Introduction to Econometrics (4th Edition)

by

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Solutions to Odd-Numbered End-of-Chapter Exercises: Chapter 15

(This version September 18, 2018)

15.1. (a) Since the probability distribution of Y_t is the same as the probability distribution of Y_{t-1} (this is the definition of stationarity), the means (and all other moments) are the same.

(b) $E(Y_t) = \beta_0 + \beta_1 (Y_{t-1}) + E(u_t)$, but $E(u_t) = 0$ and $E(Y_t) = E(Y_{t-1})$.

Thus $E(Y_t) = \beta_0 + \beta_1 E(Y_t)$, and solving for $E(Y_t)$ yields the result.

15.3. (a) To test for a stochastic trend (unit root) in ln(*IP*), the ADF statistic is the *t*-statistic testing the hypothesis that the coefficient on ln(IP_{t-1}) is zero versus the alternative hypothesis that the coefficient on ln(IP_{t-1}) is less than zero. The calculated *t*-statistic is $t = \frac{-0.0070}{0.0037} = -1.89.$ From Table 15.4, the 10% critical value with a time trend is −3.12. Because −1.89 > −3.12, the test does not reject the null hypothesis that ln(*IP*) has a unit autoregressive root at the 10% significance level. That is, the test does not reject the null hypothesis that ln(*IP*) contains a stochastic trend, against the alternative that it is stationary.

(b) The ADF test supports the specification used in Exercise 15.2. The use of first differences in Exercise 15.2 eliminates random walk trend in ln(*IP*).

15.5. (a)

$$
\begin{aligned}
E[(W-c)^2] &= E\{[W-\mu_W)+(\mu_W-c)]^2\} \\
&= E[(W-\mu_W)^2]+2E(W-\mu_W)(\mu_W-c)+(\mu_W-c)^2 \\
&= \sigma_W^2+(\mu_W-c)^2.
\end{aligned}
$$

(b) Using the result in part (a), the conditional mean squared error

$$E[(Y_t-f_{t-1})^2\,|\,Y_{t-1},Y_{t-2},...]=\sigma_{t|t-1}^2+(Y_{t|t-1}-f_{t-1})^2$$

with the conditional variance $\sigma_{t|t-1}^2=E[(Y_t-Y_{t|t-1})^2]$.This equation is minimized when the second term equals zero, or when $f_{t-1}=Y_{t|t-1}$. (An alternative is to use the hint, and notice that the result follows immediately from Appendix 2.2.)

(c) Applying Equation (2.27), we know the error u_t is uncorrelated with u_{t-1} if $E(u_t|u_{t-1}) = 0$. From Equation (15.14) for the AR(p) process, we have

$$u_{t-1}=Y_{t-1}-\beta_0-\beta_1Y_{t-2}-\beta_2Y_{t-3}- \quad -\beta_pY_{t-p-1}=f(Y_{t-1},Y_{t-2},...,Y_{t-p-1}),$$

a function of Y_{t-1} and its lagged values. The assumption $E(u_t|Y_{t-1},Y_{t-2},...)=0$ means that conditional on Y_{t-1} and its lagged values, or any functions of Y_{t-1} and its lagged values, u_t has mean zero. That is,

$$E(u_t|u_{t-1})=E[u_t\,|\,f(Y_{t-1},Y_{t-2},...,Y_{t-p-2})]=0.$$

Thus u_t and u_{t-1} are uncorrelated. A similar argument shows that u_t and u_{t-j} are uncorrelated for all $j \geq 1$. Thus u_t is serially uncorrelated.

15.7. (a) From Exercise (15.1) $E(Y_t) = 2.5 + 0.7E(Y_{t-1}) + E(u_t)$, but $E(Y_t) = E(Y_{t-1})$ (stationarity) and $E(u_t) = 0$, so that $E(Y_t) = 2.5/(1–0.7)$.

Also, because $Y_t = 2.5 + 0.7Y_{t-1} + u_t$, $var(Y_t) = 0.7^2 var(Y_{t-1}) + var(u_t) + 2 \times 0.7 \times cov(Y_{t-1}, u_t)$. But $cov(Y_{t-1}, u_t) = 0$ and $var(Y_t) = var(Y_{t-1})$ (stationarity), so that $var(Y_t) = 9/(1 - 0.7^2) = 17.647$.

