Price Impact Models & Optimal Execution

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We already saw that we should split and spread large orders, so:

- split and spread large orders, so:
- How can we capture market price impact in a model?
- What are the desirable properties of a Price Impact model?
- How can we compute optimal execution trading strategies?
- What happens when several execution strategies interact?
"Amlgren-Chriss Price Impact" Model

- **Unaffected (fair) price** given by a semi-maringale
- **Mid-price** affected by trading
  - **Permanent price impact** given by a function \( g \) of trading speed
    \[
    dP_{mid}^t = g(v(t))dt + \sigma dW_t
    \]
  - **Temporary price impact** given by function \( h \) of trading speed
    \[
    P_{trans}^t = P_{mid}^t + h(v(t))
    \]
- **Problem**: find deterministic continuous transaction path to maximize **mean-variance** reward.
  - Closed form solution when **permanent** and **instantaneous price impact** functions \( g \) and \( h \) are **linear**
  - **Efficient frontier**: Speed of trading and hence risk/return controlled by risk aversion parameter

*Widely used in industry*
Criticisms

- Mid-price $P_{t}^{\text{mid}}$ arithmetic Brownian motion + drift
  - Can become negative
  - Reasonable only for short times
- Possible issues with rate of trading in continuous time?
- **Price impact** more complex than instantaneous + permanent
- What is the link between **Price Impact** and **LOB** dynamics?
  - e.g. can we combine elegant description of risk-return trade-off in Almgren / Chriss with detail of Smith-Farmer type models?
- Empirical evidence that instantaneous price impact is stochastic in many markets
Optimal Execution

An execution algorithm has three layers:

- At the **highest level** one decides how to **slice** the order, when to trade, in what size and for how long.
- At the **mid level**, given a slice, one decides whether to place **market** or **limit** orders and at what price level(s).
- At the **lowest level**, given a limit or market order, one decides to which **venue** should this order be routed?

We shall not discuss the last bullet point here.
Optimal Execution Set-Up

Goal: sell $x_0 > 0$ shares by time $T > 0$

- $X = (X_t)_{0 \leq t \leq T}$ execution strategy
- $X_t$ position (nb of shares held) at time $t$. $X_0 = x_0$, $X_T = 0$
- Assume $X_t$ absolutely continuous (differentiable)
- $\tilde{P}_t$ mid-price (unaffected price), $P_t$ transaction price, $I_t$ price impact

$$P_t = \tilde{P}_t + I_t$$

e.g. Linear Impact A-C model:

$$I_t = \gamma[X_t - X_0] + \lambda \dot{X}_t$$

- **Objective:** Maximize some form of revenue at time $T$

Revenue $\mathcal{R}(X)$ from the execution strategy $X$

$$\mathcal{R}(X) = \int_0^T (-\dot{X}_t)P_t \, dt$$
Specific Challenges

- **First generation**: Price impact models (e.g. Almgren - Chriss)
  - Risk Neutral framework (maximize $\mathbb{E} R(X)$) versus utility criteria
  - More complex portfolios (including options)
  - Robustness and performance constraints (e.g. slippage or tracking market VWAP)

- **Second generation**: Simplified LOB models
  - Simple liquidation problem
  - performance constraints (e.g. slippage or tracking market VWAP)
  and using **both** market and limit orders
Optimal Execution Problem in A-C Model

\[ R(X) = \int_0^T (-\dot{X}_t) P_t \, dt \]

\[ = -\int_0^T \dot{X}_t \tilde{P}_t \, dt - \int_0^T \dot{X}_t I_t \, dt \]

\[ = x_0 \tilde{P}_0 + \int_0^T X_t d\tilde{P}_t - C(X) \]

with \( C(X) = \int_0^T \dot{X}_t I_t \, dt \).

