Market Design and Walrasian Equilibrium[†]

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Abstract

We establish the existence of Walrasian equilibrium for economies with many discrete goods and possibly one divisible good. Our goal is not only to study Walrasian equilibria in new settings but also to facilitate the use of market mechanisms in resource allocation problems such as school choice or course selection. We consider all economies with quasilinear gross substitutes preferences. We allow agents to have limited quantities of the divisible good (limited transfers economies). We also consider economies without a divisible good (nontransferable utility economies). We show the existence and efficiency of Walrasian equilibrium in limited transfers economies and the existence and efficiency of strong (Walrasian) equilibrium in nontransferable utility economies. Finally, we show that various constraints on minimum and maximum levels of consumption and aggregate constraints of the kind that are relevant for school choice/course selection problems can be accommodated by either incorporating these constraints into individual preferences or by incorporating a suitable production technology into nontransferable utility economies.

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1. Introduction

In this paper, we establish the existence of Walrasian equilibrium in economies with many discrete goods and either with a limited quantity of one divisible good or without any divisible goods. Our goal is not only to study Walrasian equilibria in new settings but also to facilitate the use of market mechanisms in resource allocation problems such as school choice or course selection. To this end, we develop techniques for analyzing allocation problems in economies with or without transfers and for incorporating additional constraints into allocation rules. In particular, we show that distributional requirements (for example, a rule stating that every BA student must take at least 2 science courses, 2 humanities courses and 1 social science course for credit) can be incorporated into preferences in a manner consistent with the resulting economy having a Walrasian equilibrium. Such requirements are common in US universities. We also show that aggregate constraints that restrict the total number of seats in a set of classes can be rendered consistent with the existence of Walrasian equilibrium by incorporating a suitable production technology into the ecoomy.

In Kelso and Crawford (1982)'s formulation of the competitive economy, there is a finite number of goods, and a finite number of consumers with quasilinear utility functions that satisfy the *substitutes property*.[1] Kelso and Crawford also assume that each consumer is endowed with enough of the divisible good to ensure that she can purchase any bundle of discrete goods at the equilibrium prices. This last condition would be satisfied, for example, if each consumer had more of the divisible good than the value she assigns to the aggregate endowment of indivisible goods. We call the Kelso-Crawford setting the *transferable utility economy*. Kelso and Crawford's ingenious formulation of the substitutes property facilitates their existence theorem as well as a tatonnement process/dynamic auction for computing Walrasian equilibrium. Subsequent research has identified various important properties of Walrasian equilibrium in transferable utility economies.

Our first goal is to do away with the assumption that each consumer has enough of the divisible good to purchase whatever she may wish at the equilibrium prices. In

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[1] The substitutes property requires that if x is an optimal consumption bundle at prices p and prices increase (weakly) to some $\hat{p}$ then the agent must have an optimal bundle at $\hat{p}$ which has her consuming at least as much of every good that did not incur a price increase.

particular, we allow for arbitrary positive endowments of the divisible good. We call this the *limited transfers economy*. We also consider the *nontransferable utility* economy; that is, we consider economies in which there is no divisible good. This setting is particularly well-suited for the analyzing many allocation problems such as school choice or course selection.

Substitutes preferences have been used to analyze a variety of market design problems. Then, results suggesting that the Walras equilibrium correspondence is nearly incentive compatible when there are sufficiently many agents (see, for example, Roberts and Postlewaite, (1976)) have been invoked to argue that Walrasian methods can play a role in market design. In most of these applications it is unreasonable to assume that each agent has enough of the divisible good to acquire whatever she wishes. In many applications, transfers (i.e., the divisible good or equivalently, money) are ruled out altogether and the problem is one of assigning efficiently and fairly a fixed number of objects to individuals. Hence, both the limited transfers economy and the nontransferable utility economy are of interest.

Theorem 1 establishes the existence of a Walrasian equilibrium (henceforth, equilibrium) in random allocations for limited transfers economies. In the transferable utility case, randomization is not necessary since in such economies, a random equilibrium allocation at prices p is simply a probability distribution over deterministic competitive equilibria at prices p. However, in both limited transfers and nontransferable utility economies, randomization is necessary for the existence of equilibrium. The following simple example establishes this fact.

Example 1: There are two agents and a single good. Both agent's utility for the good is 2 and both have one unit of the divisible good. Without randomization, if the price is less than or equal to 1, both agents will demand the good; if the price is greater than 1 neither will demand the good. Since there is exactly one unit of the good, there can be no deterministic equilibrium for this economy. If randomization is allowed, the equilibrium price of the indivisible good is 2 and each agent will get the indivisible good with probability $\frac{1}{2}$. Note that the above example is non-generic; that is, if the money endowments are not

equal (but smaller than 2) then there is an equilibrium without randomization in which the agent with the larger money endowment receives the good.

Example 1, above, features utilities that satisfy the substitutes property and, as a result, a competitive equilibrium exists. Example 2, below, illustrates the failure of existence whe utilities do not satisfy the substitutes property.

Example 2: The economy has three agents and three indivisible goods. Initially, agents 1 and 2 each have 1 unit of the divisible good and no divisible goods. Agent 3's initial endowment consists of the three goods and zero units of the divisible good. For agents 1 and 2,

$$u_i(A) = \begin{cases} 0 & \text{if } |A| < 2 \\ 2 & |A| \geq 2 \end{cases}$$

while $u_3(A) = 0$ for all A. Since the three goods are perfect substitutes, in equilibrium, all three must have the same price. Let r be this common price. Clearly, $r = 0$ is impossible in any equilibrium since both agents 1 and 2 would demand at least 2 goods with probability 1 and market clearing would fail. Then, in any equilibrium, $r > 0$ and agents 1 and 2 must consume the three goods with probability 1. This implies $r \cdot 3 \leq 2$ and hence $r \leq 2/3$. At $r \leq 2/3$ both agent 1 and 2 will want to consume any 2 of the 3 goods with probability $1/(2r)$ and 0 goods with probability $1 - 1/(2r)$. Market clearing requires, at a minimum, that the expected total consumption of the these two agents is 3. Hence, $2 \cdot 2 \cdot 1/(2r) = 3$ and therefore $r = 2/3$. This means that the unique optimal random consumption bundle for agent $i = 1, 2$ at these prices is the distribution that assigns her 2 goods with probability $3/4$ and zero goods with probability $1/4$. This pair of random consumptions is feasible in expectation but is not *implementable.* That is, there is no random allocation that yields this random consumption to both consumers at prices $p_1 = p_2 = p_3 = 2/3$. To see why, note that in any state of the world in which player 1 is allocated 2 goods, player 2 must be allocated either 1 good, which is never optimal for him, or 0 goods. However, consuming 0 goods with probability $3/4$ is not optimal for player 2.

The utility function of agents 1 and 2 in Example 2 does not satisfy the *substitutes property.*[2] Our main results show that examples such as the one above cannot be constructed with utility functions that do.

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[2] Consider price vector p such that $p^1 = p^2 = 0.5, p^3 = 3$. At this price, the unique optimal bundle is $\{1, 2\}$. Next, increase the price of good 2 to $q^2 = 3$ and keep the other prices unchanged. At the new price $q = (0.5, .3.3)$, the unique optimal bundle of divisible goods is $\emptyset$. Hence, the demand for good 1 decreased despite the fact that its price remained the same while another good's price increased.

The assignment of courses to students typically requires a mechanism without transfers, i.e., without a divisible good. To address this and related applications, Theorem 2 demonstrates the existence of a competitive equilibrium for the nontransferable utility economy. Hylland and Zeckhauser (1979) first proposed Walrasian equilibria as an allocation mechanism for the unit demand nontransferable utility economy. They showed that some equilibria may be Pareto inefficient because local non-satiation need not hold in this setting. Nonetheless, Hylland and Zeckhauser (1979) showed that efficient equilibria always exist. Mas-Colell (1992) coins the term *strong equilibrium* for a competitive equilibrium in which every consumer chooses the cheapest utility maximizing consumption and shows that strong equilibria are efficient. Our Theorem 2 establishes the existence of a strong and, therefore, of a Pareto efficient equilibrium.

Allocation problems often feature constraints on individual or group consumption. In course assignment problems, university rules may constrain students' course selection either by imposing distributional requirements of the kind described above or by limiting the number of courses that a student can take for credit from a specified list of courses (as in Princeton's rule of 12 discussed below). In a school choice problem, administrators may restrict parents' choices based on the location of their residence; and, finally, in office allocation problems, choices may be constrained by employee seniority. We analyze such constrained allocation problems in Theorem 3. There, we consider a broad range of constraints on individual consumption and show that our model can incorporate them. In some applications, groups of individuals may face constraints on their joint consumption or there may be aggregate constraints. For example, a university may reserve a certain number of seats in a class for those students who must take this class as a requirement of their majors. In addition, there may be aggregate constraints on lab space that limit the total number of seats available in a collection of related courses. We analyze such constraints in section 3.

1.1 Ordinal versus Walrasian Mechanisms: an Example

Many of the commonly used allocation mechanisms do not entail explicit randomization; randomization, if it takes place at all, does so only as a tie-breaking rule or to order the agents. Relatedly, outcomes of these mechanisms depend only on the ordinal

preferences of the agents and not on their attitude towards random bundles. In one-to-one matching, commonly used ordinal mechanisms such has the Gale-Shapley algorithm or the top-cycles procedure have certain efficiency properties. In the more general, multiple good setting, there is little consensus on the choice of mechanism and virtually no results regarding efficiency. Hylland and Zeckhauser (1979) note that even in unit demand economies, the existing efficiency results rely on an ordinal notion that precludes the possibility of randomization and does not take into account the agents' preferences over lotteries. The simple example below highlights the inefficiency of all ordinal mechanism; that is, mechanisms that depend only on ordinal preferences.

There are two indivisible goods (a and b), three consumers ($1, 2$ and 3) and no divisible good in the economy. The following table summarizes the utility functions of the consumers:

Consumer	$\{a,b\}$	$\{a\}$	$\{b\}$
1	11	10	8
2	11	10	2
3	11	10	2

Note that all three agents have the same ordinal ranking over consumption bundles: they all strictly prefer $\{a,b\}$ to $\{a\}$ and $\{a\}$ to $\{b\}$. Under the Walrasian mechanism, the planner endows each student with a budget of 1 (unit of fiat money) and the goods initially belong to a fictitious agent (the designer/seller).[3] This agent values the fiat money but does not value the divisible goods. Our mechanism allocates courses according to the resulting Walrasian equilibrium lotteries. In this example, the unique equilibrium price is $p_1 = 1, p_2 = p_3 = 2$. At these prices, agent 1 purchases a deterministic lottery consisting of good b. Agents 2 and 3 purchase the lottery that yields the good a with probability $1/2$. The resulting equilibrium utility of student 1 is 8 and the resulting equilibrium utility of students 2 and 3 is 5.

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[3] Alternatively, we could endow agents 1-3 with equal random allocations of the indivisible goods; that is, each agent would own each good with probability $1/3$.

All three students are better off under the Walrasian mechanism than under any ordinal mechanism since any mechanism that allocates the two goods based solely on the agents' ordinal preferences would have to give the same allocation to all three agents. The best symmetric allocation would give each agent each good with probability 1/3 and nothing with probability 1/3. This allocation yields utility 6 for agent 1 and 4 for agents 2 and 3; strictly lower utilities than the Walrasian mechanism for all three agents.

As Budish (2011) notes, the non-existence of deterministic Walrasian equilibria is non-generic. In particular, when agents have unequal money endowments a deterministic competitive equilibrium typically exists. Thus, an alternative approach to randomized allocations would be to assign agents random endowments of money so as to take advantage of the generic existence of equilibrium. While solving the existence problem, this approach would typically lead to an ex ante inefficient allocation. Suppose, in the example above, the designer draws the allocations of fiat money at random from the interval $[1, 2]$. Then, with probability 1, no two agents have identical money endowments and, therefore, a deterministic Walrasian equilibrium exists: the agent with the greatest money endowment would receive good a and the agent with the second greatest money endowment would receive good b. Therefore, deterministic Walrasian equilibria (with randomized endowments of money) would replicate the allocation of random serial dictatorship in this example. Thus, while solving the existence problem, the random allocation of money cannot achieve an ex ante Pareto efficient allocation.

1.2 Related Literature

Kelso and Crawford (1982) establish the existence of a Walrasian equilibrium using an ascending tatonnement process. They show that this process converges to a Walrasian equilibrium price vector. Gul and Stacchetti (1999) argue that, in a sense, Kelso and Crawford's substitutes property is necessary for the existence of equilibrium: given any utility function that does not satisfies the substitutes property, it is possible to construct an N-person economy consisting of an agent with this utility function and $N - 1$ agents with substitutes utility functions that has no Walrasian equilibrium.[4] Hence, their result

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[4] Yang (2017) finds an error in Gul and Stacchetti's proof and supplies an alternative proof.

shows that it is impossible to extend Kelso and Crawford's existence result to a larger class of utility functions than those that satisfy the substitutes property.

Sun and Yang (2006) provide a generalization of the Kelso-Crawford existence result that allows for some complementarities in a transferable utility economy. They circumvent Gul and Stacchetti's impossibility result by imposing joint restrictions on agent's preferences. In particular, they assume that the set of indivisible goods can be partitioned into two sets such that all agents consider goods within each element of the partition substitutes and goods in different partition elements complements.