(b) The 1st autocovariance is

$$\begin{aligned}\text{cov}(Y_t, Y_{t-1}) &= \text{cov}(2.5 + 0.7Y_{t-1} + u_t, Y_{t-1}) \\ &= 0.7\,\text{var}(Y_{t-1}) + \text{cov}(u_t, Y_{t-1}) \\ &= 0.7\sigma_Y^2 \\ &= 0.7 \times 17.647 = 12.353.\end{aligned}$$

The 2nd autocovariance is

$$\begin{aligned}\text{cov}(Y_t, Y_{t-2}) &= \text{cov}[(1+0.7)2.5 + 0.7^2 Y_{t-2} + u_t + 0.7u_{t-1}, Y_{t-2}] \\ &= 0.7^2\,\text{var}(Y_{t-2}) + \text{cov}(u_t + 0.7u_{t-1}, Y_{t-2}) \\ &= 0.7^2 \sigma_Y^2 \\ &= 0.7^2 \times 17.647 = 8.6471.\end{aligned}$$

(c) The 1st autocorrelation is

$$\text{corr}\,(Y_t, Y_{t-1}) = \frac{\text{cov}(Y_t, Y_{t-1})}{\sqrt{\text{var}(Y_t)\,\text{var}(Y_{t-1})}} = \frac{0.7\sigma_Y^2}{\sigma_Y^2} = 0.7.$$

The 2nd autocorrelation is

$$\text{corr}\,(Y_t, Y_{t-2}) = \frac{\text{cov}(Y_t, Y_{t-2})}{\sqrt{\text{var}(Y_t)\,\text{var}(Y_{t-2})}} = \frac{0.7^2\sigma_Y^2}{\sigma_Y^2} = 0.49.$$

(d) The conditional expectation of Y_{T+1} given Y_T is

$$Y_{T+1/T} = 2.5 + 0.7Y_T = 2.5 + 0.7 \times 102.3 = 74.11.$$

15.9. (a) $E(Y_t) = \beta_0 + E(e_t) + b_1E(e_{t-1}) + \cdots + b_qE(e_{t-q}) = \beta_0$ [because $E(e_t) = 0$ for all values of t].

(b)

$$\begin{aligned}\text{var}(Y_t) &= \text{var}(e_t) + b_1^2\,\text{var}(e_{t-1}) + \cdots + b_q^2\,\text{var}(e_{t-q})\\ &\quad +2b_1\,\text{cov}(e_t, e_{t-1}) + \cdots + 2b_{q-1}b_q\,\text{cov}(e_{t-q+1}, e_{t-q})\\ &= \sigma_e^2(1 + b_1^2 + \cdots + b_q^2)\end{aligned}$$

where the final equality follows from $\text{var}(e_t) = \sigma_e^2$ for all t and $\text{cov}(e_t, e_i) = 0$ for $i \neq t$.

(c) $Y_t = \beta_0 + e_t + b_1e_{t-1} + b_2e_{t-2} + \cdots + b_qe_{t-q}$ and

$Y_{t-j} = \beta_0 + e_{t-j} + b_1e_{t-1-j} + b_2e_{t-2-j} + \cdots + b_qe_{t-q-j}$ and

$\text{cov}(Y_t, Y_{t-j}) = \sum_{k=0}^{q}\sum_{m=0}^{q} b_kb_m\,\text{cov}(e_{t-k}, e_{t-j-m})$, where $b_0 = 1$.

Notice that $\text{cov}(e_{t-k}, e_{t-j-m}) = 0$ for all terms in the sum.

(d) $\text{var}(Y_t) = \sigma_e^2\left(1 + b_1^2\right)$, $\text{cov}(Y_t, Y_{t-1}) = \text{cov}(Y_t, Y_{t+1}) = \sigma_e^2 b_1$, and $\text{cov}(Y_t, Y_{t-j}) = 0$

for $|j| > 1$.

15.11. Write the model as $Y_t - Y_{t-1} = \beta_0 + \beta_1(Y_{t-1} - Y_{t-2}) + u_t$. Rearranging yields $Y_t = \beta_0 + (1+\beta_1)Y_{t-1} - \beta_1 Y_{t-2} + u_t$.

15.13 Recursive substitution yields $Y_t = Y_0 + \sum_{i=1}^{t} u_i = \sum_{i=1}^{t} u_i$.

(a) $E(Y_t) = E\left(\sum_{i=1}^{t} u_i\right) = \sum_{i=1}^{t} E(u_i) = 0.$

$\text{var}(Y_t) = \text{var}\left(\sum_{i=1}^{t} u_i\right) = \sum_{i=1}^{t} \text{var}(u_i) = t\sigma^2$, where the second equality uses $\text{cov}(u_t u_i) = 0$ for $t \neq i$.

(b) $Y_t = \sum_{i=1}^{t} u_i$ and $Y_{t-k} = \sum_{i=1}^{t-k} u_i$, so that $\text{cov}(Y_t, Y_{t-k}) = \min(t, t-k)\sigma^2$.

(c) From (a) the variance of Y_t depends on t, so Y_t is nonstationary. From (b) the $\text{cov}(Y_t, Y_{t-k})$ depends on t, so again Y_t is nonstationary.