Information

- A (finite) set states $\mathcal{S}$, probabilities $\pi_s > 0$ for each $s \in \mathcal{S}$, and dates $t = 0, 1, \ldots, T$.

- At each date t a collection of subsets of $\mathcal{S}$, $F_t = \{A_t^1, A_t^2, \cdots A_t^{k_t}\}$, such that $A_t^i \bigcap A_t^j = \emptyset$ if $j \neq i$ and $\bigcup_i A_t^i = \mathcal{S}$. (Partition)

  - $F_0 = \{\mathcal{S}\}$.

  - Each $A \in F_T$ contains exactly one state.

  - A_t^i, $i = 1, \ldots, k_t$, are the events at t.

- Information structure: $\mathcal{F} = \{F_0, F_1, \cdots F_T\}$.

- Total recall: If $A \in F_t$, and $t' < t$ there exists an $A' \in F_{t'}$ such that $A \subset A'$.

- Trees

- Stochastic process: A collection of random variables $y_t(s)$ for $t = 0, \cdots T$.

- Stochastic process is adapted to $\mathcal{F}$ if for each $A \in F_t$, $y_t(s) = y_t(s')$ for each $s \in A$ and $s' \in A$. $y_t(A) \equiv y_t(s), s \in A$.

- $y \geq 0$ (positive) if $y_t(s) \geq 0$ for each (t, s). $y > 0$ (positive and non-zero) if $y \geq 0$ and $y \neq 0$. $y >> 0$ (strictly positive) if $y_t(s) > 0$ for each (t, s).

- For each $j = 1, \ldots J$, a security is an adapted dividend process x_t^j, $t = 1, \ldots T$. The dividend paid at t is received by the agent that held the security from $t - 1$ to t.

- The (*ex dividend*) price of this security is an adapted process p_t^j.

Strategies

- A strategy consist of J adapted processes $h^1, \cdots, h^J$. h_t^j denotes the amount held from t to $t+1$ of security j.

- $\mathcal{H}$ the set of all strategies.

- The dividend of the strategy is the process $z_t^h = (h_{t-1} - h_t) \cdot p_t + h_{t-1} \cdot x_t \equiv \sum_{j=1}^J [p_t^j (h_{t-1}^j - h_t^j) + h_{t-1}^j x_t^j]$, for $t \geq 1$.

- z^h is adapted.

- The cost of strategy h is

$$z_0^h = p_0 \cdot h_0 \equiv \sum_j p_0^j h_0^j.$$

Marketed subspace

- $\mathcal{M}^p = \{y : y = z^h$ for some $h \in \mathcal{H}\}$.

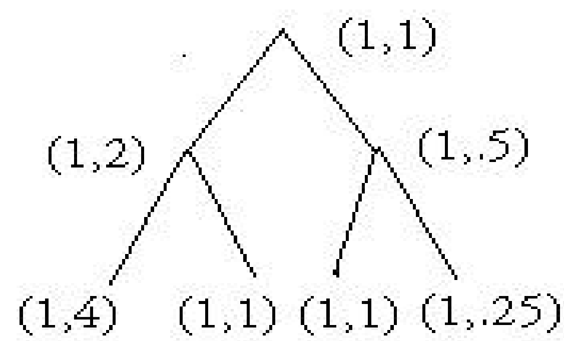
- $\mathcal{M}^p$ is a linear space.

  - Complete markets if any adapted $y \in \mathcal{M}^p$.

  - Incomplete markets.

Dynamic hedging

- No dividends paid until period 2.
- Prices of the two assets in parenthesis in periods 0,1. Dividends in parenthesis in period 2.
- Hedge: $z_0 = z_1 = 0$, $z_2 = (0,1,0,0)$



Arbitrage

- The law of one price holds if

$$z^h = z^{h'} \Rightarrow p_0 \cdot h_0 = p_0 \cdot h'_0.$$

- Law of one price $\Leftrightarrow$ every portfolio strategy with zero payoff has zero price.

- If the law of one price holds we may define a linear functional $q : \mathcal{M}^p \rightarrow R$ defined by: $q(z) = p_0 \cdot h_0$ for any strategy h such that $z^h = z$. q is the payoff pricng functional.