As noted above, a special class of substitutes preferences are the unit-demand preferences. These preferences are suitable for situations in which agents can consume at most one unit of the divisible good. Leonard (1983) studies transferable utility unit-demand economies and identifies an allocation rule that generalizes the second-price auction and has strong incentive and efficiency properties. His allocation rule is the Walrasian rule together with the lowest equilibrium prices. Hylland and Zeckhauser (1979) are the first to study what we have called a nontransferable utility unit demand economy. They establish the existence of an efficient Walrasian equilibrium in such economies. Hylland and Zeckhauser's work has led to a literature on competitive equilibrium solutions to market design problems: Ashlagi and Shi (2016) study competitive equilibrium with equal incomes in a market with continuum of agents. Le (2017), He, Miralles, Pycia and Yan (2015) and Echenique, Miralles and Zhang (2018) maintain the assumption of unit-demand preferences, but allow for general endowments, non-EU preferences[5] or priority-based allocations. Mas-Colell (1992) and McLennan (2018) study more general convex economies with production.

There are two major differences between these papers and ours. First, most of these papers introduce some notion of "slackness" into the definition of Walrasian equilibrium to guarantee its existence, while the equilibrium notion in our paper is standard. Second, they focus on convex (or convexified) economies and thus there is no implementability problem. In our set-up implementability is the key is issue. We provide a discussion of the second point after the statement of Theorem 1.

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[5] That is, in the convexified economy, they allow utility functions that are nonlinear in probabilities.

Budish, Che, Kojima and Milgrom (2013) study a variety of probabilistic assignment mechanisms. Our work relates to section 4 of their paper, where they define and show the existence of what they call pseudo-Walrasian equilibrium.[6] In that section, they consider fully separable preferences and establish the existence of efficient pseudo-Walrasian equilibrium. They also describe how individual constraints can be incorporated into pseudo-Walrasian equilibrium.

In Appendix B of their paper, they consider a richer class of preferences adopted from Milgrom (2009). These preferences amount to the closure of unit-demand preferences under satiation and convolution.[7] Ostrovsky and Paes Leme (2015) prove that the closure of unit-demand preferences under endowment and convolution yields a strict subset of substitutes preferences. They identify a rich class of preferences that belong to the latter but not the former. It is easy to check that this class of preferences is also excluded from the class described in Appendix B of Budish, Che, Kojima and Milgrom (2013). Thus, compared to Budish, Che, Kojima and Milgrom (2013), we consider a richer class of preferences and a richer class of constraints. In particular, their analysis of pseudo-Walrasian equilibrium does not include group constraints or aggregate constraints, other than bounds on the aggregate supply of each good.

Kojima, Sun and Yu (2018) study constraints in a transferable utility economy. They show that imposing upper and lower bounds on quantities consumed (i.e., interval constraints) on gross substitutes preferences preserves the gross substitutes property. They also show that a slight generalization of interval constraints are the only ones that preserve the gross substitutes property for every gross substitute utility function. Lemma 1 below and the discussion prior to it is related to their first result regarding interval constraints. Our main results focus on limited transfers and nontransferable utility economies and allow for joint restrictions on the utility function and the constraints (and hence permit a larger set of constraints than Kojima, Sun and Yu (2018)).

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[6] Presumably, the qualifier pseudo is to indicate the interjection of fiat money and also to acknowledge various additional constraints that are typically not a part on the definition of a competitive economy. By incorporating these constraints into preferences and technology and by assuming that the mechanism designer/seller values the fiat money, we are able to interpret our equilibria as proper Walrasian equilibria.

[7] See section 2.1 for a discussion of closures under substitutes preserving operations.

2. The Substitutes Property and the Limited Transfers Economy

Let $H = \{1, \ldots, L\}$ be the set of goods. Subsets of H are consumption bundles.[8] We identify each $A \subset H$ with $x \in X := \{0,1\}^L$ such that $x^j = 1$ if and only if $j \in A$. Hence, $o = (0, \ldots, 0) \in X$ is identified with the empty set.

For any $x \in X$, let $\operatorname{supp}(x) = \{k \in H \,|\, x^k = 1\}$ and $\sigma(x) = \sum_j x^j$. A utility on X is a function $u : X \to \mathbb{R} \cup \{-\infty\}$. The effective domain of u, denoted $\operatorname{dom} u$, is the set $\operatorname{dom} u = \{x \in X : -\infty < u(x)\}$. Without loss of generality, we normalize u so that $u(x) \geq 0$ for all $x \in \operatorname{dom} u$. Throughout, we adopt the following convention: $-\infty + (-\infty) = -\infty \geq -\infty$. We assume that every agent's overall utility function is quasilinear in the divisible good. Given any price vector $p \in \mathbb{R}_+^L$, we let $U_i(x, p) = u(x) - p \cdot x$ denote the agent's objective function.[9]

For $x, y \in \mathbb{R}^L$, we write $x \leq y$ to mean that each coordinate of x is no greater than the corresponding coordinate of y and let $x \wedge y$ denote $z \in X$ such that $z^j = \min\{x^j, y^j\}$ for all j. Similarly, let $x \vee y$ denote $z \in X$ such that $z^j = \max\{x^j, y^j\}$ for all j. Without risk of confusion, we sometimes refer to u as the utility function (instead of saying the utility index associated with the utility function U). We let $\chi^j \in X$ denote the good j; that is, $\chi^j(k) = 1$ if $k = j$; otherwise, $\chi^j(k) = 0$. Similarly, for any set of indivisible goods $A \subset H$, define $\chi^A \in X$ as follows:

$$\chi^A(k) = \begin{cases} 1 & \text{if } k \in A \\ 0 & \text{otherwise} \end{cases}$$

Throughout, we will assume that $\operatorname{dom} u \neq \emptyset$ and u is monotone; that is, $x \leq y$ implies $u(x) \leq u(y)$.

2.1 The Substitutes Property and the Transferable Utility Economy

Define the transferable utility demand correspondence for u as follows:

$$D_u(p) := \{x \in X \,|\, u(x) - p \cdot z \geq u(y) - p \cdot y \text{ for all } y \in X\}$$

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[8] We are assuming that there is a single unit of each good. This assumption makes the analysis of the implementability problem easier and is without loss of generality, since we can label each of the multiple units of a good as a distinct good. Equilibrium will ensure that each of these units has the same price.

[9] If the agent has an endowment of indivisible goods, the objective function is unchanged since the value of the endowment enters the utility function as a constant.

Since dom $u \neq \emptyset$ and $p \in \mathbb{R}^L_+$, $D_u(p)$ will always lie in the effective domain. The substitutes property states the following: let x be an optimal consumption bundle at prices p and assume that prices increase (weakly) to some $\hat{p}$. Then, the agent must have an optimal bundle at $\hat{p}$ which has her consuming at least as much of every good that did not incur a price increase. The formal definition is as follows:

Definition[10]: *The function u has the substitutes property if $x \in D_u(p)$, $p \leq \hat{p}$, $\hat{p}^j = p^j$ for all $j \in A$ implies there exists $y \in D_u(\hat{p})$ such that $y^j \geq x^j$ for all $j \in A$.*

Kelso and Crawford (1982) introduced the substitutes property. Since then, numerous alternative characterizations have been identified. For example, the substitutes property is equivalent to $M^\sharp$-concavity: the function u is $M^\sharp$-*concave* if for all $x, y \in \text{dom } u$, $x^j > y^j$ implies $[u(x - \chi^j) + u(y + \chi^j) \geq u(x) + u(y)$ or there is k such that $y^k > x^k$, $u(x - \chi^j + \chi^k) + u(y + \chi^j - \chi^k) \geq u(x) + u(y)]$.[11] Gul and Stacchetti (1999) show that that if u satisfies the substitutes property, then it must be submodular:[12]

$$u(x) + u(y) \geq u(x \vee y) + u(x \wedge y)$$

If the inequality above always holds with equality, then u is additive.

Perhaps the best-known subclass of substitutes utility functions are unit demand utilities. These utility functions are appropriate for situations in which each agent can consume at most one unit of the indivisible goods: u is a *unit demand utility* if

$$u(x) = \max\{u(\chi^j) \,|\, \chi^j \leq x\}$$

Below, we describe operations on gross substitutes utility functions that enable us to derive new gross substitute utility functions. In section 3, we use these operations to incorporate additional (curricular) restrictions in course selection and school choice

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[10] Generally, the substitutes property should be defined for arbitrary price vector $p \in \mathbb{R}^L$. However, as we assume monotone utility functions, it suffices to define it for nonnegative price vectors.

[11] See, for example, Shioura and Tamura (2015), Theorem 4.1.

[12] Gul and Stacchetti (1999) show this result for u such that dom $u = H$. Their result extends immediately to utilities functions u such that dom $u \neq X$.

problems. Let $k > 0$ be an integer, $z \in X$ and u, v be two substitutes utility functions. Define,

$$u^z(x) = u(x \wedge z)$$

$$u_z(x) = u(x \vee z) - u(z)$$

$$(u \odot v)(x) = \max_{y \leq x} \{u(y) + v(x - y)\}$$

$$[u]^k(x) = \max_{\substack{y \leq x \\ \sigma(y) \leq k}} u(y)$$

$$[u]_k(x) = \begin{cases} \max_{\substack{y \leq x \\ \sigma(y) \geq k}} u(y) & \text{if } \sigma(x) \geq k \\ -\infty & \text{otherwise.} \end{cases}$$

Call u^z the z-constrained u, u_z the z-endowed u, $u \odot v$ the convolution (or aggregation) of u, v, $[u]^k$ the k-satiation of u and $[u]_k$ the k-lower bound u. It is easy to verify that a z-endowed u satisfies the substitutes property whenever u does and that the convolution of u and v satisfies the substitutes property whenever u and v both satisfy the substitutes property. Similarly, verifying that a z-constrained utility satisfies the gross substitutes property whenever $\operatorname{dom} u \cap \{x \mid x \leq z\} \neq \emptyset$ is straightforward.[13]

Bing, Lehmann and Milgrom (2004) prove that the k-satiation of the substitutes utility u is a substitutes utility (provided there is at least one k-element set in $\operatorname{dom} u$). The following lemma establishes the substitutes property for k-lower bound u. All proofs are in the Appendix.

Lemma 1: *If u satisfies substitutes property and $\operatorname{dom} u \cap \{x \in X | \sigma(x) \geq k\} \neq \emptyset$, then $[u]_k$ satisfies the substitutes property.*

For any given class of utility functions, $\mathcal{U}$, and set of substitutes preserving operations, τ, let $\tau(\mathcal{U})$ denote the set of all utility functions that can be derived from the elements of $\mathcal{U}$ by repeatedly applying various operations in τ. We will call $\tau(\mathcal{U})$ the τ-closure of $\mathcal{U}$. In other words, $\tau(\mathcal{U})$ is the smallest family of utility functions that includes $\mathcal{U}$ and is closed under operations in τ. Clearly, if each element of $\mathcal{U}$ satisfies the substitutes property, then so does the τ-closure of $\mathcal{U}$.

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[13] In some cases, it is easier to verify that the new utility function satisfies $M^\sharp$-concavity which, as we noted above, is equivalent to the substitutes property.

Ostrovsky and Paes Leme (2015) show that the endowment and convolution closure of the set of unit demand preferences is a strict subset of the set of all substitutes preferences.[14] They also provide a rich class of examples that satisfy the substitutes property but are not in the endowment and convolution closure of the set of unit demand preferences. Let τ be the five substitutes preserving operations discussed above and let $\mathcal{U}$ be the set of all unit demand preferences. Then, using Ostrovsky and Paes Leme's arguments, it is easy to verify that $\tau(\mathcal{U})$ is a strict subset of substitutes preferences and excludes the same rich class of preferences that these authors have identified.

Next, we define a new class of utility functions that are useful for describing student preferences over class schedules. We call these utility functions academic preferences since they incorporate curricular requirements into the agent's utility function. The following example provides an illustration of academic preferences.

Example 3: Students are required to take at least three courses for credit and satisfy a distributional requirement by taking courses from at least four out of five categories a, b, c, d, e. Each class that a student takes for credit can potentially meet two of these four requirements. There are ten classes, $H = \{ab, ac, ad, ae, bc, bd, be, cd, ce, de\}$, each identified by the requirements that it might satisfy. Students can use at most one class to meet two different requirements simultaneously. The remaining two requirements must be met with distinct classes.[15] A student may get credit for an additional course if the four courses together meet all five requirements. Hence, a student can meet the distributional requirement by taking either three or four courses for credit, but students who take four courses for credit must cover all of the five categories.

For example, the schedules $\{ab, ac, ad\}$ and $\{ab, bc, cd, de\}$ are both feasible; the former yields 3 course credits, the latter yields 4. The schedule $\{ab, bc, ac\}$ is not feasible since it only meets 3 distributional requirements; the schedule $C = \{ab, bc, cd, da\}$ is feasible but

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[14] They conjecture that the endowment and convolution closure of the set of all weighted matroids is the set of substitutes preferences. Ostrovsky and Paes Leme (2015) note that results from Murota (1996), Murota and Shioura (1999), and Fujishige and Yang (2003) ensure that every *weighted matroid* and hence every *rank function* satisfies the substitutes property. For the definitions of weighted matroid, rank function and other relevant terms and results from matroid theory, see Appendix A where we provide a short proof that weighted matroids satisfy the substitutes property based on Fujishige and Yang (2003)'s result that the substitutes property is equivalent to $M^\natural$-concavity.

