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Methods for distance-based judgment aggregation

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Abstract Judgment aggregation theory, which concerns the translation of individual judgments on logical propositions into consistent group judgments, has shown that group consistency generally cannot be guaranteed if each proposition is treated independently from the others. Developing the right method of abandoning *independence* is thus a high-priority goal. However, little work has been done in this area outside of a few simple approaches. To fill the gap, we compare four methods based on distance metrics between judgment sets. The methods generalize the premise-based and sequential priority approaches to judgment aggregation, as well as distance-based preference aggregation. They each guarantee group consistency and implement a range of distinct functions with different properties, broadening the available tools for social choice. A central result is that only one of these methods (not previously considered in the literature) satisfies three attractive properties for all reasonable metrics.

1 Introduction

Judgment aggregation (JA) theory concerns the translation of individual judgments on logically interrelated propositions into a set of consistent group judgments. Its use extends to groups making multiple decisions or establishing supporting reasons for a decision (Pettit 2003; List 2006). JA is thus fundamental to the design of democratic procedures in politics, law, and business, as well as to information aggregation in

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Table 1 The doctrinal paradox

	a	b	$a \wedge b$
Individual 1	T	T	T
Individual 2	T	F	F
Individual 3	F	T	F
Majority	T	T	F

Consistent individuals produce inconsistent majority judgments

artificial intelligence and economics. It has also been shown to be a generalization of preference aggregation theory.

JA theory demonstrates that in most cases we cannot guarantee group consistency if aggregation satisfies *independence*, that is, if it proceeds by aggregating opinions about each proposition separately from the others. The present article investigates abandoning independence by employing distance metrics between sets of judgments. We adopt a flexible framework that encompasses many prominent aggregation methods in JA and preference aggregation theory. In particular, the methods we compare can be seen as generalizing the logic of the Kemeny, Slater, and Dodgson rules used in preference aggregation. As we show, these different distance-based methods generate distinct aggregation rules with different properties. We see our results as synthesizing three related literatures—JA, belief merging, and preference aggregation. Let us consider them in turn.

1.1 Judgment aggregation theory

The simplest JA example, called the "doctrinal paradox," originated in the legal literature (Kornhauser and Sager 1993; Brennan 2001). Consider a panel of judges who want to decide by majority vote the truth of each of three propositions: a, b, and $a \wedge b$. As shown in Table 1, the paradox concerns the fact that if the panel aggregates each proposition separately, the majority's view may be logically inconsistent. List and Pettit (2002) arrived at the first systematic result in JA by showing that no aggregation rule applied to these propositions can satisfy four appealing conditions. Additional JA results (Dietrich 2006; Dietrich and List 2007a; Dokow and Holzman 2008) have demonstrated that for any sufficiently interrelated *agenda* (the set of propositions being judged), no aggregation rule can satisfy independence, *collective rationality* (the group output is always consistent), *non-dictatorship*, and *Pareto* (the group preserves unanimous agreement). Subsequently, JA theory has generalized along several dimensions, particularly concerning the agenda and the conditions placed on the aggregation rule (List 2006; Dietrich and List 2007a). Investigators have considered more general logics (Pauly and van Hees 2006; Dietrich 2007) and have relaxed the condition that individuals must judge every proposition in the agenda (Gärdenfors 2006; Dietrich and List 2008). Others have implicitly considered infinite-sized agendas by extending the consistency requirement to the agenda's logical consequences (Gärdenfors 2006; Pauly and van Hees 2006). Several papers (Nehring 2003; List and Pettit 2004; Dietrich and List 2007a) have shown that preference aggregation theory can be captured in the JA framework and major results (like Arrow's Theorem) have been duplicated and generalized. See List and Puppe (2008) for an overview.

For the most part, JA has retained independence as a driving assumption, particularly in impossibility theorems. Although partly justified by its requirement for strategy-proofness (Dietrich and List 2007b; Nehring and Puppe 2007), independence cannot coexist with a guarantee of collective rationality. This implies that one of JA's most pressing open questions is how best to pursue aggregation without independence. To date, approaches to this task have been limited in scope. Besides the belief merging technique discussed shortly, non-independent JA has largely been limited to the premise-based and sequential priority approaches. In brief, these rules ensure consistency by deciding propositions in a predetermined order, letting previous group judgments act as logical constraints. On the doctrinal paradox agenda, for example, a group can use majority rule to decide a and b first, then let these group judgments decide $a \wedge b$. We prove that one of our aggregation methods generalizes the class of sequential majority rules (Theorem 18). Conversely, all aggregation rules derived from the remaining three methods are distinct from this class (Corollary 25). In general, our goal is a broader set of options for belief and preference aggregation.

1.2 Belief merging

Pigozzi (2006) challenges JA's limitation to independence and sequential priority by pointing to a general method of belief merging in the artificial intelligence literature. Instead of considering each proposition one by one, the approach employs distance metrics to aggregate across the entire set of judgments at once (Konieczny and Pino-Pérez 1998, 1999, 2002). In this way, it becomes straightforward to define a function, which we call a *solution rule*, that transforms any collection of individual judgments into a set of consistent group judgments.

We illustrate the distance-based approach by applying the solution rule that Pigozzi highlights to the doctrinal paradox agenda. Define the *Hamming* distance between any two sets of judgments on $\{a, b, a \wedge b\}$ to be the number of propositions on which they disagree. Thus, if I believe TTT and you believe FTF, the Hamming distance between our judgment sets is 2. Given a collection of individual judgment sets for this agenda, Pigozzi's favored rule sets the group output as the *consistent* judgment set s that minimizes the sum of Hamming distances between s and the individual judgment sets.[1] In this way, we guarantee both group consistency and closeness to individual judgments.

To generalize Pigozzi's approach, we focus here on four distance-based *solution methods*. For each solution method, choosing a different distance metric induces a different solution rule. Schematically: Method + Metric = Rule. In Pigozzi's example, the solution method (which we call *Prototype*) is to choose the consistent judgment set s that minimizes the sum of distances between s and the individual judgment sets. When combined with the Hamming metric, this method produces Pigozzi's favored solution rule. A second method, *Endpoint*, chooses the judgment set closest to the (possibly inconsistent) proposition-wise majority output. The remaining two solution methods,

[1] If the minimum is not unique, Pigozzi (2006, p. 294) recommends using a tie-breaking rule (perhaps based on unequal individual expertise) to reach a single output.

Full and *Output*, have not been considered in the JA literature to our knowledge. In similar ways, they look for the closest profile of individual judgments that yields a consistent proposition-wise majority output, and then take this output.

The resulting generality can be illustrated by visualizing the continuum of aggregation rules generated by Prototype. Starting with Hamming, suppose we gradually alter the metric so that the marginal increase in distance from disagreement on an additional proposition declines with the existing level of disagreement. At the extreme of this process is the *Drastic* metric, defined as 0 if the judgment sets are identical and 1 if they are different.[2] Combining this range of metrics with Prototype yields a continuum of solution rules from Pigozzi's favored rule (using Hamming) to plurality rule (using Drastic), which outputs the judgment set most commonly affirmed by individuals. Similarly, we could increase the distance incurred by disagreement on a particular proposition—this yields a continuum from Pigozzi's rule to an approximation of a sequential majority rule.

The belief merging literature cited by Pigozzi has been limited in three respects. First, it has concentrated on a small number of solution methods, namely: (a) Prototype, (b) a generalization of Prototype that minimizes the sum of the distances raised to a power, and (c) G_{min}, which lexicographically minimizes the maximum distance between any individual and the output. In comparison, we investigate three new solution methods. A second limitation of the literature is that it has largely restricted itself to the Hamming metric, with particular focus on its combination with Prototype, and a few other metrics. In contrast, we consider a broad class of metrics for each solution method. Finally, the literature has not fully developed the substantive characteristics of solution rules, with the exception of work on computational complexity (Konieczny et al. 2004; Everaere et al. 2005) and strategy-proofness (Konieczny and Pino-Pérez 2004). For instance, the Prototype-Hamming combination has not been defended by appeal to any specific and unique characteristics.[3] We consider several attractive properties of solution rules.

1.3 Preference aggregation theory

Our results apply to a large range of agendas, one of which is the preference agenda. Employing distances for aggregation has been explored extensively in preference aggregation theory (Kemeny 1959; Fishburn 1977; Young and Levenglick 1978; Baigent 1987; Nurmi 2004; Klamler 2004, 2005, 2008; Saari 2008), but a central constraint has been the implicit limitation of the employed distance metric. The most prominent distance-based rules—such as Kemeny, Slater, and Dodgson—measure the

[2] In some literatures, the Drastic metric is called the "Trivial metric".

[3] The Prototype-Hamming rule does agree with proposition-wise majority rule when that rule is consistent (Pigozzi 2006), a result that we generalize. However, we provide several rules with this property. For a preference agenda, Prototype-Hamming is equivalent to the Kemeny rule (Eckert and Mitlöhner 2005), which does have special properties, but these do not carry over to general agendas. One unique feature of Prototype-Hamming is that it picks the group output minimizing the total number of disagreements on judgments with the individuals. However, as we argue below, there is no a priori reason to equally weight each disagreement.

distance between preference rankings as the number of binary comparisons on which the rankings disagree. This is equivalent to limiting our solution methods to only the Hamming metric, which is restrictive. For instance, if x and y are considered highly "similar" alternatives, we might want to give small weight to disagreement on them. Consequently, we here consider a broad range of distance metrics, allowing us to generalize several distance-based preference aggregation rules. It will be shown that Output is the only one of our methods satisfying three attractive properties for all reasonable metrics (Theorem 29). We also show that the sets of aggregation rules generated by each of our methods are disjoint (Theorem 30), thereby extending the tools at hand for preference aggregation.

1.4 Goals of the present study

The principal goal of the current paper is to provide a flexible framework for distance-based JA that encompasses and broadens existing approaches to non-independent JA. We consider four solution methods, each giving rise to a specific solution rule in the presence of a distance metric over judgment sets. The four methods are compared and evaluated in terms of their satisfaction of three appealing properties.

One reason we take this highly flexible approach is that the appropriate distance metric (intuitively representing individual dissatisfaction in letting one judgment set substitute for another) may vary between situations and groups. In particular, there is no a priori reason why the Hamming distance should be suitable in every case. In many circumstances, the group will weigh disagreement on some issues more than others. The group might further consider interactions between issues, so that disagreement on one issue is weighted less if there is already disagreement on some other issue. For instance, differing on $\{a, b \wedge c\}$ might be judged more consequential than differing on $\{a, a \wedge b\}$.

A second rationale for flexibility is that merely identifying a solution rule and thereby guaranteeing group consistency does not resolve the JA problem. Consider a constant-valued solution rule that always outputs an identical consistent judgment set. Intuitively, this rule's lack of responsiveness to individual opinion makes it inadequate for decision-making, despite its guarantee of a consistent output. The merit of a given solution rule thus depends on its satisfaction of desirable properties beyond guaranteed group consistency. In what follows, we compare the properties of plausible solution rules and demonstrate that many of them lack attractive characteristics. Again, it will be seen that only one of the solution methods we investigate (Output) satisfies our three properties for all reasonable metrics (Theorem 23). We see our work as a first step toward an axiomatic characterization of a unique choice among all solution methods.

