash Flows	
Transaction	**Time 0**	**Time *T* (expiration)**
Buy $e^{-\delta T}$ Units of the Index	$-S_0 e^{-\delta T}$	$+S_T$
Borrow $S_0 e^{-\delta T}$	$+S_0 e^{-\delta T}$	$-S_0 e^{(r-\delta)T}$
Total	0	$S_T - S_0 e^{(r-\delta)T}$

  - ❑ Note that the total payoff at expiration is same as forward payoff

Creating a *synthetic* forward (cont.)

- The idea of creating synthetic forward leads to following:
  - Forward = Stock – zero-coupon bond
  - Stock = Forward + zero-coupon bond
  - Zero-coupon bond = Stock – forward
- Cash-and-carry arbitrage: Buy the index, short the forward

TABLE 5.6 Transactions and cash flows for a cash-and-carry: A market-maker is short a forward contract and long a synthetic forward contract.

	Cash Flows	
Transaction	**Time 0**	**Time *T* (expiration)**
Buy Tailed Position in Stock, Paying $S_0e^{-\delta T}$	$-S_0e^{-\delta T}$	$+S_T$
Borrow $S_0e^{-\delta T}$	$+S_0e^{-\delta T}$	$-S_0e^{(r-\delta)T}$
Short Forward	0	$F_{0,T} - S_T$
Total	0	$F_{0,T} - S_0e^{(r-\delta)T}$

Other issues in forward pricing

- Does the forward price predict the future price?
  - According the formula $F_{0,T} = S_0 e^{(r-\delta)T}$ the forward price conveys no additional information beyond what S_0, r, and δ provides
  - Moreover, the forward price underestimates the future stock price
- Forward pricing formula and cost of carry
  - Forward price =

$$\underbrace{\text{Spot price} + \text{Interest to carry the asset} - \text{asset lease rate}}_{\text{Cost of carry, } (r-\delta)S}$$

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Put-Call Parity

- For European options with the same strike price and time to expiration the parity relationship is:

Call – put = *PV* (forward price – strike price)

or

$$C(K, T) - P(K, T) = PV_{0,T}(F_{0,T} - K) = e^{-rT}(F_{0,T} - K)$$

- Intuition:
  - Buying a call and selling a put with the strike equal to the forward price ($F_{0,T} = K$) creates a synthetic forward contract and hence must have a zero price.

Parity for Options on Stocks

- If underlying asset is a stock and *Div* is the deterministic* dividend stream, then $e^{-rT} F_{0,T} = S_0 - PV_{0,T}(Div)$, therefore

$$C(K, T) = P(K, T) + [S_0 - PV_{0,T}(Div)] - e^{-rT}(K)$$

- Rewriting above,

$$S_0 = C(K, T) - P(K, T) + PV_{0,T}(Div) + e^{-rT}(K)$$

- For index options, $S_0 - PV_{0,T}(Div) = S_0 e^{-\delta T}$, therefore

$$C(K, T) = P(K, T) + S_0 e^{-\delta T} - PV_{0,T}(K)$$

*allows to stick with one period setting

Properties of option prices

- American vs. European
  - Since an American option can be exercised at anytime, whereas a European option can only be exercised at expiration, an American option must always be at least as valuable as an otherwise identical European option:

$$C_{Amer}(S, K, T) \geq C_{Eur}(S, K, T)$$

$$P_{Amer}(S, K, T) \geq P_{Eur}(S, K, T)$$

- Option price boundaries
  - Call price cannot: be negative, exceed stock price, be less than price implied by put-call parity using zero for put price:

$$S > C_{Amer}(S, K, T) \geq C_{Eur}(S, K, T) >$$
$$> max\ [0, PV_{0,T}(F_{0,T}) - PV_{0,T}(K)]$$

Properties of option prices (cont.)

- Option price boundaries
  - Call price cannot:
    - be negative
    - exceed stock price
    - be less than price implied by put-call parity using zero for put price:

$$S > C_{Amer}(S, K, T) \geq C_{Eur}(S, K, T) \geq max\ [0, PV_{0,T}(F_{0,T}) - PV_{0,T}(K)]$$

  - Put price cannot:
    - be more than the strike price
    - be less than price implied by put-call parity using zero for call price:

$$K > P_{Amer}(S, K, T) \geq P_{Eur}(S, K, T) \geq max\ [0, PV_{0,T}(K) - PV_{0,T}(F_{0,T})]$$

Properties of option prices (cont.)

- Early exercise of American options
  - ❑A non-dividend paying American call option should not be exercised early, because:

$$C_{\text{Amer}} \geq C_{\text{Eur}} \geq S_{\text{t}} - K + P_{Eur} + K(1-e^{-r(T-t)}) > S_{\text{t}} - K$$

  - ❑That means, one would lose money be exercising early instead of selling the option
  - ❑If there are dividends, it may be optimal to exercise early
  - ❑It may be optimal to exercise a non-dividend paying put option early if the underlying stock price is sufficiently low

Properties of option prices (cont.)

- Time to expiration
  - An American option (both put and call) with more time to expiration is at least as valuable as an American option with less time to expiration. This is because the longer option can easily be converted into the shorter option by exercising it early.
  - European call options on dividend-paying stock and European puts may be less valuable than an otherwise identical option with less time to expiration.
  - A European call option on a non-dividend paying stock will be more valuable than an otherwise identical option with less time to expiration.
  - When the strike price grows at the rate of interest, European call and put prices on a non-dividend paying stock increases with time.
    - Suppose to the contrary $P(T) < P(t)$ for $T>t$, then arbitrage. Buy $P(T)$ and sell $P(t)$. At t if $S_t>K_t$, $P(t)=0$, if $S_t<K_t$, payoff $S_t - K_t$. Keep stock and finance K_t. Time T-value $S_T-K_te^{r(T-t)}=S_T-K_T$.

Properties of option prices (cont.)

- Different strike prices ($K_1 < K_2 < K_3$), for both European and American options
  - ❑ A call with a low strike price is at least as valuable as an otherwise identical call with higher strike price:

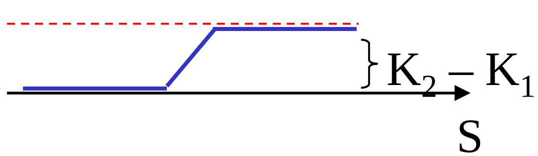
$$C(K_1) \geq C(K_2)$$

  - ❑ A put with a high strike price is at least as valuable as an otherwise identical call with low strike price:

$$P(K_2) \geq P(K_1)$$

  - ❑ The premium difference between otherwise identical calls with different strike prices cannot be greater than the difference in strike prices:

$$C(K_1) - C(K_2) \leq K_2 - K_1$$



Properties of option prices (cont.)

- Different strike prices ($K_1 < K_2 < K_3$), for both European and American options
  - The premium difference between otherwise identical puts with different strike prices cannot be greater than the difference in strike prices:

$$P(K_1) - P(K_2) \leq K_2 - K_1$$

  - Premiums decline at a decreasing rate for calls with progressively higher strike prices. (Convexity of option price with respect to strike price):

$$\frac{C(K_1) - C(K_2)}{K_2 - K_1} \geq \frac{C(K_2) - C(K_3)}{K_3 - K_2}$$

Properties of option prices (cont.)

TABLE 9.7 The example in Panel A violates the proposition that the rate of change of the option premium must decrease as the strike price rises. The rate of change from 50 to 59 is 5.1/9, while the rate of change from 59 to 65 is 3.9/6. We can arbitrage this convexity violation with an asymmetric butterfly spread. Panel B shows that we earn at least \$3 plus interest at time *T*.

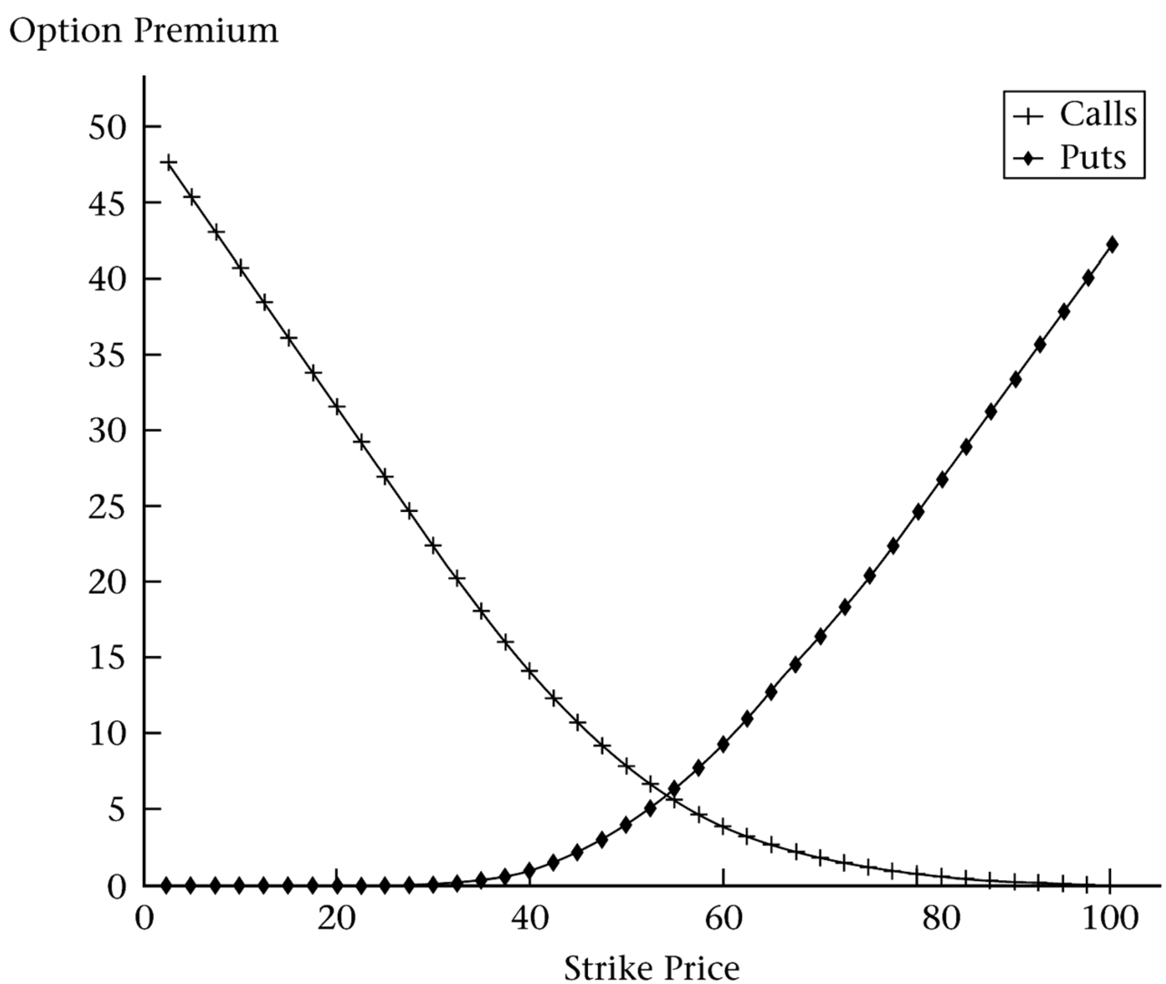
Panel A

Strike	50	59	65
Call Premium	14	8.9	5

Panel B

		Expiration or Exercise			
Transaction	**Time 0**	$S_T < 50$	$50 \le S_T \le 59$	$59 \le S_T \le 65$	$S_T > 65$
Buy Four 50-Strike Calls	−56	0	$4(S_T - 50)$	$4(S_T - 50)$	$4(S_T - 50)$
Sell Ten 59-Strike Calls	89	0	0	$10(59 - S_T)$	$10(59 - S_T)$
Buy Six 65-Strike Calls	−30	0	0	0	$6(S_T - 65)$
Lend \$3	−3	$3e^{rT}$	$3e^{rT}$	$3e^{rT}$	$3e^{rT}$
Total	0	$3e^{rT}$	$3e^{rT} + 4(S_T - 50)$	$3e^{rT} + 6(65 - S_T)$	$3e^{rT}$

Properties of option prices (cont.)



