

Experimental tests of Coulomb's Law and the photon rest mass

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Abstract

Coulomb's Law is a fundamental principle describing the electric force between isolated charges, and represents the first quantitative law achieved in electromagnetism. The degree of confidence with which the law is experimentally known to hold was investigated after the law was put forth by Coulomb in 1785. The electrodynamics for massive particles suggests that a photon with a finite rest mass will cause a deviation from the inverse square law. So, modern interpretations of the possible deviation from Coulomb's inverse square law are usually associated with the non-zero photon mass. In this article, we first give a historical review of the foundation of Coulomb's inverse square law. Then, the experimental searches for validity of Coulomb's Law, particularly in its inverse square nature, are generally introduced. Based on Proca's equations, the unique simplest relativistic generalization of Maxwell's equations, the link between the deviation from Coulomb's Law and the upper limit on the photon rest mass based on the concentric-spheres apparatus established in the classical experiment of Cavendish is reviewed. Up to now, all the experiments show no evidence for a positive value, and the experimental result was customarily expressed as an upper limit on the deviation or on the photon rest mass. As a representative method with the double mission of testing of the validity of Coulomb's Law and of the photon rest mass, possible improvements for this kind of experiment are discussed.

1. Introduction

The famous inverse square law in electrostatics, first published in 1785 by Charles Augustin de Coulomb, is known as the fundamental law of electrostatics. As the first quantitative law in the history of electricity, Coulomb's inverse square law has played a crucial role and made great contributions to the development of electricity and magnetism, and other related fields. Coulomb's Law, along with the principle of superposition, gives Gauss's Law and the conservative nature of the electric field, which may be generalized using the Lorentz transformation to obtain Maxwell's equations. Even then, the validity of Coulomb's Law has been tested continuously over the past centuries. Based on the classical ingenious scheme devised by Henry Cavendish [1, 2], modern experiments usually yield not only the result of possible

deviation from Coulomb's inverse square law, but that of the upper limit on the photon rest mass [3–7].

The photon, as the fundamental particle of electromagnetic interaction, is generally assumed to be massless. This hypothesis is based on the fact that a photon cannot stand still for ever. However, a nonzero photon mass could be so small that present-day experiments cannot probe it. Taking into account the uncertainty principle, the photon mass could be estimated using $m_\gamma \approx \hbar/(\Delta t)c^2$ to have a magnitude of about 10^{-66} g while the age of the universe is about 10^{10} years, which gives the ultimate limit for meaningful experimental measurements of the photon mass. Up to now, there is no positive result for the photon rest mass or the deviation from Coulomb's inverse square law. The experimental results just serve to set an upper bound to the photon mass and the deviation from the exponent 2 in the inverse square law. The aim of this

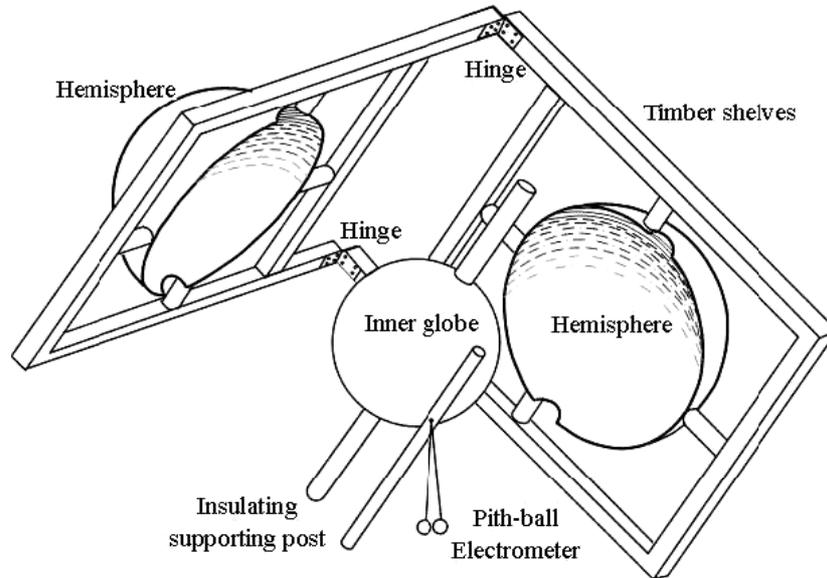


Figure 1. Cavendish's apparatus for establishing the inverse square law of electrostatics. The inner globe of 12.1 inch (307 mm) diameter was fixed on an insulated supporting post. The hollow pasteboard hemispheres, slightly larger than the globe, were mounted on timber shelves with two glass bars, respectively. The shelves were fixed with hinges so as to close the hemispheres easily. When the timber shelves were closed together, the globe and the hemispheres formed insulated concentric spheres. Both the globe and hemispheres were covered with tinfoil to make them better conductors of electricity.

paper is to give a review of the main ideas and results of the investigations intended to test Coulomb's Law and pertinently to improve the upper limit on the photon rest mass.

2. Foundation of Coulomb's Law

Early studies of electrical and magnetic phenomena began from quantitative studies of the law of the force between electric charges, which is now known as Coulomb's Law. An excellent historical development of Coulomb's inverse square law can be found in [8].

2.1. Robison's experiment

The first experimental determination of Coulomb's inverse square law was made in 1769 by John Robison [8]. The scheme is very simple. He used gravity to balance the repulsive force between two electrically charged spheres fixed on a rotating rod, and then determined the values of the electrical force from the known weight of the rod at different distances. Robison analysed his results as a possible deviation from the inverse square law, assuming that the exponent applied to distance was not exactly 2 but $2 + q$. He finally obtained a value for q of 0.06. Robison attributed this value to experimental error and disclosed that the magnitude of the electric force between two charges was inversely proportional to the square of the distance between them. Unfortunately, Robison did not report his experimental results in a timely manner. A reasonable interpretation of Robison's experiment was not given until the 20th century, which implied a limit on the photon mass of about 4×10^{-40} g.

2.2. Concentric metal spheres experiments

In 1773, Henry Cavendish [1] employed concentric spheres to search for the relation between charges indirectly. The experimental apparatus is shown in figure 1. The experiment was performed as follows: first, the globe was connected to one of the closed hemispheres with a conducting wire. Then, the outer sphere was electrically charged for a while, and the connecting wire was broken by a silk thread. Finally, the outer sphere was opened and removed to discharge absolutely. A pith-ball electrometer was used to detect the electric charge on the inner globe. The experiment showed that the pith balls of the electrometer did not separate, which indicated the absence of charge on the inner globe.