[15] Undergraduates need to satisfy distributional requirements in most universities. For example, Northwestern's version allows multiple requirements to be met with a single course in some but not all situations.

only yields 3 course credits since it only meets four distributional requirements. Let Y be the set of all feasible schedules. That is, each $z \in Y$ either has 3 courses and meets four distributional requirements or has four courses and meets five distributional requirements. Let v be an additive utility function on X. A student's objective is to choose a feasible course schedule that maximizes the total utility of the courses she takes for credit.[16] That is, her objective is to maximize the utility function u such that

$$u(x) = \begin{cases} \max_{\substack{y \in Y \\ y \leq x}} v(y) & \text{if there is } y \in Y \text{ such that } y \leq x \\ -\infty & \text{otherwise} \end{cases}$$

In Appendix A, we offer a general definition of academic preferences, show that they include the example above and that they are gross substitutes preferences.

We conclude this section by discussing the existence of equilibrium in a transferable utility economy. Let N be the number of agents in the economy, $\xi \in X^N$ be an allocation and let ξ_i denote agent i's consumption in the allocation ξ. Then (ξ, p) is a (deterministic) Walrasian equilibrium in the transferable utility economy if $\sum_{i=1}^{N} \xi_j \leq \chi^H$, $u_i(\xi_i) - p\xi_i \geq u_i(x) - px$ for all $x \in X$, i and $\sum_{i=1}^{N} \xi_i^a = 1$ if $p^a > 0$ for all $a \in H$. Kelso and Crawford (1982) showed that if preferences satisfy monotonicity and the substitutes property, and the effective domain is X, then there exists an equilibrium with (deterministic) allocation in the transferable utility economy. To see how the result can be extended to general effective domains, note that since $\operatorname{dom} u_i \neq \emptyset$, the demand satisfies $D_{u_i}(p) \subseteq \operatorname{dom} u_i$ for all i and p. Hence, for any candidate equilibrium allocation ξ, we have $\xi_i \in \operatorname{dom} u_i$ for each i. The existence of such an allocation is a necessary condition for existence of an equilibrium and we will incorporate it into the definition of a transferable utility economy.

Definition: $\mathcal{E} = \{(u_i)_{i=1}^N\}$ *is a transferable utility economy if,* u_i *satisfies the substitutes property for all* i *and there exists an allocation* ξ *such that* $\sum \xi_i \leq \chi^H$ *and* $\xi_i \in \operatorname{dom} u_i$ *for all* i.

The following lemma states that the additional condition above is also sufficient for the existence of an equilibrium when preferences satisfy the substitutes property.

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[16] The example is meant to show that a rich set of curricular restrictions can be accommodated with academic preferences. Simpler versions can easily be constructed. For example, replacing the condition "a student may get credit for an additional course if the four courses together meet all five distributional requirements" with "a student who fulfills the requirements for 3 course credits can get an additional credit by taking a fourth course" would also yield an academic preference.

Lemma 2: *The transferable utility economy has a (deterministic) equilibrium.*

Notice that efficient allocations of indivisible goods, optimal demands and Walrasian equilibria are independent of the initial endowments in the transferable utility economy. Therefore, the definition of the transferable utility economy omits them. However, endowments will matter in the limited transfers economy, defined in the next section.

2.2   The Limited Transfers Economy

Example 1 shows that when agents have limited budgets, a deterministic equilibrium may not exist even in the simplest limited transfers economies. Thus, we need to extend our definition of an allocation to allow randomness: a random consumption (of indivisible goods) $\theta : X \to [0,1]$ is a probability distribution on X; that is, $\sum_x \theta(x) = 1$. Let Θ denote the set of all random consumptions. For $\theta \in \Theta$, let $\bar{\theta} \in I\!\!R_+^L$ denote the coordinate-by-coordinate mean of θ; that is $\bar{\theta}^j = \sum_x \theta(x) \cdot x^j$. We assume that u is also the agent's von Neumann-Morgenstern utility function. Hence,

$$u(\theta) = \sum_z u(z)\theta(z)$$

The effective domain of u on Θ consists of all the random consumptions such that $\theta(x) > 0$ implies $x \in \operatorname{dom} u$.[17]

Quasilinearity allows us to ignore randomness in the consumption of the divisible good and identify each random consumption of the divisible good with its expectation. Hence, we define the von Neumann utility function U as follows:

$$U(\theta, p) = u(\theta) - p \cdot \bar{\theta}$$

Let $w_i \in X$ denote agent i's endowment of indivisible goods and let b_i denote her endowment of the divisible good. For some applications, it is useful to have an additional agent, the seller or market designer, who holds some or all of the aggregate endowment of the indivisible goods. We will sometimes refer to this agent as agent 0 and assume that

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[17]   The function $u : \Theta \to [0,1]$ is continuous on the effective domain but not on the whole domain. For example, suppose that $\operatorname{dom} u = X\backslash\{o\}$, $x \in \operatorname{dom} u$ and take a sequence of random consumptions θ^n such that $\theta^n(o) = 1/n$ and $\theta^n(x) = 1 - 1/n$. Clearly, $u(\theta^n) = -\infty$ for all n, but $\lim \theta^n = \theta$ such that $\theta(x) = 1$ and therefore, $u(\theta) \neq -\infty$.

she derives no utility from the indivisible goods; she only values the divisible good. The aggregate endowment of indivisible goods in the economy is $\chi^H := (1, \ldots, 1) \in X$ and, therefore, the seller's endowment of the indivisible goods is $w_0 = \chi^H - \sum_{i=1}^{N} w_i$. We will assume that $w_i \in \text{dom } u_i$ for each i to guarantee that agent i can afford at least one bundle in the effective domain.

Definition: $\mathcal{E}^o = \{(u_i, w_i, b_i)_{i=1}^{N}\}$ is a *limited transfers economy* if, for all i, u_i satisfies the substitutes property, $b_i > 0$ and $w_i \in \text{dom } u_i$.

A random allocation (of indivisible goods) for this economy is a probability distribution $\alpha : X^N \to [0, 1]$. For any such α, let α_i denote the i'th marginal of α; that is, $\alpha_i \in \Theta$ is the random consumption of agent i, where

$$\alpha_i(x) = \sum_{\{\xi : \xi_i = x\}} \alpha(\xi)$$

A random allocation α is *feasible* for the economy $\mathcal{E}^o$ if, for all ξ such that $\alpha(\xi) > 0$, $\xi_i \in \text{dom } u_i$ for all i and $\sum_{i=1}^{N} \xi_i \leq \chi^H$.

The budget (set) of an agent with endowment w, b at prices p is

$$B(p, w, b) = \left\{ \theta \in \theta \,\middle|\, p \cdot \bar\theta \leq p \cdot w + b \right\}$$

Then, $\theta \in B(p, w, b)$ is optimal for agent i given budget $B(p, w, b)$ if

$$U_i(\theta, p) \geq U_i(\theta_o, p)$$

for all $\theta_o \in B(p, w, b)$.

Definition: A price $p \in \mathbb{R}_+^L$ and a random allocation α is an *equilibrium* for the limited transfers economy $\mathcal{E}^o$ if

(1) α is feasible for the economy $\mathcal{E}^o$;

(2) for all i, α_i is optimal for agent i given budget $B(p, w_i, b_i)$;

(3) $p^j > 0$ and $\alpha(\xi) > 0$ imply $\sum_{i=1}^{N} \xi_i^j = 1$.

Theorem 1: *Every limited transfers economy has an equilibrium.*

Our proof relies on the existence of equilibrium in the transferable utility economy. We seek a $\lambda_i \in (0, 1]$ for each agent i and a Walrasian equilibrium (p, α) for the modified *transferable utility* economy (with random consumption) in which each u_i is replaced by

$$\tilde{u}_i = \lambda_i u_i$$

such that each agent i spends, in expectation, (1) no more that b_i on indivisible goods and (2) exactly b_i on indivisible goods if $\lambda_i < 1$. It is possible to decrease an agent's equilibrium spending as much as needed by decreasing that agent's λ_i. Hence, we can satisfy condition (1). A fixed-point argument ensures that we can also satisfy condition (2). The Walrasian equilibria for this modified economy are then shown to be equilibria of the original economy. The reason we choose this proof strategy is because it allows us to solve the implementability problem by using properties of the unconstrained economy: in the unconstrained economy every (randomized) Walrasian equilibrium must simply be a randomization over deterministic equilibria and, therefore, the allocation must be implementable.

3. Nontransferable Utility Economies and Constraints

In this section, we will consider allocation problems in settings without a divisible good. We call this type of an economy a *nontransferable utility economy*. In many applications, nontransferable utility economies impose constraints on individual consumption, or on the consumption of groups. To address some of these applications, we show how our model can incorporate a variety of individual, group, and aggregate constraints. An individual constraint restricts the number of goods that a single agent can consume from a specified set of goods. A group constraint restricts the total number of goods that can be consumed from a specified set of perfect substitutes by a particular group of agents. Finally, aggregate constraints restrict the various combinations of goods available for the entire population.

An example of an individual constraint is Princeton University's *rule of 12*. According to this rule, no more than 12 courses in a student's major may be counted towards the 31 courses needed to obtain the A.B. degree. Distribution requirements are a second

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type of individual constraint. For example, Art and Archaeology students at Princeton University must take at least one course in each of the following three areas: group 1 (ancient), group 2 (medieval/early modern), and group 3 (modern/contemporary). An example of a group constraint is the requirement that at least 50 percent of the slots in each school should go to students who live in the school's district. Similarly, the so-called "controlled choice" constraints in school assignment that require schools to balance the gender, ethnicity, income, and test score distributions among their students, are group constraints.[18] Aggregate constraints define the feasible allocations for the entire economy. For example, suppose two versions of introductory physics are being offered: Phy 101, the version that does not require calculus and Phy 103, the version that does require calculus. Suppose each of these classes can accommodate 120 students, but because both courses have lab requirements and lab facilities are limited, the total enrollment in the two courses can be no greater than 200 students.

In the next subsection, we describe the nontransferable utility economy, define a strong (Walrasian) equilibrium and establish its existence and efficiency. Section 3.2 deals with individual, group and aggregate constraints.

3.1 Nontransferable Utility Economies

In a nontransferable utility economy, each agent i has a substitutes utility function u_i and a quantity b_i of fiat (or artificial) money. Initially, the entire aggregate endowment belongs to the market designer. Each agent's utility depends only on her consumption of indivisible goods. That is, agents solve the following utility maximization problem:

$$U_i(p, b_i) = \max u_i(\theta) \text{ subject to } p \cdot \bar{\theta} \le b_i$$

Hence, U_i is the indirect utility function of agent i.

Definition: $\mathcal{E}^* = \{(u_i, b_i)_{i=1}^N\}$ is a nontransferable utility economy if, for all i, u_i satisfies the substitutes property, $o \in dom\, u_i$ and $b_i > 0$.

In the nontransferable utility setting, Walrasian mechanisms provide a rich menu of allocation rules with desirable properties. The designer may accommodate fairness

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[18] See Abdulkadiroğlu, Pathak and Roth (2005) for examples of such constraints in practice.

concerns by choosing the agents' endowments of fiat money (the b_i's) appropriately. In particular, choosing the same b_i for every agent ensures that the resulting allocations are envy-free. This is the setting for many allocations problems such as school choice, course selection or office selection (for example, when a business or a department moves into a new building). In such markets, the Walras correspondence can serve both as real allocation mechanism and as a benchmark for evaluating other mechanisms.

Hylland and Zeckhauser (1979) note that in a nontransferable utility economy with unit-demand preferences, some Walrasian equilibria are inefficient. Specifically, nontransferable utility economies may have equilibria in which some agents do not purchase the least expensive optimal option in their budget sets and equilibria with this property may be inefficient. To address this problem, Mas-Colell (1992) introduces the concept of a *strong* equilibrium; that is, a Walrasian equilibrium in which every consumer chooses the least expensive optimal bundle and proves that strong equilibria are Pareto efficient.

Definition: *A price $p \in \mathbb{R}^L_+$ and a random allocation α is a strong equilibrium for the nontransferable utility economy $\mathcal{E}^*$ if*

(1) *α is feasible for the economy $\mathcal{E}^*$;*

(2) *for all i, α_i is optimal given budget $B(p, b_i)$ and costs no more than any other optimal random consumption;*

(3) *$p^j > 0$ and $\alpha(\xi) > 0$ imply $\sum_{i \geq 1} \xi_i^j = 1$.*

The theorem below establishes the existence of a strong and, therefore, Pareto efficient equilibrium for the nontransferable utility economy.

Theorem 2: *The nontransferable utility economy has a strong equilibrium.*

Our proof of Theorem 2 relies on Theorem 1: we consider the sequence of limited transfers economies $\mathcal{E}^o_n = \{(nu_i, w_i, b_i)_{i=1}^k\}$ for $n = 1, 2 \ldots$ where $w_i^j = 0$ for all j and i. Hence, $\mathcal{E}^o_n$ is a limited transfers economy in which agent i's endowment of goods is equal to her endowment of goods in $\mathcal{E}^*$ (i.e., zero), her endowment of the divisible good is the same as her endowment of fiat money in $\mathcal{E}^*$ and her utility function is n-times her utility function in $\mathcal{E}^*$. Then, we appeal to Theorem 1 to conclude that each $\mathcal{E}^o_n$ has an equilibrium (p^n, α^n). Since this sequence lies in a compact set, it has a limit point which we show to

be an equilibrium of $\mathcal{E}^*$. This equilibrium must be a strong since it is a limit point of equilibria in which money has intrinsic value.