Summary of parity relationships

TABLE 9.9 Versions of put-call parity. Notation in the table includes the spot currency exchange rate, x_0; the risk-free interest rate in the foreign currency, r_f; and the current bond price, B_0.

Underlying Asset	Parity Relationship
Futures Contract	$e^{-rT}F_{0,T} = C(K,T) - P(K,T) + e^{-rT}K$
Stock, No-Dividend	$S_0 = C(K,T) - P(K,T) + e^{-rT}K$
Stock, Discrete Dividend	$S_0 - PV_{0,T}(Div) = C(K,T) - P(K,T) + e^{-rT}K$
Stock, Continuous Dividend	$e^{-\delta T}S_0 = C(K,T) - P(K,T) + e^{-rT}K$
Currency	$e^{-r_f T}x_0 = C(K,T) - P(K,T) + e^{-rT}K$
Bond	$B_0 - PV_{0,T}(Coupons) = C(K,T) - P(K,T) + e^{-rT}K$

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… back to the big picture

- State space (evolution of states)
- (Risk) preferences
- Aggregation over different agents
- Security structure – prices of traded securities
- *Problem:*
  - *Difficult to observe risk preferences*
  - *What can we say about* ***existence of state prices*** *without assuming specific utility functions/constraints for all agents in the economy*

Vector Notation

- Notation: $y,x \in R^n$
  - $y \geq x \Leftrightarrow y^i \geq x^i$ for each i=1,…,n.
  - $y > x \Leftrightarrow y \geq x$ and $y \neq x$.
  - $y >> x \Leftrightarrow y^i > x^i$ for each i=1,…,n.
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- Law of one price is equivalent to
  every portfolio with zero payoff has zero price.
- No arbitrage $\Rightarrow$ no strong arbitrage
  No strong arbitrage $\Rightarrow$ law of one price

Pricing

- Define for each $z \in$ <X>,

$$q(z) := \{p \cdot h : z = h'X\}$$

- If LOOP holds *q(z)* is a single-valued and linear functional. (i.e. if h’ and h’ lead to same z, then price has to be the same)
- Conversely, if *q* is a linear functional defined in <X> then the law of one price holds.

Pricing

- LOOP $\Rightarrow q(h'X) = p \cdot h$
- A linear functional Q in R^S is a valuation function if $Q(z) = q(z)$ for each z $\in$ <X>.
- Q(z) = q · z for some $q \in R^S$, where $q^s = Q(e_s)$, and e_s is the vector with $e_s^s = 1$ and $e_s^i = 0$ if $i \neq s$
  - ❑ e_s is an Arrow-Debreu security
- q is a vector of state prices

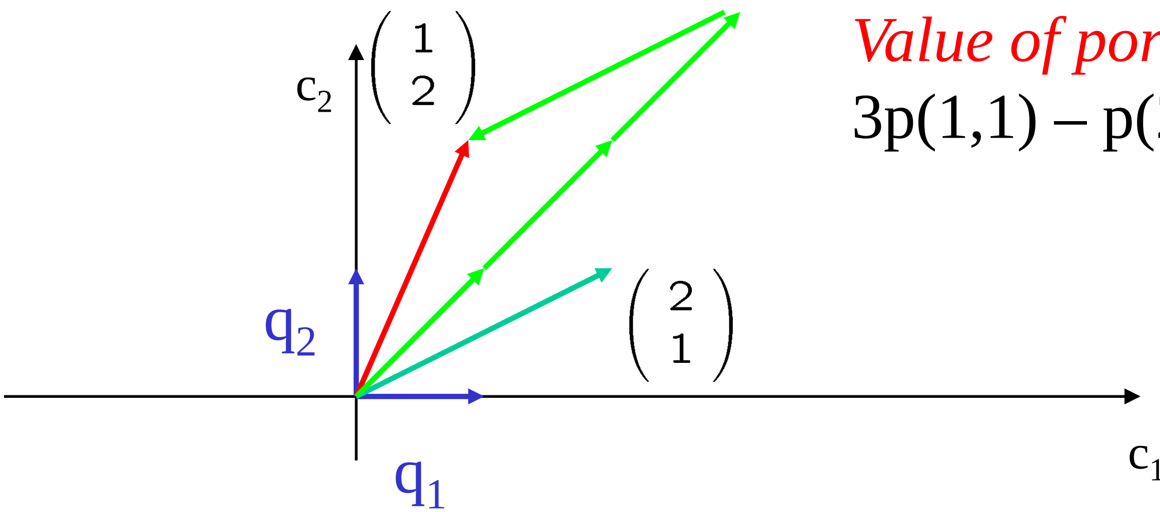
State prices q

- q is a vector of state prices if $p = X q$,
  that is $p^j = x^j \cdot q$ for each $j = 1,\ldots,J$
- If $Q(z) = q \cdot z$ is a valuation functional then q is a vector of state prices
- Suppose q is a vector of state prices and LOOP holds.
  Then if $z = h'X$ LOOP implies that

$$q(z) = \sum_j h^j p^j = \sum_j (\sum_s x_s^j q_s) h^j =$$
$$= \sum_s (\sum_j x_s^j h^j) q_s = q \cdot z$$

- $Q(z) = q \cdot z$ is a valuation functional
  $\Leftrightarrow$ q is a vector of state prices and LOOP holds

State prices q



$p(1,1) = q_1 + q_2$

$p(2,1) = 2q_1 + q_2$

Value of portfolio (1,2)

$3p(1,1) - p(2,1) = 3q_1 + 3q_2 - 2q_1 - q_2$

$= q_1 + 2q_2$

The Fundamental Theorem of Finance

- **Proposition 1.** Security prices exclude arbitrage if and only if there exists a valuation functional with $q >> 0$.
- **Proposition 1'.** Let X be an $J \times S$ matrix, and $p \in R^J$. There is no h in R^J satisfying $h \cdot p \leq 0$, $h' X \geq 0$ and at least one strict inequality if, and only if, there exists a vector $q \in R^S$ with $q >> 0$ and $p = X q$.

No arbitrage $\Leftrightarrow$ positive state prices

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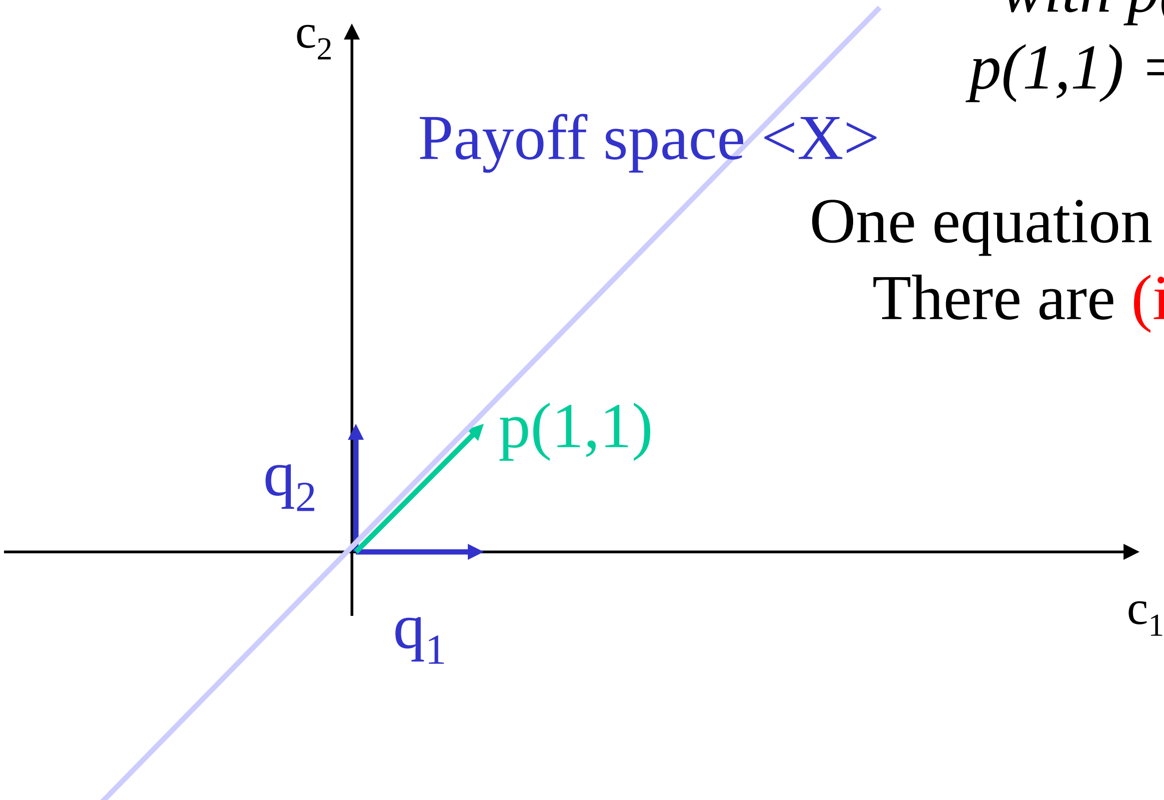
Multiple State Prices q & Incomplete Markets

bond (*1*,*1*) only



What state prices are consistent with p(1,1)?