To explain the result more elaborately Cavendish suggested the following simple model. Supposing that there was an electrically charged spherical shell with a uniform surface charge density σ , and considering a unit point charge P placed inside the shell, the force acting on the point charge P included two parts: one was from the charges on the area dS_1 which held a solid angle $d\Omega_1$ towards the point charge P , while the other was from dS_2 of solid angle $d\Omega_2$ as presented in figure 2. The net force on the unit point charge P would be

$$\frac{\sigma dS_2}{r_2^2} r_0 - \frac{\sigma dS_1}{r_1^2} r_0. \quad (2.1)$$

Considering the relation between the area and the corresponding solid angle, it is easy to obtain

$$\frac{\sigma dS_2}{r_2^2} r_0 - \frac{\sigma dS_1}{r_1^2} r_0 = \sigma (d\Omega_2 - d\Omega_1) = 0. \quad (2.2)$$

This indicated that the net force on the inside charge was exactly zero if the electric force obeyed the inverse square law. In other words, a uniformly distributed charge on the

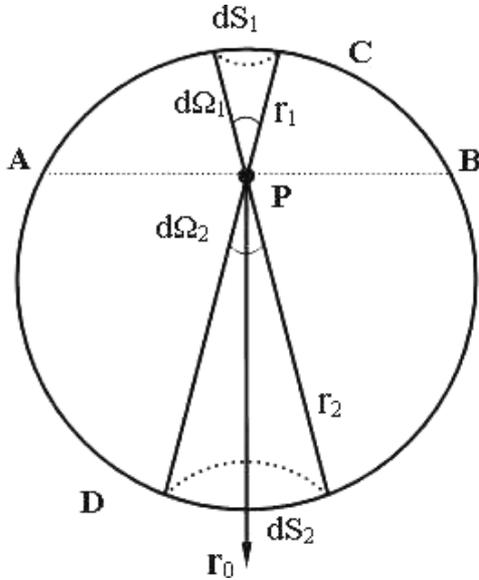


Figure 2. The electrical force was inversely proportional to the square of the distance. The electrically charged metal spherical shell was divided into two parts by the plane AB. A point charge P was placed inside the shell, which experienced force from the charge on the surface of the shell. Although the upper part ACB was smaller than the bottom one ADB, the net force acting on the charge P would be exactly zero if the electric force obeyed the inverse square law.

outer surface of the metal sphere was the necessary conclusion of the inverse square law. If there was any deviation from the inverse square law, charges would migrate through the connecting wire to the inner globe in Cavendish’s experiment. Similarly, Cavendish ascribed the experimental error to a possible breakdown of the inverse square law and concluded that the deviation q from 2 cannot exceed 0.02. A modern interpretation of Cavendish’s result gave an upper limit on the photon rest mass that was about 1×10^{-40} g. Those results were improved in 1873 by Maxwell [2], with the deviation q of less than 5×10^{-5} and the bound on the photon mass about 5×10^{-42} g. The main improvements were that the outer sphere was earthed instead of being removed, which provided a shield from outer disturbances at the cost of determining the potential of the inner shell with more difficulty. The researches of both Robison and Cavendish were not published until the late nineteenth century, when Maxwell mentioned this heuristic experiment in [2].

2.3. Coulomb’s experiments

Although numerous workers have investigated the law of the force between charges before Coulomb, and with even higher precision, it was Coulomb who first announced the inverse square law in 1785. The noteworthy point being emphasized is that the experiments performed by Coulomb were independent of other experiments. Coulomb’s experiments were divided into three parts: first, using a torsion balance Coulomb demonstrated directly that two like charges repel each other with a force that varies inversely as the square of the distance between them; second, the law of the attractive force between unlike charges was indirectly detected by an electrical torsion balance, which was inspired by the inverse square law of

gravitation; third was the relation between the force and the products of two charges. Using this, Coulomb presented the famous law of electricity which has come to be known as Coulomb’s Law. The expression for the electrostatic force from charge 1 to charge 2 was

$$F_{12} = -F_{21} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2 r_{12}}{|r_{12}|^3}. \quad (2.3)$$

In recognition of Coulomb’s work, the SI unit of charge is called the coulomb (C).

3. The photon rest mass and related experiments

3.1. General introduction

The great triumphs of Maxwellian electromagnetism and quantum electrodynamics were based on the hypothesis that the photon should be a particle with zero rest mass. The photon could carry energy and momentum from place to place and light rays would propagate in vacuum with a constant velocity c being independent of inertial frames, which was the second postulate in Einstein’s theory of special relativity. As a result, the velocity of a particle with finite mass would never reach the constant c . The fact that light could not stand still made the assumption reasonable and it was difficult to find any counter-examples in theory. Still, experimental efforts to improve the limits on the rest mass of the photon—in other words, to challenge the accepted theories of the time—have continued since the time of Cavendish or earlier, even before the concept of the photon was introduced.

A finite photon mass may be accommodated in a unique way by changing the inhomogeneous Maxwell equations to the Proca equations, the theoretical expressions of possible nonzero photon rest mass introduced by Proca [9] and de Broglie [10]. In the presence of sources ρ and \mathbf{J} , these equations may be written as (SI units)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} - \mu_\gamma^2 \phi, \quad (3.1)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (3.2)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3.3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} - \mu_\gamma^2 \mathbf{A}, \quad (3.4)$$

together with the field strengths $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$, $\mathbf{B} = \nabla \times \mathbf{A}$ and the Lorentz condition

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0, \quad (3.5)$$

where ϕ and \mathbf{A} are the scalar and the vector potentials, which uniquely determine the field, and $\mu_\gamma^{-1} = \hbar/(m_\gamma c)$ is a characteristic length, with m_γ as the photon mass. If $m_\gamma = 0$, the Proca equations would reduce to Maxwell’s equations. The Proca equations, the relativistically invariant modification of Maxwell’s equations, provide a complete and self-consistent description of electromagnetic phenomena [7].

In four-dimensional space the Proca equations can be rewritten as

$$(\square^2 - \mu_\gamma^2) A_\mu = -\mu_0 J_\mu, \quad (3.6)$$

Table 1. Several important limits on the photon rest mass m_γ .