The three properties that frame our discussion do not, of course, exhaust the potential virtues of solution rules. We focus on these three because of their intuitive attractiveness, the clear verdict they yield on our four methods, and their usefulness for proving disjointness. Alternative criteria for solution rules are suggested in the Discussion section. Likewise, our analysis is limited to the four solution methods defined below (with others invoked in the Discussion). Here we observe that two of the four (Prototype and Endpoint) are present in the existing literature, and the remaining two (Full and Output) are natural techniques for reaching group decisions with minimal deformation of individual judgments.

We proceed as follows: Sect. 2 defines the basic elements of JA and belief merging and formally introduces the four solution methods. It also defines the metric classes, solution rule characteristics, and range of agendas employed in our results. Sect. 3 reports general theorems. These yield results in Sects. 4 and 5 on specific agendas, such as the preference agenda and truth-functional agendas. Proofs are relegated to an Appendix. Discussion of our results, as well as suggestions for further solution methods and criteria of adequacy, occupy Sect. 6.

2 Definitions

2.1 Basic elements

An *agenda* A is a finite set of logical propositions. A *judgment set* (for A) is a collection of *judgments* evaluating each proposition in A as either True or False.[4] Let A° be the collection of (not necessarily logically consistent) judgment sets, and $A^\bullet$ the subset of logically consistent members of A°. For $p \in A^\circ$ and $\alpha \in A$, let $p^{\neg\alpha}$ be the judgment set that agrees with p on $A \setminus \{\alpha\}$ but disagrees on α. Similarly, $p^{\neg X}$ disagrees on each element in $X \subseteq A$.

For finite, odd $n \geq 1$, a *profile* (for A) is a vector of length n over A°, representing n individuals' judgments on the propositions in A. Let $\mathcal{P}^\circ_A$ be the set of profiles of any odd length, and $\mathcal{P}^\bullet_A$ the subset of these profiles all of whose members are consistent. Given $\mathsf{P} \in \mathcal{P}^\circ_A$, let $M(\mathsf{P})$ be the judgment set that results from majority voting on each proposition in A. Call P *majority consistent* if $M(\mathsf{P}) \in A^\bullet$.

We employ distance metrics to compare judgment sets. Intuitively, the distance expresses an individual's dissatisfaction in letting one judgment set replace another. Formally, by a *distance metric* for agenda A is meant any binary operation d such that:

$$\begin{aligned} &\text{(a)} \quad \mathsf{d} : A^\circ \times A^\circ \longrightarrow \mathbb{R} \\ &\text{(b)} \quad \mathsf{d}(p, q) = 0 \Leftrightarrow p = q \\ &\text{(c)} \quad \mathsf{d}(p, q) = \mathsf{d}(q, p) \\ &\text{(d)} \quad \mathsf{d}(p, r) \leq \mathsf{d}(p, q) + \mathsf{d}(q, r) \end{aligned} \tag{1}$$

We extend d to profiles in the obvious way: Given $n \geq 1$ and profiles $\mathsf{P} = \{p_i\}_{i \leq n}$, $\mathsf{Q} = \{q_i\}_{i \leq n} \in \mathcal{P}^\circ_A$, we let

$$\mathsf{d}(\mathsf{P}, \mathsf{Q}) = \sum_{i \leq n} \mathsf{d}(p_i, q_i).$$

[4] This assumes all judgment sets are *complete*, meaning that they include judgments for every proposition in the agenda. For relaxations of completeness, see Gärdenfors (2006) and Dietrich and List (2008).

2.2 Four solution methods

Definition 2 A *solution rule* is a function from $\mathcal{P}^{\bullet}_{A}$ to $pow(A^{\bullet}) \setminus \{\emptyset\}$.

where $pow()$ is the power set. A solution rule assigns one or more consistent judgment sets to any profile whose members are consistent (we let the output consist of multiple judgment sets to allow for ties). A solution rule may violate independence, implying that the group judgment on a proposition need not depend solely on the individuals' judgments on that proposition.

Definition 3 A *solution method* is a mapping that converts any given distance metric into a solution rule.

The four solution methods we investigate are denoted:

Full Output Endpoint Prototype

Applied to a distance metric d, the four methods yield the solution rules:

Full_{d} $\text{Output}_{\mathsf{d}}$ $\text{Endpoint}_{\mathsf{d}}$ $\text{Prototype}_{\mathsf{d}}$

We now define these methods. For a given distance metric d and agenda A, the following are defined for all odd $n \geq 1$ and $\mathsf{P} = \{p_i\}_{i \leq n} \in \mathcal{P}^{\bullet}_{A}$:

Definition 4 $\text{Full}_{\mathsf{d}}(\mathsf{P})$ is the collection of $M(\mathsf{Q}) \in A^{\bullet}$ such that Q minimizes $\mathsf{d}(\mathsf{P}, \mathsf{Q})$ over all majority consistent profiles $\mathsf{Q} \in \mathcal{P}^{\bullet}_{A}$ of length n.

Definition 5 $\text{Output}_{\mathsf{d}}(\mathsf{P})$ is the collection of $M(\mathsf{Q}) \in A^{\bullet}$ such that Q minimizes $\mathsf{d}(\mathsf{P}, \mathsf{Q})$ over all majority consistent profiles $\mathsf{Q} \in \mathcal{P}^{\circ}_{A}$ of length n. (Recall that members of $\mathcal{P}^{\circ}_{A}$ can include inconsistent judgment sets.)

Definition 6 $\text{Endpoint}_{\mathsf{d}}(\mathsf{P})$ is the collection of $q \in A^{\bullet}$ that minimize $\mathsf{d}(M(\mathsf{P}), q)$ over all $q \in A^{\bullet}$.

Definition 7 $\text{Prototype}_{\mathsf{d}}(\mathsf{P})$ is the collection of $q \in A^{\bullet}$ that minimize $\sum_{i \leq n} \mathsf{d}(p_i, q)$ over all $q \in A^{\bullet}$.

Appendix 1 illustrates the operation of each of the four methods. Although the plausibility of each of these solution methods is hopefully obvious, some supporting intuition is worth mentioning. Full, Output, and Endpoint all rely on majority rule for their definition. As we will show, Prototype is also intimately related to majority rule for a broad class of metrics. We give a central place to majority rule based on its democratic credentials as the natural choice for aggregation, as justified by axiomatic results like May's Theorem (May 1952). Although we need to abandon independence (and thus proposition-wise majority rule) to secure group consistency, an appropriate goal is to design aggregation to conform as closely as possible to this majoritarian ideal.

Full and Output seek to change individual judgment sets as little as possible to reach a majority consistent profile. The output is then the result of proposition-wise majority rule on the altered profile. These solution methods can be pictured as reflecting a process of belief revision by the individuals, perhaps as part of a deliberative process or negotiation. To our knowledge, these methods have not been investigated in the literature, although Konieczny (2004) models belief merging as a game that revises the "weakest" beliefs until a consistent output is reached. The distinction between Full and Output is that Output allows us to change the individual judgment sets to inconsistent judgment sets in order to reach a majority consistent profile. Full requires us to substitute only consistent judgment sets. This license for Output perhaps strains the interpretation of the method in terms of rational group deliberation, but surprisingly, the results to follow will offer strong reasons to favor Output.

Endpoint is the simplest solution method. It considers only the (possibly inconsistent) judgment set that results from proposition-wise majority rule, and then chooses the closest consistent judgment set(s). The informational poverty of this method cuts both ways. On the one hand, it is easy to apply, since it requires knowing only the aggregated result. On the other hand, it ignores information about individual opinions and the sizes of the majorities. As formalized below, Endpoint can therefore not be expected to yield distinct results in the face of different profiles with the same majority outcomes.

As discussed in Sect. 1, Prototype has received the most attention in the belief merging literature and in Pigozzi (2006). Informally, Prototype chooses the judgment set that minimizes total individual dissatisfaction (defined by the distance metric). It has a clear utilitarian quality, achieving global coherence with sensitivity to individual opinion. A related interpretation is that Prototype minimizes the distance from universal agreement on a chosen judgment set.

2.3 Classes of metrics

If p, q are judgment sets defined on A, let $\mathit{diff}(p, q)$ denote the agenda items on which they disagree.

Definition 8 A distance metric d is *normal* if, for all judgment sets p, q, r, s, $\mathit{diff}(p, q) \subseteq \mathit{diff}(r, s)$ implies $\mathsf{d}(p, q) \leq \mathsf{d}(r, s)$.

Normal metrics satisfy a straightforward non-decreasing property as disagreement grows. Since the normal distance between judgment sets depends only on the differing judgments,[5] we can associate a function $\delta : pow(A) \longrightarrow \mathbb{R}$ with any normal metric d such that, for all judgment sets p, q, $\mathsf{d}(p, q) = \delta(\mathit{diff}(p, q))$. Where context allows, we leave implicit the dependence of δ on d. Also, we write $\delta(a)$ in place of $\delta(\{a\})$. We limit our study to normal metrics, but also make use of two subclasses:

Definition 9 A normal distance metric d is *increasing* if for all nonempty, disjoint $X, Y \subseteq A$, $\delta(X \cup Y) > \delta(X)$.

Definition 10 A normal distance metric d is *separable* if for all judgment sets p, q, $\mathsf{d}(p, q) = \sum\{\delta(\alpha) : \alpha \in \mathit{diff}(p, q)\}$.

[5] This follows because $\mathit{diff}(p, q) = \mathit{diff}(r, s)$ implies $\mathsf{d}(p, q) = \mathsf{d}(r, s)$.

Increasing metrics define a strictly increasing distance as the disagreement between judgment sets grows. A non-increasing (but normal) metric is illustrated by the Drastic metric defined in Sect. 1. For separable metrics, the distance between judgment sets is the sum of the distances corresponding to the individual agenda elements on which they disagree. A commonly used separable metric is the Hamming metric, for which $\delta(\alpha) = 1$ for all $\alpha \in A$. Thus, $\text{Ham}(p, q) = |\{\alpha : \alpha \in \mathit{diff}(p, q)\}|$.

Let $\mathcal{N}, \mathcal{I}, \mathcal{S}$ be the classes of all normal, increasing, and separable metrics, respectively. Then we have $\mathcal{S} \subseteq \mathcal{I} \subseteq \mathcal{N}$.

Definition 11 Given a class $\mathcal{D}$ of distance metrics and solution method f, we define the set of solution rules

$$f_{\mathcal{D}} = \{f_{\mathsf{d}} : \mathsf{d} \in \mathcal{D}\}.$$

2.4 Characteristics of solution rules

Let agenda A, normal distance metric d, and solution method f be given.

Definition 12 For odd n, solution rule f_{d} is *supermajority responsive for size n (SR$_n$)* if and only if there exist profiles $\mathsf{P}, \mathsf{Q} \in \mathcal{P}_A^\bullet$ of length n such that $M(\mathsf{P}) = M(\mathsf{Q})$, but $f_{\mathsf{d}}(\mathsf{P}) \neq f_{\mathsf{d}}(\mathsf{Q})$. f_{d} is *supermajority responsive (SR)* if and only if it is SR$_n$ for some size n.