$p(1,1) = q_1 + q_2$

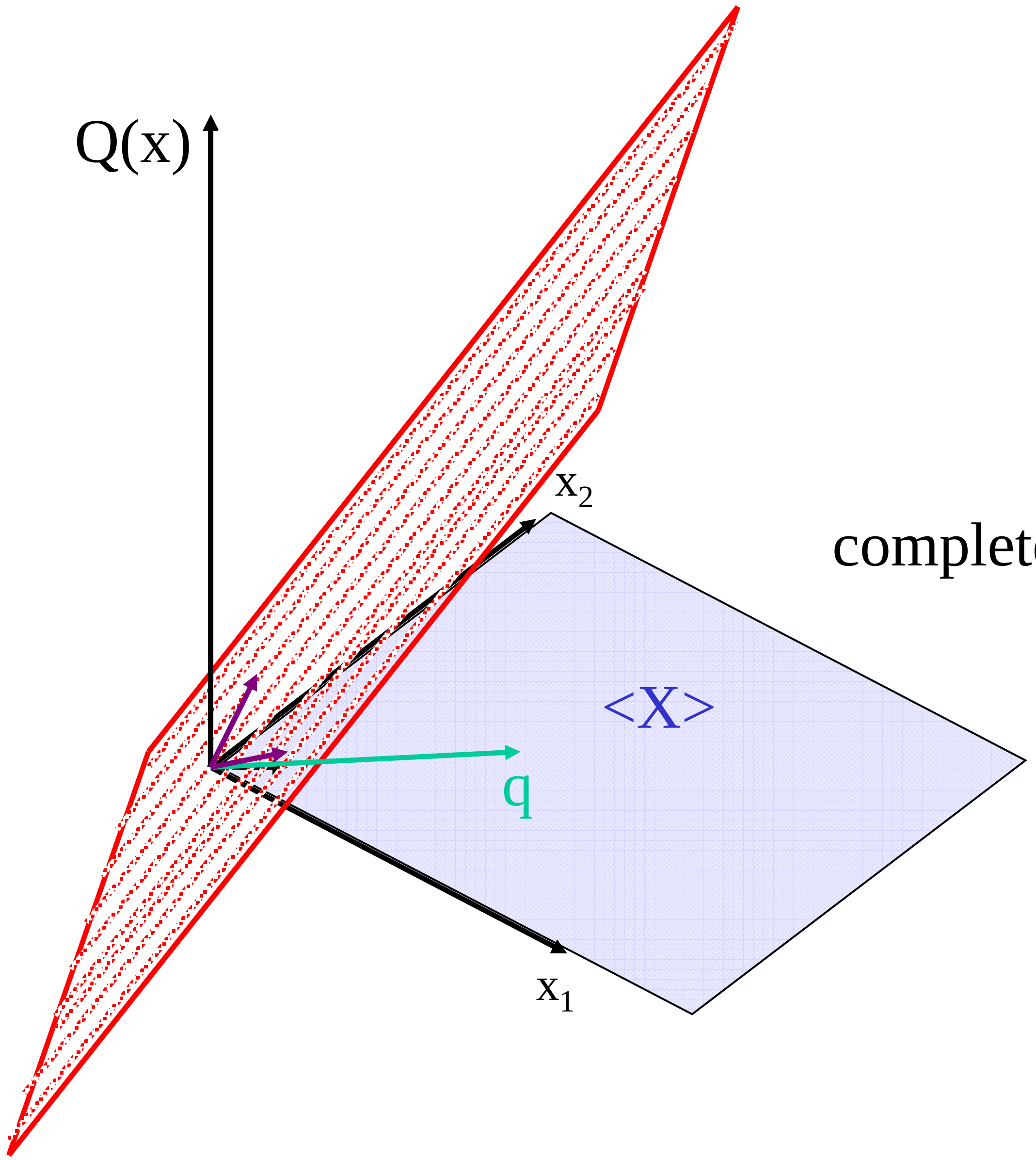
One equation – two unknowns q_1, q_2

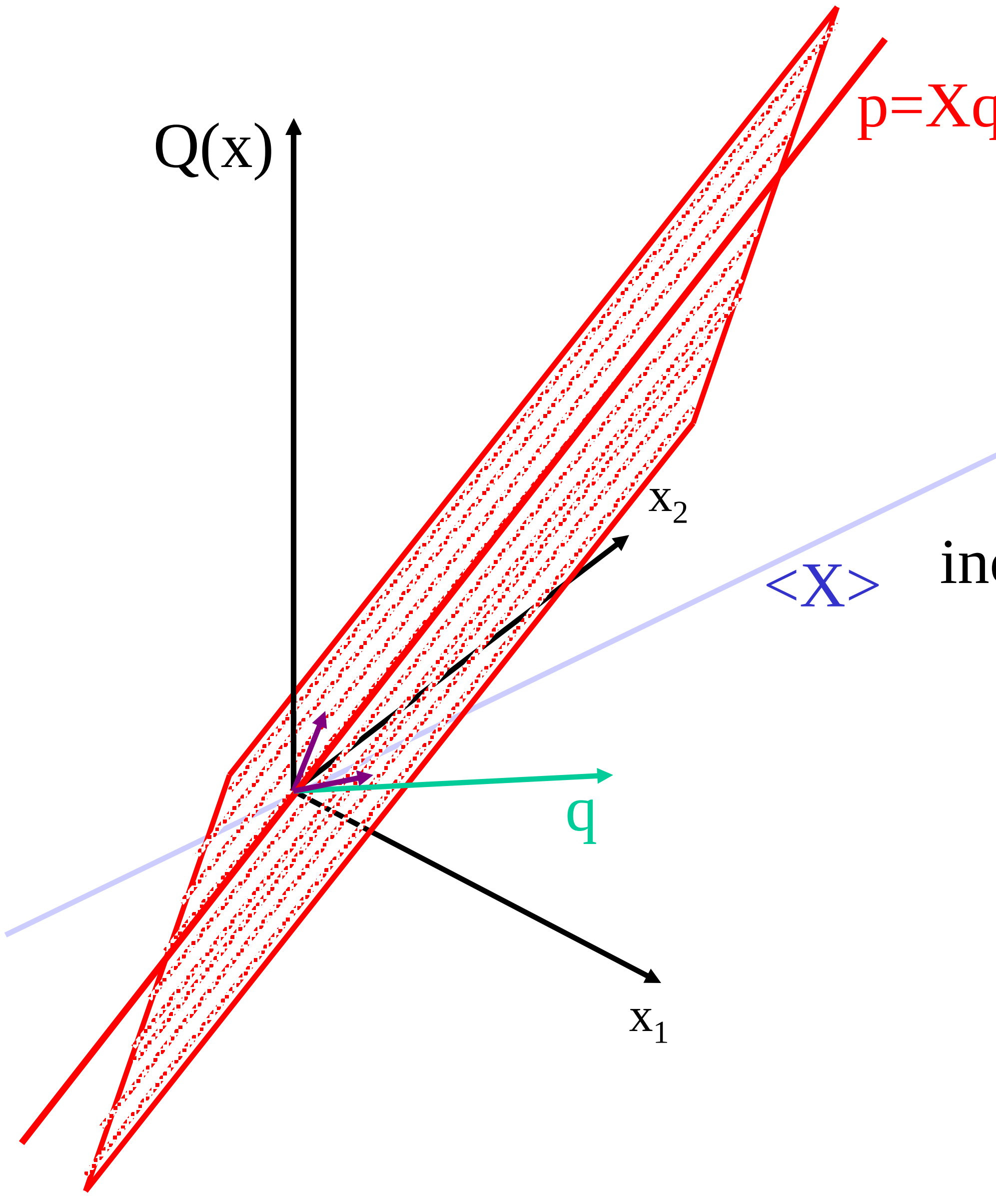
There are (infinitely) many.