Author (date)	Ref.	Experimental scheme	Upper limit on m_γ/g
<i>Terrestrial results</i>			
Goldhaber <i>et al</i> (1971)	[4]	Speed of light	5.6×10^{-42}
Williams <i>et al</i> (1971)	[13]	Test of Coulomb's law	1.6×10^{-47}
Chernikov <i>et al</i> (1992)	[14]	Test of Ampere's law	8.4×10^{-46}
Lakes (1998)	[15]	Static torsion balance	2×10^{-50}
Luo <i>et al</i> (2003)	[16]	Dynamic torsion balance	1.2×10^{-51}
<i>Extraterrestrial results</i>			
de Broglie (1940)	[10]	Dispersion of starlight	0.8×10^{-39}
Feinberg (1969)	[17]	Dispersion of starlight	10^{-44}
Schaefer (1999)	[18]	Dispersion of gamma ray bursts	4.2×10^{-44}
Davis <i>et al</i> (1975)	[19]	Analysis of Jupiter's magnetic field	8×10^{-49}
Fischbach <i>et al</i> (1994)	[20]	Analysis of Earth's magnetic field	1.0×10^{-48}
Ryutov (1997)	[21]	Solar wind magnetic field and plasma	10^{-49}
Gintsburg (1964)	[22]	Altitude dependence of geomagnetic field	3×10^{-48}
Patel (1965)	[23]	Alfvén waves in Earth's magnetosphere	4×10^{-47}
Hollweg (1974)	[24]	Alfvén waves in interplanetary medium	1.3×10^{-48}
Barnes <i>et al</i> (1975)	[25]	Hydromagnetic waves	3×10^{-50}
DeBernadis <i>et al</i> (1984)	[26]	Cosmic background radiation	3×10^{-51}
Williams <i>et al</i> (1971)	[27]	Galactic magnetic field	3×10^{-56}
Chibisov (1976)	[5]	Stability of the galaxies	3×10^{-60}

where A_μ and J_μ are the 4-vector of potential ($\mathbf{A}, i\phi/c$) and current density ($\mathbf{J}, ic\rho$), respectively. The d'Alembertian symbol \square^2 is equal to $\nabla^2 - \partial^2/\partial(ct)^2$. In free space, the above equation reduces to

$$(\square^2 - \mu_\gamma^2)A_\mu = 0, \quad (3.7)$$

which is essentially the Klein–Gordon equation for the photon. The characteristic length scale μ_γ^{-1} , namely the reduced Compton wavelength of the photon, is an effective range in which the electromagnetic interaction would exhibit an exponential damping by $\exp(-\mu_\gamma^{-1}r)$.

3.2. Effect of massive photon on the static electric field

Once the photon is provided with a finite mass, three immediate consequences may be deduced from the Proca equations: the frequency dependence of the velocity of light propagating in free space; the third state of the polarization direction, namely the 'longitudinal photon'; and some modifications in the characteristics of classical static fields. All those effects are useful approaches for laboratory experiments and cosmological observations to determine the upper bound on the photon mass.

What is of interest in this paper is the effect of a massive photon in a static electric field. In the case of a massive photon, the wave equation will be modified for all potentials (including the Coulomb potential) in the form

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \mu_\gamma^2\right)\phi = -\frac{\rho}{\epsilon_0}. \quad (3.8)$$

For a point charge and in the static case, this yields a Yukawa type potential,

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \exp(-\mu_\gamma r), \quad (3.9)$$

and the electric field

$$E(r) = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r^2} + \frac{\mu_\gamma}{r}\right) \exp(-\mu_\gamma r). \quad (3.10)$$

Inspection of equations (3.8)–(3.10) shows that if $r \ll \mu_\gamma^{-1}$, then the inverse square law of forces is a good approximation, but if $r \gg \mu_\gamma^{-1}$, then the force law departs from the prediction of Maxwell's equations. Up to now, finding the exponential deviation from Coulomb's Law provides the most reliable test for the photon rest mass in terrestrial experiments, in that those laboratory tests have the advantage of free variation of the experimental parameters [7]. As for large scale observations, the limits usually come from the analyses of astronomical data of the cosmological magnetic field. However, those results are essentially order-of-magnitude arguments due to the incomplete knowledge about the structure of the large-scale magnetic field [11, 12]. In section 4 we will review those laboratory experiments in detail.

3.3. Methods and related results

Experiments for determining an upper limit on the photon rest mass can be categorized, and the results of some important experiments are listed in table 1. All those null results gave the experimental evidence that the static large-scale field coincided with the Maxwellian field. Up to now, no experiment has proved the photon rest mass to be nonzero. However, an experiment that fails to find a finite photon mass does not prove definitely that the mass is zero. The limits on the photon mass have approached ever more closely the ultimate limit determined by the uncertainty principle. So, nobody can assert that the next experiment will not reveal evidence of a definite, nonzero mass.

4. Laboratory tests of Coulomb's inverse square law

4.1. General method and technical background

From the time of Cavendish or earlier, Coulomb's inverse square law has been tested directly or indirectly. Experiments with higher precision and involving different dimensions have been performed over the years. It is now customary to quote

tests of the inverse square law in one of the following two ways [28]:

- (a) Assume that the force varies with the distance r between two point charges according to the phenomenological formula $1/r^{2+q}$ and quote a value or limit for q , which represents departure from the Coulomb inverse square law.
- (b) Assume that the electrostatic potential has the ‘Yukawa’ form $e^{-\mu_\gamma r}/r$ instead of the Coulomb form $1/r$ and quote a value or limit for μ_γ or μ_γ^{-1} . Since $\mu_\gamma = m_\gamma c/\hbar$, the test of the inverse square law is sometimes expressed in terms of an upper limit on the photon rest mass. Geomagnetic and extraterrestrial experiments give μ_γ or m_γ , while laboratory experiments usually give q and perhaps μ_γ or m_γ .

The experimental study of the photon rest mass, or equivalently the deviation from Coulomb’s inverse square law to a certain extent, is difficult because, for an experiment confined in dimension D , the effect of finite μ_γ is of the order of $(\mu_\gamma D)^2$, a theorem proposed in 1971 by Goldhaber and Nieto [4]. This means that an experiment designed to find the effects of a massive photon or a breakdown of Coulomb’s Law must either interrogate a region of size comparable to μ_γ^{-1} or manage extraordinary precision in detecting the infinitesimal evidence of a massive photon or the deviation. The concentric sphere experiments, proposed by Cavendish, are typical of the apparatus used by successive people while the primary improvement comes from technological developments that allow detection of weak signals with enormous sensitivity. The principal advantage of this method is that all the parameters involved can be individually varied and tested, whereas astronomical observations usually involve a number of factors that are subject to interpretations due to various assumptions that are difficult to verify.