- A strong arbitrage is a strategy h with $p_0 \cdot h_0 < 0$ and $z^h \geq 0$.

- An arbitrage is a strategy h that is either a strong arbitrage or satisfies $p_0 \cdot h_0 = 0$ and $z^h > 0$.

- The payoff pricing functional is strictly positive ($q(z) > 0$ for every $z > 0$, $z \in \mathcal{M}$) $\Leftrightarrow$ there is no arbitrage.

- The payoff pricing functional is positive ($q(z) \geq 0$ for every $z \geq 0$, $z \in \mathcal{M}$) $\Leftrightarrow$ there is no strong arbitrage.

- A one-period strong arbitrage at an event A_t^i at $t < T$ is a portfolio h such that, for each $s \in A_t^i$,

$$[p_{t+1}(s) + x_{t+1}(s)] \cdot h \geq 0,$$

and $p_t(s) \cdot h < 0$.

- A one-period arbitrage at an event A_t^i at $t < T$ is a portfolio h that is either a one-period strong arbitrage or that $p_t(s) \cdot h = 0$ and $[p_{t+1}(s) + x_{t+1}(s)] \cdot h > 0$.

- No one-period arbitrage $\Leftrightarrow$ no arbitrage.

Complete markets

- The immediate successors of an event $A \in F_t$ are all the events $B \in F_{t+1}$ such that $B \subset A$.

- $\iota(A) \equiv$ the number of immediate successors of an event A.

- The one-period payoff matrix in event $A \in F_t$, $t < T$, is the matrix with entries $p^j_{t+1}(B) + x^j_{t+1}(B)$, for $j = 1, \ldots J$ and B an immediate successor of A.

- Markets are complete $\Leftrightarrow$ for all events $A \in F_t$, $t < T$ the one period payoff matrix has rank $\iota(A)$.

- $J \geq \iota(A^i_t)$ whenever $t < T$.

Event prices

- Given an event $A_t \in F_t$, $t > 1$, consider a security that has a dividend process y^{A_t} with $y_\tau^{A_t}(s) = 0$ if $\tau \neq t$, $y_t^{A_t}(s) = 0$ if $s \notin A_t$, and $y_t^{A_t}(s) = 1$ if $s \in A_t$.

- If complete markets and the law of one price prevails then $q(y^{A_t})$ is called the price of an elementary (Arrow-Debreu) claim associated with A_t, or the price of the event A_t.

- $q(y^{A_0}) = q(y^{\mathcal{S}}) = 1$.

- For any security x^j, we may write:

$$x^j = \sum_t \sum_{A_t \in F_t} x_t^j(A_t) y^{A_t}.$$

- $$q(x^j) = \sum_{t=1}^{T} \sum_{A_t \in F_t} x_t^j(A_t) q(y^{A_t}). \qquad (1)$$

- For any strategy h

$$p_0 \cdot h_0 = \sum_{t=1}^{T} \sum_{A_t \in F_t} z_t^h(A_t) q(y^{A_t}).$$

- Suppose you follow strategy of buying an asset at $t < T$ and selling at $t+1$. Then:

$$q(y^{A_t}) p^j(A_t) = \sum_{A_{t+1} \subset A_t, A_{t+1} \in F_{t+1}} q(y^{A_{t+1}})[p^j(A_{t+1}) + x^j(A_{t+1})].$$

Implications of no arbitrage

- An event price vector is a family $q_t(A_t)$ for each $A_t \in F_t$, $0 \leq t \leq T$, with $q_0(\mathcal{S}) = 1$.

- Event prices $q(A_t)$ are compatible with (x, p) if for every $j = 1, \ldots, J$ and every (t, A_t), $0 \leq t \leq T$ and $A_t \in F_t$

$$q_t(A_t)p^j(A_t) = \sum_{t<\tau\leq T} \sum_{A_\tau \subset A_t, A_\tau \in F_\tau} q_t(A_\tau)x^j(A_\tau). \tag{2}$$

- Equation (2) holds if and only if

$$q_t(A_t)p^j(A_t) = \sum_{A_{t+1} \subset A_t, A_{t+1} \in F_{t+1}} q_{t+1}(A_{t+1})[x^j(A_{t+1}) + p^j(A_{t+1})]$$

Two basic results

- **There exists a strictly positive vector of event prices consistent with** (x, p) **if and only if there is no arbitrage.**

- **If no arbitrage, markets are complete if and only if there exists a unique positive vector of event prices that are consistent with** (x, p).

