Fin 501: Asset Pricing
Pricing Models and Derivatives
Problem Set 2

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Please put your solution in Martin Oehmke's mailbox in Fisher 001

1.1 Problem

Robinson Crusoe has one fish today. Tomorrow there are two possible events. In the first event, Robinson crusoe will have two fish, and in the second event one fish. The utility function of Mr. Crusoe is given by:

$$u(c_o, c_1^1, c_1^2) = \log(c_o) + \frac{1}{2}\log(c_1^1) + \frac{1}{2}\log(c_1^2),$$

where c_o is his consumption today, c_1^1 his consumption tomorrow in event one, and c_1^2 his consumption tomorrow in event two. Fish can not be stored and Robinson Crusoe lives alone in the island (Friday is gone). As a pastime Mr. Crusoe has established a financial market. In this financial market today, claims to consumption tomorrow are traded. There are two traded claims. One pays $(1, 0)$ the other $(0, 1)$. Mr. Crusoe takes as given the price vector p for these claims and decides how much he wants to buy or sell of the claims. He has decided that his optimal action is to buy exactly zero of each claim, so the market is in equilibrium. Calculate the price vector p.

1.2 Problem

Utility function: Under certainty, any increasing monotone transformation of a utility function is also a utility function representing the same preferences. Under uncertainty, we must restrict this statement to linear transformations if we are to keep the same preference representation. Give a mathematical as well as an economic interpretation for this.

Check it with this example. Assume an initial utility function attributes the following values to 3 outcomes:

$$\begin{array}{ll} B & u(B) = 100 \\ M & u(M) = 10 \\ P & u(P) = 50 \end{array}$$

a) Check that with this initial utility function, the lottery $L = (B, M, 0.50) \succ P$.

b) The proposed transformations are $f(x) = a + bx, a \geq 0, b > 0$, and $g(x) = \ln(x)$. Check that under f, $L \succ P$, but that under g, $P \succ L$.

1.3 Problem

Inter-temporal consumption: Consider a two-date (one-period) economy and an agent with utility function over consumption:

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

at each period. Define the inter-temporal utility function as $V(c_1, c_2) = U(c_1) + U(c_2)$. Show (try it mathematically) that the agent will always prefer a smooth consumption stream to a more variable one with the same mean, that is,

$$\begin{array}{l} U(\bar{c}) + U(\bar{c}) > U(c_1) + U(c_2) \\ \text{if } \bar{c} = \dfrac{c_1 + c_2}{2}, c_1 \neq c_2. \end{array}$$

1.4 Problem

Risk aversion: Consider the following utility functions (defined over wealth Y):

$$\begin{array}{lll} (1)\ U(Y) & = & -\dfrac{1}{Y} \\ (2)\ U(Y) & = & \ln(Y) \\ (3)\ U(Y) & = & -Y^{-\gamma} \\ (4)\ U(Y) & = & -\exp(-\gamma Y) \\ (5)\ U(Y) & = & \dfrac{Y^{\gamma}}{\gamma} \\ (6)\ U(Y) & = & \alpha Y - \beta Y^2 \end{array}$$

a) Check that they are well behaved ($U' > 0, U'' < 0$) or state restrictions on the parameters so that they are (utility functions (1) – (5)). For utility function (6), take positive α and β, and give the range of wealth over which the utility function is well behaved.

b) Compute the absolute and relative risk aversion coefficients.

c) What is the effect of the parameter γ (when relevant)?

d) Classify the functions as increasing/decreasing risk aversion utility functions (both absolute and relative).

1.5 Problem

Certainty equivalent:

$$
\begin{aligned}
(1)\ U(Y) &= -\frac{1}{Y} \\
(2)\ U(Y) &= \ln(Y) \\
(3)\ U(Y) &= \frac{Y^{\gamma}}{\gamma}
\end{aligned}
$$

Consider the lottery $L_1 = (50,000; 10,000; 0.50)$. Determine the lottery $L_2 = (x; 0; 1)$ that makes an agent indifferent to lottery L_1 with utility functions $(1), (2)$, and (3) as defined. For utility function (3), use $\gamma = \{0.25, 0.75\}$. What is the effect of changing the value of γ? Comment on your results using the notions of risk aversion and certainty equivalent.

1.6 Problem

Risk premium: A businesswoman runs a firm worth CHF 100,000. She faces some risk of having a fire that would reduce her net worth according to the following three states, $i = 1, 2, 3$, each with probability $\pi(i)$ (Scenario A).

State	*Worth*	$\pi(i)$
1	1	0.01
2	50,000	0.04
3	100,000	0.95

Of course, in state 3, nothing detrimental happens, and her business retains its value of CHF 100,000.

a) What is the maximum amount she will pay for insurance if she has a logarithmic utility function over final wealth? (Note: The insurance pays CHF 99,999 in the first case; CHF 50,000 in the second; and nothing in the third.)

b) Do the calculations with the following alternative probabilities:

	Scenario B	*Scenario C*
$\pi(1)$	0.01	0.02
$\pi(2)$	0.05	0.04
$\pi(3)$	0.94	0.94

Is the outcome (the comparative change in the premium) a surprise? Why?

1.7 Problem

Risk aversion and portfolio choice: Consider an economy with two types of financial assets: one risk-free and one risky asset. The rate of return offered by the risk-free asset is r_f. The rate of return of the risky asset is $\tilde{r}$. Note that the expected rate of return $E(\tilde{r}) > r_f$.

Agents are risk-averse. Let Y_0 be the initial wealth. The purpose of this exercise is to determine the optimal dollar amount a to be invested in the risky asset as a function of the Arrow-Pratt measure of absolute risk aversion.

The objective of the agents is to maximize the expected utility of terminal wealth:

$$\max_a E(U(Y))$$

where E is the expectation operator, $U(\cdot)$ is the utility function with $U' > 0$ and $U'' < 0$, Y is the wealth at the end of the periond, and a is the dollar amount being invested in the risky asset.

a) Determine the final wealth as a function of a, r_f, and $\tilde{r}$.

b) Compute the f.o.c. (first order condition). Is this a maximum or a minimum?

c) We are interested in determining the sign of da^*/dY_0. Calculate first the total differential of the f.o.c. as a function of a and Y_0. Write the expression for da^*/dY_0. Show that the sign of this expression depends on the sign of its numerator.

d) You know that R_A, the absolute risk aversion coefficient, is equal to $-U''(\cdot)/U'(\cdot)$. What does it mean if $R'_A = dR_A/dY < 0$?

e) Assuming $R'_A < 0$, compare $R_A(Y)$ and $R_A(Y_0(1+r_f))$: Is $R_A(Y) > R_A(Y_0(1+r_f))$ or vice-versa? Don't forget there are two possible cases: $\tilde{r} \geq r_f$ and $\tilde{r} < r_f$.

f) Show that $U''(Y_0(1+r_f) + a(\tilde{r} - r_f))(\tilde{r} - r_f) > -R_A(Y_0(1+r_f)) \times U'(Y_0(1+r_f) + a(\tilde{r} - r_f))(\tilde{r} - r_f)$ for both cases in part e).

g) Finally, compute the expectation of $U''(Y)(\tilde{r} - r_f)$. Using the f.o.c., determine its sign. What can you conclude about the sign of da^*/dY_0? What was the key assumption for the demonstration?