4.2. Static experiments

The original concentric spheres experiment performed by Cavendish in 1773 gave an upper limit on q of $|q| \leq 0.02$, and this result was improved to $|q| \leq 5 \times 10^{-5}$ in 1873 by Maxwell, corresponding to the limits on the photon rest mass being 1×10^{-40} g and 5×10^{-42} g, respectively, as mentioned in section 2. In the above experiments, the electric potential on the inner conducting sphere was measured. Maxwell [3] first derived the relation between the deviation q and the potential, and a more detailed interpretation was revisited by Fulcher and Telljohann [29].

Suppose that the electrostatic force between two unit charges is an arbitrary function $F(r)$ of the distance r between them. The electrostatic potential is given by

$$U(r) = \int_r^\infty F(s) ds. \tag{4.1}$$

Then, considering a uniform distribution of a unit charge over a conducting shell of radius a , the potential at a distance r from the centre of the sphere is readily determined to be [30]

$$V(r) = \frac{f(r+a) - f(|r-a|)}{2ar}, \tag{4.2}$$

where

$$f(r) \equiv \int_0^r sU(s) ds. \tag{4.3}$$

Now, apply these expressions to the Cavendish type experiments, and suppose that the radii of the two concentric spheres are R_1 and R_2 ($R_1 < R_2$) with the charges Q_1 and Q_2 spread uniformly over them, respectively. Using equation (4.2), then, one could get the potential on the inner shell

$$V(R_1) = \frac{Q_1}{2R_1^2} f(2R_1) + \frac{Q_2}{2R_1R_2} [f(R_1+R_2) - f(|R_1-R_2|)] \tag{4.4}$$

and the potential on the outer shell

$$V(R_2) = \frac{Q_2}{2R_2^2} f(2R_2) + \frac{Q_1}{2R_1R_2} [f(R_1+R_2) - f(|R_1-R_2|)]. \tag{4.5}$$

After the outer shell was charged by a potential V_0 , a part of the charge would pass through the connecting wire into the inner shell until the equilibrium of $V(R_1) = V(R_2) \equiv V_0$ was satisfied. Then, the charges piled up on the inner shell could be determined:

$$Q_1 = 2R_1V_0 \frac{R_1 f(2R_2) - R_2 [f(R_1+R_2) - f(|R_1-R_2|)]}{f(2R_1) f(2R_2) - [f(R_2+R_1) - f(|R_1-R_2|)]^2}. \tag{4.6}$$

If Coulomb’s Law is valid, the potential of a unit charge has the form $U(r) = 1/r$, hence $f(r) \equiv r$. Because the two shells had been connected by a wire, the charge originally given to the inner shell will end up at the outer surface of the outer one, i.e. $Q_1 = 0$, which is the essential consequence of Coulomb’s Law. Actually, for a conductor of an arbitrary shape, the charge distribution is so arranged that the electric field inside the conductor vanishes. The aim of the experiments to verify Coulomb’s Law is to find traces of the residual fields inside the outer shell.

In Cavendish’s experiment, after the two shells are charged and the connecting wire broken, the outer shell was removed to infinity. The potential of the inner shell became

$$V_C(R_1) = \frac{Q_1}{2R_1^2} f(2R_1). \tag{4.7}$$

In Maxwell’s case, after the connecting wire was broken, the outer shell was then earthed instead of being removed, which meant $V(R_2) \equiv 0$. So the potential of the inner shell can be expressed as

$$V_M(R_1) = V_0 \left[1 - \left(\frac{R_2}{R_1} \right) \frac{f(R_2+R_1) - f(|R_1-R_2|)}{f(2R_2)} \right]. \tag{4.8}$$

Following Maxwell, suppose that the exponent in Coulomb’s inverse square law is not 2, but $2+q$ with $|q| \ll 1$. In this case, for a unit point charge, we get

$$U(r) = \frac{1}{1+q} \frac{1}{r^{1+q}} \approx \frac{1}{r^{1+q}}, \tag{4.9}$$

and to first order in q ,

$$f(r) \approx r(1 - q \ln r). \tag{4.10}$$

Substituting (4.10) into (4.7) and (4.8), the link between the potential and the deviation q would be given by the following formulae in the experiments of Cavendish and Maxwell, respectively,

$$V_C(R_1) \approx \left(\frac{R_2}{R_2 - R_1} \right) q V_0 M(R_1, R_2), \quad (4.11)$$

$$V_M(R_1) \approx q V_0 M(R_1, R_2) \quad (4.12)$$

where $M(R_1, R_2)$ is a dimensionless geometrical factor of order unity,

$$M(R_1, R_2) = \frac{1}{2} \left[\frac{R_2}{R_1} \ln \frac{R_2 + R_1}{R_2 - R_1} - \ln \frac{4R_2^2}{R_2^2 - R_1^2} \right]. \quad (4.13)$$

Obviously, if $q = 0$, the potential of the inner sphere is just zero, which is the case predicted by Coulomb's inverse square law. So, by detecting the potential on the inner shell directly, one could obtain the deviation from Coulomb's inverse square law.

4.3. Dynamic experiments

In order to obtain a higher precision, modern experiments using a similar arrangement were performed with high alternating voltage applied to the outer shell accompanied by phase-sensitive technology to detect the relative potential difference between the shells. The motivation for concentric sphere experiments since 1971 has been dominated by the possibility that the photon rest mass may not be exactly zero rather than testing for q . In this case, the relative potential difference between the two spheres, according to (4.2) and (4.10), can be expressed as

$$\frac{V(R_2) - V(R_1)}{V(R_2)} = q M(R_1, R_2), \quad (4.14)$$

with the same expression for the geometrical factor $M(R_1, R_2)$, so that q is essentially the quotient of the measured potential difference $V(R_2) - V(R_1)$ and the applied voltage $V(R_2)$.

By now, the expected relation between the potential difference and the deviation q has been obtained. In order to determine the photon rest mass from experimental results, it is also necessary to find the relation between μ_γ and the potential difference.

