Optimal asset management contracts with hidden savings

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PRELIMINARY

Abstract

We study the role of hidden savings in optimal contracts for delegated asset management. The principal uses the agent’s access to capital to manipulate his precautionary motive and reduce the cost of providing incentives. After bad outcomes, the agent’s consumption is somewhat insured, and he is punished instead with less access to capital and lower growth. As a result, in addition to an equity constraint, the optimal contract requires a leverage constraint to be implemented. Hidden investment limits the principal’s ability to provide incentives, but doesn’t change the contract’s qualitative features. We provide a sufficient analytical condition for the validity of the first-order approach: if the agent’s precautionary motive falls after bad outcomes, the contract is globally incentive compatible. This condition holds in the optimal contract and in a broader class of contracts.

1 Introduction

Delegated asset management plays an important role in modern economies, from financial intermediaries such as fund managers, to CEOs or entrepreneurs who manage real capital assets. However, financial frictions limit the efficient allocation of capital to its most productive users. Asset managers must keep an equity stake to provide incentives, which makes it costly to delegate capital. Providing incentives is particularly difficult when the agent has access to hidden savings, which he can use to undo the incentive scheme. We study the role of hidden savings in the optimal dynamic contract for an agent who manages capital.
We consider a classical investment setting. An agent with CRRA preferences over consumption can continuously invest in risky capital and obtain an excess return. He would like to raise funds and share risk with a complete financial market, but he faces a moral hazard problem: he can secretly divert funds and has access to hidden savings. The full commitment contract specifies compensation and capital under management contingent on the agent’s returns.

With hidden savings, the agent’s precautionary motive plays a central role. Giving capital to the agent is attractive because he can obtain an excess return. But because of the fund diversion problem, the agent must keep some “skin in the game” to provide incentives, which exposes him to risk. This is costly because the agent is risk averse, but also because it creates a precautionary motive for savings that distorts intertemporal consumption smoothing and makes fund diversion more attractive. When the agent expects a risky consumption stream in the future, the temptation to divert funds and save to self insure is larger. The principal must therefore take into account the agent’s precautionary motive, and manipulate it to his advantage by controlling his access to capital.

First, the principal restricts the agent’s access to capital over time (up to a steady state). By promising him a less risky contract in the future, the principal relaxes the agent’s precautionary motive. This reduces the cost of giving capital to the agent today, in exchange for an inefficiently low amount of capital in the future. Second, the optimal contract provides some insurance for the agent’s consumption, and instead punishes him with less access to capital and lower consumption growth after bad outcomes. Intuitively, the agent is most concerned about the risk he will face when his consumption is low and his marginal utility high. By promising him a less risky continuation contract after bad outcomes, he relaxes the agent’s precautionary motive and reduces the cost of providing incentives. On the upside, after good outcomes the agent gets a riskier contract, with more capital and greater consumption growth. Since these contracts are more efficient, the principal finds it attractive to use them when he must deliver more utility to the agent.

The optimal contract can be implemented as a consumption-portfolio problem where the agent consumes and invests in risky capital subject to history dependent financial frictions. First, there is an equity constraint: the agent must keep an equity stake, which gives him some “skin in the game” and exposes him to part of the risk in his capital. Hidden savings, however, create a second type of financial friction: a leverage constraint that limits the agent’s investment in capital. The leverage constraint arises purely from the presence of hidden savings - the optimal contract without hidden savings only requires an equity
constraint - and reflects a completely different logic than models of limited commitment, such as Kiyotaki and Moore (1997). Here, the principal uses the leverage constraint to weaken the agent’s precautionary motive and reduce the cost of providing incentives. Indeed, the leverage constraint allows the principal to relax the equity constraint and improve risk sharing. After bad outcomes the agent is punished not only with less net worth, but also with tighter financial frictions.