3.2 Group Constraints

In many applications, one group is given priority over another. For example, suppose that the maximal enrollment in a particular physics class is n and there are $m < n$ physics majors who are required to take that class. Thus, at most $n - m$ non-majors can enroll in the class. More generally, a group constraint (A, n) for the group $I \subset \{1, \ldots, N\}$ states that the agents in I can collectively consume at most n units from the set A, where A is a collection of *perfect substitutes* (for all agents).

To accommodate this constraint, pick any $|A| - n$ element subset B of A. Then, replace each u_i for $i \in I$ with u_i' such that

$$u_i'(x) = u_i \left(x \wedge \chi^{B^c} \right)$$

Thus, the new utility for members of group I is the their original utility restricted to the complement of B. As we noted above, restrictions of utilities to a subset of choices satisfy the substitutes property if the original utility satisfies the substitutes property. Moreover, since elements of A are perfect substitutes, restricting members of group I to B^c is equivalent to restricting their aggregate consumption of the good represented by the elements of A. Thus, a group constraint can be accommodated by modifying utility functions of the group's members.

3.3 Individual Constraints

The simplest individual constraints are bounds on the number of goods an agent may consume from a given set of goods. For example, a student may be required to take 4 classes each semester, but may be barred from enrolling in more than 6. We can incorporate this constraint by modifying the student's unconstrained utility function u as follows:

$$u^m(x) = \max_{\substack{y \leq x \\ \sigma(y) \leq 6}} \hat{u}(y)$$

where

$$\hat{u}(x) = \begin{cases} u(x) & \text{if } \sigma(x) \geq 4 \\ -\infty & \text{if } \sigma(x) < 4 \end{cases}$$

The modified utility u^m incorporates the lower bound constraint by restricting the effective domain of u to those bundles that satisfy the constraint. It incorporates the upper bound by imposing satiation above the constraint. Next, we generalize these constraints and impose bounds on overlapping subsets of goods. To preserve the gross substitutes property, we require the utility function to be separable across subsets of goods that must satisfy a constraint. A collection of goods, $A \subset H$, is a *module* for the utility u if

$$u(x) = u(x \wedge \chi^A) + u(x \wedge \chi^{A^c})$$

Note that this condition is symmetric: if A is a module, then so is A^c. For example, suppose that A is the set of all humanities courses and A^c is the set of all other courses. If a student's utility for various combination of humanities courses is independent of her utility over various combinations of the other courses, then A is a module. A collection of sets, $\mathcal{H}$, is a *hierarchy* if $A, B \in \mathcal{H}$ and $A \cap B \neq \emptyset$ implies $A \subset B$ or $B \subset A$. Given any u, we say that the hierarchy $\mathcal{H}$ is *modular* if each element of $\mathcal{H}$ is a module of u.

A modular constraint places bounds on the agent's consumption for subsets of items that form a modular hierarchy. The collection $c = \{(A(k), (l(k), h(k))_{k=1}^K\}$ is a *constraint* if $l(k), h(k)$ are integers, $A(k) \subset H$, and $X_c \cap X^c \neq \emptyset$, where

$$X_c := \{x \in X \,|\, \forall k, \sigma(x \wedge \chi^{A(k)}) \geq l(k)\}$$

are consumptions that satisfy the lower bound and

$$X^c = \{x \in X \,|\, \forall k, \sigma(x \wedge \chi^{A(k)}) \leq h(k)\}$$

are consumptions that satisfy the upper bound. The constraint $c = \{(A(k), (l(k), h(k))_{k=1}^K\}$ is a *modular constraint* for u if $\mathcal{H} = \{A(1), \ldots, A(K)\}$ is a modular hierarchy for u and $(X^c \cap X_c) \cap \operatorname{dom} u \neq \emptyset$.

As an example of a modular constraint, suppose that students must take at least 3 humanities classes and at least 4 social science classes; moreover, each student is required to take at least 8 but no more than 12 classes overall. In this case, the constraint is modular if the student's utility over combinations of science courses is independent of her utility over combinations of humanities courses.

Given a utility u, let $\hat{u}_c$ be the utility function with effective domain X_c; that is,

$$\hat{u}_c(y) = \begin{cases} u(y) & \text{if } y \in X_c \\ -\infty & \text{otherwise.} \end{cases}$$

Finally, define $u(c, \cdot)$ as follows:

$$u(c, x) = \max\left\{\hat{u}_c(y) \,\middle|\, y \in X^c, y \leq x\right\}$$

Then, the effective domain of $u(c, \cdot)$ is $\operatorname{dom} u(c, \cdot) = X_c \cap \operatorname{dom} u \neq \emptyset$.

Lemma 3: *If u satisfies the substitutes property and c is a modular constraint for u, then $u(c, \cdot)$ satisfies the substitutes property.*

To see how the substitutes property may fail if the constraints are not modular, consider the utility function described in equation (1) below. Let $H = \{0, 1, 2, 3\}$. Then,

$$u(x) = \begin{cases} 2 & \text{if } x^j \cdot x^{j \oplus 1} > 0 \text{ for some } j \in H \\ 0 & \text{if } x = o \\ 1 & \text{otherwise} \end{cases} \tag{1}$$

where $\oplus$ denotes addition modulo 4. Note that u satisfies the substitutes property.[19] Let $A = \{0, 1\}$ and suppose that the agent is restricted to consuming at most one unit from A. To see that the resulting utility function does not satisfy the substitutes property, set $p^0 = p^1 = p^2 = p^3 = 1/2$. Then, $\{0, 3\}$ is an optimal consumption set at prices p. The substitutes property fails since at prices q such that $q^3 = 2$ and $q^j = p^j$ for $j \neq 3$, there is no optimal bundle that contains item 0.

To see how the substitutes property may fail if the modular constraints do not form a hierarchy, consider the following utility: $u(x) = \sigma(x)$. Let $H = \{0, 1, 2, 3\}$, then any subset of H is a module of u. Suppose the constraints are $(\{1, 2\}, 0, 1)$, $(\{0, 1\}, 0, 1)$ and $(\{0, 1, 2, 3\}, 0, 2)$. Then, at $p^j = 1/2$ for all $j \in H$, $\{1, 3\}$ is an optimal consumption set at p. Again, the substitutes property fails since at prices q such that $q^3 = 2$ and $q^j = p^j$ for $j \neq 3$, there is no optimal consumption set that contains 1.

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[19] It is a convolution of the two unit demand preferences v and $\hat{v}$ where v takes the value 1 at any x such that $x^0 > 0$ or $x^2 > 0$ and is equal to zero otherwise and $\hat{v}$ takes the value 1 at any x such that $x^1 > 0$ or $x^3 > 0$ and is equal to zero otherwise.

To establish existence of an equilibrium that meets all of the constraints, we must not only assume the existence of an allocation that yields a consumption in the effective domain of every each agent's modified utility but we must also guarantee that the "interior" of each agent's budget set contains some consumption in the effective domain of her modified utility. To address this issue, we add two assumptions to our earlier model. First, we assume that aggregate resources can be divided into $N+1$ consumption bundles that meet every consumer's lower bound constraint. Second, we assume that all agents have equal endowments of fiat money, which we normalize to 1.

Definition: $\mathcal{E}_c^* = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N\}$ is a nontransferable utility economy with modular constraints if,

(i) for all i, u_i satisfies the substitutes property;

(ii) for all i, c_i is a modular constraint for u_i;

(iii) there is $x_1, \cdots, x_{N+1}$ such that $\sum_{k=1}^{N+1} x_k \leq \chi^H$ and $x_k \in dom\, u_i(c_i, \cdot)$ for all k, i.

Clearly, an equilibrium of $\mathcal{E}_c^*$ will satisfy all of the lower bound constraints.[20] If $\alpha(\xi) > 0$ and ξ_i violates some upper bound constraint, then there must be some $x \leq \xi_i$ that satisfies all of the constraints such that $u_i(c_i, x) = u_i(c_i, \xi_i)$. Then, $p^j = 0$ for all j such that $\xi_i^j > x^j$ since otherwise ξ could not be a *strong* equilibrium. Therefore, we can replace α with a random allocation β that satisfies all of the upper bound (and lower bound) constraints such that (p, β) is also a strong equilibrium of $\mathcal{E}_c^*$. Hence, the existence of a strong equilibrium ensures the existence of a strong equilibrium that satisfies all of the constraints in c.

Theorem 3: *A nontransferable utility economy with modular constraints has a strong equilibrium.*

The role of (iii) and equal money endowments is to ensure that every agent can afford a consumption in the effective domain of her utility function. Alternatively, we could assume that there is a subset of goods in abundant supply (goods that have zero price in equilibrium) and agents can choose a consumption in the effective domain from that

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[20] Since all agents have the same budget, if one agent cannot afford any x_i, then the aggregate endowment must cost more than $N+1$. Hence, some goods with positive prices are left in the hands of agent 0, contradicting the definition of equilibrium.

subset. In our course selection application, it may be the case that a subset of classes is never oversubscribed and students can choose courses that meet the requirements from that subset. In that case, item (iii) in the definition above and the assumption of equal budgets could be dispensed with.

3.4 Aggregate Constraints

Suppose an economics department schedules classes in labor economics, intermediate microeconomics and in corporate finance. There are two types of TAs, those that can cover labor economics and microeconomics, and those that can cover microeconomics and corporate finance. There are 60 TAs of each type. TA time is fungible across different classes so that at most 60 students can enroll in labor economics, at most 60 students can enroll in corporate finance and at most 120 students can enroll in any of the three types of classes. Alternatively, consider a company that must design a new office building. The building has at most 200 offices each either small, medium or large; no more than 10 can be large and no more than 25 can be large or medium. In these examples, we can describe the aggregate constraint as a hierarchy $\mathcal{H}$ that limits the supply of available items. That is, the aggregate constraint has the form $c = \{(A(k), n(k))_{k=1}^{K}\}$ such that the $A(k) \subseteq H$ for all k, $\{A(k)\}_{k=1}^{K}$ is a hierarchy and each $n(k)$ is a natural number describing the maximal quantity of indivisible goods that can be supplied from set $A(k)$.

We will re-interpret aggregate constraints as a production technology; that is, define a production possibility set for the economy that includes only the output combinations consistent with the desired constraints. To see how we can embed a collection of aggregate restrictions into a production set, let (A, n) denote a single aggregate restriction. Hence, the set of feasible production plans given any X and the restriction (A, n) is:

$$X(A, n) = \{x \in X \mid \sigma(x \wedge \chi^{A}) \le n\}$$

We can nest aggregate constraints the same way that we nested individual and group constraints; that is, we can construct a hierarchy of aggregate constraints. Given any hierarchy of aggregate restrictions $d = \{(A(k), n(k))_{k=1}^{K}\}$, let $\mathcal{I}_d$ denote the set of all production plans consistent with d; that is,

$$\mathcal{I}_d = \bigcap_{a \in d} X(a)$$

Hierarchical constraints are a special case of constraints that can be captured by a submodular upper bound. Let $f : X \to \{0, 1 \ldots\}$ satisfy the following three properties: (1) $f(0) = 0$; (2) $f(x) \geq f(y)$ if $x \geq y$ and (3) $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$. A collection of consumption bundles $\mathcal{I} \subset X$ has a submodular bound if there is a function f that satisfies properties (1)-(3) such that $\mathcal{I} = \{x \in X \mid \sigma(x) \leq f(x)\}$.[21]

Lemma 4: *If d is a hierarchical collection of aggregate constraints, then $\mathcal{I}_d$ has a submodular bound.*

Production sets with submodular bounds are more general than those obtained with hierarchical constraints as the following example illustrates:

Example: Let $N = \{1, \ldots, 6\}$, let $A = \{1, 2\}, B = \{3, 4\}, C = \{5, 6\}$ and let $\mathcal{I}$ be the set defined by the constraints

$$\sum_{A \cup B} x^j \leq 2; \sum_{A \cup C} x^j \leq 2; \sum_{A} x^j \leq 1; \sum_{N} x^j \leq 3$$

Then, $\mathcal{I}$ has a submodular bound but the constraint is not a hierarchy.

Definition: $\tilde{\mathcal{E}} = \{(u_i, 1)_{i=1}^{N}, \mathcal{I}\}$ *is a production economy with nontransferable utility if $\mathcal{I}$ has a submodular bound and for all i, u_i satisfies the substitutes property and $o \in \operatorname{dom} u_i$.*

In the economy with production, a random allocation α is a probability distribution over $X^N \times \mathcal{I}$. For any such α, the marginal α_i is the random consumption for agent $i = 1, \ldots, N$ and the marginal α_{N+1} is the production plan for the producer or seller. A random allocation α is *feasible* for the economy $\tilde{\mathcal{E}} = \{(u_i, 1)_{i=1}^{N}, \mathcal{I}\}$ if, for all (ξ, z) such that $\alpha(\xi, z) > 0$, $\sum_{i=1}^{N} \xi_i \leq z$, $z \in \mathcal{I}$ and $\xi_i \in \operatorname{dom} u_i$ for all i. The definitions of budget sets and consumer optimality remain unchanged. The random allocation α is *producer optimal* if $\alpha(\xi, z) > 0$ implies $pz \geq pz'$ for all $z' \in \mathcal{I}$. Let $\mathcal{B}$ be the production possibility frontier of the technology $\mathcal{I}$; that is, $\mathcal{B} = \{x \in \mathcal{I} \mid y \geq x, y \in \mathcal{I} \text{ implies } y = x\}$.