**Interpretation**

- \( x_0 \tilde{P}_0 \) (initial) **face value** of the portfolio to liquidate
- \( \int_0^T X_t d\tilde{P}_t \) **volatility risk** for selling according to \( X \) instead of immediately!
- \( C(X) \) **execution costs** due to market impact
Special Case: the Linear A-C Model

\[ \mathcal{R}(X) = x_0 \tilde{P}_0 + \int_0^T X_t d\tilde{P}_t - \lambda \int_0^T \dot{X}_t^2 dt - \frac{\gamma}{2} x_0^2 \]

**Easy Case:** Maximizing \( \mathbb{E}[\mathcal{R}(X)] \)

\[ \mathbb{E}[\mathcal{R}(X)] = x_0 P_0 - \frac{\gamma}{2} x_0^2 - \lambda \mathbb{E} \int_0^T \dot{X}_t^2 dt \]

Jensen’s inequality & constraints \( X_0 = x_0 \) and \( X_T = 0 \) imply

\[ \dot{X}_t^* = -\frac{x_0}{T} \]

trade at a constant rate **independent of volatility**!

**Bertsimas - Lo (1998)**
More Realistic Problem

Almgren - Chriss propose to maximize

$$\mathbb{E}[\mathcal{R}(X)] - \alpha \text{var}[\mathcal{R}(X)]$$

($\alpha$ risk aversion parameter – late trades carry volatility risk)

For DETERMINISTIC trading strategies $X$

$$\mathbb{E}[\mathcal{R}(X)] - \alpha \text{var}[\mathcal{R}(X)] = x_0 P_0 - \frac{\gamma}{2} x_0^2 - \int_0^T \left( \frac{\alpha \sigma^2}{2} X_t^2 + \lambda \dot{X}_t^2 \right) dt$$

maximized by (standard variational calculus with constraints)

$$\dot{X}_t^* = x_0 \frac{\sinh \kappa (T - t)}{\sinh \kappa T} \quad \text{for} \quad \kappa = \sqrt{\frac{\alpha \sigma^2}{2 \lambda}}$$

For RANDOM (adapted) trading strategies $X$, more difficult as Mean-Variance not amenable to dynamic programming
Maximizing Expected Utility

Choose $U : \mathbb{R} \rightarrow \mathbb{R}$ increasing concave and

$$\text{maximize} \quad \mathbb{E}[U(R(X_T))]$$

Stochastic control formulation over a state process $(X_t, R_t)_{0 \leq t \leq T}$.

$$v(t, x, r) = \sup_{\xi \in \Xi(t, x)} \mathbb{E}[u(R_T) | X_t = x, R_T = r]$$

value function, where $\Xi(t, x)$ is the set of admissible controls

$$\left\{ \xi = (\xi_s)_{t \leq s \leq T}; \text{progressively measurable}, \int_t^T \xi_s^2 ds < \infty, \int_t^T \xi_s ds = x \right\}$$

$$X_s = X_{s}^{\xi} = x - \int_t^s \xi_u du, \quad \dot{X}_s = -\xi_s, \ X_t = x$$

and (choosing $\tilde{P}_t = \sigma W_t$)

$$R_s = R_s^{\xi} = R + \sigma \int_t^s X_u dW_u - \lambda \int_t^s \xi_u^2 du, \quad dR_s = \sigma X_s dW_s - \lambda \xi_s^2 ds, \ R_t = r$$
Finite Fuel Problem

Non Standard Stochastic Control problem because of the constraints

$$\int_0^T \xi_s ds = x_0.$$ 

Still, one expects
- For any admissible $\xi$, $[v(t, X_t^\xi, R_t^\xi)]_{0 \leq t \leq T}$ is a super-martingale
- For some admissible $\xi^*$, $[v(t, X_t^{\xi^*}, R_t^{\xi^*})]_{0 \leq t \leq T}$ is a true martingale