Definition 13 Solution rule f_{d} satisfies *preservation* if and only if for all majority consistent $\mathsf{P} \in \mathcal{P}_A^\bullet$, $f_{\mathsf{d}}(\mathsf{P}) = M(\mathsf{P})$.[6]

Definition 14 Solution rule f_{d} is *sensible on* d if and only if for all profiles $\mathsf{P} \in \mathcal{P}_A^\bullet$ and $\alpha \in A$, $f_{\mathsf{d}}(\mathsf{P}) = M(\mathsf{P})^{\neg\alpha}$ as long as $M(\mathsf{P})^{\neg\alpha}$ is consistent and the following conditions hold:

(a) $\delta(\alpha) < \mathsf{d}(M(\mathsf{P}), r)$ for all $r \in A^\bullet$ (with $r \neq M(\mathsf{P})^{\neg\alpha}$).
(b) For all $\beta \in A$ (with $\beta \neq \alpha$), $|\{p_i \in \mathsf{P} : p_i \text{ and } M(\mathsf{P}) \text{ agree on } \alpha\}| < |\{p_i \in \mathsf{P} : p_i \text{ and } M(\mathsf{P}) \text{ agree on } \beta\}|$.

When majority rule leads to inconsistency, it becomes necessary to choose which group judgments will be altered. Supermajority responsiveness (SR) requires that the output depends on more than $M(\mathsf{P})$, implying that some consideration is given to the profile at the individual level. For instance, the number of individuals in favor of a judgment may factor into the output.

Preservation requires that when proposition-wise majority rule leads to a consistent group judgment set, the output is this judgment set. Requiring preservation recognizes majority rule as an ideal which is to be altered only in the event of inconsistency.

Finally, sensibility (on a metric d) provides a plausible set of criteria for when a judgment set should be chosen. It requires that if a consistent judgment set q differs from proposition-wise majority rule on only one proposition and meets two conditions,

6 As a convention, if $f_{\mathsf{d}}(\mathsf{P})$ is a single element q, we write $f_{\mathsf{d}}(\mathsf{P}) = q$ instead of $\{q\}$.

then q is the output. The two conditions are (a) q is strictly closest to the majority rule judgment set over all consistent judgment sets, and (b) the majority margin for the proposition that is changed in q is strictly smallest among all propositions. The conditions together imply that q can be reached with a minimal number of changes to the profile, each of minimal distance.

2.5 Connected agendas

Although many of our results apply to all agendas, we derive further results on a very general set of *connected* agendas, which includes the preference agenda, truth-functional agendas, and several others featured in the JA and social choice literature.

Definition 15 A subagenda $X \subseteq A$ is *minimally complex* if $|X| \geq 3$ and there exists an inconsistent judgment set p such that for any $a \in X$, $p^{\neg a}$ is consistent. An agenda is *connected* if it contains a minimally complex subagenda.

For example, the doctrinal paradox agenda $\{a, b, a \wedge b\}$ is itself minimally complex since the judgment set $\{T, T, F\}$ is inconsistent, but changing any one judgment makes it consistent. The agenda is thus connected. For the preference agenda, Condorcet's Paradox is evidence of connectedness. Note that connectedness is only slightly more restrictive than the widely used agenda condition of "minimal connectedness" (e.g., Dietrich 2007; List and Puppe 2008).[7] To see the wide applicability of connected agendas, note the following result (derived from Nehring and Puppe (2007) and discussed in List and Puppe (2008)) that all agendas of interest "contain" a connected agenda.

Proposition 16 (Nehring and Puppe 2007) *If majority rule on an agenda A can lead to group inconsistency, then there exists a subagenda $X \subseteq A$ such that X is connected when specified as its own agenda.*

3 Theorems for general and connected agendas

3.1 Generalizing the sequential majority approach

As a preliminary, we provide the promised result that Endpoint generalizes the premise-based and sequential majority approaches to JA, which are the most commonly recommended ways of abandoning independence (List and Pettit 2006; List 2006; Mongin 2008). The other three solution methods generalize these approaches on profiles up to a given size.

Definition 17 A *sequential majority rule f* specifies an ordering of the agenda elements $(a_1, a_2, \ldots, a_k)$ and defines the social judgments in the following way: a_1 is

[7] In addition to one other property that we can pass over, minimal connectedness assumes A contains a subagenda X (with $|X| \geq 3$) such that X is minimally complex when specified as its own agenda (i.e., not considering $A \setminus X$). An example in which this condition is satisfied, but connectedness is not, is illustrated by $A = \{a, b, a \wedge b, \neg(a \wedge b)\}$. Then $X = \{a, b, a \wedge b\}$ is minimally complex when specified as its own agenda, but A contains no minimally complex subagenda.

determined by majority rule. If a truth value for a_i is entailed by the judgments for $a_1, \ldots, a_{i-1}$, then a_i takes this value. Otherwise, it is determined by majority rule.[8]

Theorem 18 *For any sequential majority rule f and odd n, there exists a normal metric d such that:*

(a) $\text{Endpoint}_{\mathsf{d}} = f$ *for all profiles.*
(b) $\text{Full}_{\mathsf{d}} = \text{Output}_{\mathsf{d}} = \text{Prototype}_{\mathsf{d}} = f$ *for profiles up to size n.*

Since the equivalence holds for Endpoint on any profile size, it follows that the sequential majority rules form a subset of Endpoint's solution rules. The converse, however, does not hold—there exist solution rules derived from Endpoint that are not equivalent to sequential majority rules.

3.2 Characteristics

We now compare the four solution methods across various classes of metrics, concentrating on two modes of comparison. First, we demonstrate which solution methods and metrics induce solution rules satisfying the characteristics of Sect. 2.4. Second, in the following subsection, we show that for broad classes of metrics, the sets of solution rules derived from different solution methods are disjoint. The link between the two kinds of comparisons is captured by the following remark.

Remark 19 Consider distance metric classes $\mathcal{C}$ and $\mathcal{D}$ and solution methods f, g. If f_{c} satisfies either SR or preservation for all $\mathsf{c} \in \mathcal{C}$ and g_{d} fails that characteristic for all $\mathsf{d} \in \mathcal{D}$, then $f_{\mathcal{C}}$ and $g_{\mathcal{D}}$ are disjoint.

Unless noted otherwise, the following results apply to all agendas. Moreover, all assumptions made on agendas are satisfied by at least one of the major agendas of interest, such as the preference agenda and truth-functional agendas.

SR Endpoint fails SR_n by design, since it depends only on the proposition-wise majority output. The other three solution methods, however, satisfy SR_n for all metrics with comparable distances incurred by different elements. This comparability allows the margin of victory to matter for whether a proposition is endorsed in the output. The three methods besides Endpoint thus satisfy SR for essentially all metrics, ensuring that they take into account judgments at the individual level. These points are captured in the following proposition.

Proposition 20 *The following hold for all connected agendas:*

(a) *For all metrics d, $\text{Endpoint}_{\mathsf{d}}$ fails SR_n for all n.*

[8] In principle, the aggregation rule for each proposition could be anything, but we limit our comparison to sequential *majority* rules. The premise-based approach is the special case of sequential majority that prioritizes a logically independent subset of propositions (the premises) whose truth values imply those of the remaining members of the agenda. Mongin (2008) recommends that the prioritized set include all atomic propositions.

(b) *For $m > 0$, let $\mathcal{D}(m)$ be the class of normal metrics for which $\delta(\alpha) < m\delta(\beta)$ for all $\alpha, \beta \in A$. Let odd $n \geq 5$ be given.*
   i. *For all* $\mathsf{d} \in \mathcal{D}(\frac{n-1}{2})$, $\mathrm{Output}_{\mathsf{d}}$ *satisfies* SR_n.
   ii. *For all* $\mathsf{d} \in \mathcal{D}(\frac{n-1}{4})$, $\mathrm{Full}_{\mathsf{d}}$ *satisfies* SR_n.
   iii. *For all* $\mathsf{d} \in \mathcal{D}(n-2) \cap \mathcal{S}$, $\mathrm{Prototype}_{\mathsf{d}}$ *satisfies* SR_n.

(c) *For all* $\mathsf{d} \in \mathcal{N}$, $\mathrm{Full}_{\mathsf{d}}$ *and* $\mathrm{Output}_{\mathsf{d}}$ *satisfy SR.*

(d) *For all* $\mathsf{d} \in \mathcal{I}$, $\mathrm{Prototype}_{\mathsf{d}}$ *satisfies SR.*

Preservation By design, solution rules derived from Full, Output, and Endpoint guarantee preservation since a majority consistent profile requires no change. It has been observed that $\mathrm{Prototype}_{\mathrm{Ham}}$ also coincides with majority rule when this is consistent (Pigozzi 2006). We generalize this to all separable metrics. We also demonstrate in the following proposition the surprising result that Prototype may fail preservation given a specific agenda condition.

Proposition 21 (a) *For all metrics* d, $\mathrm{Full}_{\mathsf{d}}$, $\mathrm{Output}_{\mathsf{d}}$, *and* $\mathrm{Endpoint}_{\mathsf{d}}$ *satisfy preservation. For all* $\mathsf{d} \in \mathcal{S}$, $\mathrm{Prototype}_{\mathsf{d}}$ *satisfies preservation.*

(b) *For* $\mathsf{d} \in \mathcal{N}$, *Prototype*$_{\mathsf{d}}$ *satisfies preservation only if* $\delta(Y \cup Z) = \delta(Y) + \delta(Z)$ *for all disjoint* $Y, Z \subseteq A$ *such that there exist* $q, q^{\neg Y}, q^{\neg Z} \in A^{\bullet}$.

The second part of this proposition indicates that $\mathrm{Prototype}_{\mathsf{d}}$ satisfies preservation only if d is, in a sense, sufficiently close to separable. Depending on the richness of the agenda, Prototype will typically fail preservation for non-separable metrics.

Sensibility Unlike Output or Endpoint, Full fails sensibility on some connected agendas, as shown by the following proposition. The reason is that Full is affected by logical constraints at the individual level, leading to (arguably) sub-optimal aggregation from a group perspective.

Proposition 22 (a) *For all* $\mathsf{d} \in \mathcal{N}$, $\mathrm{Output}_{\mathsf{d}}$ *and* $\mathrm{Endpoint}_{\mathsf{d}}$ *satisfy sensibility on* d. *For all* $\mathsf{d} \in \mathcal{S}$, $\mathrm{Prototype}_{\mathsf{d}}$ *satisfies sensibility on* d.

(b) *For some connected agendas and some* $\mathsf{d} \in \mathcal{I}$, *Full*$_{\mathsf{d}}$ *fails sensibility on* d.

These three propositions together establish that only Output satisfies all three characteristics for all normal metrics and all connected agendas.

Theorem 23 *For all connected agendas and all* $\mathsf{d} \in \mathcal{N}$, *Output*$_{\mathsf{d}}$ *satisfies SR, preservation, and sensibility on* d. *There exist connected agendas such that for each* $f \in \{\mathrm{Full}, \mathrm{Endpoint}, \mathrm{Prototype}\}$, f_{e} *fails one of these characteristics for some* $\mathsf{e} \in \mathcal{I}$.

3.3 Disjointness

We now consider the extent to which solution methods define distinct solution rules, which gives an idea of the richness of distance-based aggregation. The preceding theorems lead directly to the following result, which divides Endpoint from the remaining three solution methods.

Theorem 24 *For connected agendas:* $\text{Endpoint}_{\mathcal{N}}$ *is disjoint from* $\text{Full}_{\mathcal{N}}$*,* $\text{Output}_{\mathcal{N}}$*, and* $\text{Prototype}_{\mathcal{I}}$*.*

The theorem is remarkable for its generality. No matter what normal metric is chosen, the solution rule induced by Endpoint is distinct from *every* solution rule induced by Full and *any* normal metric. Similar remarks apply to Output and Prototype. Theorem 24 further implies the following.