Risk-free return and discount factors

- If $p_t^j(s) > 0$,

$$r_{t+1}^j(s) = \frac{p_{t+1}^j(s) + x_{t+1}^j(s)}{p_t^j(s)}$$

  is the rate of return between t and $t+1$ in state s.

- $r_{t+1}^j(s)$ only depend on the event $A_{t+1} \in F_{t+1}$ that contains s.

- An asset j is risk free at (t, s) if $r_{t+1}^j(s)$ only depends on the event $A_t \in F_t$ that contains s.

- If no arbitrage, all risk-free assets at a given (t, s) must have the same $\bar{r}_t(s) > 0$.

- If $A_t \in F_t$, let $\bar{r}_t(A_t) = \bar{r}_t(s)$, for any $s \in A_t$.

- $$q_t(A_t) = \bar{r}_{t+1}(A_{t+1}) \sum_{A_{t+1} \subset A_t, A_{t+1} \in F_{t+1}} q_{t+1}(A_{t+1})$$

- If a risk-free security exists at each (t, s), the discount factor between zero and t in state s is $\rho_0(s) = 1$ and,

$$\rho_t(s) = \Pi_{\tau=1}^{t} (\bar{r}_\tau(s))^{-1},\ t \geq 1$$

  - $\rho_t(s)$ only depends on the event $A_{t-1} \in F_{t-1}$ that contains s.

  - $\rho_t(s) = \rho_{t+1}(s)\bar{r}_{t+1}(s)$

- If $A_t \in F_t$ let $\rho_t(A_t) = \rho_t(s)$ for any $s \in A_t$.

An example: model of interest rate

- State $s = (r_0, r_1, \ldots, r_T)$, where r_t is one plus the one period interest rate prevailing at t.

- A one year bond issued at t pays a dividend of 1 at $t+1$. The price at t is $\frac{1}{r_t(s)}$.

- A τ year zero coupon bond issued at t pays a dividend of 1 at $t+\tau$.

  - A τ year zero coupon issued at t is a $\tau - j$ bond in period $t+j$.

- $P_t^\tau(s)$ the price of a τ year zero coupon bond as of t, ($P_t^0(s)$=1). If $A_t \in F_t$,

$$P_t^\tau(A_t)q_t(A_t) = \sum_{A_{t+1} \subset A_t,\, A_{t+1} \in F_{t+1}} P_{t+1}^{\tau-1}(A_{t+1})q_{t+1}(A_{t+1}).$$

Risk-neutral probabilities

- Assume no arbitrage or equivalently that you have a set of positive state prices, and that at each (t, s) a one period risk free security exists.

- Recall that $\{s\} \in F_T$, and set $q(s) = q_T(\{s\})$.

- Let

$$\pi^*(s) = \frac{q(s)}{\rho_T(s)} > 0,$$

- For every $A \subset \mathcal{S}$ let

$$\pi^*(A) = \sum_{s \in A} \pi^*(s).$$

- If $A_{T-1} \in F_{T-1}$, then $q_{T-1}(A_{T-1}) = \bar{r}_T(s) \sum_{s \in A_{T-1}} q(s)$

- $q_{T-1}(A_{T-1}) = \frac{\rho_{T-1}(s)}{\rho_T(s)} \sum_{s \in A_{T-1}} q(s) = \rho_{T-1}(A_{T-1}) \sum_{s \in A_{T-1}} \frac{q(s)}{\rho_T(s)} = \rho_{T-1}(A_{T-1}) \pi^*(A_{T-1})$.

- $\pi^*(A_{T-1}) = \frac{q_{T-1}(A_{T-1})}{\rho_{T-1}(A_{T-1})}$.

- If $A_t \in F_t$ then

$$\pi^*(A_t) = \frac{q_t(A_t)}{\rho_t(A_t)}.$$

- $\pi^*(\mathcal{S}) = \frac{q_0(\mathcal{S})}{\rho_0(\mathcal{S})} = 1$

- π^* is a probability.