Since the principal uses the agent’s access to capital to provide incentives, it is natural to ask how hidden investment affects results. If the agent can secretly invest his hidden savings in risky capital (for example by investing more than indicated by the contract), the principal finds it harder to provide incentives. However, the principal still has some power because he can provide some insurance for the capital invested through the contract, while the agent must bear all the risk on the capital he invests on his own. The optimal contract with hidden investment can be characterized using the same tools with an extra incentive compatibility constraint for hidden investment, and has the same qualitative features as the optimal contract without hidden investment. In particular, the agent’s access to capital is still restricted over time and after bad outcomes, and the contract can be implemented with history dependent equity and leverage constraints. In fact, we can also add aggregate/market risk that commands a premium into our setting, and allow the agent to also invest his hidden savings in the market. Our agency model can therefore be fully embedded within the standard setting of continuous-time dynamic asset pricing theory, e.g., see Duffie (?).

A second concern is that the optimal contract requires full commitment. The principal relaxes the agent’s precautionary motive by promising an inefficiently low amount of capital, over time and after bad outcomes. It is therefore tempting to renegotiate and “start over”. We also characterize the optimal renegotiation-proof contract. The dynamic behavior disappears, but the principle of restricting the agent’s access to capital to reduce the cost of providing incentives remains. In particular, by imposing a leverage constraint, the principal can relax the equity constraint and improve risk sharing.

One of the main methodological contributions of this paper is to provide an analytical verification of the validity of the first order approach. Contractual environments where the agent has access to hidden savings are often difficult to analyze because we need to ensure incentive compatibility with respect to double deviations.1 Dealing with single deviations is

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1See for example Kocherlakota (2004).
relatively straightforward. We can deter the agent from stealing and immediately consuming the proceeds by giving him some skin in the game. Likewise, we can ensure that he will not secretly save his recommended consumption for later by incorporating his Euler equation as a constraint on the contract design. But what if the agent both steals and saves the proceeds for later? Since stealing makes bad outcomes more likely, the agent expects to be punished with lower consumption in the future. In other words, he expects a high marginal utility in the future, so stealing and secretly saving the proceeds for later could potentially be an attractive double deviation.

We prove the validity of the first order approach analytically by establishing an upper bound on the agent’s continuation utility from any valid deviation, after any history. The crucial sufficient condition is that after bad outcomes the agent’s precautionary motive becomes weaker. To understand why, note that if after some history the agent expects his legitimate consumption to be very risky looking forward, he will have a large precautionary motive for saving and will place a large marginal value on hidden savings that can help him self insure. As a result, if after bad outcomes the contract became more risky, and the agent’s precautionary motive larger, stealing and saving the proceeds for later would be an attractive double deviation, since the agent would expect to have hidden savings precisely when they are most valuable to him. It is here that the contract’s dynamic behavior helps ensure incentive compatibility. After bad outcomes, the contract becomes less risky and the precautionary motive weaker, so hidden savings become less valuable to the agent. It is important to note that this verification argument applies beyond the optimal contract, to any contract in which the precautionary motive weakens after bad outcomes.

Literature Review. This paper fits within the literature on dynamic agency problems. It builds upon the standard recursive techniques, but adds the problem of persistent private information, in our case, about the agent’s hidden savings.

There is extensive literature that uses recursive methods to characterize optimal contracts, including Spear and Srivastava (1987), Phelan and Townsend (1991), Sannikov (2008), He (2011), Biais et al. (2007), and Hopenhayn and Clementi (2006). The agency problem we study is one of cash flow diversion, as in DeMarzo and Sannikov (2006) and DeMarzo et al. (2012), but unlike their models we have CRRA (as opposed to risk neutral) preferences and no investment frictions, i.e. there are no costs to adjusting the size of the project. CRRA utility allows us to study the effects of risk aversion and elasticity

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2See Werning (2001).
of intertemporal substitution. With risk-neutral preferences, the optimal contracts with
and without hidden savings are the same. Once concave preferences are introduced, the
principal has incentives to front load consumption in order to reduce the private benefit of
cash diversion and relax the risk sharing problem.

