ORF 307: Lecture 14

Linear Programming: Chapter 14: Network Flows: Algorithms

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• Primal Network Simplex Method

• Dual Network Simplex Method

• Two-Phase Network Simplex Method

• One-Phase Primal-Dual Network Simplex Method

• Planar Graphs

• Integrality Theorem
Primal Network Simplex Method

Used when all primal flows are nonnegative (i.e., primal feasible).

**Pivot Rules:**

- **Entering arc:** Pick a nontree arc having a negative (i.e. infeasible) dual slack.

- **Leaving arc:** Add entering arc to make a cycle. Leaving arc is an arc on the cycle, pointing in the opposite direction to the entering arc, and of all such arcs, it is the one with the smallest primal flow.
Primal Method—Second Pivot

Explanation of leaving arc rule:

- Increase flow on (d,e).
- Each unit increase produces a unit \textit{increase} on arcs pointing in the \textit{same} direction.
- Each unit increase produces a unit \textit{decrease} on arcs pointing in the \textit{opposite} direction.
- The first to reach zero will be the one pointing in the opposite direction and having the smallest flow among all such arcs.
Primal Method—Third Pivot

Entering arc: (c,g)
Leaving arc: (c,e)

Obj value = 335

Entering arc: (c,e)
Leaving arc: (c,g)

Obj value = 316

Optimal!
Dual Network Simplex Method

Used when all dual slacks are nonnegative (i.e., dual feasible).

Pivot Rules:

*Leaving arc:* Pick a tree arc having a negative (i.e. infeasible) primal flow.

*Entering arc:* Remove leaving arc to split the spanning tree into two subtrees. Entering arc is an arc reconnecting the spanning tree with an arc in the opposite direction, and, of all such arcs, is the one with the smallest dual slack.
Dual Network Simplex Method—Second Pivot

Leaving arc: (d,a)
Entering arc: (b,c)

Obj value = 106

Leaving arc: (d,a)
Entering arc: (b,c)

Obj value = 316

Optimal!
Recall initial tree solution:

Leaving arc: \((g,a)\)
Entering arc: \((d,e)\)

- Remove leaving arc. Need to find a reconnecting arc.
- Since the leaving arc has a negative flow, there is a net supply at the subtree attached to the head node and a net demand at the subtree attached to the tail node.
- So, reconnecting with an arc that spans in the same direction does not improve anything.
- Hence, only consider arcs spanning the two subtrees in the opposite direction.

- Consider a potential arc reconnecting in the opposite direction, say \((b,c)\).
  - Its dual slack will drop to zero.
  - All other reconnecting arcs pointing in the same direction will drop by the same amount.
  - To maintain nonnegativity of all the others, must pick the one that drops the least.
Example.

- Turn off display of dual slacks.
- Turn on display of artificial dual slacks.
Two-Phase Method–First Pivot

Use dual network simplex method.
Leaving arc: (d,e)  Entering arc: (e,f)

Primal Feasible!
• Turn off display of artificial dual slacks.
• Turn on display of dual slacks.
Two-Phase Method—Second Pivot

Entering arc: (g, b)
Leaving arc: (g, f)

Obj value = 500

Obj value = 290
Two-Phase Method–Third Pivot

Entering arc: (f,c)
Leaving arc: (f,a)

Obj value = 290

Obj value = -46

Final Solution: Optimal!
Click here (or on any displayed network) to try out the online network simplex pivot tool.
• Artificial flows and slacks are multiplied by a parameter $\mu$.
• In the Figure, $6,1$ represents $6 + 1\mu$.
• Question: For which $\mu$ values is dictionary optimal?
• Answer:

\[
\begin{align*}
1 + \mu & \geq 0 (a,b) \\
-2 + \mu & \geq 0 (a,c) \\
\mu & \geq 0 (a,d) \\
-3 + \mu & \geq 0 (a,g) \\
\mu & \geq 0 (b,c) \\
3 + \mu & \geq 0 (b,d) \\
\mu & \geq 0 (c,a) \\
-2 + \mu & \geq 0 (c,e) \\
-9 + \mu & \geq 0 (g,d) \\
12 + \mu & \geq 0 (f,c) \\
6 + \mu & \geq 0 (g,e) \\
\end{align*}
\]

• That is, $9 \leq \mu < \infty$.
• Lower bound on $\mu$ is generated by arc $(g,d)$.
• Therefore, $(g,d)$ enters.
• Arc $(a,d)$ leaves.
Second Iteration

- Range of $\mu$ values: $2 \leq \mu \leq 9$.
- Entering arc: (a,c)
- Leaving arc: (b,c)

New tree:
Third Iteration

- Range of $\mu$ values: $1.5 \leq \mu \leq 2.$
- Leaving arc: (a, g)
- Entering arc: (g, e)

New tree:
Fourth Iteration

- Range of $\mu$ values: $1 \leq \mu \leq 1.5$.
- A tie:
  - Arc (f,b) enters, or
  - Arc (f,c) leaves.
- Decide arbitrarily:
  - Leaving arc: (f,c)
  - Entering arc: (f,b)
Fifth Iteration

- Range of $\mu$ values: $1 \leq \mu \leq 1$.
- Leaving arc: $(f, b)$
- Nothing to Enter.

Primal Infeasible!
Click here (or on any displayed network) to try out the online network simplex pivot tool.
**Definition.** Network is called **planar** if can be drawn on a plane without intersecting arcs.

**Theorem.** Every planar network has a geometric dual—dual nodes are faces of primal network.

Notes:
- Dual node $A$ is "node at infinity".
- Primal spanning tree shown in red.
- Dual spanning tree shown in blue (don’t forget node $A$).

**Theorem.** A dual pivot on the primal network is exactly a primal pivot on the dual network.
**Theorem.** Assuming integer data, every basic feasible solution assigns integer flow to every arc.

**Corollary.** Assuming integer data, every basic optimal solution assigns integer flow to every arc.