Definition: *A price $p \in I\!R_+^L$ and a random allocation α is a *strong equilibrium* for the production economy with nontransferable utility $\tilde{\mathcal{E}}$ if*

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[21] Sets with submodular bounds form a matroid. In Appendix A, we offer two alternative definitions and a few other relevant notions and results from matroid theory.

(1) α is feasible for $\tilde{\mathcal{E}}$;

(2) for all i, α_i is optimal given budget $B(p,1)$ and costs no more than any other optimal random consumption;

(3) α is producer optimal;

(4) $p^j > 0$ and $\alpha(\xi, z) > 0$ imply $\sum_{i=1}^{N} \xi_i^j = z^j$.

Hence, with production, a Walrasian equilibrium specifies prices, a random allocation and a random production plan. The implied random consumption and production plans must be feasible and optimal for both the consumers and the producer. The definition of a strong equilibrium is as in the previous section: the Walrasian equilibrium (p, α) is a strong equilibrium if for each agent i, α_i is the cheapest optimal random consumption for i given the budget constraint.

Theorem 4: *The production economy with nontransferable utility has a strong equilibrium.*

Our definition of a production economy includes the assumption of equal budgets for all consumers. This assumption is for convenience only and we could allow arbitrary positive budgets b_i. Finally, we can add the modular constraints to the production economy:

Definition: $\tilde{\mathcal{E}}_c = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N, \mathcal{I}\}$ *is a production economy with nontransferable utility and modular constraints if*

(1) u_i satisfies the substitutes property for all i;

(2) c_i is a modular constraint for u_i for all i;

(3) for some $z \in \mathcal{I}$, there is $x_1, \cdots, x_{N+1}$ such that $\sum_{k=1}^{N+1} x_k \leq z$ and $x_k \in \mathrm{dom}\, u_i(c_i, \cdot)$ for all k, i.

Part (3) of the definition above requires that some production plan can be divided into $N + 1$ consumptions that satisfy every consumers lower bound constraint. This mirrors a similar assumption in Theorem 3.

Theorem 5: *A strong equilibrium for the production economy with nontransferable utility and modular constraints* $\tilde{\mathcal{E}}_c = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N, \mathcal{I}\}$ *exists if* $\mathcal{I}$ *is a matroid technology. Strong equilibrium allocations are Pareto efficient.*

In Theorems 3 and 5, the assumption of equal money endowments ensures that every consumer can afford some element in the effective domain of her utility function. If money endowments were arbitrary, then we would need to add an assumption that preserves this feature. By contrast, the assumption of equal money holdings plays no role in Theorem 4 because in that case all non-negative consumption plans are in the effective domain of the consumer's utility function. In fact, Theorem 4 still holds if we allow the money endowments to be arbitrary $b_i > 0$ for $i = 1, ..., N$.

4. Conclusion

Our results suggest that Walrasian methods can be employed in a variety of market design problems whenever preferences satisfy the substitutes property. Gul and Stacchetti (1999) show that given any utility function that does not satisfy the substitutes property, it is possible to construct a transferable utility economy with N agents, one with the preference in question and $N - 1$ with a substitutes preference such that no equilibrium exists. Hence, it seems unlikely that a general existence result for the nontransferable utility economy that permits a larger set of preferences than the substitutes class can be proved.

However, Sun and Yang (2006) provide a generalization of the Kelso-Crawford existence result that allows for some complementarities in consumption. In particular, they show that if the goods can be partitioned into two classes such that all agents consider goods within each element of the partition substitutes and consider goods in different elements complements, then a Walrasian equilibrium exists in the corresponding transferable utility economy. A generalization of this result is offered in Shioura and Yang (2015). One possible extension of the current work would be the see if equilibrium also exists with Sun-Yang preferences in the limited transfers and nontransferable utility economies.

5. Appendix A

Unless indicated otherwise, the definitions and results below can be found in Oxley (2011):

A *matroid* $\mathcal{I} \subset X$ is a collection of sets such that (I1) $\emptyset \in \mathcal{I}$, (I2) $y \in \mathcal{I}$, $x \le y$ implies $x \in \mathcal{I}$ and (I3) $x, y \in \mathcal{I}$, $\sigma(x) < \sigma(y)$ implies there is j such that $x^j < y^j$ and $x + \chi^j \in \mathcal{I}$.

There are various alternative ways to describe a matroid. One is the definition in the text with a non-decreasing, submodular function $f : X \to \{0, 1, \ldots\}$ that satisfies $f(0) = 0$. The matroid corresponds to the set $\mathcal{I} = \{x \in X \,|\, \sigma(y) \le f(y), \forall y \le x\}$.

Alternatively, we can describe the matroid $\mathcal{I}$ via its maximal elements. Let $\mathcal{B}(\mathcal{I}) = \{x \in \mathcal{I} \,|\, y \ge x \text{ and } y \in \mathcal{I} \text{ implies } y = x\}$ be the set of all maximal elements of $\mathcal{I}$. Then, $\mathcal{B}(\mathcal{I})$ is a *basis system*; that is, (B1) $\mathcal{B}(\mathcal{I})$ is nonempty and (B2) $x, y \in \mathcal{B}(\mathcal{I})$ and $x^j > y^j$ implies there is k such that $y^k > x^k$ and $x - \chi^j + \chi^k \in \mathcal{B}(\mathcal{I})$.

If $\mathcal{B} \subset X$ satisfies (B1) and (B2), then $\mathcal{I} = \{x \in X \,|\, x \le y \text{ for some } y \in \mathcal{B}\}$ is a matroid and $\mathcal{B} = \mathcal{B}(\mathcal{I})$. Every basis system $\mathcal{B}$ satisfies the following stronger version of (B2): (B2*) $x, y \in \mathcal{B}(\mathcal{I})$ and $x^j > y^j$ implies there is k such that $y^k > x^k$ and $x - \chi^j + \chi^k, y - \chi^k + \chi^j \in \mathcal{B}(\mathcal{I})$. Also, all elements of a basis system have the same cardinality; that is, if $x, y \in \mathcal{B}$ and $\mathcal{B}$ is a basis system, then $\sigma(x) = \sigma(y)$. Hence, for any matroid $\mathcal{I}$, $\mathcal{B}(I)$ is the set of elements of $\mathcal{I}$ with the maximal cardinality; $\mathcal{B}(\mathcal{I}) = \{x \in \mathcal{I} \,|\, y \in \mathcal{I} \text{ implies } \sigma(x) \ge \sigma(y)\}$.

Gul and Stacchetti (2000) show that if u satisfies the substitutes property, then the set of elements of $D_u(p)$ with the smallest cardinality is a basis system for every p.

For any $\mathcal{B}$, let $\mathcal{B}^\perp = \{\chi^H - x \,|\, x \in \mathcal{B}\}$. If $\mathcal{B}$ is a basis system, then $\mathcal{B}^\perp$ is also a basis system and is called the *dual* of $\mathcal{B}$.

A function $r : X \to I\!N$ is a *rank function* if (R1) $0 \le r(x) \le \sigma(x)$, (R2) $x \le y$ implies $r(x) \le r(y)$ and (R3) $r(x \vee y) + r(x \wedge y) \le r(x) + r(y)$. For any rank function, r, the set of all minimal (in the natural order on $I\!R^L$) maximizers of r is a basis system. Also, given any matroid $\mathcal{I}$, the function r defined by $r(x) = \max\{\sigma(y) \,|\, y \le x, y \in \mathcal{I}\}$ is a rank function.

A *weighted matroid* is a function ρ, defined as follows: given an additive and monotone utility function v and matroid $\mathcal{I}$, let $\rho(x) = \max_{\substack{y \le x \\ y \in \mathcal{I}}} v(y)$. A rank function is a special case of a weighted matroid, one in which $v(x) = \sigma(x)$.

To define academic preferences, we adopt the following concept from Yokote (2017): $Y \subset X$ is an $M^\sharp$-*convex set* if $x, y \in Y$ and $x^j > y^j$ implies either $x - \chi^j, y + \chi^j \in Y$ or there is k such that $y^k > x^k$ and $x - \chi^j + \chi^k, y - \chi^k + \chi^j \in Y$. It is easy to see that a set Y is $M^\sharp$-convex if and only if the function I_Y^* define below is $M^\sharp$-concave:

$$I_Y^*(x) = \begin{cases} t & \text{if } x \in Y \\ -\infty & \text{otherwise} \end{cases}$$

for some $t \in I\!\!R_+$.

The utility function u is an academic preference if there exists an additive and monotone utility function v and an $M^\sharp$-convex set Y such that

$$u(x) = \begin{cases} \max\limits_{\substack{y \in Y \\ y \leq x}} v(y) & \text{if there is } y \leq x,\, y \in Y \\ -\infty & \text{otherwise} \end{cases}$$

Fact: Every academic preference satisfies the substitutes property.

Proof: Murota (2009) shows that a weighted matroid is $M^\sharp$-concave. The same argument establishes that an academic preference is $M^\sharp$ concave. Since $M^\sharp$-concavity is equivalent to the substitute property, the fact follows. $\square$

We will conclude Appendix A by showing that the utility function in example 3 is an academic preference. Identify H with the edges of an undirected graph with vertices $\{a, b, c, d, e\}$. Then, the set of feasible schedules, Y, is the collection of all sets of edges with 3 or 4 elements that contain no cycles. To prove that the utility function in example 3 is an academic preference, we we need to show that Y is $M^\sharp$-convex. Let Z be the set of all subsets of H that contain no cycles. It is well-known that Z is a matroid. Then, let r be the rank function of the matroid Z. Murota (2009) shows that a weighted matroid (and in particular, a rank function) is $M^\sharp$-concave. Hence, by Lemma 1 (its proof is in Appendix B), $[r]_3$, the 3-lower bound of r satisfies the substitutes property. Then, by Bing, Lehman and Milgrom (2004), $I_Y^* = [[r]_3]^3$, the 3-satiation of $[r]_3$, satisfies the substitutes property; that is $M^\sharp$-concavity. Then, by the observation above, Y is $M^\sharp$-convex.

6. Appendix B

6.1 Proof of Lemma 1

First, we will extend the definition of the single improvement property (SI) (Gul and Stacchetti, 1999) to include u such that $o \notin \text{dom}\, u$ as follows:

Definition: *The function u has the single improvement property (SI) if for all p such that $D_u(p) \subset \text{dom}\, u$ and all $x \in \text{dom}\, u - D_u(p)$, there is y such that $U(x, p) < U(y, p)$, $|\text{supp}\,(x) - \text{supp}\,(y)| \leq 1$ and $|\text{supp}\,(y) - \text{supp}\,(x)| \leq 1$.*

Theorem 4.1 and Theorem 5.1 in Shioura and Tamura (2015) establish that the substitutes property, (SI) and $M^{\sharp}$-concavity are equivalent. Also, a utility function u is submodular if it satisfies the substitutes property. Gul and Stacchetti (1999) show that (SI) is equivalent to the substitutes property for the effective domain X. Their proof reveals that the above modified definition of (SI) is equivalent to the substitutes property for a general effective domain.

The following proof is similar to the proof that k-satiation preserves substitutes property in Bing, Lehmann and Milgrom (2004). We first prove two auxiliary lemmas. Lemma B1 provides an alternative characterization of $M^{\sharp}$-concavity.

Lemma B1: *Let u be a utility that satisfies the substitutes property. If $x, y \in \text{dom}\, u$ with $x \not\geq y$ and $y \not\geq x$, then there is j, k such that $x^j > y^j$, $y^k > x^k$ and $u(x - \chi^j + \chi^k) + u(y + \chi^j - \chi^k) \geq u(x) + u(y)$.*

Proof: Since u satisfies the substitutes property, u is $M^{\sharp}$-concave. Since $y \not\geq x$, there exists j with $x^j > y^j$. Since $x, y \in \text{dom}\, u$, $M^{\sharp}$-concavity implies that either there is k such that $y^k > x^k$ and $u(x - \chi^j + \chi^k) + u(y + \chi^j - \chi^k) \geq u(x) + u(y)$, hence we are done, or that $u(x - \chi^j) + u(y + \chi^j) \geq u(x) + u(y)$. That is,

$$u(y + \chi^j) - u(y) \geq u(x) - u(x - \chi^j)$$

Similarly, since $x \not\geq y$, there exists l with $y^l > x^l$. It follows from $M^{\sharp}$-concavity that either there is k with $x^k > y^k$ such that $u(x - \chi^k + \chi^l) + u(y + \chi^k - \chi^l) \geq u(x) + u(y)$ and we are done, or

$$u(x + \chi^l) - u(x) \geq u(y) - u(y - \chi^l)$$

The above two inequalities together with the submodularity of u imply

$$u(x - \chi^j + \chi^l) - u(x - \chi^j) \geq u(x + \chi^l) - u(x) \geq u(y) - u(y - \chi^l)$$

$$u(y - \chi^l + \chi^j) - u(y - \chi^l) \geq u(y + \chi^j) - u(y) \geq u(x) - u(x - \chi^j)$$

Hence,

$$u(x - \chi^j + \chi^l) + u(y - \chi^l + \chi^j) \geq u(x) + u(y)$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following lemma states that if a bundle with n elements does not maximize utility among all bundles with at least n elements, then we can increase its utility either by adding an element to it or replacing one of its elements with a different one.