Considering an idealized geometry of two concentric, conducting, spherical shells of radii R_1 and R_2 ($R_1 < R_2$) with an inductor across this spherical capacitor in which there is no charge inside, an alternating potential of $V_0 e^{i\omega t}$ is applied to the outer shell. In this case, a solution of the massive electromagnetic field equation (3.8) can be written as $\phi(r, t) = \phi(r) e^{i\omega t}$. Then, the wave equation reduces to

$$(\nabla^2 + k^2) \phi(r) = 0, \quad (4.15)$$

where

$$k^2 = \frac{\omega^2}{c^2} - \mu_\gamma^2. \quad (4.16)$$

The exact result of the potential with the proper boundary conditions is

$$\phi(r) = V_0 \frac{R_2}{r} \frac{e^{ikr} - e^{-ikr}}{e^{ikR_2} - e^{-ikR_2}} \quad (r \leq R_2) \quad (4.17)$$

Choosing a spherical Gaussian surface at radius r between the two shells and then using the expression (4.17) for the interior region, the closed integral of the Proca equation (3.1) over the volume from the interior to the Gaussian surface becomes

$$\int dV \nabla \cdot E = -\mu_\gamma^2 \int dV \cdot \phi(r). \quad (4.18)$$

Then, a complete solution of the field inside a uniformly charged single sphere of radius R_2 can be obtained:

$$E(r, t) = \frac{\mu_\gamma^2}{k^2 r^2} \frac{V_0 R_2}{e^{ikR_2} - e^{-ikR_2}} [ikr (e^{ikr} + e^{-ikr}) - (e^{ikr} - e^{-ikr})] e^{i\omega t} \cdot \frac{\mathbf{r}}{r}. \quad (4.19)$$

Making a power series expansion of the electric field, keeping in mind that $kr < 1$ and $\omega/c > \mu_\gamma$, and neglecting the second-order term for the performed experiments, the above electric field reduces to

$$E(r, t) \approx -\frac{1}{3} \mu_\gamma^2 r (V_0 e^{i\omega t}) \frac{\mathbf{r}}{r}. \quad (4.20)$$

Then, the relative potential difference between the inner and outer shells is given as

$$\frac{V(R_2) - V(R_1)}{V(R_2)} = -\frac{1}{6} \mu_\gamma^2 (R_2^2 - R_1^2). \quad (4.21)$$

This result shows emphatically that the relative potential difference $\Delta V/V$ is independent of the frequency of the applied alternating voltage, i.e. the boundary condition problems in the dynamic experiments are the same as those in the static experiments. Meanwhile, it hints that the quadratic dependence of the potential difference on μ_γ makes this method insensitive to a small value of μ_γ . The first use of equation (4.21) and a description of the Cavendish type experiments in terms of μ_γ were given in 1971 [13].

The first dynamic experiment dates back to Plimpton and Lawton (1936) [31]. The electrostatic experiments of Cavendish and Maxwell with concentric metal globes were replaced by a quasistatic method, in which the difficulties due to spontaneous ionization and contact potentials were overcome by placing the detector inside the inner globe and connecting it permanently to detect any evidence of the potential difference between globes (figure 3). The detector was employed as a resonance electrometer with a frequency of about 2 Hz, which could improve sensitivity and reduce the inductive effects due to simply opening and closing the circuits of the applied voltage on the outer globe. As a consequence of the potential difference between the two globes induced by the voltage applied to the outer globe, any resonant motion of the galvanometer could be observed through the conducting window at the top of the outer globe using a mirror and telescope. The conducting window, claimed by the authors to be one of the essential points to the success of the experiment, was a glass-bottomed vessel threaded into the outer globe. It contained a solution of ordinary salt in water with its surface flush with the top surface of the outer globe, in which a disc of fine wire gauze, covering the glass and soldered to the threaded rim of the vessel, was used to ensure perfect conductivity. A harmonically alternating high potential of about 3000 V, generated by a specially designed condenser

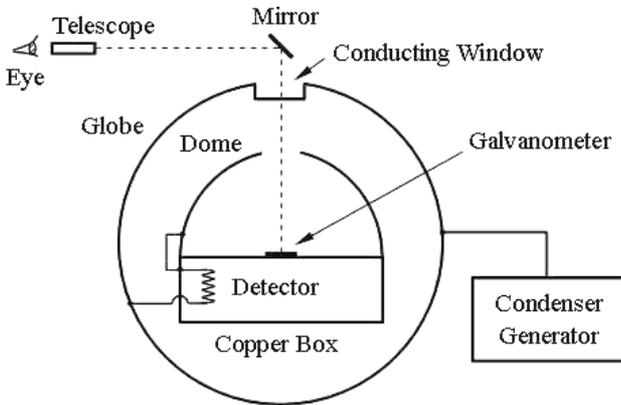


Figure 3. Experimental equipment of Plimpton and Lawton for testing the inverse square law of force between charges.

operating at the low resonance frequency of the galvanometer, was finally applied to the outer globe. Tests were made to detect the change in the potential of the dome relative to the outer globe. The experimental result showed that no change in the small heat motion of the galvanometer could be detected during the course of the whole experiment for sensitivities of the detector up to $1 \mu\text{V}$. The radii of the two globes were 2.5 feet and 2.0 feet (760 and 610 mm), respectively. Substitution of the experimental parameter in the expressions (4.14) and (4.21) then yielded the limit on the deviation from Coulomb's inverse square law of $q < 2 \times 10^{-9}$, and from this the limit on the photon rest mass of $m_\gamma < 3.4 \times 10^{-44} \text{ g}$ can be estimated.

Cochran and Franken's experiment [32] employed concentric cubic conductors instead of concentric spheres, due to the cost and awkwardness of constructing and using large spheres. The most significant improvement was the application of the 'lock-in amplifier' for detecting the minute potential difference between the conducting surfaces, with an ability to detect voltage amplitudes of $2 \times 10^{-9} \text{ V}$. An alternating voltage of about 1200 V with frequency ranging from 100 Hz to 500 Hz was applied to the outer box. The authors finally gave two values for the new bound on the deviation q from Coulomb's inverse square law: the first (limit of error) representing a bound greater than the worst possible value obtained under all conditions was $|q| < 4.6 \times 10^{-11}$, and the second (70% confidence) representing a bound with a probable error of 70% was $|q| < 9.2 \times 10^{-12}$, expressing it as the corresponding bound $m_\gamma \leq 3 \times 10^{-45} \text{ g}$ on the photon rest mass. However, due to the large amount of calculation involved in the cubic configuration, it is impossible here to document explicitly the errors the authors quoted.