Corollary 25 *For connected agendas:* $\text{Full}_{\mathcal{N}}$*,* $\text{Output}_{\mathcal{N}}$*, and* $\text{Prototype}_{\mathcal{I}}$ *are disjoint from the set of sequential majority rules.*

Once again, we emphasize the generality of this result. No matter what normal metric is chosen for Full, the resulting solution rule is distinct from *every* sequential majority rule (and similarly for Output and Prototype). Corollary 25 ensures that distance-based JA extends well beyond the sequential majority approach.

The following proposition compares Output and Prototype, demonstrating that they coincide only for metrics satisfying a knife-edge criterion.

Proposition 26 *Given* $\mathsf{d} \in \mathcal{S}$ *and* $\mathsf{e} \in \mathcal{N}$*,* $\text{Prototype}_{\mathsf{d}} = \text{Output}_{\mathsf{e}}$ *only if the following condition is satisfied: For every minimally complex subagenda* X *and every* $\alpha, \beta \in X$*,* $\delta_{\mathsf{d}}(\alpha) = \delta_{\mathsf{d}}(\beta)$ *and* $\delta_{\mathsf{e}}(\alpha) = \delta_{\mathsf{e}}(\beta)$*.*

Because disjointness results for Full depend on the logical structure of the agenda, we consider them in the context of specific agendas.

4 Preference agendas

The general distance-based JA we have developed applies immediately to many agendas in the social choice literature. By far the most prominent is the *preference agenda*, in which individuals' strict linear orderings over $k \geq 3$ alternatives are aggregated into a group ordering. It is straightforward to capture preference orderings in a JA framework.[9] Let xPy indicate the proposition "x is strictly preferred to y." We stipulate that $xPy \Rightarrow \neg yPx$, hence for each pair of alternatives x and y, we assume either $xPy \in A$ or $yPx \in A$. The only requirement for consistency is *transitivity*, meaning $xPy \wedge yPz \Rightarrow xPz$ for general x, y, z. In this framework, JA's independence condition is equivalent to Arrow's independence of irrelevant alternatives. The following remark ensures that the results of Sect. 3 apply to the preference agenda:

Remark 27 All preference agendas are connected, and every minimally complex subagenda is of the form $\{xPy, yPz, xPz\}$.

There is ample room for cross-fertilization between distance-based JA and distance-based preference aggregation theory. Any solution rule can be applied to a preference agenda to derive consistent group preferences, substantially generalizing

9 For full expositions of preference aggregation within a JA framework and derivations of central preference aggregation results (like Arrow's Theorem), see Nehring (2003), List and Pettit (2004), and Dietrich and List (2007a).

existing theory. As mentioned before, most of this literature gives equal weight to the difference on each pair of alternatives, which is equivalent to limiting our study to the Hamming metric. In fact, it has been shown that $\text{Prototype}_{\text{Ham}}$ is equivalent to the Kemeny rule (Eckert and Mitlöhner 2005), and it is also evident that $\text{Endpoint}_{\text{Ham}}$ generates the Slater rule.[10] Full_{Ham} and $\text{Output}_{\text{Ham}}$ are similar in spirit, but not equivalent, to the Dodgson rule.[11] However, Hamming may be inappropriate if, for example, it is more acceptable to reverse similar alternatives compared to disparate ones. We view our approach as generalizing the logic of Kemeny, Slater, and Dodgson to capture such intuitions.

A central result (Young and Levenglick 1978) is that Kemeny is the only rule satisfying three attractive properties, one of which is a stronger form of preservation.[12] The following complementary result shows that preservation alone considerably restricts the space of Prototype's metrics.

Theorem 28 *For any preference agenda and* $\mathsf{d} \in \mathcal{N}$*, Prototype*$_{\mathsf{d}}$ *satisfies preservation if and only if* $\mathsf{d} \in \mathcal{S}$.

If we add to preservation a second assumption of Young and Levenglick (1978), neutrality between alternatives, then d must equal Ham. Hence, these two properties alone can isolate Kemeny among all of Prototype's solution rules.

We are interested, however, in identifying a solution method with attractive properties no matter what metric is deemed appropriate. Extending the conclusion of Sect. 3, the following theorem gives some support for using Output on the preference agenda.

Theorem 29 *For any preference agenda and* $\mathsf{d} \in \mathcal{N}$*,* $\text{Output}_{\mathsf{d}}$ *satisfies SR, preservation, and sensibility on* d*. Given a preference agenda with* $k \geq 5$ *alternatives, for each* $f \in \{\text{Full}, \text{Endpoint}, \text{Prototype}\}$*,* f_{e} *fails one of these characteristics for some* $\mathsf{e} \in \mathcal{I}$.

Pairwise comparisons of prominent rules like Kemeny, Slater, and Dodgson take up a great deal of the distance-based preference aggregation literature (Saari and Merlin 2000; Nurmi 2004; Klamler 2004, 2005). Results show not only that the rules are distinct, but often that the winning alternative under one rule can appear at any point in the ranking of another. The following theorem adds to these distinctness results by showing that virtually *all* aggregation rules generated by our four solution methods are distinct from one another.

10 Speaking informally, Kemeny chooses the consistent group ordering minimizing the total number of "flips" in preferences when compared to the individual orderings. Slater chooses the consistent ordering minimizing the flips from the (possibly inconsistent) ordering under majority rule.

11 Dodgson ranks each alternative by the number of flips within individual orderings necessary to make it a *Condorcet winner* (a majority winner against every other alternative) (Klamler (2004) gives a slightly different version). Full_{Ham} and $\text{Output}_{\text{Ham}}$ instead look for the minimal number of flips within individual orderings to get a majority agreement on *every* ranking in a group ordering, not just for the top-ranked alternative. To see that they are distinct, take the profile with 2 orderings of xyz (meaning x is preferred to y is preferred to z), 2 of zxy, and 1 of yzx. Dodgson produces a tie between xzy and zxy, whereas Full_{Ham} and $\text{Ouput}_{\text{Ham}}$ produce a tie between xyz and zxy.

12 The properties are (a) *Condorcet* (if a majority favors x over y, then y does not immediately precede x in the group ranking), which implies, but is not implied by, preservation, (b) *neutrality* between different alternatives, and (c) *consistency* (if the output of two subgroups is the same ordering, then the output of the union of the subgroups is this ordering).

Theorem 30 *For a preference agenda with $k \geq 5$ alternatives,* $\text{Full}_{\mathcal{I}}$, $\text{Output}_{\mathcal{N}}$, $\text{Endpoint}_{\mathcal{N}}$*, and* $\text{Prototype}_{\mathcal{N}}$ *are mutually disjoint.*

The proof also establishes the following: if $k \geq 4$, $\text{Prototype}_{\mathcal{N}}$ is disjoint from $\text{Output}_{\mathcal{N}}$, $\text{Full}_{\mathcal{N}}$, and $\text{Endpoint}_{\mathcal{N}}$. If $k = 3$, the only intersection between $\text{Prototype}_{\mathcal{N}}$ and either $\text{Full}_{\mathcal{N}}$ or $\text{Output}_{\mathcal{N}}$ is $\text{Prototype}_{\text{Ham}}$. It is also straightforward to prove that there exist an infinite number of Prototype solution rules which are distinct from one another, and the same applies to Full and Output. Hence, Theorem 30 establishes in one fell swoop an infinite number of pairwise comparisons among distance-based preference aggregation rules.

5 Other agendas

To illustrate the versatility of our approach, we now apply our results to other prominent agendas. Each of the following agendas is connected, hence the results of Sect. 3 carry over. For instance, applying Corollary 25, the class of sequential majority rules is disjoint from $\text{Full}_{\mathcal{N}}$, $\text{Output}_{\mathcal{N}}$, and $\text{Prototype}_{\mathcal{I}}$ for each agenda discussed below.

Truth-functional Truth-functional agendas model a decision-making situation in which the judgment on a conclusion is determined by judgments on several logically independent premises (Dokow and Holzman 2008; Nehring and Puppe 2008; Miller 2008).[13] The doctrinal paradox agenda is a simple example. The truth-functional structure also has wide applicability to real-world decisions. In Nehring and Puppe (2008), the choice to hire a job candidate is based on several independent evaluations. In Miller (2008), opinions about policies determine a citizen's vote for a political representative.

Definition 31 An agenda A is *truth-functional* if and only if it consists of $k + 1 \geq 3$ propositions $\{a_1, a_2, \ldots, a_k, b\}$ such that the following are satisfied:

(a) The *premises* $\{a_1, a_2, \ldots, a_k\}$ are logically independent.
(b) Any truth values for $\{a_1, a_2, \ldots, a_k\}$ determine a value for the *conclusion* b (i.e., b is a Boolean function of the premises).
(c) There does not exist a proposition $a \in A$ such that for every $p \in A^\bullet$, we also have $p^{\neg a} \in A^\bullet$ (i.e., every proposition matters).

In case it is not obvious, we prove the following:

Proposition 32 *All truth-functional agendas are connected.*

Again, Output stands as the only solution method that satisfies all three properties for all metrics.

[13] Nehring and Puppe (2008) allow interconnections among the premises. Dokow and Holzman (2008) allow the conclusion to form a multi-proposition set.

Proposition 33 *For any truth-functional agenda and all* $\mathsf{d} \in \mathcal{N}$*,* $\mathrm{Output}_{\mathsf{d}}$ *satisfies SR, preservation, and sensibility on* d*. Given any truth-functional agenda, for each* $f \in \{\mathrm{Full}, \mathrm{Endpoint}, \mathrm{Prototype}\}$*,* f_{e} *fails one of these characteristics for some* $\mathsf{e} \in \mathcal{I}$*.*

It is also straightforward to derive tighter results on truth-functional agendas that are *monotonic* in the following sense: If the conclusion is held to be true, then switching any premise to true does not imply the conclusion is false. On a monotonic truth-functional agenda, we again find that Prototype satisfies preservation only for separable metrics. In addition, with the possible exception of $\mathrm{Prototype}_{\mathrm{Ham}}$, $\mathrm{Prototype}_{\mathcal{I}}$ is disjoint from $\mathrm{Output}_{\mathcal{N}}$, $\mathrm{Full}_{\mathcal{I}}$, and $\mathrm{Endpoint}_{\mathcal{N}}$.

Many-worlds and identification The "many-worlds agenda" was put into a JA framework in List (2008), based on work in Kasher and Rubinstein (1997). Consistency for this agenda requires that we judge some, but not all, states of the world to be possible. An application of this framework is the "group identification" problem, in which individuals classify themselves into social groups.

Voting Suppose there are $c \geq 3$ candidates for k spots (with $0 < k < c$). These may be political candidates, job hires, or the like. Each individual on a committee picks k preferred candidates and the judgments are aggregated to a group decision. This is equivalent to an agenda for which consistency requires affirmation on exactly k propositions. Any proposition-wise voting rule can lead to inconsistency (i.e., more than or fewer than k candidates are chosen) and a sequential priority rule is clearly inappropriate. A natural aggregation rule is to pick the k candidates with the most votes, which corresponds to both $\mathrm{Prototype}_{\mathrm{Ham}}$ and $\mathrm{Output}_{\mathrm{Ham}}$.