The combination of CRRA preferences and frictionless investment technology affords
scale invariance properties that make the optimal contract particularly tractable. Because
capital can be continuously adjusted, after a very good history the agent will be managing a
large amount of capital, but he will not be retired nor outgrow the moral hazard problem as
in Sannikov (2008) or Hopenhayn and Clementi (2006) respectively. Since there is no outside
option and the project can be scaled down, neither will the agent retire after sufficiently
bad outcomes as in DeMarzo and Sannikov (2006): the contract just gives him a very
small amount of capital and consumption, but there is always the chance of recovery. In
fact, the long-run behavior of the contract features a non-degenerate stationary distribution.
Without hidden savings, the optimal contract can be characterized in closed form. For more
details, see Di Tella (2014) who adopts a version of our setting without hidden savings and
allows aggregate shocks and Epstein-Zin preferences. The focus of Di Tella (2014) is on
optimal financial regulation in a general equilibrium setting. Our results stand in contrast
to Cole and Kocherlakota (2001), who show risk-free debt is optimal in an environment with
hidden income and hidden savings. In our investment setting the principal can use access
to capital to provide incentives. However, we recover their result in the special case with
hidden investment where fund diversion does not destroy any resources (a pure misreporting
problem).

Hidden savings are a natural but difficult problem. Werning (2001) introduces the
first-order approach in the context of unemployment insurance. The first order approach
captures the agent’s incentives to save with the Euler equation, and opens the door to
questions of distortions in the optimal contract and validity of the first-order approach.
The first order approach could fail if the agent’s benefits of effort reduction are linear, as in
Kocherlakota (2004), so the agent’s payoff is not globally concave. We provide a sufficient
condition for the validity of the first order approach, which holds in our optimal contract
and beyond. Distortions don’t arise in CARA settings, such as He (2011), and Williams
(2013), where the ratio of the agent’s current utility to continuation utility is invariant to
contract design. Likewise, the dynamic incentive accounts of Edmans et al. (2011) exhibit
no distortions either, where hidden action enters multiplicatively and project size is fixed.
In contrast, we are able to characterize the optimal way that the principal can control the
precautionary motive through distortions to risk exposure.

Our paper is also related to the literature on persistent private information, since the agent has private information about savings. The growing literature in this area includes the fundamental approach of Fernandes and Phelan (2000), who propose to keep track of the agent’s entire off-equilibrium value function, and the first-order approach, such as DeMarzo and Sannikov (2016), who use a recursive structure that includes the agent’s “information rent,” i.e. the derivative of the agent’s payoff with respect to private information. In our case, information rent is the marginal utility of consumption (of an extra unit of savings). While the connection to our work may appear subtle, it is actually quite direct - key issues are (1) the way that information rents enter the incentive constraint, (2) distortions that arise from this interaction and (3) forces that affect the validity of the first-order approach. In our case, the precautionary motive hurts incentives, therefore the principal distorts future risk exposure down to reduce the precautionary motive, and first-order approach is valid if precautionary motive falls after bad outcomes. In the setting of Farhi and Werning (2013) with unobservable skill shocks, information rents reduce truth telling incentives, therefore dynamic distortions are also in the downward direction, and first-order approach is verified by computing the agent’s off-equilibrium payoff numerically. Other papers that study dynamic distortions include Garrett and Pavan (2015), Cisternas (2015), He et al (2015) and Prat and Jovanovic (2014).

This paper is organized as follows. Section 2 presents the model in the absence of the stock market. Section 3 solves for the optimal contract and provides a suitable sufficient condition to verify the validity of the first-order approach. Section 4 discusses the properties of the optimal contract in terms of its implementation as a portfolio problem subject to financial frictions. Section 5 incorporates both aggregate risk and hidden investment into the setting, and Section 6 introduces renegotiation. Section 7 concludes.