A repetition of Plimpton and Lawton's experiment was reported in 1970 by Bartlett *et al* [33]. Instead of two concentric spheres, this experiment adopted five concentric spheres in order to improve the sensitivity and to help eliminate errors introduced by stray charges. The voltage was applied between the two outermost spheres, and the induced signal between the two innermost was measured. The middle one served as a shield. A potential difference of 40 kV at 2500 Hz was induced between the outer two spheres. A lock-in detector with a sensitivity of about 0.2 nV measured the potential difference between the inner two spheres. Finally, the authors

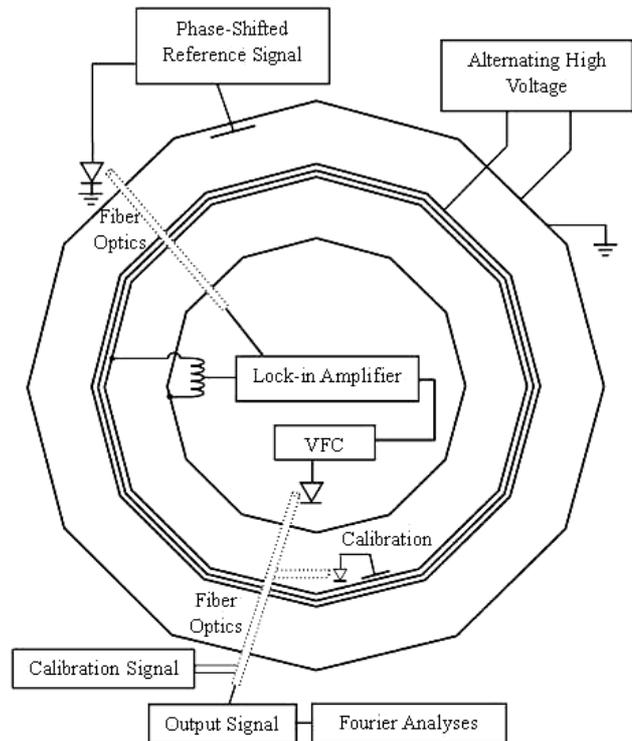


Figure 4. Schematic set-up of Williams *et al* (1971) at Wesleyan University. The 4 MHz and 10 kV in peak-to-peak voltage, obtained by pumping energy into a resonant circuit formed by the two outer shells and a high- Q water-cooled coil, is applied across the outer two shells. A battery-powered lock-in amplifier, located inside the innermost shell, is employed to search for any trace of this signal appearing across the inner two shells. Three sets of optic fibres are used for data transmitting, which are the reference signal for the phase detector, the output of the voltage-to-frequency converter (VFC) from the lock-in amplifier and the calibration voltage periodically introduced into the system. The output signal is finally analysed for evidence of a violation of Coulomb's Law and the existence of the photon mass.

obtained $|q| \leq 1.3 \times 10^{-13}$, expressed as the limit of the photon rest mass being $m_\gamma \leq 3 \times 10^{-46} \text{ g}$.

The experiment performed by Williams *et al* [13] was representative of extraordinary precision within the laboratory range. The experimental apparatus (figure 4) consisted of five concentric icosahedra. The two outermost, with the inner one of about 1.5 m in diameter, were charged to 10 kV peak to peak with a 4 MHz sinusoidal voltage. Any deviation from the $1/r^2$ force law would be detected by measuring the line integral of the electric field between the two innermost spheres, with a detection sensitivity of about 10^{-12} V peak to peak. Data were transmitted in and out by fibre optics, through holes in the icosahedra. In order to prevent penetration of outer fields through the holes, the fibres were used as a waveguide, whose diameters were smaller than the cut-off frequency. Similar to Bartlett *et al* the middle shell was added in order to prevent stray electric and magnetic fields inside the sphere. A lock-in amplifier was employed to enhance the sensitivity of the detectable potential difference. To ensure the system worked properly, a calibration voltage was periodically introduced into the system on a third light beam while the reference beam was working. In the experiment, a high-frequency voltage

Table 2. Results of experimental tests of Coulomb's law and the photon rest mass.

Author (date)	Experimental scheme	Deviation of q	Upper limit on m_γ/g
Robison (1769)	Gravitational torque on a pivot arm	6×10^{-2}	4×10^{-40}
Cavendish (1773)	Two concentric metal spheres	2×10^{-2}	1×10^{-40}
Coulomb (1785)	Torsion balance	4×10^{-2}	$\sim 10^{-39}$
Maxwell (1873)	Two concentric spheres	5×10^{-5}	1×10^{-41}
Plimpton and Lawton (1936)	Two concentric spheres	2×10^{-9}	3.4×10^{-44}
Cochran and Franken (1967)	Concentric cubical conductors	9.2×10^{-12}	3×10^{-45}
Bartlett <i>et al</i> (1970)	Five concentric spheres	1.3×10^{-13}	3×10^{-46}
Williams <i>et al</i> (1971)	Five concentric icosahedra	$(2.7 \pm 3.1) \times 10^{-16}$	1.6×10^{-47}
Fulcher (1985)	Improved on Williams' experiment	$(1.0 \pm 1.2) \times 10^{-16}$	1.6×10^{-47}
Crandall <i>et al</i> (1983)	Three concentric icosahedra	6×10^{-17}	8×10^{-48}
Ryan <i>et al</i> (1985)	Cryogenic experiment		$(1.5 \pm 1.4) \times 10^{-42}$

was used in order to reduce the skin depth, which varied as $\delta \propto 1/\sqrt{\omega}$. The experimental result of three days of data was statistically consistent with the assumption that the photon rest mass is identically zero. Expressing the result as a deviation from Coulomb's Law in the form $1/r^{2+q}$, they gave $q \leq (2.7 \pm 3.1) \times 10^{-16}$, alternatively the limit on the photon rest mass being $m_\gamma \leq 1.6 \times 10^{-47}$ g.

