

Chaotic Pendulum

In this experiment you will explore the behavior of a simple driven pendulum in a non-linear regime. In this regime it exhibits a surprisingly rich set of behaviors, including bi-stability, period doubling, and chaos. Before starting the lab, you need to get some background on non-linear oscillations and chaos. The textbooks used in our classical mechanics courses (Analytical Mechanics by Hand and Finch or Classical Dynamics by Thornton and Marion) have nice chapters devoted to the subject.

The mechanical system is quite simple, as shown in the figure. The pendulum consists of a disk free to rotate around its axle and an extra mass on the bottom at the edge of the disk. Torque is applied to the disk by springs through a pulley located behind the disk. The pendulum also has adjustable damping provided by a permanent magnet located behind the disk. The magnet creates eddy currents in the aluminum disk, which result in a damping torque proportional to the velocity of the disk. The damping strength can be adjusted by bringing the magnet closer or further away from the disk with a screw. If the magnet is far away from the disk, there is still small friction in the axle, which can be modeled as a constant torque that always opposes the direction of motion.

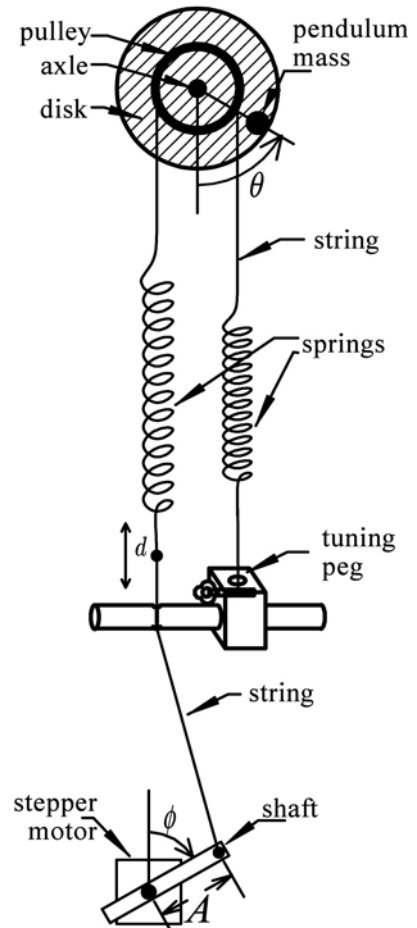
The system is driven by a stepper motor which is controlled by the computer. The amplitude of the drive can be adjusted by sliding the lever attached to the stepper motor and the frequency of the drive is set by the speed of the motor. The computer program allows one to slowly scan the frequency of the drive. The angular position of the disk is recorded by the computer using a rotational encoder attached to the axle. The position is recorded 400 times per revolution of the motor. The encoder output for one complete revolution of the pendulum is 1440 counts.

The equilibrium position of the pendulum can be adjusted with a tuning peg on the bottom. The extra pendulum mass (a screw with nuts) can be changed or removed altogether. When adjusting the mass, make sure all the nuts are tight, so they don't vibrate during motion.

The oscillator can be operated in a variety of modes:

Linear torsional oscillator: If you remove the extra mass on the edge, the oscillator becomes a simple harmonic system (to the extent that the springs obey Hook's law).

Simple pendulum: If the amplitude of the drive is small, the system can be thought of as a simple pendulum undergoing harmonic motion but with an extra restoring torque provided by springs.



Non-linear pendulum: If the amplitude of the drive is large, the behavior becomes much richer because of the non-linear nature of the torque due to the extra mass.

Inverted bi-stable pendulum: If you screw the extra mass at the top, instead of the bottom of the disk, the system will have two equilibrium points, with a potential maximum at the top. Small oscillations around each equilibrium point are still described by harmonic motion, but for larger oscillation amplitude the system behavior is more complicated.