Lemma B2: *Let u be a utility that satisfies the substitutes property. Let A, B be such that $|B| \geq n = |A|$, $\chi^A \in \operatorname{dom} u$ and $U(\chi^B, p) > U(\chi^A, p)$. Then, either there exists $l \notin A$ such that $U(\chi^A + \chi^l, p) > U(\chi^A, p)$ or there exists $k \in A$, $l \notin A$ such that $U(\chi^A + \chi^l - \chi^k, p) > U(\chi^A, p)$.*

Proof: Let $\mathcal{D}$ denote the utility maximizing bundles, at price p, among all bundles with at least n elements. Let B^* minimize the Hausdorff distance $(d(\hat{A}, \hat{B}) = |\hat{A} - \hat{B}| + |\hat{B} - \hat{A}|)$ from A among the elements of $\mathcal{D}$. By assumption, $U(\chi^{B^*}, p) > U(\chi^A, p)$. Clearly $B^* - A \neq \emptyset$, otherwise, since $|A| = n, |B^*| \geq n$ and $B^* \subseteq A$, we have $A = B^*$, a contradiction. Since $\chi^A \in \operatorname{dom} u$, we have $\chi^{B^*} \in \operatorname{dom} u$.

First, assume $A - B^* \neq \emptyset$. By Lemma B1, there exists k, l with $k \in A - B^*$, $l \in B^* - A$ such that

$$u(\chi^A) + u(\chi^{B^*}) \leq u(\chi^A - \chi^k + \chi^l) + u(\chi^{B^*} - \chi^l + \chi^k)$$

Since the total cost of bundles on either side of the above inequality is the same, we have,

$$U(\chi^A, p) + U(\chi^{B^*}, p) \leq U(\chi^A - \chi^k + \chi^l, p) + U(\chi^{B^*} - \chi^l + \chi^k, p)$$

By assumption, $U(\chi^{B^*}, p) \geq U(\chi^{B^*} - \chi^l + \chi^k, p)$. If $U(\chi^{B^*}, p) = U(\chi^{B^*} - \chi^l + \chi^k, p)$, then $\chi^{B^*} - \chi^l + \chi^k$ is also optimal at price p among all bundles with at least n elements and

$d(A, B^* \cup \{k\} - \{l\}) < d(A, B^*)$, which contradicts the definition of B^*. Thus, $U(\chi^{B^*}, p) > U(\chi^{B^*} - \chi^l + \chi^k, p)$ and by the inequality above, $U(\chi^A, p) < U(\chi^A - \chi^k + \chi^l, p)$.

Second, assume $A - B^* = \emptyset$. Then, since $A \neq B^*$, A is a strict subset of B^* and $|B^*| \geq n + 1$. For any $j \in B^* - A$, $d(B^* - \{j\}, A) < d(B^*, A)$ and $|B^* - \{j\}| \geq n$. Then, $U(\chi^{B^*}, p) > U(\chi^{B^*} - \chi^j, p)$ and therefore,

$$p^j < u(\chi^{B^*}) - u(\chi^{B^*} - \chi^j)$$

Since u is submodular u has decreasing marginal returns. Recall that $\chi^A + \chi^j \in \text{dom } u$, $\chi^A \leq \chi^{B^*}$ and $j \notin A$. Hence,

$$p^j < u(\chi^{B^*}) - u(\chi^{B^*} - \chi^j) \leq u(\chi^A + \chi^j) - u(\chi^A)$$

That is, $U(\chi^A, p) < U(\chi^A + \chi^j, p)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proof of Lemma 1: Since $M^\sharp$-concavity is equivalent to (SI), it suffices to show that $[u]_k$ satisfies the latter. Since the effective domain of $[u]_k$ is nonempty, there is x^* such that $\sigma(x^*) \geq k$, $u(x^*) = [u]_k(x^*)$ and x^* is optimal at price p for the utility function $[u]_k$.

Take any $x \in \text{dom}\,[u]_k - D_{[u]_k}(p)$. Hence, $\sigma(x) \geq k$. We need to show there exists y such that $[U]_k(x, p) < [U]_k(y, p)$, $|\text{supp}\,(x) - \text{supp}\,(y)| \leq 1, |\text{supp}\,(y) - \text{supp}\,(x)| \leq 1$.

If $\sigma(x) = k$, then since $\sigma(x^*) \geq k$ and $U(x^*, p) > U(x, p)$, Lemma B2 yields the desired conclusion.

If $\sigma(x) > k$, then $\sigma(z) \geq k$ for any z such that $|\text{supp}\,(x) - \text{supp}\,(z)| \leq 1$ and $|\text{supp}\,(z) - \text{supp}\,(x)| \leq 1$. Recall that u satisfies (SI) and $x \notin D_u(p)$, $x \in \text{dom}\,[u]_k \subseteq \text{dom}\,u$. Then, there exists y such that $U(x, p) < U(y, p)$, $|\text{supp}\,(x) - \text{supp}\,(y)| \leq 1, |supp(y) - \text{supp}\,(x)| \leq 1$. Since $\sigma(y), \sigma(x) \geq k$, $[U]_k(x, p) = U(x, p)$, $[U]_k(y, p) = U(y, p)$ and therefore, $[U]_k(x, p) < [U]_k(y, p)$. $\qquad\qquad\qquad\qquad\qquad$ □

6.2   Proof of Lemma 2

This is a corollary of Theorem 8.2 in Shioura and Tamura (2015). They assume $\{o, \chi^H\} \in \text{dom } u_i$, we instead assume that there exists a feasible allocation ξ such that

$\sum \xi_i \leq \chi^H$ and $\xi_i \in \text{dom}\, u_i$ for all i. By monotonicity, our assumption implies that there exists ξ^* such that $\xi_i^* \in \text{dom}\, u_i$, $\sum_{i=1}^{N} \xi_i^* = \chi^H$ and

$$\sum_{i=1}^{N} u_i(\xi_i^*) = \max \left\{ \sum_{i=1}^{N} u_i(x_i) \,\middle|\, x_i \in \text{dom}\, u_i, \forall i, \sum_{i=1}^{N} x_i = \chi^H \right\}$$

Then, the remainder of the proof follows Theorem 8.2 in Shioura and Tamura (2015). $\square$

6.3   Proof of Theorem 1

Our existence proof relies on Lemma 2, a modification of Kelso and Crawford's proof of existence of an equilibrium for the transferable utility economy with substitutes. In a transferable utility economy, the set of equilibrium prices and the set of equilibrium allocations of divisible goods are independent of the initial endowments and we can state the consumers' problem as maximizing (over x)

$$U_i(x, p) = u_i(x) - p \cdot x$$

By assumption, $w_i \in \text{dom}\, u_i$ for each i. Then, Lemma 2 establishes the existence of an equilibrium in deterministic allocations for the transferable utility gross substitutes economy. It is easy to see that given an equilibrium allocation α, the set of prices that support α; that is, p such that (p, α) is a an equilibrium, is defined by a finite set of linear weak inequalities and therefore is a compact and convex set. Since we are in a transferable utility setting, any Pareto efficient allocation must maximize total surplus. It is also easy to verify that if (p, ξ) is a deterministic equilibrium, and $\hat{\xi}$ is a social surplus maximizing allocation, then $(p, \hat{\xi})$ is also a Walrasian equilibrium. The following *exchangeability property* is a consequence of the last two observations: if (p, α) and $(\hat{p}, \hat{\alpha})$ are both equilibria, then $(p, \hat{\alpha})$ is also an equilibrium. Then, it also follows that the set of random equilibrium allocations is simply the convex hull of the set of deterministic equilibrium allocations and hence, the set of equilibrium prices for random allocations is the same as the set of equilibrium prices for deterministic allocations. It follows that for any transferable utility economy, there is a set of prices P^* and a set of random an allocations $\mathcal{A}^*$ such that the set of equilibria is $P^* \times \mathcal{A}^*$.

Since every price in P^* supports the same allocation, P^* is a nonempty, convex and compact set as it is defined by a finite set of linear weak inequalities. Since $\mathcal{A}^*$ is the set of surplus maximizing allocations, it is also a a nonempty, convex and compact set. We summarize these observation in Lemma B3 below.

Lemma B3: *For any transferable utility gross substitutes economy $\mathcal{E}$, the set of equilibria is $P^* \times \mathcal{A}^*$ for some nonempty compact convex set of prices P^* and the nonempty compact convex set of total surplus-maximizing random allocations $\mathcal{A}^*$.*

For the transferable utility economy $\mathcal{E}$, let $\mathcal{A}_o$ be the set of all feasible random allocations such that $\alpha(\xi) > 0$ implies $\xi_i \in \mathrm{dom}\, u_i$ for all i. Restricting attention to $\mathcal{A}_o$ is without loss of generality; for any $\alpha \notin \mathcal{A}_o$, there must be some agent with utility $-\infty$ which cannot be efficient in a transferable utility economy. Since the set of deterministic feasible allocations is finite and $w_i \in \mathrm{dom}\, u_i$ for each i, $\mathcal{A}_o$ is a nonempty, compact, convex subset of a Euclidian space. For any $\lambda = (\lambda_1, \ldots, \lambda_N) \in [0,1]^N$, we define the maximization problem:

$$M(\lambda) = \max_{\alpha \in \mathcal{A}_o} \sum_i \lambda_i u_i(\alpha_i)$$

We set $-\infty \times 0 = -\infty$; that is, when $\lambda_i = 0$, $\lambda_i u_i(\cdot)$ has the same effective domain as u_i and is 0 on the effective domain. Note that $M(\lambda)$ is a linear programming problem. Let $\Delta(\lambda)$ denote the set of solutions to this problem.

For $\lambda \in [0,1]^N$, define the transferable utility economy $\mathcal{E}(\lambda) = \{\lambda_1 u_1, \ldots, \lambda_N u_N\}$. Note that the transferable utility demand of u^j at price p is the same as the transferable utility demand of $\lambda_i u_i$ at price $\lambda_i p_i$ and hence $\mathcal{E}(\lambda)$ is a transferable utility gross substitutes economy.

By Lemma B3, $\Delta(\lambda)$ is the set of equilibrium allocations for the economy $\mathcal{E}(\lambda)$. Let $a = \sum_i u_i(\chi^H)$ and $\mathcal{P} = [0,a]^N$. Hence, any equilibrium price p of the transferable utility economy $\mathcal{E}(\lambda)$ must be in $\mathcal{P}$. Let $P^*(\lambda)$ be the set of all equilibrium prices for $\mathcal{E}(\lambda)$. Let $\mathcal{E}(\lambda)$ denote the transferable utility economy in which each agent i has utility $\lambda_i u_i$. Since the set of Walrasian equilibria in a transferable utility economy does not depend on initial endowments, we suppress them.

Lemma B4: *For any limited transfers economy $\mathcal{E}^o$, there exists $\lambda \in [0,1]^N$ and an equilibrium (p, α) of the corresponding transferable utility economy $\mathcal{E}(\lambda)$ such that if $\lambda_i < 1$, then $p\bar{\alpha}_i = b_i$ and if $\lambda_i = 1$, then $p\bar{\alpha}_i \leq b_i$.*

Proof: Let U_i^λ be consumer i's utility function in the transferable utility economy $\mathcal{E}(\lambda)$; that is, $U_i^\lambda(\theta, p) = \lambda_i u_i(\theta) - p\bar{\theta}$.

By Lemma B3, the correspondences Δ and P^* are nonempty, compact and convex valued. Since $\Delta(\lambda)$ is also the solution of the maximization problem defined above, Berge's Theorem ensures that Δ is uhc. Next, we will show that P^* is uhc as well. Since P^* compact-valued, it is enough to show that $\lambda(t) \in [0,1]^N$, $p(t) \in P^*(\lambda(t))$ for all $t = 1, 2, \ldots$, $\lim \lambda(t) = \lambda$ and $\lim p(t) = p$ implies $p \in P^*(\lambda)$.

Choose $\alpha(t) \in \Delta(\lambda(t))$ for all t. Since $\mathcal{A}$ is compact, we can assume, by passing to a subsequence if necessary, that $\alpha(t)$ converges. Let $\alpha = \lim \alpha(t)$. Since Δ is uhc, $\alpha \in \Delta(\lambda)$. Let θ be any random consumption. Then, since $\alpha(t)$ and is an equilibrium random allocation for the transferable utility economy $\mathcal{E}(\lambda(t))$,

$$U_i^{\lambda(t)}(\alpha_i(t), p(t)) = \lambda_i(t)u_i(\alpha_i(t)) - p(t) \cdot \bar{\alpha}_i(t) \geq \lambda_i(t)u_i(\theta) - p(t) \cdot \bar{\theta} = U_i^{\lambda_i(t)}(\theta, p(t))$$

Then, the continuity of U_i ensures that $U_i^{\lambda(t)}(\alpha_i, p) \geq U_i^{\lambda(t)}(\theta, p)$ for all θ and for all i. This implies that p is an equilibrium price for the transferable utility economy $\mathcal{E}(\lambda(t))$ and establishes the upper hemi-continuity of P^*.