If $v$ is smooth, and we set $V_t = v(t, X_t^\xi, R_t^\xi)$, Itô's formula gives

$$dV_t = \left( \partial_t v(t, X_t, R_t) + \frac{\sigma^2}{2} \partial_{rr}^2 v(t, X_t, R_t) 
- \lambda \xi_t^2 \partial_r v(t, X_t, R_t) - \xi_t \partial_x v(t, X_t, R_t) \right) dt 
+ \sigma \partial_x v(t, X_t, R_t) dW_t$$
One expects that $v$ solves the HJB equation (nonlinear PDE)

$$
\partial_t v + \frac{\sigma^2}{2} \partial^2_{xx} v - \inf_{\xi \in \mathbb{R}} \left( \xi^2 \lambda \partial_r v + \xi \partial_x v \right) = 0
$$

in some sense, with the (non-standard) terminal condition

$$
v(T, x, r) = \begin{cases} 
U(r) & \text{if } x = X_0 \\
-\infty & \text{otherwise}
\end{cases}
$$
Solution for CARA Exponential Utility

For \( u(x) = -e^{-\alpha x} \) and \( \kappa \) as before

\[
v(t, x, r) = e^{-\alpha r + x_0^2 \alpha \lambda \kappa \coth \kappa (T - t)}\]

solves the HJB equation and the unique maximizer is given by the DETERMINISTIC

\[
\xi_t^* = x_0 \kappa \frac{\cosh \kappa (T - t)}{\sinh \kappa T}
\]

Schied-Schöneborn-Tehranchi (2010)

- Optimal solution same as in Mean - Variance case
- **Schied-Schöneborn-Tehranchi**’s trick shows that optimal trading strategy is generically deterministic for exponential utility
- Open problem for general utility function
- Partial results in infinite horizon versions
Shortcomings

- Optimal strategies
  - are DETERMINISTIC
  - do not react to price changes
  - are time inconsistent
  - are counter-intuitive in some cases

- Computations require
  - solving nonlinear PDEs
  - with singular terminal conditions
Recent Developments


- In the spirit of Almgren-Chriss mean-variance criterion, maximize

\[ \mathbb{E} \left[ \mathcal{R}(X) - \tilde{\lambda} \int_0^T X_t P_t \, dt \right] \]

- The solution happens to be **ROBUST**
  - \( \tilde{P}_t \) can be a semi-martingale, optimal solution does not change
Recent Developments

Almgren - Li (2012), Hedging a large option position

- $g(t, \tilde{P}_t)$ price at time $t$ of the option (from Black-Scholes theory)
- Revenue

$$\mathcal{R}(X) = g(T, \tilde{P}_T) + X_T \tilde{P}_T - \int_0^T \tilde{P}_t \dot{X}_t dt - \lambda \int_0^T \dot{X}_t^2 dt$$

- Using Itô’s formula and the fact that $g$ solves a PDE,

$$\mathcal{R}(X) = R_0 + \int_0^T [X_t + \partial_x g(t, \tilde{P}_t)] dt - \lambda \int_0^T \dot{X}_t^2 dt \quad R_0 = x_0 \tilde{P}_0 + g(0, \tilde{P}_0)$$

- Introduce $Y_t = X_t + \partial_x g(t, \tilde{P}_t)$ for hedging correction

$$\begin{cases} 
  d\tilde{P}_t = \gamma \dot{X}_t dt + \sigma dW_t \\
  dY_t = [1 + \gamma \partial_{xx} g(t, \tilde{P}_t)] dt + \sigma \partial_{xx} g(t, \tilde{P}_t) dW_t
\end{cases}$$

- Minimize

$$\mathbb{E} \left[ G(\tilde{P}_T, Y_T) + \int_0^T \left( \frac{\sigma^2}{2} Y_t^2 - \gamma \dot{X}_t Y_t + \lambda \dot{X}_t^2 \right) dt \right]$$

Explicit solution in some cases (e.g. $\partial_{xx} g(t, x) = c$, $G$ quadratic)
Transient Price Impact