The latest Coulomb null experiment with a similar principle was proposed in 1982 by Crandall [34], who introduced a slightly upgraded approach adapted for physics students at several different college levels in view of a flexible budget for the total cost. The main distinction in contrast with earlier experiments was the three-sphere arrangement, instead of five concentric spheres, and the geometry was inside-out with respect to that of Williams *et al*. The motivation for this change was to provide increased convenience for studies by students. The radii of the three icosahedral spheres were 0.2 m, 0.5 m and 1.0 m respectively. The applied alternating voltage on the two innermost spheres was 500 V peak to peak with a frequency of 500 kHz. Differences in configuration only resulted in differences in cost, as students in different grades could obtain the corresponding bounds on the deviation from Coulomb's inverse square law and the limit on the photon rest mass. The author and his collaborators claimed that they had improved the values to $q \leq 6 \times 10^{-17}$ for the deviation from Coulomb's Law and $m_\gamma \leq 8 \times 10^{-48}$ g for the photon rest mass using traditional geometry.

In 1985, a completely different experiment to determine the photon mass using Coulomb's Law was conducted at a temperature of 1.36 K by Ryan *et al* [35]. The motivation was to understand the origin of parity and weak interactions which had led to concepts such as spontaneous symmetry breaking. The basic idea behind this proof for a massive photon was based on the concept that particles were massless above a critical temperature and would acquire a mass below this temperature. Their null result established that the photon at 1.36 K had a mass less than $(1.5 \pm 1.4) \times 10^{-42}$ g, several orders less sensitive than results of earlier experiments.

Finally, the results of experiments to test Coulomb's inverse square law are listed in table 2 as a comparison. Coulomb's Law is a fundamental law of electromagnetism, and it seems pertinent to inquire to what extent this law is experimentally known to hold, in particular its inverse square nature. The above experimental results reveal that the validity of its inverse square nature can be unassailable almost to a certainty at the macroscopic level, the length scale of which

has been shown to be of the order of 10^{13} cm by laboratory and geophysical tests reviewed above. As for the microcosmic scale, the well-known Rutherford experiments on the scattering of alpha particles by a thin metal foil gave an indication that Coulomb's Law would be valid at least down to distances of about 10^{-11} cm, which is roughly the size of the nucleus. Modern high energy experiments on the scattering of electrons and protons proved that Coulomb's inverse square law was successful even down to the fermi range [36]. Thus, the evidence from experimental results reveals that the inverse square Coulomb's Law is valid not only over the classical range, but deep into the quantum domain also, a total length scale spanning 26 orders of magnitude or more: this range is impressive but still finite.

5. Discussions

5.1. Effect of irregular configuration

Equation (4.14), used to estimate the deviation q , could be directly deduced from the results derived by Maxwell for two concentric globes of radii R_1 and R_2 , in which the globes were assumed to be exact and perfect spheres and the charges were uniformly distributed over them without any variation. All those assumptions were made to facilitate the computation. In fact, the configuration of the experiments mentioned above changed the conditions somewhat, such as concentric icosahedra instead of concentric spheres, holes made in the globes, and so on.

As for irregularities in the shapes of those globes, it apparently is not of crucial importance, since there would be no electric field inside a cavity of any shape unless Coulomb's Law is invalid. However, Shaw [37] has proposed a conjecture by virtue of the symmetry of the problem and the well-known uniqueness theorem, which states that charge will distribute itself uniformly over the surface of an isolated conducting sphere. But the uniqueness theorem has proved to hold only for two cases: Coulomb's Law and the Yukawa potential $\phi(r) = \exp(-kr)/r$, both of which satisfy a second-order field equation—Laplace's equation $\nabla^2\phi = 0$ for the Coulomb potential and the Helmholtz equation $\nabla^2\phi = k^2\phi$ for the Yukawa potential. Excluding these two cases, charge may be distributed on an isolated spherical conductor in any number of non-equilibrium ways, or perhaps even none [37]. Obviously, the distribution of charge between concentric conducting shells has been at the heart of the most sensitive tests of the exponent

in Coulomb's Law since the days of Cavendish. Spencer [38] has found spherically symmetric solutions under the assumption that the potential of a point charge varies either as $\exp(-kr/r)$ or as $1/r^n$. One might imagine that the induced potential on an inner globe would depend on some particular charge distribution that is not spherically symmetric, and then by superposition of all possible distributions one can construct an equilibrium distribution that is symmetric. Hence, the irregularities in the spherical surface would be illuminated by the spectrum of observed distributions.

As regards the holes made in the globes, calculations show that the effect of the hole shows a quartic dependence on the solid angle subtended by the hole at the centre of the sphere, when no improving measures are adopted. However, this effect will be significantly weakened to a negligible level by taking technical precautions, such as covering the hole with a lid, or by employing something like the 'conducting window' used in the experiment of Plimpton and Lawton in 1936.

5.2. Improved result for Williams's experiment

The geometrical factor $M(R_1, R_2)$ occurring in (4.14) was derived by Maxwell based on the special configuration of two concentric globes with radii of R_1 and R_2 . However, Plimpton-Lawton modified Cavendish-Maxwell's apparatus by replacing the inner globe with a hemispherical dome on top and a box containing the detector at the bottom, while Bartlett *et al* used five concentric spheres and Williams *et al* employed five concentric icosahedra instead of two globes. So, a more refined calculation should be done to give an accurate result. The improved result for the experiment of Williams *et al* was reviewed in 1986 by Fulcher [39]. The exact expression for the potential at a distance r from a sphere of radius a containing a uniformly distributed charge Q would be

$$V(r) = \frac{Q}{2ar(1-q^2)} [(r+a)^{1-q} - |r-a|^{1-q}]. \quad (5.1)$$

Applying this equation to the four spheres (the outermost two and the innermost two), with radii $r_4 > r_3 > r_2 > r_1$, and the outermost spheres with charges Q for r_4 and $-Q$ for r_3 , a refinement of the original expression (4.14) can be obtained as

$$\frac{V(r_2) - V(r_1)}{V(r_3) - V(r_4)} \approx \frac{qr_4}{r_4 - r_3} [M(r_3, r_2) - M(r_3, r_1)] - \frac{qr_3}{r_4 - r_3} [M(r_4, r_2) - M(r_4, r_1)] = qF(r_1, r_2, r_3, r_4), \quad (5.2)$$

wherein the approximation of the first order was used to refine the expression for the potential difference, and the geometrical factor M was replaced by the conjunct geometrical factor F dependent on the radii of the four spheres.