6 Discussion

6.1 Comparing the four methods

We have investigated a broad class of JA rules that employ distance metrics between judgment sets to guarantee group consistency. Our conclusions have run along two dimensions. First, we identified three appealing characteristics of individual responsiveness that solution rules can satisfy—supermajority responsiveness, preservation, and sensibility. We then showed that Prototype, the most prominent distance-based JA method, fails preservation on many agendas (including the preference agenda) if the employed metric is non-separable (Proposition 21(b) and Theorem 28). For most agendas of interest, we further proved that of the four methods we investigate, only Output satisfies all three characteristics for all normal metrics (Theorems 23, 29, and Proposition 33). As a second theme of our results, we have mapped out the incommensurability of the four methods. In particular, each method induces distinct solution rules for large classes of metrics (Theorems 24 and 30), a fact that greatly expands the toolbox for JA and preference aggregation. Moreover, Endpoint generalizes the class of sequential majority rules (Theorem 18), proving disjointness between these rules and ones derived from the remaining three methods. Thus, it matters greatly whether we aggregate solely at the group level (as in Endpoint) or incorporate individual-level beliefs.[14]

A comparison of Output and Prototype is instructive. For many agendas, Prototype fails preservation for non-separable metrics, whereas Output never fails preservation. For separable metrics, however, the two methods are virtually identical. Consider a profile for which only one majority-rule proposition needs to be changed to get group consistency. For proposition α, let m_α be the margin of victory for the group judgment on α. As usual, for separable metric d, let $\delta_{\mathsf{d}}(\alpha)$ be the distance incurred by disagreement on α. Then Prototype changes the proposition α that minimizes $m_\alpha \delta_{\mathsf{d}}(\alpha)$, whereas Output minimizes $\frac{m_\alpha+1}{2}\delta_{\mathsf{d}}(\alpha)$. The difference makes them technically, but not substantively, distinct. Since Output is more appealing than Prototype for non-separable metrics (assuming we care about preservation) and is identical in spirit for separable metrics, a tentative case is made for the superiority of Output relative to Prototype.

This raises the question of whether Output is an appropriate method for making real-world decisions. Besides satisfying several attractive characteristics, Output has an intuitive interpretation as the reflection of a negotiation process. On the negative side, it requires calculations over inconsistent judgment sets, which is troubling for both philosophical and practical reasons. Are we to imagine individuals believing in inconsistent judgments for the purpose of group consistency? Thus, Output is appealing as a method of calculating a group judgment set for the purpose of compromise, but may fall short if we demand explanatory rigor.

6.2 Extensions

The comparison we have undertaken is one of many potential approaches to evaluating and distinguishing among distance-based JA rules. A feature of the present study is that we compare solution methods according to whether they satisfy characteristics across *all* normal metrics. Yet some classes of metrics may be considered more natural than others. A comparison restricted to only the separable metrics, for example, might find reason to support Full instead of Output. Thus, we must be mindful that evaluations of solution methods are always relative to the class of metrics under consideration. Although it is tempting to restrict our attention to simple metrics (like Hamming), it is also desirable for a solution method to apply to as many situations as possible, some of which may weigh certain issues above others. We have chosen the normal metrics as a suitable balance between these demands.

Our study concentrates on four solution methods, but at least five other types of aggregation (not all distance-based) are worth comment: (1) Various methods similar to Prototype are possible; specifically, we can employ any real-valued function from the set of individual distances and choose the judgment set that minimizes this value (Konieczny et al. 2004). (2) Hybrid rules are conceivable that, for example, use Prototype for separable metrics and Output for other metrics. (3) A variant of the sequential

[14] Another potential avenue of comparison is to extend work on the truth-tracking properties of the premise-based approach (List 2005; Bovens and Rabinowicz 2006) to distance-based JA.

priority approach can prioritize propositions not by a predetermined ordering, but by the margin of their victory. (4) More generally, we can derive a distance metric from the profile itself and apply the metric to a given solution method. This defines a solution rule that goes directly (i.e., without a supplied metric) from the profile to a consistent group judgment set. One possibility is to have the distance for a proposition increase with its majority margin, representing the group's confidence in that proposition's truth-value. This may improve Endpoint, which in its present form does not account for margins of victory. (5) Finally, it is worth considering the method that looks for the largest majority consistent subset of the profile and takes the majority rule output of this sub-profile, an approach similar to the Young rule used in preference aggregation (Fishburn 1977).

Numerous properties of solution rules, beyond the three discussed here, are worth investigation. We sketch some possibilities: (1) *Anonymity* (the output is the same under any permutation of the individuals) is satisfied by all solution rules discussed here. Note that *non-dictatorship* is implied by preservation. (2) We can judge a solution rule's *uniqueness* by how often it yields a non-tied output. (3) *Proximity preservation* (Baigent 1987) holds that smaller changes in individual judgments should yield smaller changes in the group judgments. This can be adapted to an evaluative criteria for how "sensitive" a solution rule is to individual changes. (4) *Consistency* holds that if a solution rule outputs p for two separate profiles, then it should also output p when applied to the combined profile (with an additional copy of p to keep the profile size odd). (5) *Monotonicity* holds that if a solution rule outputs p for a profile, then it should also output p for an identical profile appended with any number of copies of p. (6) Finally, other criteria might compare outputs across agendas. For instance, given profile P, a solution rule applied to the agenda $\{a, b\}$ may provide group judgments that conflict with those for an extension of P to $\{a, b, a \wedge b\}$. It is preferable for a solution rule to be as consistent as possible across agendas.[15]

A challenge thus exists to find the most satisfactory solution method and/or rule, keeping in mind that this classification is relative to a class of metrics and agenda types. Future research should specify further desired characteristics and demonstrate which rules satisfy them, with the ultimate goal of an axiomatic characterization of a unique and general-purpose JA method.

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Appendix 1: Illustration of the four solution methods

Consider the leftmost profile P shown in Table 2, defined for the doctrinal paradox agenda $\{a, b, a \wedge b\}$. Observe that there are exactly four consistent judgments sets over this agenda, namely, TTT, TFF, FTF, and FFF. We illustrate the four solution methods by applying each of them to P using a metric d defined as follows.

[15] However, no non-independent method of JA can *guarantee* such inter-agenda consistency. This is a consequence of the impossibility of strategy-proofness without independence (Dietrich and List 2007b; Nehring and Puppe 2007).

Table 2 A majority inconsistent profile on the doctrinal paradox agenda along with two alterations

	a	b	$a \wedge b$	a	b	$a \wedge b$	a	b	$a \wedge b$
1	T	T	T	T	T	T	T	T	T
2	T	T	T	T	T	T	T	T	T
3	T	F	F	T	F	F	**T**	**F**	**T**
4	T	F	F	T	F	F	T	F	F
5	F	T	F	**F**	**F**	**F**	F	T	F
M	T	T	F	T	F	F	T	T	T

For judgment sets p, q, let $\mathrm{Ham}(p, q)$ denote the Hamming distance (i.e., the number of agenda items on which p and q disagree). Define:

$$\mathsf{d}(p, q) = \begin{cases} 0.9 & \text{if } p \text{ and } q \text{ disagree only on } a \wedge b \\ \sqrt{\mathrm{Ham}(p, q)} & \text{otherwise.} \end{cases}$$

It is easily verifiable that d is a metric as defined in Sect. 2.1.

Full Full looks for the closest consistent profile that yields a consistent majority output. Starting at P, we can change the fifth judgment set from FTF to FFF, as shown in the middle profile of Table 2. This incurs a distance of $\mathsf{d}(FTF, FFF) = 1$, and yields a (consistent) majority rule output of TFF. No other consistent change yields a majority consistent profile with as small an overall distance, hence $\mathrm{Full}_{\mathsf{d}}(\mathsf{P}) = TFF$.

Output The distinction with Full is that Output allows individual judgment sets to be replaced by inconsistent judgment sets. Starting at P, we can change the third judgment set from TFF to TFT (which is inconsistent), as shown in the rightmost profile of Table 2. This incurs a distance of $\mathsf{d}(TFF, TFT) = 0.9$, and yields a majority rule output of TTT. No other change yields a majority consistent profile with as small an overall distance (and a different majority rule output), hence $\mathrm{Output}_{\mathsf{d}}(\mathsf{P}) = TTT$.

Endpoint $\mathrm{Endpoint}_{\mathsf{d}}(\mathsf{P})$ equals the consistent judgment set that minimizes the distance with $M(\mathsf{P}) = TTF$. TTT incurs a distance of $\mathsf{d}(TTF, TTT) = 0.9$, the minimal possible distance. Hence, $\mathrm{Endpoint}_{\mathsf{d}}(\mathsf{P}) = TTT$.

Prototype For $\mathrm{Prototype}_{\mathsf{d}}(\mathsf{P})$, we find the consistent judgment set q that minimizes the sum of distances between q and each judgment set in P. This sum is $3\sqrt{2}$ for TTT, $3\sqrt{2}$ for TFF, $4\sqrt{2}$ for FTF, and $2\sqrt{3} + 3$ for FFF. We have a tie for the smallest sum, hence $\mathrm{Prototype}_{\mathsf{d}}(\mathsf{P}) = \{TTT, TFF\}$.

Appendix 2: Proofs

We first develop some notation and facts that will be generally useful for the following proofs.

Definition 34 Let $\mathsf{P} = \{p_i\}_{i \leq n} \in \mathcal{P}^\bullet_A$, distance metric d, and $s \in A^\bullet$ be given. Define

$$\begin{aligned}
\text{cost}(s \in \text{Full}_{\mathsf{d}}(\mathsf{P})) &= \min_{\mathsf{Q} \in \mathcal{P}^\bullet_A,\, M(\mathsf{Q})=s} \mathsf{d}(\mathsf{P}, \mathsf{Q}) \\
\text{cost}(s \in \text{Output}_{\mathsf{d}}(\mathsf{P})) &= \min_{\mathsf{Q} \in \mathcal{P}^\circ_A,\, M(\mathsf{Q})=s} \mathsf{d}(\mathsf{P}, \mathsf{Q}) \\
\text{cost}(s \in \text{Endpoint}_{\mathsf{d}}(\mathsf{P})) &= \mathsf{d}(M(\mathsf{P}), s) \\
\text{cost}(s \in \text{Prototype}_{\mathsf{d}}(\mathsf{P})) &= \textstyle\sum_{i \leq n} \mathsf{d}(p_i, s)
\end{aligned}$$

Intuitively, $\text{cost}(s \in \text{Full}_{\mathsf{d}}(\mathsf{P}))$ is the minimal distance incurred by $\text{Full}_{\mathsf{d}}(\mathsf{P})$ to obtain s as one of its outputs, implying the following:

Lemma 35 *For solution method f, $f_{\mathsf{d}}(\mathsf{P}) = \arg\min \text{cost}(s \in f_{\mathsf{d}}(\mathsf{P}))$ among all $s \in A^\bullet$.*

For each proof on connected agendas, fix a connected agenda A with minimally complex subagenda X. Let $a, b, c \in X$. Let inconsistent p be such that $p^{\neg a}$, $p^{\neg b}$, $p^{\neg c} \in A^\bullet$. Say that $D = A \backslash \{a, b, c\}$, where D may be empty.

Definition 36 A *complex triple* is any triple (n_1, n_2, n_3) of positive integers such that for distinct $i, j, k \in \{1, 2, 3\}$, $n_i + n_j > n_k$.