Flexible price impact model

- **Resilience function** $G : (0, \infty) \rightarrow (0, \infty)$ measurable bounded
- Admissible $X = (X_t)_{0 \leq t \leq T}$ cadlag, adapted, **bounded variation**
- Transaction price

\[
P_t = \tilde{P}_t + \int_0^t G(t - s) \, dX_s
\]

- Expected cost of strategy $X$ given by

\[
-x_0 P_0 + \mathbb{E}[C(X)]
\]

where

\[
C(X) = \int \int G(|t - s|) \, dX_s \, dX_t
\]
Transient Price Impact: Some Results

- No **Price Manipulation** in the sense of Huberman - Stanzl (2004) if $G(|\cdot|)$ positive definite
- Optimal strategies (if any) are **deterministic**
- Existence of an optimal $X^*$ $\iff$ solvability of a Fredholm equation
- Exponential Resilience $G(t) = e^{-\rho t}$

\[
dX_t^* = -\frac{x_0}{\rho T + 2} \left( \delta_0(dt) + \rho dt + \delta_T(dt) \right)
\]

- $X^*$ purely discrete measure on $[0, T]$ when $G(t) = (1 - \rho t)^+$ with $\rho > 0$
  - $dX_t^* = -\frac{x_0}{2} \left[ \delta_0(dt) + \delta_T(dt) \right]$ if $\rho < 1/T$
  - $dX_t^* = -\frac{x_0}{n+1} \sum_{i=0}^{n} \delta_{iT/n}(dt)$ if $\rho < n/T$ for some integer $n \geq 1$

Obizhaeva - Wang (2005), Gatheral - Schied (2011)
Optimal Execution in a LOB Model

- Unaffected price $\tilde{P}_t$ (e.g. $\tilde{P}_t = P_0 + \sigma W_t$)
- Trader places only market sell orders
  - Placing buy orders is not optimal
- Bid side of LOB given by a function $f : \mathbb{R} \to (0, \infty)$ s.t.
  \[ \int_0^\infty f(x)dx = \infty. \] At any time $t$
  \[ \int_a^b f(x)dx = \text{bids available in the price range } [\tilde{P}_t + a, \tilde{P}_t + b] \]
- The shape function $f$ does not depend upon $t$ or $\tilde{P}_t$

Optimal Execution in a LOB Model (cont.)

- **Price Impact**: process $D = (D_t)_{0 \leq t \leq T}$ adapted, cadlag
  
  At time $t$ a market order of size $A$ moves the price from $\tilde{P}_t + D_{t-}$ to $\tilde{P}_t + D_t$ where
  
  \[ \int_{D_{t-}}^{D_t} f(x)dx = A \]

- **Volume Impact**: $Q_t = F(D_t)$ where $F(x) = \int_0^x f(x')dx'$.

- **LOB Resilience**: $Q_t$ and $D_t$ decrease between trades, e.g.
  
  \[ dQ_t = -\rho Q_t dt, \quad \text{for some } \rho > 0 \]

- At time $t$, a sell of size $A$ will bring
  
  \[ \int_{D_{t-}}^{D_t} (\tilde{P}_t + x)f(x)dx = A\tilde{P}_t + \int_{D_{t-}}^{D_t} xdf(x) \]
  
  \[ = A\tilde{P}_t + \int_{Q_{t-}}^{Q_t} \psi(x)dx = A\tilde{P}_t + \psi(Q_t) - \psi(Q_{t-}) \]

  if $\psi = F^{-1}$ and $\psi(x) = \int_0^x \psi(x')dx'$. 