Another noteworthy point was the determination of the effective radius to incorporate the differences between the spheres and the icosahedra. There were two means to deal with the problem: one in which the icosahedron was made to have the same surface area as the sphere, which gave $R = 0.83L$ with L being the length of the triangular sides, and the other which adopted the radius of the inscribed sphere as the effective radius, which gave $R = 0.76L$. Due to the fact that the factors

appearing in the conjunct geometrical factor F depended on the ratio of the radii, one would obtain the same value for the ratio of the potential difference occurring in (5.2). Finally, using the surface area criterion to define the effective radii, Fulcher gave an improved result for the deviation from Coulomb's inverse square law,

$$q = (1.0 \pm 1.2) \times 10^{-16}, \quad (5.3)$$

which was a factor of about 2.5 times smaller than that reported by Williams *et al*. As for the upper limit on the photon rest mass, the analyses of Williams *et al* for the sensitivity of their experiment were correct [39].

5.3. Possible improvements

For convenience of discussion, equation (4.14) is rewritten as

$$q = \frac{\Delta V}{V_0 M(n)}, \quad (5.4)$$

where V_0 and ΔV denote the applied high voltage and the potential difference between conducting spheres, respectively. From equation (5.4), it is evident that there are potentially three ways to improve the experiment: (1) choose a larger ratio of the sphere radii to increase the geometrical factor $M(n)$; (2) detect a smaller potential difference ΔV ; and (3) increase the applied voltage V_0 .

As for the geometrical factor, $M(R_1, R_2)$ is revised as $M(n)$ with $0 \leq (n = R_1/R_2) \leq 1$, so

$$M(n) = \frac{1}{2} \left[\frac{1}{n} \ln \frac{1+n}{1-n} - \ln \frac{4}{1-n^2} \right]. \quad (5.5)$$

$M(n)$ is a monotonic slowly changing function of n and only reaches its maximum at $n = 0$, where $M(0) = 0.3069$, while its minimum $M(1) = 0$ is shown in figure 5. So even using a smaller value for n , this does not offer a significant improvement on the deviation q from Coulomb's Law. However, equation (4.21) shows that the sensitivity to μ_γ^{-1} increases linearly with the size of the shells.

For further experiments, a crucial improvement in the accuracy of the Cavendish method can be obtained by reducing the Johnson noise (thermal noise) voltage and hence detecting a smaller potential difference, which is given by the Nyquist Theorem [40, 41],

$$\langle V_{\text{noise}}^2 \rangle = 4kT \Delta f \text{Re } Z, \quad (5.6)$$

where k is the Boltzmann constant, T the absolute temperature, Δf the bandwidth, and $\text{Re } Z$ the real part of the input impedance. In the case of high frequency ω [4],

$$\text{Re } Z \approx [R(\omega C)^2]^{-1}, \quad (5.7)$$

where R is the input resistance and C is the parallel capacitance. From the expressions (5.6) and (5.7), the approaches to decrease the Johnson noise are obvious: low temperature, long observation time, high frequency alternating voltage applied to the concentric spheres, and increase in the input resistance. The most promising approach is to apply a superconducting apparatus at low temperature, which would

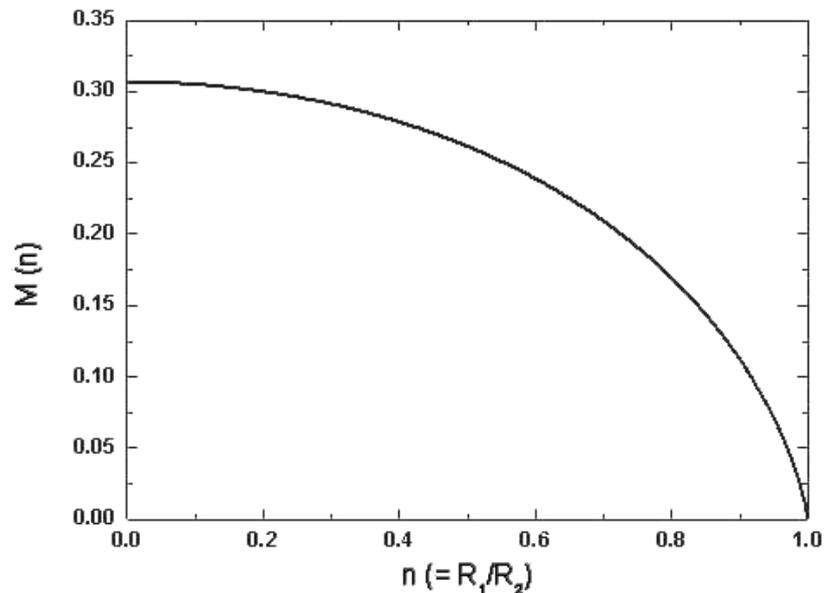


Figure 5. The geometrical factor $M(n)$ for spheres. The parameter n denotes the ratio of the radii between the inner sphere and the outer one.

indeed lower the noise voltage by several orders, but cooling such a large apparatus would be a stupendous task.

The amplitude of the voltage applied to the outer conducting sphere is not limited by theoretical restrictions but by practical considerations. However, the frequency is limited by the approximation used to deduce (4.20), namely $kr < 1$ and $\omega/c > \mu_\gamma$. Obviously, the rough range of the frequency available should be $c\mu_\gamma < \omega < c/r$, which gives an ultimate frequency of the order of 10^7 Hz for the general dimensions of the laboratory experiment.

The investigation of the photon mass, or the deviation of Coulomb's Law in a sense, is not theoretical, but experimental. As mentioned before, the limitations of experimental results on the photon mass and the deviation of Coulomb's Law are becoming lower and lower. A question that arises naturally is: is it necessary to strive continuously to lower the upper limit in order to convince ourselves that the photon rest mass is zero or non-zero? Or do we really need to continue to infinity the succession of these upper limits if there are no theoretical grounds appropriate for specifying the microscopic origin of the photon rest mass at present? For experimental physicists at least, the answer obviously is yes. But there is an ultimate limit for meaningful experimental measurement of the photon rest mass, which is dependent on the age and the dimensions of the Universe. The limit below this ultimate bound, say about 10^{-66} g estimated by the uncertainly principle, would be thought of as meaningless or just zero. However, there is a great gap between the current limits and the ultimate one, and we could not guarantee that future attempts to improve this limit will not lead to unexpected results about the photon rest mass and the deviation from Coulomb's inverse square law.

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