e.g. *if p(1,1)=.9*

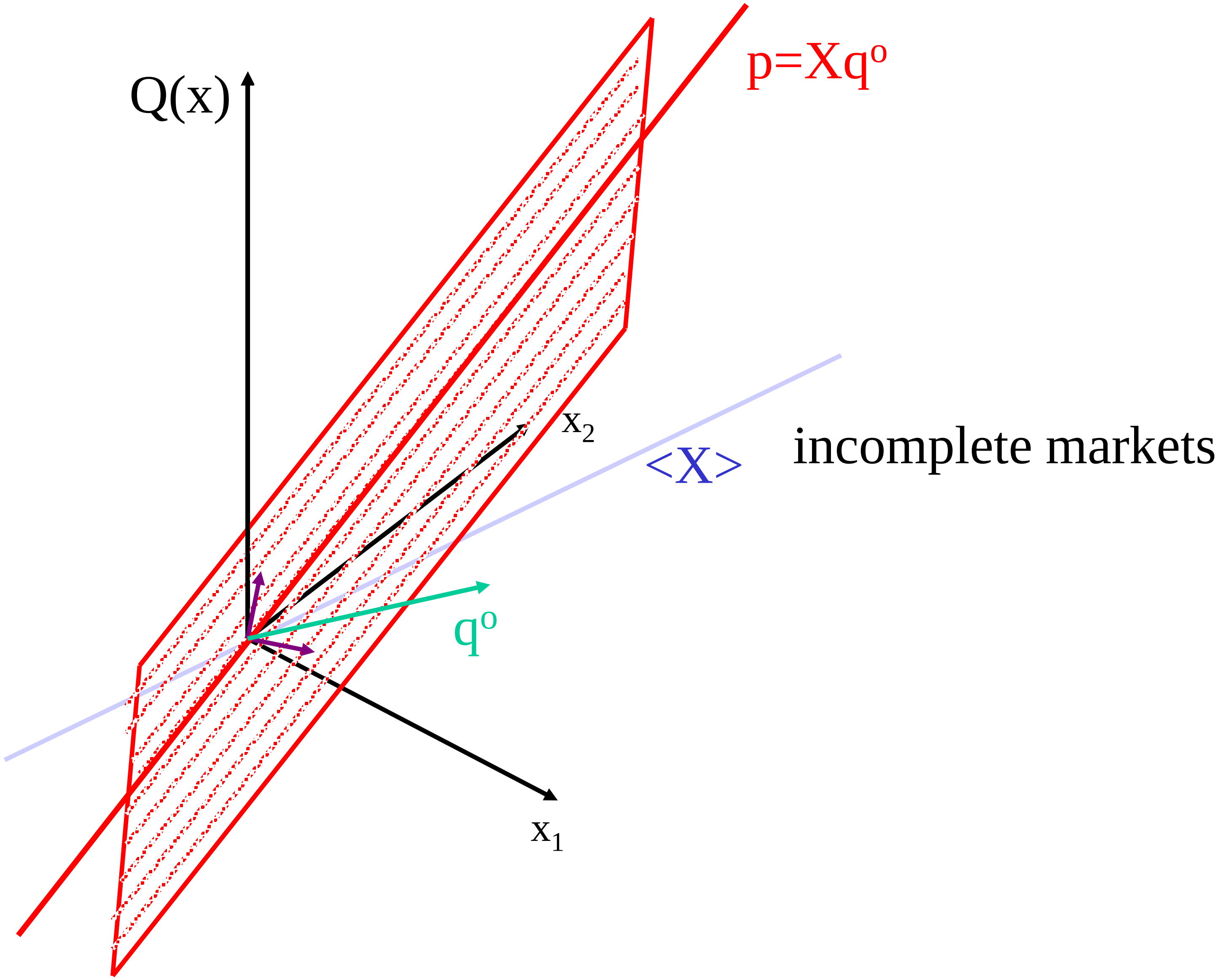
$q_1 = .45,\ q_2 = .45$

or $q_1 = .35,\ q_2 = .55$

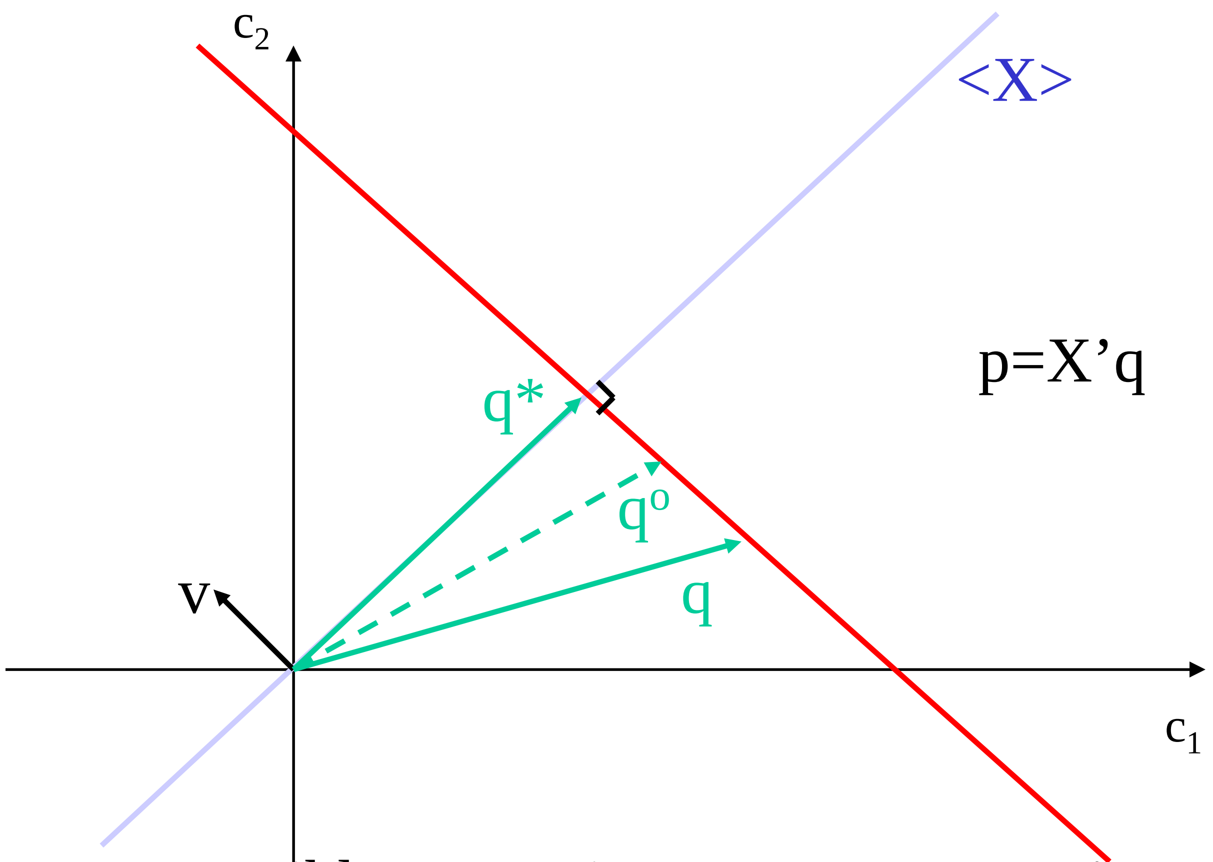

Q(x)
x
2
complete markets
<X>
q
x
1



incomplete markets


p=Xq^o
Q(x)
x_2
incomplete markets
<X>
q^o
x_1

Multiple q in incomplete markets



Many possible state price vectors s.t. $p=X'q$.
One is special: q* - it can be replicated as a portfolio.

Uniqueness and Completeness

- **Proposition 2.** If markets are complete, under no arbitrage there exists a *unique* valuation functional.

- If markets are not complete, then there exists $v \in R^S$ with $0 = Xv$.
  Suppose there is no arbitrage and let $q >> 0$ be a vector of state prices. Then $q + \alpha v >> 0$ provided α is small enough, and $p = X(q + \alpha v)$. Hence, there are an infinite number of strictly positive state prices.

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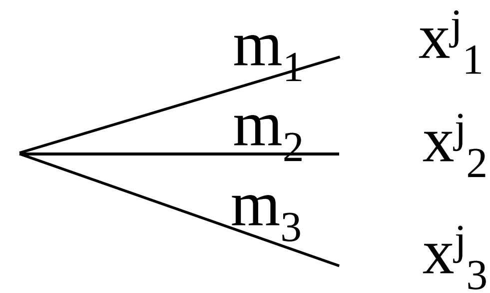
Four Asset Pricing Formulas

1. State prices $p^j = \sum_s q_s \, x_s^j$
2. Stochastic discount factor $p^j = E[m x^j]$



3. Martingale measure $p^j = 1/(1+r^f) \, E_{\hat{\pi}}[x^j]$

   (reflect risk aversion by
   over(under)weighing the "bad(good)" states!)

4. State-price beta model $E[R^j] - R^f = \beta^j \, E[R^* - R^f]$

   (in returns $R^j := x^j / p^j$)

1. State Price Model

- … so far price in terms of Arrow-Debreu (state) prices

$$p^j = \sum_s q_s x_s^j$$

2. Stochastic Discount Factor

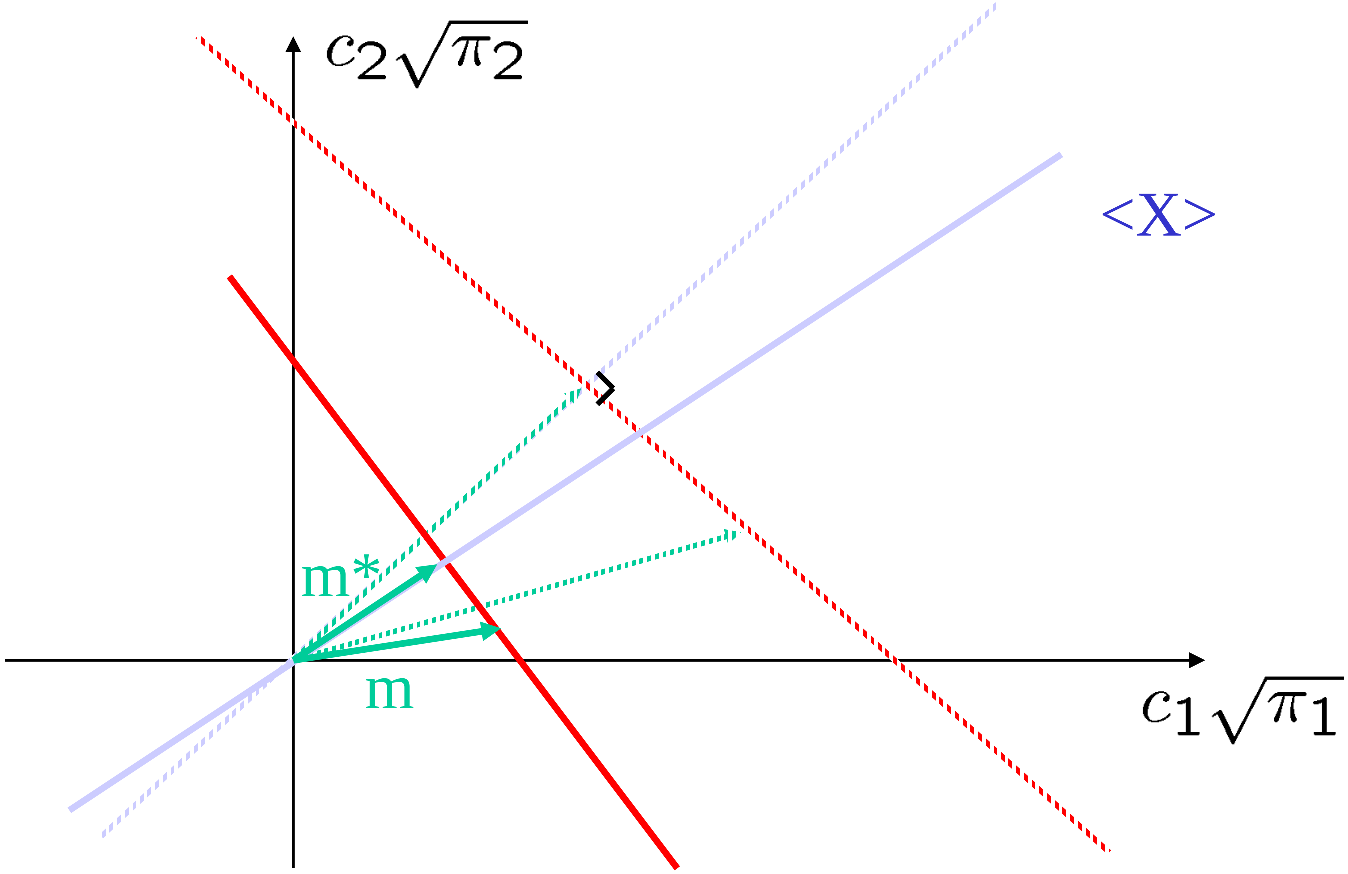
$$p^j = \sum_s q_s x_s^j = \sum_s \pi_s \underbrace{\frac{q_s}{\pi_s}}_{m_s} x_s^j$$

- That is, stochastic discount factor $m_s = q_s/\pi_s$ for all s.

$$p^j = E[mx^j]$$

2. Stochastic Discount Factor

shrink axes by factor $\sqrt{\pi_s}$



Risk-adjustment in payoffs

$p = E[mx^j] = E[m]E[x] + Cov[m,x]$

Since 1=E[mR], the risk free rate is $R^f = 1/E[m]$

$$\mathbf{p = E[x]/R^f + Cov[m,x]}$$

Remarks:

(i) If risk-free rate does not exist, R^f is the shadow risk free rate

(ii) In general $Cov[m,x] < 0$, which lowers price and increases return

3. Equivalent Martingale Measure

- Price of any asset $p^j = \sum_s q_s x_s^j$
- Price of a bond $p^{\text{bond}} = \sum_s q_s = \frac{1}{1+r^f}$

$$p^j = \sum_{s'} q_{s'} \sum_s \frac{q_s}{\sum_{s'} q_{s'}} x_s^j$$

$$p^j = \frac{1}{1+r^f} \sum_s \underbrace{\frac{q_s}{\sum_{s'} q_{s'}}}_{:=\hat{\pi}_s} x_s^j$$

$$p^j = \frac{1}{1+r^f} E_{\hat{\pi}}[x^j]$$

… in Returns: $R^j = x^j/p^j$

$E[mR^j]=1$ $\qquad R^f E[m]=1$

$\Rightarrow E[m(R^j-R^f)]=0$

$E[m]\{E[R^j]-R^f\} + Cov[m,R^j]=0$

$$\mathbf{E[R^j] - R^f = - Cov[m,R^j]/E[m]} \qquad \mathbf{(2)}$$

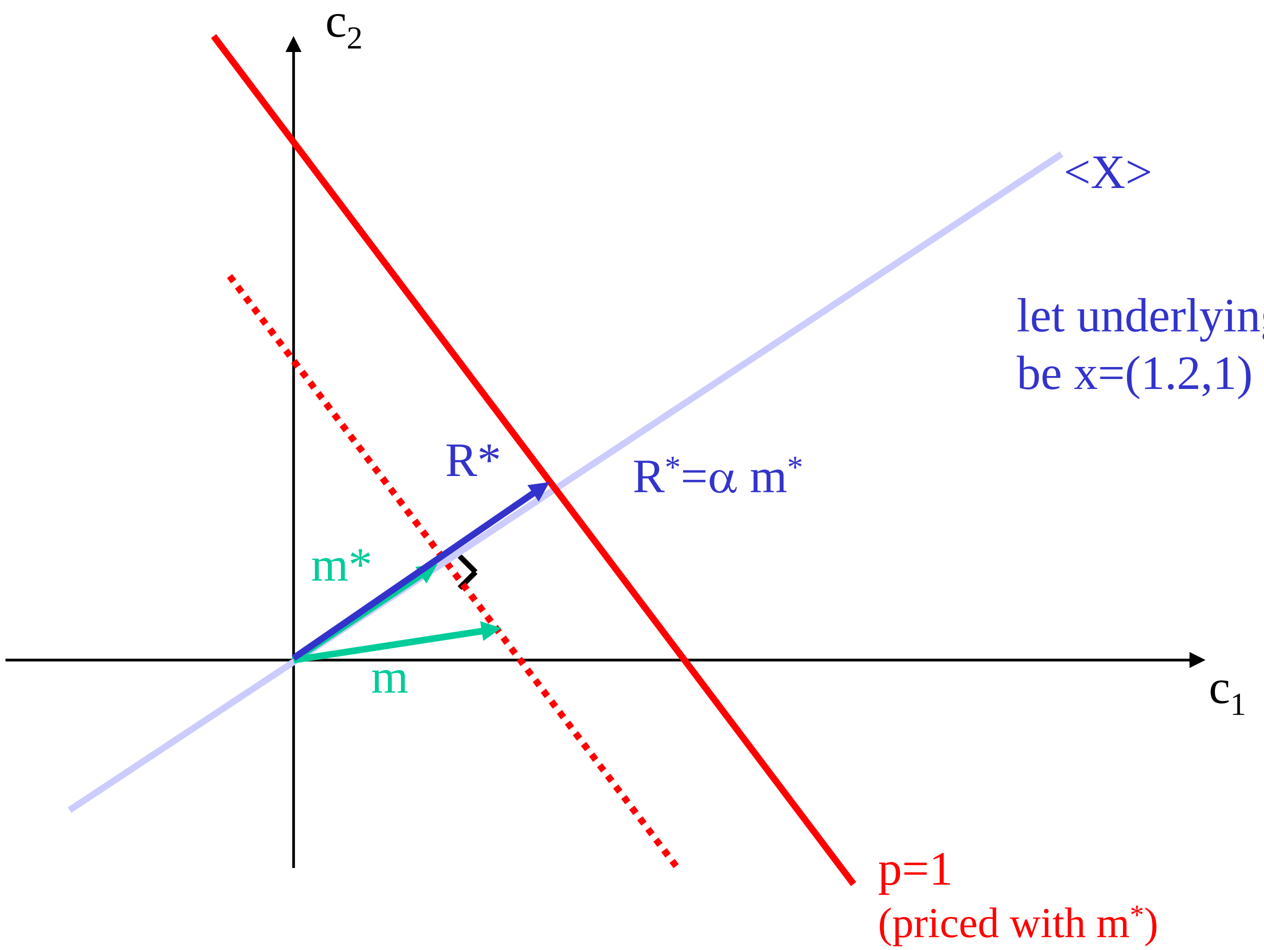
also holds for portfolios *h*

Note:

- risk correction depends only on Cov of payoff/return with discount factor.
- Only compensated for taking on systematic risk not idiosyncratic risk.

4. State-price BETA Model

shrink axes by factor $\sqrt{\pi_s}$



4. State-price BETA Model

$E[R^j] – R^f = - Cov[m,R^j]/E[m]$ (2)

also holds for all portfolios *h* and

we can replace m with m^*

Suppose (i) $Var[m^*] > 0$ and (ii) $R^* = \alpha\, m^*$ with $\alpha > 0$

$$E[R^h] – R^f = - Cov[R^*,R^h]/E[R^*] \qquad (2')$$

Define $\boldsymbol{\beta^h := Cov[R^*,R^h]/ Var[R^*]}$ for any portfolio *h*

4. State-price BETA Model

(2) for R^*: $E[R^*]-R^f=-Cov[R^*,R^*]/E[R^*]$
$=-Var[R^*]/E[R^*]$

(2) for R^h: $E[R^h]-R^f=-Cov[R^*,R^h]/E[R^*]$
$= - \beta^h \; Var[R^*]/E[R^*]$

$$\mathbf{E[R^h] - R^f = \beta^h \, E[R^* - R^f]}$$

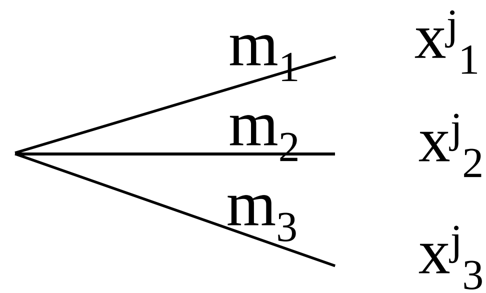
where $\beta^h := Cov[R^*,R^h]/Var[R^*]$

very general – but what is R^* in reality?

Regression $R^h_s = \alpha^h + \beta^h (R^*)_s + \varepsilon_s$ with $Cov[R^*,\varepsilon]=E[\varepsilon]=0$

Four Asset Pricing Formulas

1. State prices $1 = \sum_s q_s R_s^j$
2. Stochastic discount factor $1 = E[mR^j]$



3. Martingale measure $1 = 1/(1+r^f)\, E_{\hat{\pi}}[R^j]$

   (reflect risk aversion by
   over(under)weighing the "bad(good)" states!)

4. State-price beta model $E[R^j] - R^f = \beta^j E[R^* - R^f]$

   (in returns $R^j := x^j / p^j$)

What do we know about q, m, $\hat{\pi}$, R*?

- Main results so far
  - ❑Existence iff no arbitrage
    - ➡ Hence, single factor only
      - but doesn't famos Fama-French factor model has 3 factors?
    - ➡ multiple factor is due to time-variation
(wait for multi-period model)
  - ❑Uniqueness if markets are complete

Different Asset Pricing Models

$p_t = E[m_{t+1} x_{t+1}]$ $\Rightarrow$ $E[R^h] - R^f = \beta^h E[R^* - R^f]$

where $\beta^h := Cov[R^*, R^h]/Var[R^*]$

where $m_{t+1} = f(\cdot, \ldots, \cdot)$

$f(\cdot)$ = asset pricing model

General Equilibrium

$f(\cdot) = MRS / \pi$

Factor Pricing Model

$a + b_1 f_{1,t+1} + b_2 f_{2,t+1}$

CAPM

$a + b_1 f_{1,t+1} = a + b_1 R^M$

CAPM

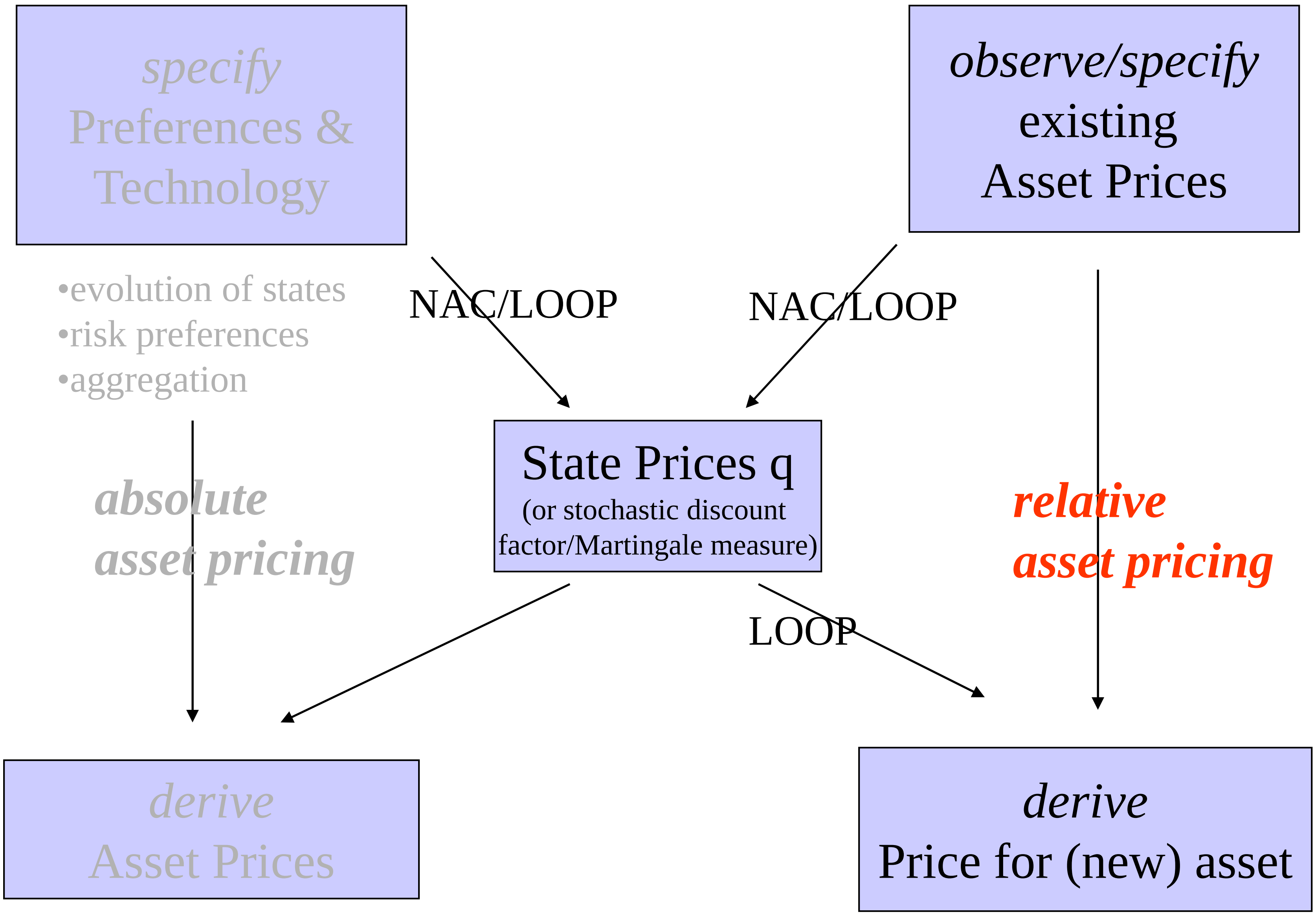
$R^* = R^f (a + b_1 R^M)/(a + b_1 R^f)$

where R^M = return of market portfolio

Is $b_1 < 0$?