Next, define correspondence Γ_i as follows:

$$\Gamma_i(p, z) = \begin{cases} [0,1] & \text{if } p(z - w_i) = b_i \\ 0 & \text{if } p(z - w_i) > b_i \\ 1 & \text{if } p(z - w_i) < b_i \end{cases}$$

Clearly, Γ_i is nonempty, convex and compact valued, and uhc.

Let $S = \mathcal{P} \times \mathcal{A}_o \times [0,1]^N$ and let

$$f(p, \alpha, \lambda) = P^*(\lambda) \times \Delta(\lambda) \times \Gamma_1(p, \bar{\alpha}_1) \times \cdots \times \Gamma_N(p, \bar{\alpha}_N)$$

Since P^*, Δ and the Γ_i's are nonempty, convex and compact valued, and uhc and the mapping $\alpha \to \bar{\alpha}_i$ is continuous, f is also nonempty, convex and compact valued, and

uhc. Then, by Kakutani's Fixed-Point Theorem, there is an $s^* = (p^*, \alpha^*, \lambda^*)$ such that $f(s^*) = s^*$. Thus, (p^*, α^*) is a Walrasian equilibrium of the transferable utility economy $\mathcal{E}(\lambda^*)$.

We claim that $\lambda_i^* > 0$ for all i. Suppose $\lambda_i^* = 0$ for some i. Then, agent i's utility is identically 0 on the effective domain of u_i and since s^* is a fixed point of f, $p(\bar{\alpha}_i^* - w_i) \geq b_i > 0$. But, since $w_i \in \text{dom}\, u_i$ and every consumption in the effective domain gives the agent utility 0, spending more than pw_i is inconsistent with utility maximization. Hence, α_i cannot be optimal, a contradiction.

To complete the proof of the lemma, we will show that $p^*(\bar{\alpha}_i^* - w_i) \leq b_i$ for all i and that the inequality is an equality whenever $\lambda_i^* < 1$. Since s^* is a fixed point of f and $\lambda_i^* > 0$, we must have $p^*(\bar{\alpha}_i^* - w_i) \leq b_i$. Similarly, since s^* is a fixed point of f, if $\lambda_i^* < 1$, we must have $p^*(\bar{\alpha}_i^* - w_i) = b_i$. $\qquad\square$

To conclude the proof of Theorem 1, we will show that (p^*, α^*) is an equilibrium of the limited transfers economy $\mathcal{E}^o$. Consider any θ that i can afford (in the limited transfers economy). The optimality of α_i^* for i in the transferable utility economy implies

$$\lambda_i^*(u_i(\theta) - u_i(\alpha_i^*)) \leq p^* \cdot \bar{\theta} - p^* \cdot \bar{\alpha}_i^* \tag{2}$$

If $\lambda_i^* = 1$, then equation (2) implies that α_i is solves the utility maximization problem of agent i in the limited transfers economy $\mathcal{E}^o$. If $\lambda_i^* < 1$, since s^* is a fixed-point of f, the right-hand side of equation (2) must be less than or equal to zero. Then, we have $\lambda_i^*(u_i(\theta) - u_i(\alpha_i^*)) \leq 0$ and hence $u_i(\theta) - u_i(\alpha_i^*) \leq \lambda_i^*(u_i(\theta) - u_i(\alpha_i^*)) \leq p^*\bar{\theta} - p^*\bar{\alpha}_i^*$ proving, again, that α_i is solves the utility maximization problem of agent i in $\mathcal{E}^o$. $\qquad\square$

6.4   Proof of Theorem 2

Fix the nontransferable utility economy $\mathcal{E}^* = \{(u_i, b_i)_{i=1}^N\}$. Let $\mathcal{E}_n^o = \{(nu_i, w_i, b_i)_{i=1}^N\}$ for $n = 1, 2 \ldots$ be a sequence of limited transfer economies such that $w_i^j = 0$ for all j, i. Since $o \in \text{dom}\, u_i$ for all i, by Theorem 1, each $\mathcal{E}_n^o$ has an equilibrium (p^n, α^n).[22] By monotonicity, $o \in \text{dom}\, u_i$ implies that $\text{dom}\, u_i = X$. Let $P = [0, \sum_i b_i]^L$. Note that p^n

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must be an element of P. Hence, the sequence (p^n, α^n) lies in a compact set and therefore has a limit point, (p, α). By passing to a subsequence if necessary, we may assume that (p, α) is its limit.

To complete the proof of Theorem 2, we will show that (p, α) is a strong equilibrium of $\mathcal{E}^*$. Clearly, α is feasible for $\mathcal{E}^*$. Since $p^n \bar{\alpha}_i^n \leq b_i^n$ for all n, $p \bar{\alpha}_i \leq b_i$, α_i is affordable for i in $\mathcal{E}^*$. Take any other affordable random allocation θ for i in $\mathcal{E}^*$; that is, $p\bar{\theta} \leq b_i$. We need to show that $u_i(\theta) \leq u_i(\alpha_i)$. First, assume $p\bar{\theta} < b_i$. Then, there exists $\epsilon > 0$ such that $p'\bar{\theta} < b_i$ for any $p' \in B_\epsilon(p) \cap \mathbb{R}^L_+$. Since $\lim p^n = p$, we can find $M > 0$ such that for all $n \geq M$, $p^n \bar{\theta} < b_i$. Hence, θ is affordable for i in $\mathcal{E}_n^o$ for $n \geq M$. Since α_i^n is an optimal consumption, $nu_i(\theta) - p^n\bar{\theta} \leq nu_i(\alpha_i^n) - p\bar{\alpha}_i^n$; that is, $u_i(\theta) - u_i(\alpha_i^n) \leq (p^n\bar{\theta} - p^n\bar{\alpha}_i^n)/n \leq b_i/n$ for all $n \geq M$. Then, the continuity of u_i ensures $u_i(\theta) \leq u_i(\alpha_i)$ as desired.

Next, assume $p\bar{\theta} = b_i$. Choose $\epsilon \in (0, 1)$. Then, $\theta_\epsilon := (1 - \epsilon)\theta + \epsilon\delta_o$ satisfies $p\bar{\theta}_\epsilon < b_i$ and, therefore, by the argument in the previous paragraph $u_i(\theta_\epsilon) \leq u_i(\alpha_i)$. Since $u_i(\theta_\epsilon)$ is continuous in ϵ and ϵ was arbitrary, it follows that $u_i(\theta) \leq u_i(\alpha_i)$. Thus, α_i is optimal for i at prices p in $\mathcal{E}^*$.

To prove that all goods with strictly positive prices are allocated to the agents, it is enough to show that $p^j(1 - \sum_{i=1}^N \bar{\alpha}_i^j) = 0$ whenever $p^j > 0$. This follows since $p^{nj}(1 - \sum_{i=1}^N \bar{\alpha}_i^{nj}) = 0$ for all j, n and hence, the same equality holds in the limit as n goes to infinity. Thus, (p, α) is an equilibrium of $\mathcal{E}^*$.

To conclude, we will show that (p, α) is a strong equilibrium; that is, for all i, $u_i(\theta) = u_i(\alpha_i)$ implies $p\bar{\theta} \geq p\bar{\alpha}_i$. If not, assume that $p\bar{\theta} < p\bar{\alpha}_i$ for some θ such that $u_i(\theta) = u_i(\alpha_i)$ and consider two cases: (1) agent i is satiated at θ; that is, $u_i(\theta) = u_i(\alpha_i) = u_i(\chi^H)$ or (2) she is not satiated at θ. If (1) holds, then for sufficiently large n, purchasing θ instead of α_i is affordable for i at all p^n and $b_i - p\bar{\theta} > b_i - p\bar{\alpha}_i^n \geq 0$, contradicting the optimality of α_i^n for i in $\mathcal{E}_n^o$. If (2) holds, then choose $0 < r < 1$ such that $p(r\chi^H + (1-r)\bar{\theta}) < p\bar{\alpha}_i$. Again, for n sufficiently large, the random consumption $r\delta_{\chi^H} + (1-r)\theta$, where δ_{χ^H} is the degenerate lottery that yields χ^H for sure, is affordable at p^n and yields a higher utility than α_i^n, contradicting its optimality in $\mathcal{E}_n^o$. $\qquad\square$

6.5   Proof of Lemma 3 and Lemma 4

Proof of Lemma 3:   By assumption, the effective domain of $u(c, \cdot)$ is nonempty. Recall that the operations that take u to u^z (the z-constrained u), $[u]^k$ (the k-satiation of

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u) and $[u]_k$ (the k-lower bound u) all preserve the substitutes property. Similarly, the binary operation $u \odot v$ (the convolution of u, v) also preserves the substitutes property. Then, to complete the proof of the lemma, we note that given any modular constraint $c = \{(A(k), (l(k), h(k))_{k=1}^{K}\}$ for u, we can express $u(c, \cdot)$ as a finite composition of these operations applied to u. This is straightforward; for example, let $c = \{\{(A(k), (l(k), h(k))_{k=1}^{4}\}$ where $A_1, A_2, A_3 \subset H$ are disjoint sets and $A_4 = A_1 \cup A_2$. Then, define

$$\hat{u} = \left([[v \odot w]^{h_4}]_{l_4} \odot [[u^{z_3}]^{h_3}]_{l_3} \right) \odot u^{z_5} \qquad \text{where}$$

$$v = [[u^{z_1}]^{h_1}]_{l_1}$$

$$w = [[u^{z_2}]^{h_2}]_{l_2}$$

$z_i = \chi^{A_i}$ for all $i = 1, 2, 3, 4$ and $z_5 = \chi^{H-A}$ for $A = \bigcup_{i=1}^{4} A_i$. Since each utility function on the right-hand side of the equation above satisfies the substitutes property and all of the operations applied to them preserve the substitutes property, $\hat{u}$ satisfies the substitutes property as well, and since c is a modular constraint for u, $\hat{u} = u(c, \cdot)$. $\qquad \square$

Proof of Lemma 4: Clearly, $o \in \mathcal{I}_d$ and $x, y \in \mathcal{I}_d$ and $x \leq y$ implies $x \in \mathcal{I}_d$. Hence, we need only prove that $x, y \in \mathcal{I}_d$ and $\sigma(x) < \sigma(y)$ implies there is j such that $x^j < y^j$ and $x^j + \chi^j \in \mathcal{I}_d$.

We order d, the hierarchy of constraints, in the obvious way: $(A, k) \succ (B, n)$ if $B \subset A$. Call j a *free* element in d if j is not an element of any A such that $(A, n) \in d$ for some n. Otherwise, call j a constraint element. Let $F \subset H$ be the set of free elements in d and let $F^c = H \backslash F$ be the set of constraint elements. Suppose there is $j \in F$ such that $y^j > x^j$. Then, clearly $x + \chi^j \in \mathcal{I}_d$ and we are done. Otherwise, $x^j \geq y^j$ for all $j \in F$ and hence there must be some $\succ$-maximal constraint, (A, k), such that $\sigma(y \wedge \chi^A) > \sigma(x \wedge \chi^A)$. Let $A_1 = A$ and $n_1 = k$.

Then, there is either $j \in A_1$ such that $y^j > x^j$, $j \notin B$ for any B such that $(A_1, n_1) \succ (B, n')$ in which case we have $x + \chi^j \in \mathcal{I}_d$ and we are done, or there is no such j. In the latter case, there must be an immediate predecessor of (A_1, n_1) such that $\sigma(y \wedge \chi^B) > \sigma(x \wedge \chi^B)$. Let (A_2, n_2) be this immediate predecessor and continue in this fashion until we end up with (A_l, n_l) and $j \in A_l$ such that $\sigma(y \wedge \chi^{A_k}) > \sigma(x \wedge \chi^{A_k})$ for all $k = 1, \ldots, l$ and $y^j > x^j$. Hence, $x + \chi^j \in \mathcal{I}_d$. $\qquad \square$

6.6 Proof of Theorems 3-5

We prove Theorem 5 and then show that Theorems 3 and 4 follows as special cases. Fix the nontransferable utility economy with production and modular constraints

$$\tilde{\mathcal{E}}_c = \{(\tilde{u}_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N, \mathcal{I}\}$$

Let $\mathcal{B}$ be the production possibility frontier of $\mathcal{I}$. Hence, $\mathcal{B}$ is a basis system (Appendix A). Define

$$H = \{1, \ldots, L\} = \{i \,|\, x_i > 0 \text{ for some } x \in \mathcal{B}\}$$

Let $\mathcal{B}^\perp = \{\chi^H - x \,|\, x \in \mathcal{B}\}$. Then $\mathcal{B}^\perp$ is a basis system since $\mathcal{B}$ is a basis system (Appendix A). Let r be the rank function associated with $\mathcal{B}^\perp$; that is,

$$r(x) = \max\{\sigma(x \wedge y) \,|\, y \in \mathcal{B}^\perp\}$$

Since every weighted matroid satisfies the substitutes property, so does r (Appendix A). Define the utilities $(u_1, \ldots, u_{N+1})$ such that

$$u_i = \tilde{u}_i(c_i, \cdot) \text{ for } i = 1, \ldots N$$

$$u_{N+1} = 2Nr$$

By Lemma 3 and the argument above, u_i satisfies the substitutes property for all $i = 1, \ldots, N+1$. Define the following nontransferable utility economy with general preferences and endowments:

$$\hat{\mathcal{E}}^* = \{(u_i, w_i, 1)_{i=1}^I, (u_{N+1}, w_{N+1}, 0)\}$$

where $w_i = 0$ for $i = 1, \ldots, I, w_{N+1} = \chi^H$. The difference between $\hat{\mathcal{E}}^*$ and a nontransferable utility economy $\mathcal{E}^*$ defined in section 3.1 is that in $\hat{\mathcal{E}}^*$, $o \notin \text{dom } u_i$ is possible and agents are allowed to have endowments of divisible goods. Thus, the budget constraint of agent i is $B_i(p) = \{\theta \in \text{dom } u_i \,|\, p\theta \leq pw_i + b_i\}$. The definition of a strong equilibrium is the same as in section 3.1.