Definition 37 Given positive integers (n_1, n_2, n_3), let P_{n_1,n_2,n_3} be the profile consisting of n_1 copies of $p^{\neg a}$, n_2 copies of $p^{\neg b}$, and n_3 copies of $p^{\neg c}$. If (n_1, n_2, n_3) is a complex triple, call P_{n_1,n_2,n_3} a *complex profile*.

Note that $(1, \frac{n-1}{2}, \frac{n-1}{2})$ and its permutations are complex triples for odd $n > 1$. It is easy to see that if P_{n_1,n_2,n_3} is a complex profile, then $M(\mathsf{P}_{n_1,n_2,n_3}) = p$. The situation can be pictured as follows, where a indicates agreement with p on a, and $\neg a$ indicates disagreement:

P_{n_1,n_2,n_3}	a	b	c	D
n_1 copies	$\neg a$	b	c	D
n_2 copies	a	$\neg b$	c	D
n_3 copies	a	b	$\neg c$	D
$M(\mathsf{P}_{n_1,n_2,n_3})$	a	b	c	D

(38)

Definition 39 Given a profile P and $\alpha \in A$, define m_α to be the number of judgment sets in P holding the majority judgment on α minus the number holding the minority judgment.

Intuitively, m_α represents the margin of victory for α. A little algebra (which we omit for space) gives us the bounds of these margins:

Lemma 40 *Consider distinct $\alpha, \beta, \gamma \in \{a, b, c\}$ and complex profile P_{n_1,n_2,n_3} of size n.*

$$\begin{aligned}
& m_\alpha \textit{ can take any odd value in } [1, n-2] \\
\textit{With } m_\alpha \textit{ fixed, } \; & m_\beta \textit{ can take any odd value in } [1, n - m_\alpha - 1] \\
\textit{With } m_\alpha, m_\beta \textit{ fixed, } \; & m_\gamma = n - m_\alpha - m_\beta
\end{aligned}$$

Consider complex profile P_{n_1,n_2,n_3}. For metric d (with associated δ), the cost for each output and solution method under consideration is given in 41. The only uncertainty is the cost for Full, which depends on the specific logical constraints within A. As such, for $\alpha \in A$, we specify $\hat{\delta}(\alpha)$ to be the smallest distance incurred by a change within P_{n_1,n_2,n_3} that reduces m_α. Note that $\hat{\delta}(a) \leq \min\{\delta(a,b), \delta(a,c)\}$, and similarly for $\hat{\delta}(b)$ and $\hat{\delta}(c)$.

Cost	Full_d	Output_d	$\text{Endpoint}_\mathsf{d}$	$\text{Prototype}_\mathsf{d}$
$p^{\neg a}$	$\frac{m_a+1}{2}\hat{\delta}(a)$	$\frac{m_a+1}{2}\delta(a)$	$\delta(a)$	$n_2\delta(a,b)+n_3\delta(a,c)$
$p^{\neg b}$	$\frac{m_b+1}{2}\hat{\delta}(b)$	$\frac{m_b+1}{2}\delta(b)$	$\delta(b)$	$n_1\delta(a,b)+n_3\delta(b,c)$
$p^{\neg c}$	$\frac{m_c+1}{2}\hat{\delta}(c)$	$\frac{m_c+1}{2}\delta(c)$	$\delta(c)$	$n_1\delta(a,c)+n_2\delta(b,c)$

(41)

Theorems

Proof of Theorem 18 Suppose that sequential majority rule f prioritizes the propositions in A in the order $(a_1, a_2, a_3, \ldots, a_k)$, with a_1 given the highest priority. Consider $g \in \{\text{Full, Output, Endpoint, Prototype}\}$ and odd n. Also consider a separable d such that $\delta(a_1) > n(\delta(a_2) + \cdots + \delta(a_k))$, $\delta(a_2) > n(\delta(a_3) + \cdots + \delta(a_k))$, and so on. It should be clear that for all profiles up to size n, the output of g_d will agree with the majority on a_1 (since changing *every* other proposition incurs less distance). It will also agree with the majority on a_2, unless this assignment is inconsistent with the assignment on a_1, and so on. $\text{Endpoint}_\mathsf{d}$ does not depend on profile size, hence $\text{Endpoint}_\mathsf{d} = f$ for all profiles. □

Characteristics

Note that Proposition 20(a) is trivial. To prove the remaining SR_n results, we assume that a solution rule fails SR_n and derive conditions on the metric.

Lemma 42 *Let metric* d *and solution method* f *be given. If* f_d *fails* SR_n*, then* $f_\mathsf{d}(\mathsf{P}_{n_1,n_2,n_3})$ *is identical for any complex profile* P_{n_1,n_2,n_3} *of size* n.

Proof It follows directly from Definition 36 that for all complex triples (n_1, n_2, n_3), $M(\mathsf{P}_{n_1,n_2,n_3}) = p$. Because f_d fails SR_n, its output must be the same for all such profiles of size n. □

Proof of Proposition 20(b) Let $n \geq 5$.

Output Let d be such that Output_d fails SR_n. It suffices to prove that $\mathsf{d} \notin \mathcal{D}(\frac{n-1}{2})$. Suppose $\mathsf{d} \in \mathcal{N}$, since otherwise this follows immediately. Suppose $q \in \text{Output}_\mathsf{d}(\mathsf{P})$ for every complex profile P of size n. Consider proposition $\beta \in \mathit{diff}(p,q)$ and assume $a \neq \beta$. From Lemma 40, there exists a complex profile Q of size n for which $m_a = 1$ and $m_\beta \geq n-2$. Then $\text{cost}(p^{\neg a} \in \text{Output}_\mathsf{d}(\mathsf{Q})) = \delta(a) \geq \text{cost}(q \in \text{Output}_\mathsf{d}(\mathsf{Q})) \geq \frac{(n-2)+1}{2}\delta(\beta) = \frac{n-1}{2}\delta(\beta)$, which implies $\mathsf{d} \notin \mathcal{D}(\frac{n-1}{2})$. □

Full Let d be such that Full_d fails SR_n. Using a parallel logic to that for Output (and recalling $\hat{\delta}(\alpha)$'s definition given above (41)), we arrive at $\hat{\delta}(a) \geq \frac{n-1}{2}\hat{\delta}(\beta) \geq \frac{n-1}{2}\delta(\beta)$.

Since $\hat{\delta}(a) \leq \delta(a) + \delta(b) \leq \max\{2\delta(a), 2\delta(b)\}$, we get that $\delta(a) \geq \frac{n-1}{4}\delta(\beta)$ or $\delta(b) \geq \frac{n-1}{4}\delta(\beta)$ and thus $\mathsf{d} \notin \mathcal{D}(\frac{n-1}{4})$. □

Prototype Let d be such that $\text{Prototype}_{\mathsf{d}}$ fails SR_n. As before, suppose that $q \in \text{Output}_{\mathsf{d}}(\mathsf{P})$ for every complex profile P of size n. To prove $\mathsf{d} \notin \mathcal{D}(n-2) \cap \mathcal{S}$, it suffices to show that there exist $\alpha, \beta, \gamma \in A$ such that $\frac{n-3}{n-1}\delta(\alpha, \gamma) + \delta(\beta, \gamma) \leq \delta(\alpha, \beta)$. This suffices since, if $\mathsf{d} \in \mathcal{S}$, then $\frac{2n-4}{n-1}\delta(\gamma) \leq \frac{2}{n-1}\delta(\alpha) \Leftrightarrow (n-2)\delta(\gamma) \leq \delta(\alpha)$. There are four cases to consider, depending on the number of a, b, c in *diff* (p, q).

To save space, we show the argument for exactly one of $\{a, b, c\}$ in *diff* (p, q), with the other cases receiving similar treatment. Suppose $c \in$ *diff* (p, q). Then $\frac{n-1}{2}\delta(a, c) + \frac{n-1}{2}\delta(b, c) \leq \text{cost}(q \in \text{Prototype}_{\mathsf{d}}(\mathsf{P}_{\frac{n-1}{2}, \frac{n-1}{2}, 1})) \leq \text{cost}(p^{\neg a} \in \text{Prototype}_{\mathsf{d}}$ $(\mathsf{P}_{\frac{n-1}{2}, \frac{n-1}{2}, 1})) = \frac{n-1}{2}\delta(a, b) + \delta(a, c)$. This implies $\frac{n-3}{n-1}\delta(a, c) + \delta(b, c) \leq \delta(a, b)$. □

Proof of Proposition 20(c) and (d) The cases for Output and Full follow immediately from Proposition 20(b) since, for sufficiently large n, every normal metric is in $\mathcal{D}(\frac{n-1}{2})$.

From the proof of the Prototype case in Proposition 20(b), we derived that if $\text{Prototype}_{\mathsf{d}}$ fails SR_n, then there exist $\alpha, \beta, \gamma \in A$ such that $\frac{n-3}{n-1}\delta(\alpha, \gamma) + \delta(\beta, \gamma) \leq \delta(\alpha, \beta)$. Now suppose $\mathsf{d} \in \mathcal{I}$. Since $\delta(\alpha, \beta) \leq \delta(\alpha) + \delta(\beta) < \delta(\alpha, \gamma) + \delta(\beta, \gamma)$, it follows that for sufficiently large n, $\delta(\alpha, \beta) < \frac{n-3}{n-1}\delta(\alpha, \gamma) + \delta(\beta, \gamma)$, proving that $\text{Prototype}_{\mathsf{d}}$ satisfies SR. □

Proof of Proposition 21(a) The results for Full, Output, and Endpoint are trivial. Consider $\mathsf{d} \in \mathcal{S}$ and a majority consistent profile P. For each $\alpha \in A$ and judgment set s, define $\phi_s(\alpha)$ to be the number of judgment sets $q \in \mathsf{P}$ such that s and q disagree on α. Since $\mathsf{d} \in \mathcal{S}$, it is clear that

$$\text{cost}\left(s \in \text{Prototype}_{\mathsf{d}}(\mathsf{P})\right) = \sum_{\alpha \in A} \delta(\alpha)\phi_s(\alpha)$$

Since $M(\mathsf{P})$ is consistent and minimizes $\phi_s(\alpha)$ for each $\alpha \in A$, $\text{Prototype}_{\mathsf{d}}(\mathsf{P}) = M(\mathsf{P})$. Hence, $\text{Prototype}_{\mathsf{d}}$ satisfies preservation. □

Proof of Proposition 21(b) Suppose $\text{Prototype}_{\mathsf{d}}$ satisfies preservation and for disjoint $Y, Z \subseteq A$ there exist $q, q^{\neg Y}, q^{\neg Z} \in A^{\bullet}$. We show that $\delta(Y \cup Z) = \delta(Y) + \delta(Z)$. Let Q_{n_1,n_2,n_3} be the profile consisting of n_1 copies of q, n_2 copies of $q^{\neg Y}$, and n_3 copies of $q^{\neg Z}$. If (n_1, n_2, n_3) is a complex triple, then $M(\mathsf{Q}_{n_1,n_2,n_3}) = q$ and thus $\text{Prototype}_{\mathsf{d}}(\mathsf{Q}_{n_1,n_2,n_3}) = q$.