- Let $A_t \in F_t$ and $A_{t+1} \subset A_t$, $A_{t+1} \in F_{t+1}$.

- $$\pi^*(A_{t+1}|A_t) \equiv \frac{\pi^*(A_{t+1})}{\pi^*(A_t)}$$

- $$\pi^*(A_{t+1}|A_t) = \frac{q_{t+1}(A_{t+1})}{q_t(A_t)} \bar{r}_{t+1}(A_{t+1}).$$

- $$p^j(A_t) = [\bar{r}_{t+1}(A_{t+1})]^{-1} \times \sum \pi^*(A_{t+1}|A_t)[p^j_{t+1}(A_{t+1}) + x^j_{t+1}(A_{t+1})],$$

  where the sum is over all $A_{t+1} \subset A_t$ and $A_{t+1} \in F_{t+1}$.

- $$p_t^j = \frac{E_t^*[p_{t+1}^j + x_{t+1}^j]}{\bar{r}_{t+1}} \qquad (3)$$

- All securities have same expected returns under π^*.

- $$p_0^j = \sum_t E^*(\rho_t x_t^j).$$

- Prices equal discounted expected payoffs.

- Risk-neutral probabilities.

Black-Scholes model

- $\rho_t = (\bar{r})^{-t}$. If A_t is event in which there are exactly ℓ downs between 0 and t,

$$q_t(A_t) = \left(\frac{u-\bar{r}}{\bar{r}(u-d)}\right)^{\ell} \left(\frac{\bar{r}-d}{\bar{r}(u-d)}\right)^{t-\ell},$$

- $$\pi^*(A_t) = \left(\frac{u-\bar{r}}{u-d}\right)^{\ell} \left(\frac{\bar{r}-d}{u-d}\right)^{t-\ell}.$$

- European call option with strike K: No dividend before T and if ℓ downs occur until T, dividend at T is $(u^{T-\ell}d^{\ell} - K)^{+}$.

- Price of option

$$\sum_{\ell=0}^{T} \binom{T}{\ell} \frac{1}{(\bar{r})^T} (u^{T-\ell}d^{\ell} - K)^{+} \left(\frac{u-\bar{r}}{u-d}\right)^{\ell} \left(\frac{\bar{r}-d}{u-d}\right)^{T-\ell}.$$

A model for interest rates

- $s = (r_0, r_1, \ldots, r_T)$, where r_t is one plus the one period interest rate prevailing at t. Conditional on the interest rate prevailing today, there are two possible interest rates in the next period $i_u > i_d > 0$.

- The short rate volatility at t in state s is:

$$\sigma_t(s) = \frac{\ln\left(\frac{i_u}{i_d}\right)}{2}$$

- $\sigma_t(s)$ equals the standard deviation of the log of the short rate that will prevail tomorrow, assuming that the probability that $i_{t+1} = i_u$ conditional on today's rate is .5.

- Probabilities are the risk-neutral probabilities.

- Example: $i_0 = .04$, the price of a two period bond $p^2 = .925$, and $\sigma_0 = 10\%$.

- $$r_u = r_d e^{2\sigma_0}.$$

- $$p_2 = \left(\frac{1}{1+i_0}\right)\frac{1}{2}\left[\frac{1}{1+i_u} + \frac{1}{1+i_d}\right]$$

- Two equations on two unknowns. Solution $r_d = 0.0356$, $r_u = 0.0435$.

- Suppose $i_{ud} = i_{du}$, and you know the price of a three period bond p^3. Still need to determine 3 rates and you have only one equation.

- One possible solution:

$$\sigma_1(i_u) = \sigma_1(i_d) = \sigma_1, \text{ given.}$$

-

$$\frac{i_{uu}}{i_{ud}} = \frac{i_{du}}{i_{dd}}.$$

-

$$i_{ud} = \sqrt{i_{uu} i_{dd}}.$$

- Works for any T.