Driven or free motion: The system is designed for driven motion, however if you turn off the power switch on the control box, the motor will stop while the encoder will still continue to record the position of the disk. You can then excite disk oscillations by hand and record their decay due to damping.

Your goal in this lab is to develop a quantitative picture for various modes of the pendulum behavior. Start by writing the differential equation for the motion of the pendulum. You will find that you need to know the following parameters: the moment of inertia of the disk I , the extra mass m and its distance from the axle l , the spring constant of the springs k , and the radius of the pulley R . Using provided scale and ruler, you can determine all these parameters. To measure the spring constant you can remove the bracket from the motor and hang an extra weight from it while holding the disk with your hand so it doesn't rotate. From the differential equations find the frequency of small oscillations of the pendulum with and without the extra mass.

The data are acquired using program **Acquire Chaotic Pendulum Data**. The program saves the data to a binary file (default: data.bin, you can rename the file if it contains interesting data and you want to use it for further analysis). First set the frequency of the motor to 1 Hz and leave frequency scanning off. Start the program with the motor box OFF.

To display the data use the program **DisplayI16**. It reads the binary file even while it is being recorded and allows basic analysis. Start the program and select the name of the binary file. Make sure the *real time updating* is on. Excite the pendulum with your hand and you should see its oscillations on the screen.

You can select a particular region of data using cursors on the upper graph. The data will be plotted on the lower graph when you press *update tabbed graphs*. The program plots the coordinate of the disk θ and the velocity of the disk ω . The velocity is calculated numerically by taking the derivative of position using a local smoothing algorithm. You can adjust the degree of smoothing by changing the number of points to fit and the order of the fit. The coordinate θ and the velocity of the disk ω are the two variables that form the Phase Space, a very useful concept in the analysis of the data. It will also be useful to look at the Fourier transform of the data. You can save θ , ω and acceleration of the disk α to an Excel spreadsheet for more detailed analysis. The data are recorded at a rate of 400 per one motor revolution, you can select fewer points if you wish. Plots of phase space trajectory can be saved directly by clicking on the graph and exporting it.

First study free motion of the pendulum without the drive. Record and save decaying oscillations for the pendulum *without* the extra mass. Make a few measurements for different distances of the

permanent magnet from the disk. Using your previous measurements of the mechanical parameters of the system you should be able to predict the frequency of the oscillations for weak damping and determine the damping constant as a function of the magnet distance.

Record decaying oscillations of the pendulum *with* the extra mass. Make sure you excite only small oscillations, so the pendulum motion is simple harmonic. The oscillation frequency will now be different, it should agree with your predictions.

Now you can study the driven motion. Turn the motor box on. Adjust the bracket on the motor for a small drive amplitude, about 1 cm. Turn on frequency scanning option and start from 0.4 Hz with step increment of 0.003 Hz per revolution or less. You should see the amplitude of the motion change as you would expect for a driven oscillator as you step through the resonance. You can vary the distance to the magnet to adjust the damping of the resonance.

Remove the extra mass, increase the amplitude of the drive and perform another scan. Be careful not to excite too large oscillations, the disk should rotate no more than 1 turn in each direction. How are the results different?

Now add back the extra mass, set the amplitude of the drive to 3-4 cm and perform another frequency scan. Now the behavior should be more complicated. You can also see what happens if the frequency is scanned down, going from 1.2 Hz down in steps of 0.003 Hz or less. Even if you make the frequency step size very small, you will see sudden changes in the amplitude of the oscillations and observe hysteresis when reversing the direction of the scan.

Go to a frequency near the resonance, about 0.7-0.8 Hz, and stop the frequency scan. By hand, stop the motion of the disk and then let it come to an equilibrium oscillation amplitude. In the phase space plot you expect to see an ellipse after the transients decay. You can also excite the disk to a large amplitude and wait for the transient to decay. You should find that there are two different stable motions of the oscillator. This is a manifestation of the bi-stability of a non-linear driven oscillator. Each of the stable orbits are called attractors or limit cycles, they represent the motion of the system after transient effects have decayed away. The initial phase space is divided into basins of attraction for each limit cycle. Depending on the starting conditions, the motion will converge to one of the two attractors. The system has undergone a bifurcation from one to two attractors at a certain drive amplitude. As you increase the amplitude of the drive further you will find more bifurcations of different kinds.