We suppose $\delta(Y \cup Z) \neq \delta(Y) + \delta(Z)$ and derive a contradiction. By metric properties, this means $\delta(Y \cup Z) = \delta(Y) + \delta(Z) - \varepsilon$ for $\varepsilon > 0$. We have:

$$\begin{aligned}
\text{cost}(q \in \text{Prototype}_{\mathsf{d}}(\mathsf{Q}_{n_1,n_2,n_3})) &= n_2\delta(Y) + n_3\delta(Z) \\
\text{cost}(q^{\neg Y} \in \text{Prototype}_{\mathsf{d}}(\mathsf{Q}_{n_1,n_2,n_3})) &= n_1\delta(Y) + n_3\delta(Y \cup Z) = (n_1 + n_3)\delta(Y) \\
&\quad + n_3\delta(Z) - n_3\varepsilon
\end{aligned}$$

This implies that for any complex triple (n_1, n_2, n_3), we must have $n_2\delta(Y) + n_3\delta(Z) < (n_1 + n_3)\delta(Y) + n_3\delta(Z) - n_3\varepsilon$, which implies $n_3\varepsilon < (n_1 + n_3 - n_2)\delta(Y)$. But consider the complex triple $(1, \frac{n-1}{2}, \frac{n-1}{2})$, which entails $\frac{n-1}{2}\varepsilon < (1 + \frac{n-1}{2} - \frac{n-1}{2})\delta(Y) = \delta(Y)$. For sufficiently large n, this inequality will not be satisfied, a contradiction. Thus, $\delta(Y \cup Z) = \delta(Y) + \delta(Z)$. □

Proof of Proposition 22 The case for Endpoint is trivial, since it chooses the consistent judgment set minimally distant from $M(\mathsf{P})$. The case for Output follows since $\text{cost}(M(\mathsf{P})^{\neg\alpha} \in \text{Output}_{\mathsf{d}}(\mathsf{P})) = \frac{m_\alpha+1}{2}\delta(\alpha)$ and thus $\text{Output}_{\mathsf{d}}(\mathsf{P}) = M(\mathsf{P})^{\neg\alpha}$ if both m_α and $\delta(\alpha)$ are strictly minimal. From the proof of Proposition 21(a), if $\mathsf{d} \in \mathcal{S}$, then $\text{cost}(M(P)^{\neg\alpha} \in \text{Prototype}_{\mathsf{d}}(\mathsf{P})) = \sum_{\beta\in A}\delta(\beta)\phi_{M(\mathsf{P})}(\beta) + \delta(\alpha)m_\alpha$. Then $\text{Prototype}_{\mathsf{d}}(\mathsf{P}) = M(\mathsf{P})^{\neg\alpha}$ if both m_α and $\delta(\alpha)$ are strictly minimal.

The case for Full follows by considering a connected agenda with $p^{\neg a,\neg b}$ and $p^{\neg a,\neg c}$ inconsistent, plus a metric with $\delta(a)$ strictly minimal and $\delta(a,b) = \delta(a,c) > \delta(b,c)$. □

Disjointness

Proof of Proposition 26 Suppose $\text{Prototype}_{\mathsf{d}} = \text{Output}_{\mathsf{e}}$ for $\mathsf{d} \in \mathcal{S}$ and $\mathsf{e} \in \mathcal{N}$. Then it is impossible to choose (m_a, m_b, m_c) so that $\text{Prototype}_{\mathsf{d}}(\mathsf{P}_{n_1,n_2,n_3}) \neq \text{Output}_{\mathsf{e}}(\mathsf{P}_{n_1,n_2,n_3})$ for complex profile P_{n_1,n_2,n_3}.

For n sufficiently large, we can choose m_c such that the output is either $p^{\neg a}$ or $p^{\neg b}$. Our choice of (m_a, m_b) cannot then satisfy both of the following:

$$m_b\delta_{\mathsf{d}}(b) < m_a\delta_{\mathsf{d}}(a) \quad \text{and} \quad \frac{m_a+1}{2}\delta_{\mathsf{e}}(a) < \frac{m_b+1}{2}\delta_{\mathsf{e}}(b) \tag{43}$$

Both equations are satisfied if $\frac{\delta_{\mathsf{d}}(b)}{\delta_{\mathsf{d}}(a)} < \frac{m_a}{m_b} < \frac{m_a+1}{m_b+1} < \frac{\delta_{\mathsf{e}}(b)}{\delta_{\mathsf{e}}(a)}$. With m_c fixed, Lemma 40 reveals that $\frac{m_a+1}{m_b+1}$ can be any value between $\frac{1}{n-m_c}$ and $\frac{n-m_c}{2}$, in steps on the order of $\frac{2}{m_b+1}$. With n large enough, it follows that we can choose (m_a, m_b) so that (43) is satisfied, unless $\frac{\delta_{\mathsf{e}}(b)}{\delta_{\mathsf{e}}(a)} \leq \frac{\delta_{\mathsf{d}}(b)}{\delta_{\mathsf{d}}(a)}$. Reversing the inequalities in (43) similarly yields that $\frac{\delta_{\mathsf{e}}(b)}{\delta_{\mathsf{e}}(a)} \geq \frac{\delta_{\mathsf{d}}(b)}{\delta_{\mathsf{d}}(a)}$, hence $\frac{\delta_{\mathsf{e}}(b)}{\delta_{\mathsf{e}}(a)} = \frac{\delta_{\mathsf{d}}(b)}{\delta_{\mathsf{d}}(a)}$. Call this ratio r. Note that (43) is also violated if $\frac{m_a+1}{m_b+1} < r < \frac{m_a}{m_b}$, which will be met for some (m_a, m_b) given n large enough unless $r = 1$, implying $\delta_{\mathsf{d}}(a) = \delta_{\mathsf{d}}(b)$ and $\delta_{\mathsf{e}}(a) = \delta_{\mathsf{e}}(b)$. □

Preference agendas

Let $wxyz$ indicate the ordering that ranks w highest, followed by x, and so on. We also let this indicate sub-orderings when the ranking of other alternatives is clear.

Proof of Theorem 28 Since the *if* part follows from Proposition 21(a), we suppose that $\text{Prototype}_{\mathsf{d}}$ satisfies preservation for $\mathsf{d} \in \mathcal{N}$ and show that $\mathsf{d} \in \mathcal{S}$. Consider subagenda Y. We need to show that $\delta(Y) = \sum_{\alpha\in Y}\delta(\alpha)$. Say that an alternative *appears in Y* if it is contained in a relation contained in Y. Consider $Z \subseteq A$ which contains every relation between alternatives appearing in Y. We show that $\delta(Z) = \sum_{\alpha\in Z}\delta(\alpha)$, which establishes the case for $\delta(Y)$ since $\delta(Z) \leq \delta(Y) + \sum_{\alpha\in Z\setminus Y}\delta(\alpha)$.

Consider $X \subseteq A$ which contains every relation between alternatives in a set C. We proceed by induction on $|C|$ to show that $\delta(X) = \sum_{\alpha\in X}\delta(\alpha)$. This is trivial for $|C| = 2$. Assume it holds for $|C| = k \geq 2$ and consider C of size $k+1$. We make repeated use of Proposition 21(b).

Take any alternative $x \in C$. Consider $p^1, p^2, p^3 \in A^\bullet$ that rank alternatives in C above any others. p^1 ranks x highest; p^2 reverses from p^1 all rankings within $C\setminus\{x\}$;

and p^3 is identical to p^1 but ranks x lowest. Say that X^1 includes all relations within $C \setminus \{x\}$ and X^2 includes all relations between x and any member of $C \setminus \{x\}$. Thus, $X = X^1 \cup X^2$. Since $\mathit{diff}(p^1, p^2) = X^1$ and $\mathit{diff}(p^1, p^3) = X^2$, it follows that $\delta(X) = \delta(X^1) + \delta(X^2)$.

Take an alternative $y \in C$, with $y \neq x$. Consider the consistent judgment set that ranks y highest, followed by x, followed by $C \setminus \{x, y\}$ in some order. There exists a consistent judgment set that reverses only the ordering of x and y, as well as one that only differs in placing x below $C \setminus \{x, y\}$. As above, it follows that $\delta(X^2) = \delta(xPy) + \delta(\{xPz : z \in C \setminus \{x, y\}\})$. We can repeat this argument to show that $\delta(X^2) = \sum_{w \in C \setminus \{x\}} \delta(xPw)$. Since X^1 contains every relation between k alternatives, we have $\delta(X) = \sum_{v,w \in C \setminus \{x\}} \delta(vPw) + \sum_{w \in C \setminus \{x\}} \delta(xPw) = \sum_{\alpha \in X} \delta(\alpha)$. □

For the proof of Theorem 29, see below, following the proof of Theorem 30.

Proof of Theorem 30 The Endpoint result follows from Proposition 21(a) and Theorem 28, combined with Theorem 24.

Prototype versus Output/Full First consider a preference agenda with 3 alternatives. Suppose $\text{Prototype}_{\mathsf{d}} = \text{Output}_{\mathsf{e}}$ for $\mathsf{d}, \mathsf{e} \in \mathcal{N}$. By Proposition 21(a) and Theorem 28, $\mathsf{d} \in \mathcal{S}$. By Proposition 26, $\mathsf{d} = \text{Ham}$ and $\delta_{\mathsf{e}}(\alpha) = \delta_{\mathsf{e}}(\beta)$ for $\alpha, \beta \in A$. Multiple changes to a single judgment set are unnecessary, so $\text{Output}_{\mathsf{e}} = \text{Output}_{\text{Ham}}$. The same logic holds for Full_{e}, since no logical constraint comes into play.

Now consider a preference agenda with $k \geq 4$ alternatives. Suppose $\text{Prototype}_{\mathsf{d}} = \text{Output}_{\mathsf{e}}$ for $\mathsf{d}, \mathsf{e} \in \mathcal{N}$. The case for Full can receive parallel treatment. Consider the set of profiles in which 3 alternatives are universally ranked above the remaining alternatives. The relationship between these profiles and the output will be analogous to the relationship for the agenda with exactly 3 alternatives. By considering each triple in turn, it follows that $\mathsf{d} = \text{Ham}$ and $\delta_{\mathsf{e}}(\alpha) = \delta_{\mathsf{e}}(\beta)$ for $\alpha, \beta \in A$. It remains to show that $\text{Prototype}_{\mathsf{d}} \neq \text{Output}_{\mathsf{e}}$ given these conditions. Consider a profile P that universally ranks alternatives w, x, y, z above all others. There are two cases to consider:

Case 1: $\delta_{\mathsf{e}}(wPx, wPy) = \delta_{\mathsf{e}}(wPx) + \delta_{\mathsf{e}}(wPy)$. Suppose P contains 2 orderings $zwxy$, $\frac{n-3}{2}$ orderings $wxyz$, and $\frac{n-1}{2}$ orderings $xyzw$. $M(\mathsf{P}) = wxyz$, except z is ranked above w (which implies $M(\mathsf{P})$ fails transitivity). Consistency can be reached by either reversing zPw or both wPx and wPy (other possibilities are too costly for large enough n). $m_{zPw} = 3$ and $m_{wPx} = m_{wPy} = 1$, hence $\text{Prototype}_{\text{Ham}} = xyzw$ and $\text{Output}_{\mathsf{e}} = \{xyzw, wxyz\}$, a contradiction.