Lemma B5: *If (p, α) is a strong equilibrium of $\hat{\mathcal{E}}^*$, then $(p, \hat{\alpha})$ such that $\hat{\alpha}(\xi, z) = \alpha(\xi, \chi^H - z)$ is a strong equilibrium of $\tilde{\mathcal{E}}_c$.*

Proof: Let (p, α) be a strong equilibrium of $\hat{\mathcal{E}}^*$. Note that agent $N + 1$ is satiated at her initial endowment χ^H; that is, $2Nr$ is maximal. Moreover, z' maximizes $r(\cdot)$ if and only if $z' = \chi^H - z \geq y' = \chi^H - y$ for some $y' \in \mathcal{B}^\perp$; that is, if and only if $z \in \mathcal{I}$. Thus, consumer optimality of agent $N + 1$ implies $z \in \mathcal{I}$ whenever $\hat{\alpha}(\xi, z) > 0$. Since the equilibrium is strong, it follows that $p(\chi^H - z) \leq p(\chi^H - z')$ for all $z' \in \mathcal{I}$ and, therefore, $pz \geq pz'$ for all $z' \in \mathcal{I}$. Parts (1), (2) and (4) of the definition of a strong equilibrium of $\tilde{\mathcal{E}}_c$, then follow from the definition of a strong equilibrium of $\hat{\mathcal{E}}^*$. $\qquad\square$

By assumption, there exists $z = \sum_{k=1}^{N+1} x_k \in \mathcal{I}$ such that $x_k \in \operatorname{dom} u_i$ for all $i = 1, ..., N$, $k = 1, ..., N + 1$. Define the random consumptions θ^0, θ^1 as follows:

$$\theta^0(x_k) = {}^1\!/_N, \text{ for all } k = 1, \ldots, N - 1 \text{ and } \theta^0(x_N + x_{N+1}) = 1/N$$

$$\theta^1(x_k) = {}^1\!/_{N+1} \text{ for all } k = 1, \ldots, N + 1$$

Let $H' = \{j \in H | z^j = 1\}$ and let $\bar{H} = H - H'$. Note that $\bar{\theta}^{0j} = 1/N$ and $\bar{\theta}^{1j} = 1/(N + 1)$ for all $j \in H'$. By monotonicity, every realization of θ^0 and θ^1 lies in the effective domain of every $u_i, i = 1, \ldots N$. Finally, define the random endowment θ_{N+1} as follows:

$$\theta_{N+1} = \frac{1}{N+1} \sum_{k=1}^{N+1} \delta_{\chi^{\bar{H}} + x_k}$$

It is straightforward to verify that there exists an allocation α with marginals θ^1 for $i = 1, \ldots, N$ and marginal θ_{N+1} for $i = N + 1$. Consider the $N + 1$ person limited transfer economy with random endowments

$$\mathcal{E}^o = \{(u_i, \theta^1, 1/(N+1))_{i=1}^N, (u_{N+1}, \theta_{N+1}, N - N/(N+1)))\}$$

Extending the definition of limited transfer economies to include random endowments is immediate and does not alter the definition of equilibrium.

Lemma B6: The economy $\mathcal{E}^o$ has an equilibrium (p, α) such that $p_j \in [0, 2N]$ for all $j \in H$, $\sum_{j \in H'} p_j \leq N$ and $\alpha(\xi) > 0$ implies $\chi^H - \xi_{N+1} \in \mathcal{I}$.

Proof: Theorem 1 implies that the economy $\mathcal{E}^o$ has an equilibrium (α, p). Note that every element in the support of θ_{N+1} yields the maximal possible value of r. Therefore, the equilibrium utility of agent $N + 1$ must be at least $u_{N+1}(\theta_{N+1}) = 2Nr(\chi^{\bar{H}}) =$

$2N \max_{x \in \{0,1\}^H} r(x)$. If $\alpha(\xi, z) > 0$ for some $\xi = (\xi_1, \ldots, \xi_{N+1})$ such that ξ_{N+1} is not r−maximal, then $2Nr(\theta_{N+1}) - 2Nr(\xi_{N+1}) \geq 2N$. Since the total money endowment of the economy is N, the preceding inequality implies that agent $N + 1$, with positive probability, receives total utility less than the utility of her endowment which is inconsistent with her utility maximization. It follows that $\alpha(\xi, z) > 0$ for some $\xi = (\xi_1, \ldots, \xi_{N+1})$ implies $r(\xi_{N+1})$ is maximal and therefore $\chi^H - z \in \mathcal{I}$.

Next, observe that the equilibrium utility of agent $N + 1$ is bounded above by

$$2N \max_{x \in H} r(x) + N$$

Hence,

$$u_{N+1}(\chi^{\bar{H}}) + p \cdot \theta^1 + N - \frac{N}{N+1} \leq 2N \max_{x \in H} r(x) + N$$

Since $\chi^{\bar{H}} = \chi^H - \chi^{H'}$ maximizes r, the display equation above implies

$$0 \geq p\theta^1 - \frac{N}{N+1} = \frac{1}{N+1} \sum_{j \in H'} p_j - \frac{N}{N+1}$$

and therefore, $\sum_{j \in H'} p_j \leq N$.

Next, note that $p_j \leq 2N$ for all $j \in \bar{H}$. To see why this is true, note that if p_j were greater than $2N$, agent $N + 1$ would consume j with probability zero since its marginal utility is never greater than $2N$. However, since the total endowments of agents 1-N are worth less than $2N$, agent $N + 1$ must consume j with strictly positive probability in any equilibrium. $\square$

Lemma B7: *The economy $\hat{\mathcal{E}}^*$ has a strong equilibrium*

Proof: Let (p, α) be the equilibrium of $\mathcal{E}^o$ shown to exist in Lemma B6. Set $b = p\theta^1$ and consider the limited transfers economy

$$\hat{\mathcal{E}}^o = \{(u_i, o, b)_{i=1}^N, (u_{N+1}, \chi_H, N(1-b))\}$$

Hence, agent $i \leq N$ only has some of the divisible good initially. Note that at prices p the endowment of each agent i in $\hat{\mathcal{E}}^o$ is worth the same as her endowment in $\mathcal{E}^o$ and therefore

(p, α) is also an equilibrium of $\hat{\mathcal{E}}^o$. By Lemma B6, $p\bar{\theta}^1 \leq N/(N+1)$ and, therefore, the divisible good endowment $b = 1/(N+1) + p\bar{\theta}^1$ is in the interval $[1/(N+1), 1]$.

Next, consider the sequence of limited transfer economies

$$\mathcal{E}_n^o = \{(nu_i, b^n)_{i=1}^N, (u_{N+1}, \chi^H, N(1 - b^n))\}$$

Arguing as above, we conclude that for a suitable choice of $\{b^n\}$ such that $b^n \in [1/(N+1), 1]$, every economy in this sequence of has an equilibrium with the properties stated in Lemma B6. Since $p \in [0, 2N]^I$ and $b^n \in [1/(N+1), 1]$, the sequence (α^n, p^n, b^n) has a convergent subsequence with limit (α, p, b^*) for some $b^*n \in [1/(N+1), 1]$ and $p \in [0, 2N]^I$.

To complete the proof of Lemma B7, we will show that $(\alpha, p/b^*)$ is a strong equilibrium of $\hat{\mathcal{E}}^*$. Clearly, α is feasible for $\hat{\mathcal{E}}^*$. Since $p^n \bar{\alpha}_i^n \leq b^n$ for all n, $p\bar{\alpha}_i \leq b^*$ and therefore, α_i is affordable for $i = 1, \ldots, N$ in $\hat{\mathcal{E}}^*$. Take any other affordable random allocation θ for i in $\hat{\mathcal{E}}^*$; that is, $p\bar{\theta} \leq b^*$. We need to show that $u_i(\theta) \leq u_i(\alpha_i)$. First, assume $p\bar{\theta} < b^*$. Then, there exists $\epsilon > 0$ such that for any $p' \in B_\epsilon(p) \cap \mathbb{R}_+^L$, where $B_\epsilon(p)$ is the ϵ-ball centered at p. Since $\lim p^n = p$, we can find $M > 0$ such that for all $n \geq M$, $p^n \bar{\theta} < b^*$. Hence, θ is affordable for i in $\mathcal{E}_n^o$ for $n \geq M$. Since α_i^n is an optimal consumption in the limited transfers economy $\mathcal{E}_n^o$, $nu_i(\theta) - p^n\bar{\theta} \leq nu_i(\alpha_i^n) - p\bar{\alpha}_i^n$; that is, $u_i(\theta) - u_i(\alpha_i^n) \leq (p^n\bar{\theta} - p^n\bar{\alpha}_i^n)/n \leq b^n/n$ for all $n \geq M$. Then, the continuity of u_i ensures $u_i(\theta) \leq u_i(\alpha_i)$ as desired.

Next, assume $p\bar{\theta} = b^*$. Recall that $p\bar{\theta}^1 < p\bar{\theta}^1 + 1/(N+1) = b^*$. Choose $\epsilon \in (0, 1)$. Then, for $\theta_\epsilon := (1 - \epsilon)\theta + \epsilon\theta_i^1$, we haves $p\bar{\theta}_\epsilon < b^*$ and, therefore, by the argument in the previous paragraph $u_i(\theta_\epsilon) \leq u_i(\alpha_i)$. Since $u_i(\theta_\epsilon)$ is continuous in ϵ and ϵ was arbitrary, it follows that $u_i(\theta) \leq u_i(\alpha_i)$. Thus, α_i is optimal for i at prices p in $\hat{\mathcal{E}}^*$ for all $i = 1, \ldots, N$.

For agent $N+1$, consumer optimality follows from the fact that $r(\alpha_{N+1}^n)$ is maximal for all n (by Lemma B6) and, therefore, $r(\alpha_{N+1})$ is maximal as well.

To prove that all goods with strictly positive prices are allocated to the agents, it is enough to show that $p^j(1 - \sum_{i=1}^{N+1} \bar{\alpha}_i^j) = 0$ whenever $p^j > 0$. This follows since $p^{nj}(1 - \sum_{i=1}^{N+1} \bar{\alpha}_i^{nj}) = 0$ for all j, n and hence, the same equality holds in the limit as n goes to infinity. Thus, (p, α) is an equilibrium of $\hat{\mathcal{E}}^*$.

To conclude, we will show that (p, α) is a strong equilibrium; that is, for all i, $u_i(\theta) = u_i(\alpha_i)$ implies $p\bar{\theta} \geq p\bar{\alpha}_i$. If not, assume that $p\bar{\theta} < p\bar{\alpha}_i$ for some θ such that $u_i(\theta) = u_i(\alpha_i)$

and consider two cases: (1) agent i is satiated at θ; that is, $u_i(\theta) = u_i(\alpha_i) = u_i(\chi^H)$ or (2) she is not satiated at θ.

If (1) holds, then for sufficiently large n, purchasing θ instead of α_i is affordable for i at all p^n and $b_i - p\bar{\theta} > b_i - p\bar{\alpha}_i^n \geq 0$, contradicting the optimality of α_i^n for i in $\mathcal{E}_n^o$. If (2) holds, then choose $0 < r < 1$ such that $p(r\chi^H + (1-r)\bar{\theta}) < p\bar{\alpha}_i$. Again, for n sufficiently large, the random consumption $r\delta_{\chi^H} + (1-r)\theta$, where δ_{χ^H} is the degenerate lottery that yields χ^H for sure, is affordable at p^n and yields a higher utility than α_i^n, contradicting its optimality in $\mathcal{E}_n^o$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proofs of Theorems 3-5: Theorem 5 follows from Lemmas B5 and B7. To prove Theorem 4, first note that the theorem is trivial if $\mathcal{I} = \{o\}$. If there exists $x \in \mathcal{I}$ such that $x \neq o$, then we will show that $\tilde{\mathcal{E}}$ is a special case of $\tilde{\mathcal{E}}_c$. First, assume that each agent is unconstrained; that is, $c_i = c = \{H, 0, L\}$. Next, choose $x_1 = x, x_2 = \ldots = x_{N+1} = o$ and note that condition (iii) in the definition of $\tilde{\mathcal{E}}_c$ is satisfied. Thus, Theorem 5 applies.

To prove Theorem 3, let $\mathcal{E}_c^* = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N\}$ be a nontransferable utility economy with modular constraints and let $\mathcal{I} = \{z \leq \chi^H\}$. Clearly, any strong equilibrium of $\tilde{\mathcal{E}}_c = \{(\tilde{u}_i, 1)_{i=1}^N, c, \mathcal{I}\}$, a nontransferable utility production economy with modular constraints, is a strong equilibrium of $\mathcal{E}_c^*$. Hence, the desired result follows from Theorem 5. $\square$

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