Different Asset Pricing Models

- Theory
  - ❑All economics and modeling is determined by
$$m_{t+1} = a + \mathbf{b'}\ \mathbf{f}$$
  - ❑Entire content of model lies in restriction of SDF
- Empirics
  - ❑m^* (which is a portfolio payoff) prices as well as m (which is e.g. a function of income, investment etc.)
  - ❑ measurement error of m^* is smaller than for any m
  - ❑Run regression on *returns* (portfolio payoffs)! (e.g. Fama-French three factor model)



Only works as long as market completeness doesn't change

Overview: Pricing - one period model

1. LOOP, No arbitrage
2. Forwards
3. Parity relationship between options
4. No arbitrage and existence of state prices
5. Market completeness and uniqueness of state prices
6. Pricing kernel q^*
7. Four pricing formulas (state prices, SDF, EMM, beta-pricing)
8. Recovering state prices from options

Recovering State Prices from Option Prices

- Suppose that S_T, the price of the underlying portfolio (we may think of it as a proxy for price of "market portfolio"), assumes a "continuum" of possible values.
- Suppose there are a "continuum" of call options with different strike/exercise prices $\Rightarrow$ markets are complete
- Let us construct the following portfolio:
  for some small positive number $\varepsilon>0$,
  - ❑ Buy one call with $E = \hat{S}_T - \frac{\delta}{2} - \varepsilon$
  - ❑ Sell one call with $E = \hat{S}_T - \frac{\delta}{2}$
  - ❑ Sell one call with $E = \hat{S}_T + \frac{\delta}{2}$
  - ❑ Buy one call with $E = \hat{S}_T + \frac{\delta}{2} + \varepsilon$

Recovering State Prices … (ctd.)

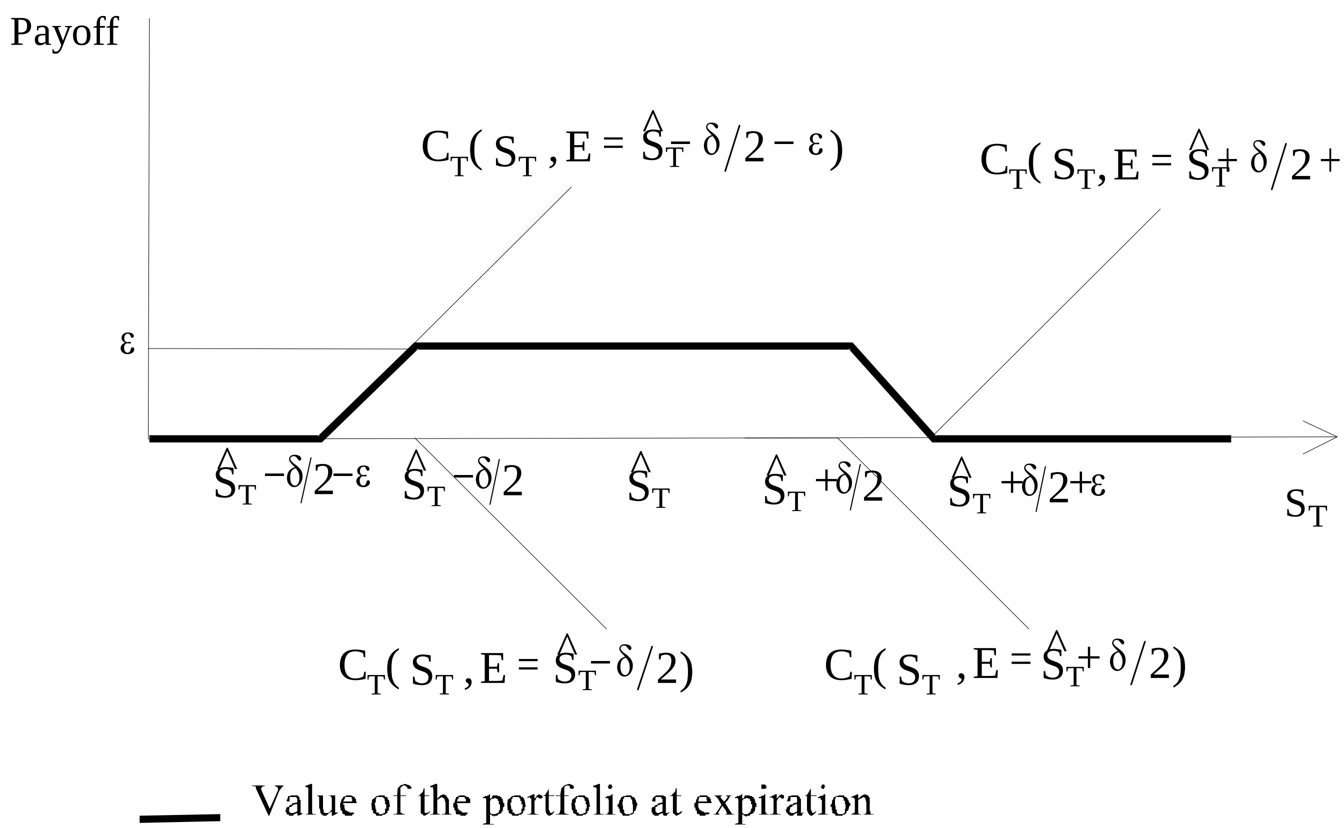


Figure 8-2 Payoff Diagram: Portfolio of Options

Recovering State Prices … (ctd.)

- Let us thus consider buying $1/_{\varepsilon}$ units of the portfolio. The total payment, when $\hat{S}_T - \frac{\delta}{2} \le S_T \le \hat{S}_T + \frac{\delta}{2}$, is $\varepsilon \cdot \frac{1}{\varepsilon} \equiv 1$, for any choice of ε. We want to let $\varepsilon \mapsto 0$, so as to eliminate the payments in the ranges $S_T \in (\hat{S}_T - \frac{\delta}{2} - \varepsilon, \hat{S}_T - \frac{\delta}{2})$ and $S_T \in (\hat{S}_T + \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2} + \varepsilon)$. The value of $1/_{\varepsilon}$ units of this portfolio is :

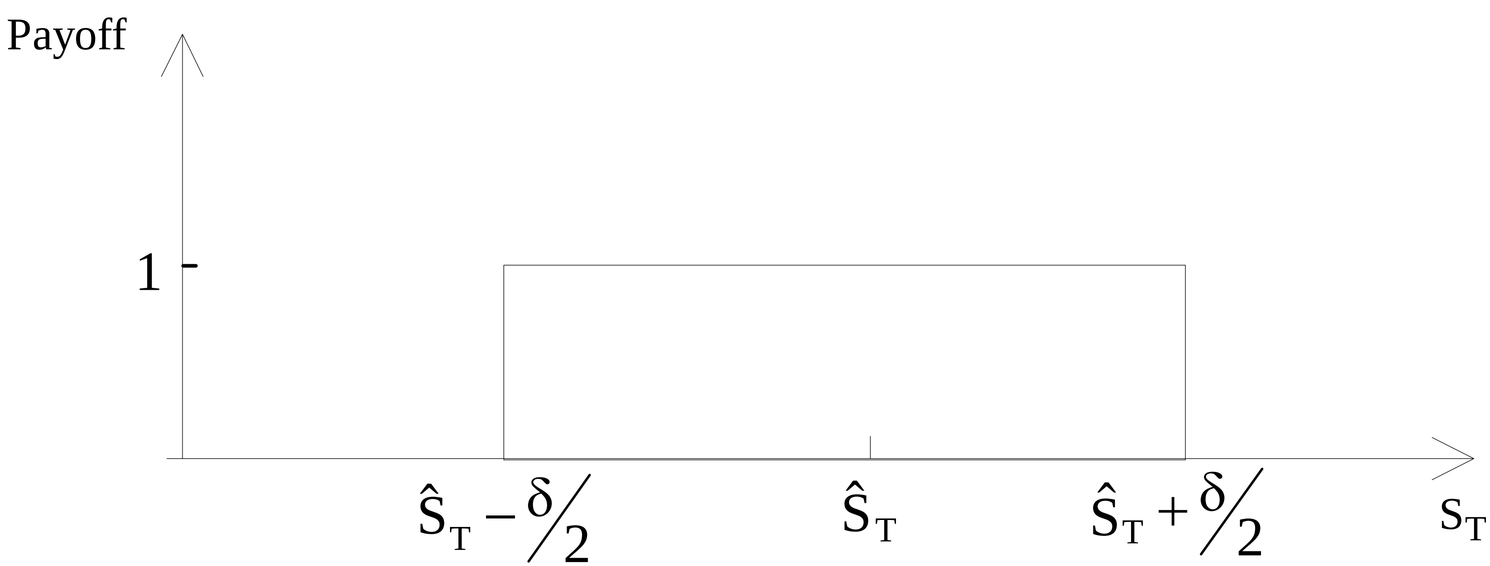
$$\frac{1}{\varepsilon}\left\{C\left(S, E = \hat{S}_T - \tfrac{\delta}{2} - \varepsilon\right) - C\left(S, E = \hat{S}_T - \tfrac{\delta}{2}\right) - \left[C\left(S, E = \hat{S}_T + \tfrac{\delta}{2}\right) - C\left(S, E = \hat{S}_T + \tfrac{\delta}{2} + \varepsilon\right)\right]\right\}$$

Taking the limit $\varepsilon \to 0$

$$\lim_{\varepsilon \mapsto 0} \tfrac{1}{\varepsilon}\left\{C\left(S, E=\hat{S}_T - \tfrac{\delta}{2} - \varepsilon\right) - C\left(S, E=\hat{S}_T - \tfrac{\delta}{2}\right) - \left[C\left(S, E=\hat{S}_T + \tfrac{\delta}{2}\right) - C\left(S, E=\hat{S}_T + \tfrac{\delta}{2} + \varepsilon\right)\right]\right\}$$

$$= -\lim_{\varepsilon \mapsto 0}\Bigg\{\underbrace{\frac{C\left(S, E=\hat{S}_T - \frac{\delta}{2} - \varepsilon\right) - C\left(S, E=\hat{S}_T - \frac{\delta}{2}\right)}{-\varepsilon}}_{\leq 0}\Bigg\} + \lim_{\varepsilon \mapsto 0}\Bigg\{\underbrace{\frac{C\left(S, E=\hat{S}_T + \frac{\delta}{2} + \varepsilon\right) - C\left(S, E=\hat{S}_T + \frac{\delta}{2}\right)}{\varepsilon}}_{\leq 0}\Bigg\}$$

$$= -\frac{\partial C}{\partial E}\left(S, E = \hat{S}_T - \frac{\delta}{2}\right) + \frac{\partial C}{\partial E}\left(S, E = \hat{S}_T + \frac{\delta}{2}\right)$$



Divide by δ and let $\delta \to 0$ to obtain state price **density** as $\partial^2 C / \partial E^2$.