Case 2: $\delta_{\mathsf{e}}(wPx, wPy) = \delta_{\mathsf{e}}(wPx) + \delta_{\mathsf{e}}(wPy) - \varepsilon$ (with $\varepsilon > 0$ by the properties of a metric). Suppose P contains $\frac{3m+1}{4}$ orderings $zwxy$, $\frac{n-m}{2}$ orderings $wxyz$, and $\frac{2n-m-1}{4}$ orderings $xyzw$, where m is chosen to make each value an integer. $M(\mathsf{P}) = wxyz$, except z is ranked above w. Consistency can be reached by either reversing zPw or both wPx and wPy (other possibilities are too costly for large enough n). $m_{zPw} = m$ and $m_{wPx} = m_{wPy} = \frac{m+1}{2}$, hence $\text{Prototype}_{\text{Ham}} = wxyz$. Thus, $\text{cost}(wxyz \in \text{Output}_{\mathsf{e}}(\mathsf{P})) - \text{cost}(xyzw \in \text{Output}_{\mathsf{e}}(\mathsf{P})) = \frac{m+1}{2}\delta_{\mathsf{e}}(wPz) - \frac{m+3}{4}\delta_{\mathsf{e}}(wPx, wPy) = \frac{m+1}{2}\delta_{\mathsf{e}}(wPz) - \frac{m+3}{2}\delta_{\mathsf{e}}(wPx) + \frac{m+3}{4}\varepsilon = \frac{m+3}{4}\varepsilon - \delta_{\mathsf{e}}(wPz)$. Since this is positive for sufficiently large m, $\text{Output}_{\mathsf{e}}(\mathsf{P}) \neq wxyz$, a contradiction. □

Output versus Full Consider a preference agenda with $k \geq 5$ alternatives. Suppose $\mathsf{d} \in \mathcal{I}$, $\mathsf{e} \in \mathcal{N}$, and $\text{Full}_{\mathsf{d}} = \text{Output}_{\mathsf{e}}$. $\{xPy, yPz, xPz\}$ is minimally complex for any 3 alternatives x, y, z. Following the logic in the proof of Proposition 26, we get that $\frac{\delta_{\mathsf{d}}(xPy)}{\delta_{\mathsf{d}}(xPz)} = \frac{\delta_{\mathsf{e}}(xPy)}{\delta_{\mathsf{e}}(xPz)}$. Now consider the set of profiles in which 5 alternatives v, w, x, y, z are ranked highest by all individuals and there is a complex triple of the orderings $xvywz, zvxwy, vyzxw$. For each P in this set, $M(\mathsf{P}) = vxyzw$, except z is ranked above x. Because x and y are not consecutive in any individual ordering, to reduce m_{xPy}, Full_{d} must reverse both an individual xPy and another ranking. Hence, $\hat{\delta}_{\mathsf{d}}(xPy) > \delta_{\mathsf{d}}(xPy)$ since $\mathsf{d} \in \mathcal{I}$. In comparison, $\hat{\delta}_{\mathsf{d}}(xPz) = \delta_{\mathsf{d}}(xPz)$. Again applying the logic of Proposition 26, we get that $\frac{\hat{\delta}_{\mathsf{d}}(xPy)}{\delta_{\mathsf{d}}(xPz)} = \frac{\delta_{\mathsf{e}}(xPy)}{\delta_{\mathsf{e}}(xPz)}$. From above, this implies $\frac{\hat{\delta}_{\mathsf{d}}(xPy)}{\delta_{\mathsf{d}}(xPz)} = \frac{\delta_{\mathsf{d}}(xPy)}{\delta_{\mathsf{d}}(xPz)}$, a contradiction. □

Proof of Theorem 29 It must be shown that Full_{e} fails sensibility on e for some $\mathsf{e} \in \mathcal{I}$. Consider the final example given in the previous proof (for $k \geq 5$ alternatives). Suppose $\delta_{\mathsf{e}}(xPy) < \min\{\delta_e(yPz), \delta_e(xPz)\}$, but $\hat{\delta}_{\mathsf{e}}(xPy) > \delta_{\mathsf{e}}(xPz)$. Then there exists a profile P for which m_{xPy} is strictly minimal, but (with $m_{xPz} = m_{xPy} + 1$) $\text{Full}_{\mathsf{e}}(\mathsf{P}) \neq M(\mathsf{P})^{\neg xPy}$. Hence, Full_{e} fails sensibility on e. □

Truth-functional agendas

Proof of Proposition 32 Consider truth-functional agenda $A = \{a_1, a_2, \ldots, a_k, b\}$, and take any $a \in \{a_1, a_2, \ldots, a_k\}$. By a condition of truth-functionality, there exists a consistent judgment set p such that $p^{\neg a}$ is inconsistent. In addition, p is consistent if and only if $p^{\neg b}$ is inconsistent. Hence, $p^{\neg a, \neg b}$ is consistent. Since b does not depend solely on a, it follows that there is a truth-value for a that is consistent with either truth-value of b. (That is, if this truth-value for a is True, then a and b are True in some consistent judgment set on A, and a is True and b is False in another consistent judgment set on A.) Without loss of generality, assume a has this value in p.

Consider the inconsistent judgment set $q = p^{\neg b}$. By assumption, $q^{\neg a}$ and $q^{\neg b}$ are consistent, and there exists a consistent judgment set r that agrees with q on a and b. Thus, there exists a sequence of judgment sets $\{q = q_1, q_2, \ldots, q_j = r\}$ such that q_i and q_{i+1} differ on a single proposition $c_i \notin \{a, b\}$. There are then two possibilities: (a) For some inconsistent q_i, we have $q_i^{\neg a}, q_i^{\neg b}$, and $q_i^{\neg c_i}$ all consistent, in which case we are done, or (b) For some inconsistent q_i, we have $q_i^{\neg a}$ and $q_i^{\neg b}$ consistent, but for inconsistent q_{i+1}, $q_{i+1}^{\neg a}$ is inconsistent. (To see this, note that if (b) is violated, then changing a makes consistent every inconsistent judgment set in the sequence, and at some point we reach a transition in the sequence from an inconsistent to a consistent judgment set.) Suppose (b) is satisfied. Consider the inconsistent judgment set $s = q_i^{\neg a, \neg b}$. Then $s^{\neg a}$ and $s^{\neg b}$ are consistent by the assumption on q_i. Also, $q_{i+1}^{\neg a} = q_i^{\neg a, \neg c_i}$ is inconsistent, hence $s^{\neg c_i} = q_i^{\neg a, \neg c_i, \neg b}$ is consistent. It follows that $\{a, c_i, b\}$ is minimally complex and thus A is connected. □

Proof of Proposition 33 This follows from Proposition 21(b) and the proof of Proposition 22(b). □

References

Baigent N (1987) Preference proximity and anonymous social choice. Q J Econ 102:161–170
Bovens L, Rabinowicz W (2006) Democratic answers to complex questions—an epistemic perspective. Synthese 150:131–153
Brennan G (2001) Collective coherence. Int Rev Law Econ 21:197–211
Dietrich F (2006) Judgment aggregation: (im)possibility theorems. J Econ Theory 126:286–298
Dietrich F (2007) A generalised model of judgment aggregation. Soc Choice Welfare 28:529–565
Dietrich F, List C (2007a) Arrow's theorem in judgment aggregation. Soc Choice Welfare 29:19–33
Dietrich F, List C (2007b) Strategy-proof judgment aggregation. Econ Philos 23:269–300
Dietrich F, List C (2008) Judgment aggregation without full rationality. Soc Choice Welfare 31:15–39
Dokow E, Holzman R (2008) Aggregation of binary evaluations. J Econ Theory
Eckert D, Mitlöhner J (2005) Logical representation and merging of preference information. Multidisciplinary IJCAI-05 workshop on advances in preference handling, pp 85–87
Everaere P, Konieczny S, Marquis P (2005) Quota and Gmin merging operators. In: Proceedings of 19th international joint conference on artificial intelligence
Fishburn PC (1977) Condorcet social choice functions. SIAM J Appl Math 33:469–489
Gärdenfors P (2006) A representation theorem for voting with logical consequences. Econ Philos 22:181–190
Kasher A, Rubinstein A (1997) On the question 'Who is a J', a social choice approach. Logique et Analyse 160:385–395
Kemeny J (1959) Mathematics without numbers. Daedalus 88:571–591
Klamler C (2004) The Dodgson ranking and its relation to Kemeny's method and Slater's rule. Soc Choice Welfare 23:91–102
Klamler C (2005) Borda and Condorcet: some distance results. Theory Decis 59:97–109
Klamler C (2008) A distance measure for choice functions. Soc Choice Welfare 30:419–425
Konieczny S (2004) Belief base merging as a game. J Appl Non-classical Logics 14:275–294
Konieczny S, Lang J, Marquis P (2004) DA^2 merging operators. Artif Intell 157:49–79
Konieczny S, Pino-Pérez R (1998) On the logic of merging. Proceedings of KR'98, pp 488–498
Konieczny S, Pino-Pérez R (1999) Merging with integrity constraints. In: Proceedings of ECSQARU'99, pp 233–244
Konieczny S, Pino-Pérez R (2002) Merging information under constraints: a logical framework. J Logic Comput 12:773–808
Konieczny S, Pino-Pérez R (2004) On merging strategy-proofness. In: Proceedings of the ninth conference on principles of knowledge representation and reasoning (KR'04)
Kornhauser LA, Sager LG (1993) The one and the many: adjudication in collegial courts. California Law Rev 81:1–59
List C (2005) The probability of inconsistencies in complex collective decisions. Soc Choice Welfare 24: 3–32
List C (2006) The discursive dilemma and public reason. Ethics 116:362–402
List C (2008) Which worlds are possible? a judgment aggregation problem. J Philos Logic 37:57–65
List C, Pettit P (2002) Aggregating sets of judgments: an impossibility result. Econ Philos 18:89–110
List C, Pettit P (2004) Aggregating sets of judgments: two impossibility results compared. Synthese 140:207–235
List C, Pettit P (2006) Group agency and supervenience. Southern J Philos 44:85–105
List C, Puppe C (2008) Judgment aggregation: a survey. In: Anand P, Puppe C, Pattanaik P (eds) Oxford handbook of rational and social choice. Oxford University Press, NY, USA (forthcoming)
May KO (1952) A set of independent necessary and sufficient conditions for simple majority decision. Econometrica 20:680–684
Miller MK (2008) Judgment aggregation and subjective decision-making. Econ Philos 24:205–231
Mongin P (2008) Factoring out the impossibility of logical aggregation. J Econ Theory 141:100–113
Nehring K (2003) Arrow's theorem as a corollary. Econ Lett 80:379–382
Nehring K, Puppe C (2007) The structure of strategy-proof social choice. Part I. General characterization and possibility results on median spaces. J Econ Theory 135:269–305
Nehring K, Puppe C (2008) Consistent judgment aggregation: the truth-functional case. Soc Choice Welfare 31:41–57
Nurmi H (2004) A comparison of some distance-based choice rules in ranking environments. Theory Decis 57:5–24

Pauly M, Hees Mvan (2006) Logical constraints on judgment aggregation. J Philos Logic 35:569–585
Pettit P (2003) Groups with minds of their own. In: Schmitt F (ed) Socializing metaphysics, pp 167–194. Rowan and Littlefield, New York
Pigozzi G (2006) Belief merging and the discursive dilemma: an argument-based account to paradoxes of judgment aggregation. Synthese 152:285–298
Saari DG (2008) Disposing dictators, demystifying voting paradoxes: social choice analysis. Cambridge University Press, NY, USA
Saari DG, Merlin VR (2000) A geometric examination of Kemeny's rule. Soc Choice Welfare 17:403–438
Young HP, Levenglick A (1978) A consistent extenstion of Condorcet's election principle. SIAM J Appl Math 35:285–300