Security gains as martingales

- The gains process of security j is the process given by:

$$g_t^j(s) = p_t^j(s) + \sum_{\tau=1}^{t} \frac{\rho_\tau(s)}{\rho_t(s)} x_\tau^j(s)$$

- The discounted gain process for security j is:

$$d_t^j(s) = \rho_t(s) g_t^j(s) = \rho_t(s) p_t^j(s) + \sum_{\tau=1}^{t} \rho_\tau(s) x_\tau^j(s)$$

-

$$\begin{aligned} d_{t+1}^j(s) - d_t^j(s) &= \\ \rho_{t+1}(s)(x_{t+1}^j(s) + p_{t+1}^j(s)) &- \rho_t(s) p_t^j(s). \end{aligned}$$

- Multiplying equation (3) by $\rho_t(s)$, we obtain:

  $$E_t^*[\rho_{t+1}(s)(x_{t+1}^j(s)+p_{t+1}^j(s))] = \rho_t(s)p_t^j(s).$$

- Hence:

  $$E_t^*[d_{t+1}^j(s) - d_t^j(s)] = 0.$$

- $$E_t^*[d_{t+1}^j] = d_t^j.$$

- If $\tau > t$,

  $$E_t^*[d_\tau^j] = d_t^j$$

- d^j is a martingale.

- Also works for portfolios of securities.

Forwards

- A payoff $W(s)$ at time T.

- A forward contract is an agreement struck at $t < T$ to pay an amount F_t, the forward price at T in exchange for W. Nothing else is exchanged.

- $$0 = E_t^*[\rho_T(F_t - W)]$$

- If the discount factor is non-random,

$$F_t = E_t^*[W].$$

  - F_t is a martingale.
  - Short rate of interest is deterministic.

Futures

- A futures price process is a process ϕ_t with $\phi_T = W$. A futures contract has a dividend stream that equals $\phi_t - \phi_{t-1}$, for each $t \geq 1$. The price of a futures contract is always zero.

- $$0 = E_t^*[\phi_{t+1} - \phi_t].$$

- $$\phi_t = E_t^*[\phi_{t+1}].$$

- $$\phi_t = E_t^*[W].$$

- Future price process is always a martingale.

The pricing kernel

- The pricing kernel is a process $k \in \mathcal{M}^p$ such that for any $z \in \mathcal{M}^p$,

$$q(z) = \sum_{t=1}^{T} E[k_t z_t]$$

- If $s \in A_t$ let $k_t(A_t) \equiv k_t(s)$.

- If complete markets,

$$k_t(A_t) = \frac{q_t(A_t)}{\pi(A_t)}$$

- Applying this equation to the strategy "buy security j at t in every $s \in A_t$ and sell it at $t+1$" we obtain:

$$k_t p_t^j = E_t[k_{t+1}(p_{t+1}^j + x_{t+1}^j)].$$

- $$k_t = E_t[k_{t+1} r^j_{t+1}]$$

- $$k_t = \bar{r}_{t+1} E_t[k_{t+1}].$$

- $$E_t[k_{t+1}(x^j_{t+1} + p^j_{t+1} - \bar{r}_{t+1} p^j_t)] = 0.$$

- The process $\tilde{g}^j$ defined as:

  $$\tilde{g}^j_t(s) = g^j_t(s) k_t(s),$$

  is a martingale.

- Also works for portfolios.

Utility maximization

- The consumption process c an adapted process (sometimes assumed non-negative.)

- The utility function u associating to each c a real number.

- A set of tradeable assets and an endowment process w.

-
$$\max_{c,h} u(c), \text{ subject to}$$
$c_0 = w_0 - p_0 \cdot h_0$ and $c_t = w_t + z_t^h$.

- If we have an interior solution then at any state $s \in A_t$

$$p(A_t) = \sum [p(A_{t+1}) + x(A_{t+1})] \frac{\frac{\partial U}{\partial c_{t+1}(A_{t+1})}}{\frac{\partial U}{\partial c_t(A_t)}},$$

where the sum is over all $A_{t+1} \subset A_t$, and $A_{t+1} \in F_{t+1}$.

- If complete markets prevail then the only constraint is $q \cdot c = q \cdot w$.

- $$q_t(A_t) = \frac{\frac{\partial U}{\partial c_t(A_t)}}{\frac{\partial U}{\partial c_0}}.$$