Recovering State Prices … (ctd.)

Evaluating following cash flow

$$\tilde{CF}_T = \begin{cases} 0 \text{ if } S_T & \notin \left[\hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2}\right] \\ 50000 \text{ if } S_T & \in \left[\hat{S}_T - \frac{\delta}{2}, \hat{S}_T + \frac{\delta}{2}\right] \end{cases}.$$

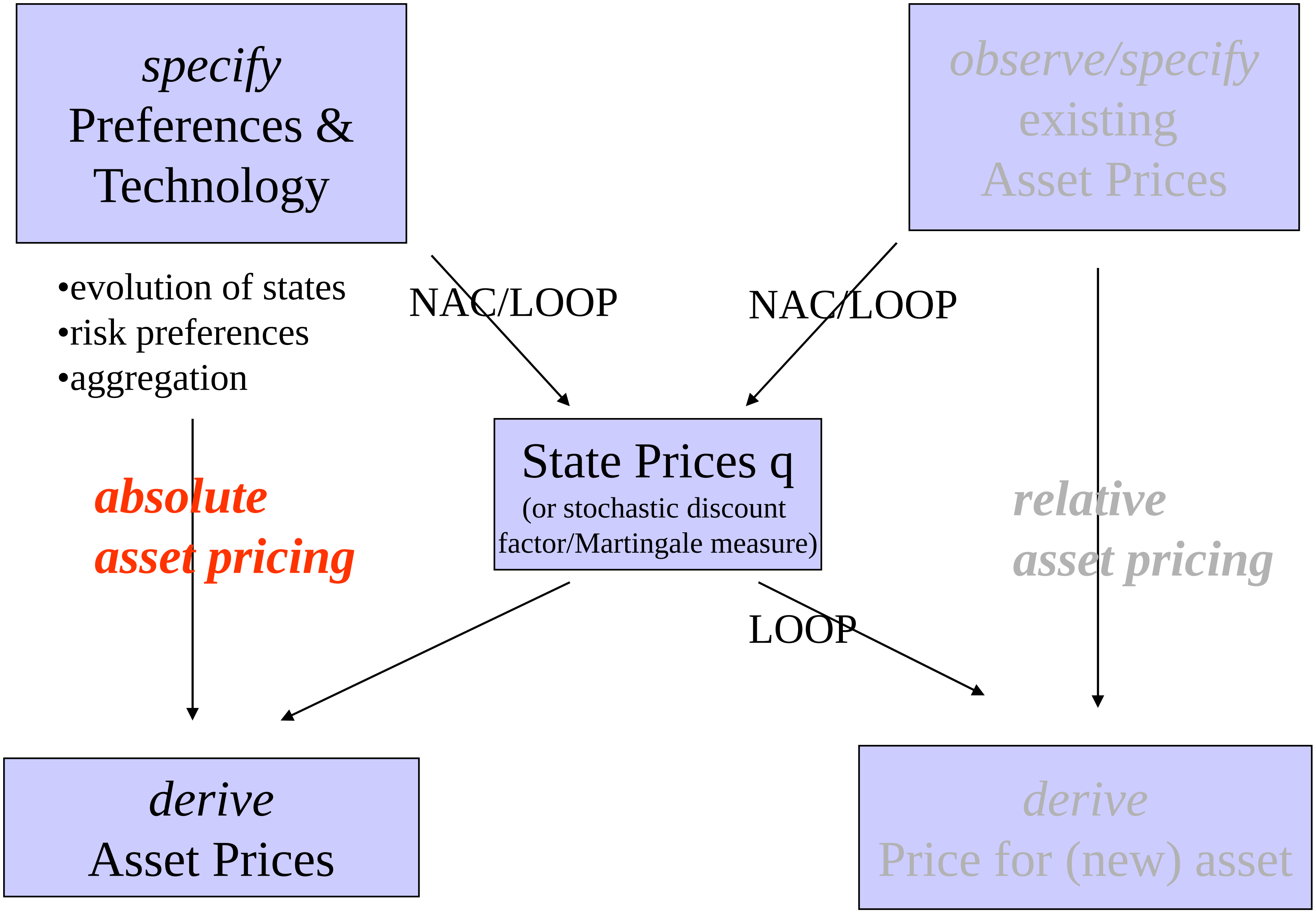
The value today of this cash flow is :

$$50000\left[\frac{\partial C}{\partial E}(S, E = \widehat{S}_T + \tfrac{\delta}{2}) - \frac{\partial C}{\partial E}(S, E = \widehat{S}_T - \tfrac{\delta}{2})\right]$$

$$q(S_T^1, S_T^2) = \frac{\partial C}{\partial E}(S, E = S_T^2) - \frac{\partial C}{\partial E}(S, E = S_T^1)$$

Table 8.1 Pricing an Arrow-Debreu State Claim

E	C(S,E)	Cost of position	Payoff if S_T =							ΔC	$\Delta(\Delta C) = q_s$
			7	8	9	10	11	12	13		
7	3.354										
										-0.895	
8	2.459										0.106
										-0.789	
9	1.670	+1.670	0	0	0	1	2	3	4		0.164
										-0.625	
10	1.045	-2.090	0	0	0	0	-2	-4	-6		**0.184**
										-0.441	
11	0.604	+0.604	0	0	0	0	0	1	2		0.162
										-0.279	
12	0.325										0.118
										-0.161	
13	0.164										
		0.184	0	0	0	1	0	0	0		



Only works as long as market completeness doesn't change

The following is for later lecture

Futures contracts

- Exchange-traded “forward contracts”
- Typical features of futures contracts
  - ❑ Standardized, with specified delivery dates, locations, procedures
  - ❑ A clearinghouse
    - Matches buy and sell orders
    - Keeps track of members’ obligations and payments
    - After matching the trades, becomes counterparty
- Differences from forward contracts
  - ❑ Settled daily through the mark-to-market process → low credit risk
  - ❑ Highly liquid → easier to offset an existing position
  - ❑ Highly standardized structure → harder to customize

Example: S&P 500 Futures

- WSJ listing:

	OPEN	HIGH	LOW	SETTLE	CHANGE	LIFETIME HIGH	LIFETIME LOW	OPEN INT.
S&P 500 Index (CME)-$250 times index								
Mar	112350	112370	109100	109530	– 2810	134960	94100	474,811
June	111950	111950	109350	109730	– 2830	170550	95030	17,224
Dec	111580	111580	110020	110390	– 2930	150070	96130	304

Est vol 79,914; vol Fri 65,250; open int 502,626, –701.
Idx prl: Hi 1122.20; Lo 1092.25; Close 1094.44, –27.76.

- Contract specifications:

FIGURE 5.2

Specifications for the S&P 500 index futures contract.

Underlying	S&P 500 index
Where traded	Chicago Mercantile Exchange
Size	$250 × S&P 500 index
Months	Mar, Jun, Sep, Dec
Trading ends	Business day prior to determination of settlement price
Settlement	Cash-settled, based upon opening price of S&P 500 on third Friday of expiration month

Example: S&P 500 Futures (cont.)

- Notional value: $250 x Index
- Cash-settled contract
- Open interest: total number of buy/sell pairs
- Margin and mark-to-market
  - Initial margin
  - Maintenance margin (70-80% of initial margin)
  - Margin call
  - Daily mark-to-market
- Futures prices vs. forward prices
  - The difference negligible especially for short-lived contracts
  - Can be significant for long-lived contracts and/or when interest rates are correlated with the price of the underlying asset

Example: S&P 500 Futures (cont.)

- Mark-to-market proceeds and margin balance for 8 long futures:

TABLE 5.8 Mark-to-market proceeds and margin balance over 10 weeks from long position in 8 S&P 500 futures contracts. The final row represents expiration of the contract.

Week	Multiplier ($)	Futures Price	Price Change	Margin Balance($)
0	2000.00	1100.00	—	220,000.00
1	2000.00	1027.99	−72.01	76,233.99
2	2000.00	1037.88	9.89	96,102.01
3	2000.00	1073.23	35.35	166,912.96
4	2000.00	1048.78	−24.45	118,205.66
5	2000.00	1090.32	41.54	201,422.13
6	2000.00	1106.94	16.62	234,894.67
7	2000.00	1110.98	4.04	243,245.86
8	2000.00	1024.74	−86.24	71,046.69
9	2000.00	1007.30	−17.44	36,248.72
10	2000.00	1011.65	4.35	44,990.57

Example: S&P 500 Futures (cont.)

- S&P index arbitrage: comparison of formula prices with actual prices:

TABLE 5.9 S&P 500 index futures prices and interest rate information from the *Wall Street Journal*, February 1, 2002. The closing S&P 500 spot price was 1130.20. Treasury-bill yields are reported yields on Treasury bills expiring in the same month as the futures contract. LIBOR rates are constructed from Eurodollar prices. The theoretical forward prices are constructed for each maturity from equation (5.7) using the interest rate in the preceding row and assuming a 1.3% dividend yield.

Expiration Month:	March	June	December
Days to Expiration:	**42**	**140**	**322**
S&P 500 Index Futures Price	1130.4	1132.5	1140.3
Treasury-Bill Yield	0.0167	0.017	0.0218
Theoretical Forward Price	1130.68	1131.93	1139.01
LIBOR	0.0187	0.0201	0.0240
Theoretical Forward Price	1130.94	1133.28	1141.22

Uses of index futures

- Why buy an index futures contract instead of synthesizing it using the stocks in the index? Lower transaction costs
- Asset allocation: switching investments among asset classes
- Example: Invested in the S&P 500 index and temporarily wish to temporarily invest in bonds instead of index. What to do?
  - ❑ Alternative #1: Sell all 500 stocks and invest in bonds
  - ❑ Alternative #2: Take a short forward position in S&P 500 index

TABLE 5.10 Effect of owning the stock and selling forward, assuming that $S_0 = \$100$ and $F_{0,1} = \$110$.