Continue to slowly increase the amplitude of the drive by sliding the bracket on the motor. For each setting, you can measure the behavior of the large and the small amplitude limit cycles by exciting large amplitude or stopping the pendulum. You might notice that for some drive amplitude the path described by the oscillator in phase space becomes asymmetric. The asymmetry could be due to misalignment of the pendulum equilibrium, which can be adjusted by the tuning peg. However, there is also a genuine phenomenon of symmetry breaking in this system, where two asymmetric solutions are possible for a symmetric potential. By varying the initial conditions, you should be able to generate two asymmetrical limit cycles in phase space which are mirror images of each other. This is another bifurcation, spontaneous breaking of the reflection symmetry.

As you increase the drive amplitude further you will find at some point that in the phase space plot the limit cycle is no longer a single closed loop. The path will close onto itself after two or more revolutions. In the Fourier transform you will see peaks at a fraction of the drive frequency and in the time domain the motion will repeat itself after 2 or more periods. Here you have reached a bifurcation resulting in period doubling, common in non-linear systems.

If you continue to make small steps in the drive amplitude, you should see successive bifurcations to 2, 4 and 8 periods. The system is quite sensitive to various parameters at this point. Changes in the behavior can be seen from small adjustments of the tuning peg, which can also generate odd number of periods due to slight asymmetry of the potential. Wait until the motion settles to the attractor and save interesting phase space limit cycles.

Further increase of the drive amplitude will result in chaotic motion. For chaotic motion the path in phase space never closes onto itself and the frequency of motion no longer has a discrete spectrum. It is possible to achieve the situation where the large-amplitude oscillation is chaotic, but the small-amplitude is not. You can investigate the behavior for different initial excitations and convince yourself that the pendulum still settles into one of two “stable” modes, one of which is chaotic. In this case one of the attractors is chaotic or *strange*. For even larger drive amplitudes chaotic motion will result from all initial conditions.

Think about what values of the extra mass on the rim will result in chaotic motion. What happens if it is too small or too large? Try a few different values of the mass.

You can also investigate the motion of an inverted pendulum. For small oscillations it should execute simple harmonic motion around one of the equilibria points with a frequency that you should be able to predict from the differential equations. Larger excitation will result again in a series of period doublings followed by chaos.

When you obtain an interesting chaotic motion you can let the system run overnight, collecting a longer record. For a long run it is easier to display data as a Poincaré plot, which displays only one point per drive period. In the phase plot you can change the reading/revolution to 1 and plot style to points/no line to see a Poincaré section.

In order to gain insight into the series of bifurcations that you have observed, it is instructive to numerically simulate the differential equations of motion. You can use the program **Simulate Pendulum** to do so. The program converts the differential equations to a dimensionless form and allows one to set a number of parameters:

Γ – damping torque proportional to velocity

Γ' – kinetic friction torque that is constant but depends on the sign of the velocity

κ – spring restoring torque

μ – torque from the extra mass on the edge of the pendulum

δ – drive amplitude

Ω – drive angular frequency

θ_0 – imbalance in the torque from the springs

Show that your differential equations reduce to the form used by the program and calculate the values of the parameters from your mechanical measurements. Try to reproduce some of the trajectories that you have observed experimentally with the simulation. You can also adjust the initial conditions for the pendulum to select between different limit cycles. Using the program you can find more precisely the drive amplitudes corresponding to various bifurcations.

With a long chaotic data set you can also confirm several interesting aspects of the chaotic behavior.