Stochastic Control Formulation

Holding trajectories / Trading strategies

\[ \Xi(t, x) = \left\{ (\Xi_s)_{t \leq s \leq T} : \text{càdlàg, adapted, bounded variation, } \Xi_t = x \right\} \]

\[ \Xi_{ac}(t, x) = \left\{ (\Xi_s)_{t \leq s \leq T} : \Xi_s = x + \int_t^s \xi_r \, dr \text{ for } (\xi_s)_{t \leq s \leq T} \text{ bounded adapted} \right\} \]

\[
\begin{align*}
    dX_t &= -d\Xi_t \\
    dQ_t &= -d\Xi_t - \rho Q_t \, dt \\
    dR_t &= -\rho Q_t \psi(Q_t) \, dt - \sigma \Xi_t \, dW_t
\end{align*}
\]
Value Function Approach

State space process \( Z_t = (X_t, Q_t, R_t) \), value function

\[
v(t, x, q, r) = v(t, z) = \sup_{\xi \in \Xi(t, x)} \mathbb{E}[U(R_T - \Psi(Q_T))]
\]

First properties

\( U(r - \Psi(q + r)) \leq v(t, x, q, r) \leq U(r - \Psi(q)) \)

\( v(t, x, q, r) = U(r - \Psi(q + r)) \) for \( x = 0 \) and \( t = T \)

Functional approximation arguments imply

\[
v(t, x, q, r) = \sup_{\xi \in \Xi(t, x)} \mathbb{E}[U(R_T - \Psi(Q_T))]
\]

\[
= \sup_{\xi \in \Xi_{ac}(t, x)} \mathbb{E}[U(R_T - \Psi(Q_T))]
\]

\[
= \sup_{\xi \in \Xi_{d}(t, x)} \mathbb{E}[U(R_T - \Psi(Q_T))]
\]
QVI Formulation

As before

- Assume $v$ smooth and apply Itô’s formula to $v(t, X_t, Q_t, R_t)$
- $v(t, X_t, Q_t, R_t)$ is a super-martingale for a typical $\xi$ implies

\[
\partial_t v + \frac{\sigma^2}{2} x^2 \partial^2_{rr} v - \rho q \psi(q) \partial_r v - \rho q \partial_q v \geq 0
\]

- $\partial_x v - \partial_q v \geq 0$

QVI (Quasi Variational Inequality) instead of HJB nonlinear PDE

\[
\min[\partial_t v + \frac{\sigma^2}{2} x^2 \partial^2_{rr} v - \rho q \psi(q) \partial_r v - \rho q \partial_q v, \partial_x v - \partial_q v] = 0
\]

with terminal condition $v(T, x, q, r) = U(r - \Psi(x + q))$

Existence and Uniqueness of a viscosity solution

### Special Cases

**Assuming a flat LOB** $f(x) = c$ and $U(c) = x$

$$v(t, x, q, r) = r - \frac{q^2(1 - e^{-2\rho s})}{2c} - \frac{(x + qe^{-\rho s})^2}{c(2 + \rho(T - t - s))}$$

with $s = (T - t) \wedge \inf\{u \in [0, T]; (1 + \rho(T - t - u))qe^{-\rho u} \leq x\}$

**Still with** $f(x) = c$ **but for a CARA utility** $U(x) = -e^{-\alpha x}$

$$v(t, x, q, r) = -\exp \left[ -\alpha r - \frac{\alpha}{2c} (\alpha c \sigma^2 xx^2 + q^2 (1 - e^{-2\rho s}) + \varphi(t+s)(x+qe^{-\rho s})^2) \right]$$

where $\varphi$ is the solution of the Riccati’s equation

$$\dot{\varphi}(t) = \frac{\rho^2}{2\rho + \alpha c \sigma^2} \varphi(t)^2 + \frac{2\rho \alpha c \sigma^2}{2\rho + \alpha c \sigma^2} \varphi(t) - \frac{2\rho \alpha c \sigma^2}{2\rho + \alpha c \sigma^2}, \quad \varphi(T) = 1$$

and

$s = (T - t) \wedge \inf\{u \in [0, T]; (\alpha c \sigma^2 + \rho \varphi(t + u))x \geq \rho(2 - \varphi(t + u))qe^{-\rho u}\}$