	Cash Flows		
Transaction	**Today**	**1 year, $S_1 = \$80$**	**1 year, $S_1 = \$130$**
Own Stock @ \$100	−\$100	\$80	\$130
Short Forward @ \$110	0	\$110 − \$80	\$110 − \$130
Total	−\$100	\$110	\$110

Uses of index futures (cont.)

- $100 million portfolio with β of 1.4 and r_f = 6 %

1. Adjust for difference in $ amount
   - 1 futures contract $250 x 1100 = $275,000
   - Number of contracts needed $100mill/$0.275mill = 363.636
2. Adjust for difference in β

   363.636 x 1.4 = 509.09 contracts

Uses of index futures (cont.)

- Cross-hedging with perfect correlation

TABLE 5.11 Results from shorting 509.09 S&P 500 index futures against a $100m portfolio with a beta of 1.4.

S&P 500 Index	Gain on 509 Futures	Portfolio Value	Total
900	33.855	72.145	106.000
950	27.491	78.509	106.000
1000	21.127	84.873	106.000
1050	14.764	91.236	106.000
1100	8.400	97.600	106.000
1150	2.036	103.964	106.000
1200	−4.327	110.327	106.000

- Cross-hedging with imperfect correlation
- General asset allocation: futures overlay
- Risk management for stock-pickers

Currency contracts

- Widely used to hedge against changes in exchange rates
- WSJ listing:

CURRENCY

	OPEN	HIGH	LOW	SETTLE	CHANGE	LIFETIME HIGH	LIFETIME LOW	OPEN INT.
Japan Yen (CME)-12.5 million yen; $ per yen (.00)								
Mar	.7528	.7600	.7508	.7576	+ .0050	.8760	.7416	105,563
June	.7570	.7635	.7547	.7611	+ .0050	.8776	.7453	20,834
Est vol 11,091; vol Fri 25,220; open int 126,860, −1,352.								
Canadian Dollar (CME)-100,000 dlrs.; $ per Can $								
Mar	.6282	.6287	.6264	.6266	− .0016	.6725	.6170	60,355
June	.6280	.6287	.6263	.6264	− .0016	.6700	.6180	3,952
Sept	.6265	.6282	.6265	.6266	− .0016	.6590	.6175	1,331
Dec	.6274	.6280	.6265	.6269	− .0016	.6555	.6190	1,075
Est vol 5,343; vol Fri 7,699; open int 66,818, −652.								
British Pound (CME)-62,500 pds.; $ per pound								
Mar	1.4112	1.4206	1.4106	1.4186	+ .0058	1.4700	1.3810	33,978
June	1.4066	1.4140	1.4038	1.4102	+ .0058	1.4550	1.3910	89
Est vol 5,315; vol Fri 4,859; open int 34,067, −174.								
Swiss Franc (CME)-125,000 francs; $ per franc								
Mar	.5836	.5902	.5825	.5892	+ .0061	.6370	.5540	47,600
June	.5866	.5906	.5830	.5895	+ .0061	.6320	.5813	238
Est vol 5,676; vol Fri 6,330; open int 47,871, −587.								
Australian Dollar (CME)-100,000 dlrs.; $ per A.$								
Mar	.5076	.5102	.5074	.5098	+ .0031	.5300	.4810	21,597
June	.5050	.5069	.5046	.5070	+ .0031	.5218	.4885	456
Est vol 944; vol Fri 1,871; open int 22,079, −518.								
Mexican Peso (CME)-500,000 new Mex. peso, $ per MP								
Mar	.10843	.10850	.10810	.10835	+ 00005	.10940	.09770	28,070
June	....	....	....	.10645	+ 00010	.10750	.09730	1,509
Sept	.10450	.10450	.10450	.10453	+ 00010	.10500	.09930	508
Est vol 1,940; vol Fri 2,817; open int 30,163, +626.								
Euro FX (CME)-Euro 125,000; $ per Euro								
Mar	.8601	.8694	.8593	.8686	+ .0085	.9630	.8336	103,910
June	.8663	.8663	.8564	.8657	+ .0085	.9275	.8365	1,492
Dec	.8600	.8600	.8600	.8616	+ .0085	.9175	.8390	236
Est vol 14,904; vol Fri 17,547; open int 105,729, −560.								

Currency contracts: pricing

- Currency prepaid forward
  - ❑ Suppose you want to purchase ¥1 one year from today using $s
  - ❑ $F^P_{0,T} = x_0 e^{-r_y T}$
    - where x_0 is current ($/ ¥) exchange rate, and r_y is the yen-denominated interest rate
    - Why? By deferring delivery of the currency one loses interest income from bonds denominated in that currency
- Currency forward
  - ❑ $F_{0,T} = x_0 e^{(r-r_y)T}$
    - r is the $-denominated domestic interest rate
    - $F_{0,T} > x_0$ if $r > r_y$ (domestic risk-free rate exceeds foreign risk-free rate)

Currency contracts: pricing (cont.)

- Example 5.3:
  - ❑ ¥-denominated interest rate is 2% and current ($/ ¥) exchange rate is 0.009. To have ¥1 in one year one needs to invest today:
    - 0.009/¥ x ¥1 x $e^{-0.02}$ = $0.008822

- Example 5.4:
  - ❑ ¥-denominated interest rate is 2% and $-denominated rate is 6%. The current ($/ ¥) exchange rate is 0.009. The 1-year forward rate:
    - $0.009e^{0.06-0.02} = 0.009367$

Currency contracts: pricing (cont.)

- Synthetic currency forward: borrowing in one currency and lending in another creates the same cash flow as a forward contract
- Covered interest arbitrage: offset the synthetic forward position with an actual forward contract

TABLE 5.12 Synthetically creating a yen forward contract by borrowing in dollars and lending in yen. The payoff at time 1 is ¥1 − $0.009367.

	Cash Flows			
	Year 0		Year 1	
Transaction	**$**	**¥**	**$**	**¥**
Borrow $x_0 e^{-r_y}$ Dollar at 6% ($)	+0.008822	—	−0.009367	—
Convert to Yen @ 0.009 $/¥	−0.008822	+0.9802	—	—
Invest in Yen-Denominated Bill (¥)	—	−0.9802	—	1
Total	0	0	−0.009367	1

Eurodollar futures

- WSJ listing

	OPEN	HIGH	LOW	SETTLE	CHANGE	YIELD	CHANGE	OPEN INT.
Treasury Bills (CME)-$1 mil.; pts of 100%								
Mar	....	....	....	98.26	+ .01	1.74	−.01	749
Est vol 17; vol Fri 3; open int 749, −10.								
Eurodollar (CME)-$1 Million; pts of 100%								
Feb	98.08	98.09	98.08	98.09	+ .01	1.91	−.01	37,493
Mar	98.02	98.05	98.01	98.04	+ .02	1.96	−.02	758,791
Apr	97.94	97.96	97.94	97.96	+ .04	2.04	−.04	4,565
June	97.67	97.74	97.65	97.73	+ .06	2.27	−.06	684,053
Sept	97.17	97.30	97.17	97.28	+ .09	2.72	−.09	629,125
Dec	96.58	96.74	96.58	96.72	+ .12	3.28	−.12	709,233
Mr03	96.05	96.15	96.05	96.14	+ .12	3.86	−.12	399,549
June	95.57	95.66	95.57	95.64	+ .11	4.36	−.11	260,328
Sept	95.19	95.26	95.19	95.26	+ .11	4.74	−.11	232,148
Dec	94.85	94.92	94.84	94.90	+ .10	5.10	−.10	168,299
Mr04	94.64	94.70	94.63	94.69	+ .09	5.31	−.09	112,715
June	94.42	94.47	94.22	94.46	+ .09	5.54	−.09	116,246
Sept	94.22	94.28	94.22	94.27	+ .09	5.73	−.09	109,477
Dec	94.03	94.08	94.02	94.07	+ .09	5.93	−.09	70,835
Mr05	93.97	94.02	93.97	94.01	+ .08	5.99	−.08	75,902
June	93.86	93.90	93.85	93.90	+ .08	6.10	−.08	67,857
Sept	93.77	93.82	93.76	93.81	+ .08	6.19	−.08	79,848
Dec	93.64	93.69	93.63	93.67	+ .08	6.33	−.08	54,328
Mr06	93.63	93.68	93.62	93.67	+ .07	6.33	−.07	46,271
June	93.59	93.61	93.58	93.61	+ .07	6.39	−.07	35,319
Sept	93.54	93.56	93.54	93.55	+ .06	6.45	−.06	44,724
Dec	93.43	93.45	93.42	93.44	+ .06	6.56	−.06	32,917
Ju07	93.38	93.42	93.38	93.42	+ .05	6.58	−.05	18,263
Sept	93.35	93.39	93.35	93.38	+ .05	6.62	−.05	14,107
Dec	93.24	93.28	93.24	93.27	+ .04	6.73	−.04	13,227
Ju08	93.23	93.26	93.23	93.25	+ .04	6.75	−.04	11,239
Ju09	93.04	93.09	93.04	93.08	+ .02	6.92	−.02	2,406

Est vol 568,992; vol Fri 1,183,059; open int 4,864,582, +85,457.

- Contract specifications

FIGURE 5.7

Specifications for the Eurodollar futures contract.

Where traded	Chicaco Mercantile Exchange
Size	3-month Eurodollar time deposit, $1 million principal
Months	Mar, Jun, Sep, Dec, out 10 years, plus 2 serial months and spot month
Trading ends	5 A.M. (11 A.M. London) on the second London bank business day immediately preceding the third Wednesday of the contract month.
Delivery	Cash settlement
Settlement	100 − British Banker's Association Futures Interest Settlement Rate for 3-Month Eurodollar Interbank Time Deposits. (This is a 3-month rate annualized by multiplying by 360/90.)

Introduction to Commodity Forwards

- *Commodity* forward prices can be described by the same formula as that for *financial* forward prices:

$$F_{0,T} = S_0\, e^{(r-\delta)T}$$

- For financial assets, δ is the dividend yield. For commodities, δ is the commodity lease rate. The lease rate is the return that makes an investor willing to buy and then lend a commodity.

- The lease rate for a commodity can typically be estimated only by observing the forward prices.

Introduction to Commodity Forwards

- The set of prices for different expiration dates for a given commodity is called the **forward curve** (or the **forward strip**) for that date.

- If on a given date the forward curve is upward-sloping, then the market is in **contango**. If the forward curve is downward sloping, the market is in **backwardation**.

  - ❑Note that forward curves can have portions in backwardation and portions in contango.