1) *Fractal attractor*

For a damped system the motion converges to a limit cycle, which is a closed loop in phase space. Thus at long times the volume that can be occupied by the system in phase space for any initial conditions goes to zero, as one would expect for a line of finite length and vanishingly small thickness. However, when we have chaotic motion, the length of the limit cycle is infinite, as it never repeats. Hence there is a seemingly contradictory situation of a phase space with infinite length but zero volume. Despite infinite length, the orbit of a chaotic pendulum will never completely cover any area of the θ - ω Poincaré section and will occupy a zero volume in θ - ω - ϕ space. This is a result of the fractal nature of the chaotic attractor.

To confirm the fractal nature of the attractor for chaotic motion you can use the program ***Box Counting Dimension***, written by Prof. Robert DeSerio of University of Florida. The following is his description of the algorithm:

The attractor for chaotic motion is called “strange” because it has a strange shape called a fractal. A fractal is a self-similar object, meaning it looks similar on ever decreasing scales. One invariant of a strange attractor is its fractal dimension. Since our attractor is dimensionally the same as an extruded Poincaré section, its fractal dimension is one plus the fractal dimension of any of its Poincaré sections. Consequently, it is the fractal dimension of a Poincaré section that will be sought. While there are several different varieties of fractal dimensions, the most common is the capacity dimension. The box-counting algorithm used to determine the capacity dimension also demonstrates the self similarity of a fractal Poincaré section.

The algorithm for a two dimensional fractal is as follows. Determine the bounds in the plane over which the fractal extends. Divide that area into $M \times rM$ grid boxes ($r = 1$ is fine but not required). Count the number of grid boxes N that are covered by any part of the fractal. Repeat the procedure as M is increased, making the grid boxes smaller and smaller. The slope of the $\log N$ vs. $\log M$ graph (in the limit as $M \rightarrow \infty$) is the capacity dimension.

The procedure above is for the idealized case of an infinite trajectory—a Poincaré section from the complete attractor. It will need to be modified for an experimental Poincaré section obtained from a finite trajectory. The phrase “covered by any part of the fractal” is modified to “having at least one phase point in it.” The phrase “in the limit as $M \rightarrow \infty$ ” must be dropped because if the size of boxes becomes too small, N will saturate at the total number of phase points in the Poincaré section. Thus, one must simply check that the graph of $\log N$ vs. $\log M$ appears to be a straight line before N saturates. For our 50,000-point Poincaré sections, $r = 1/2$ (twice as many grid lines in θ as in ω) and M in a range from 10 to 300 or so works well.

Think about what would be slope of the $\log N$ vs. $\log M$ line in the box counting algorithm if the Poincaré section was a point (as it is for non-chaotic motion), a continuous line, or if it filled a box of finite area. Run the box counting program on a long data file with chaotic motion and confirm that the slope is indeed a non-integer. The self-similarity of the fractal attractor is manifested by the fact that no matter how much you zoom onto an area that appears densely filled with points, the density of points does not increase.

2) Lyapunov Exponent

A hallmark of chaotic motion is its sensitivity to initial conditions. Even though the motion is completely deterministic, minor changes in the initial conditions will result in drastically different position after a period of time. Lyapunov exponent describes how two points that started at similar positions in phase space will evolve with time. In three-dimensional phase space there are actually three Lyapunov exponents corresponding to possible directions of initial separation in phase space. For damped non-chaotic motion all Lyapunov exponents are negative or zero because differences resulting from initial conditions decay with time. For chaotic motion, at least one of the Lyapunov exponents becomes positive, meaning that initial differences grow in time exponentially.

The algorithm for calculating Lyapunov exponents from the time record of motion is fairly complicated, see references. Fortunately, Prof. Robert DeSerio also shared with us the program for calculating the Lyapunov exponents. The program looks at successive Poincaré sections to determine local evolution of the phase space trajectory and then averages the data to determine global Lyapunov exponents.

References

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