2 The model

Let $(\Omega, P, \mathcal{F})$ be a complete probability space equipped with filtration $\mathcal{F}$ generated by a brownian motion $Z$, with the usual conditions. Throughout, all stochastic processes are adapted to $\mathcal{F}$. There is a complete financial market with equivalent martingale measure $Q$. The risk-free interest rate is $r > 0$ and $Z$ is idiosyncratic risk and therefore not priced by the market. In the baseline setting there is no aggregate risk so $Q = P$, but later we will allow them to differ.
The agent can manage capital to obtain a risky return that exceeds the required return of \( r \), but he may also get a private monetary benefit by diverting returns. If the diversion rate is \( a_t \), the observed return per dollar invested in capital is

\[
dR_t = (r + \alpha - a_t) \, dt + \beta \, dZ_t
\]

where \( \alpha > 0 \) is the excess return and \( \beta > 0 \) is the volatility. \( Z \) is agent-specific idiosyncratic risk. If we think of the agent as a fund manager, it represents the outcome of his particular investment/trading activity.\(^3\) If we take the agent to be an entrepreneur it represents the outcome of his particular project.

If capital is \( k_t \geq 0 \), diversion of \( a_t \geq 0 \) gives the agent a flow of \( \phi a_t k_t \). For each stolen dollar, the agent keeps only fraction \( \phi \in (0, 1) \). If the agent also receives payments \( c_t \geq 0 \) from the principal and consumes \( \tilde{c}_t \geq 0 \) then his hidden savings \( h_t \geq 0 \) evolve according to\(^4\)

\[
dh_t = (rh_t + c_t - \tilde{c}_t + \phi k_t a_t) \, dt.
\]

The agent invests his hidden savings at the risk free rate \( r \). Later we will introduce hidden investment, and allow the agent to also invest his hidden savings in risky capital.

The agent wants to raise funds and share risk with the market, which we refer to as the principal. The principal observes returns \( R \) but not the agent’s diversion \( a \), consumption \( \tilde{c} \), or hidden savings \( h \). The principal can commit to a fully history-dependent contract \( C = (c, k) \) that specifies payments to the agent \( c_t \) and capital \( k_t \) as a function of the history of realized returns \( R \) up to time \( t \). After signing the contract \( C \) the agent can choose a strategy \( (\tilde{c}, a) \) that specifies \( \tilde{c}_t \) and \( a_t \), also as a history of returns up to time \( t \).

The agent has CRRA preferences. Given contract \( C \), under strategy \( (\tilde{c}, a) \) the agent gets utility

\[
U_{0}^{\tilde{c}, a} = \mathbb{E}^{a} \left[ \int_{0}^{\infty} e^{-r t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} \, dt \right] \tag{1}
\]

where superscript \( a \) indicates that the expectation is computed under the distribution over returns \( R \) that is induced by the stealing strategy \( a \). Given contract \( C \), we say a strategy \( (\tilde{c}, a) \) is feasible if 1) utility \( U_{0}^{\tilde{c}, a} \) is finite, and 2) \( h_t \geq 0 \) always. Let \( \mathbb{S}(C) \) be the set of

\(^3\)If we give $1 to invest to two fund managers, they will obtain different returns depending on exactly which assets they buy or sell, and the exact timing and price of their trades.

\(^4\)We don’t allow hidden debt, \( h_t \geq 0 \). This is without loss of generality if the contract can exhaust the agent’s credit capacity.
feasible strategies \((\hat{c}, a)\) given contract \(C\).

The principal pays for the agent’s consumption, but keeps the excess return \(\alpha\) on the capital that the agent manages. He tries to minimize the cost of delivering utility \(u_0\) to the agent

\[
J_0 = \mathbb{E}^Q \left[ \int_0^{\infty} e^{-rt} (c_t - k_t \alpha) \, dt \right]
\]

A standard argument in this setting implies that the optimal contract must implement no stealing, i.e. \(a = 0\). In addition, without loss of generality and for analytic convenience, we can restrict attention to contracts in which \(h_0 = 0\) and \(\hat{c} = c\), i.e. the principal saves for the agent.\(^5\) Of course, the optimal contract has many equivalent and more natural forms, in which the agent maintains savings, but all these forms can be deduced easily from the optimal contract with \(\hat{c} = c\).

We say a contract \(C = (c, k)\) is admissible if 1) utility \(U^{c,0}_0\) is finite, and 2)\(^6\)

\[
\mathbb{E}^Q \left[ \int_0^{\infty} e^{-rt} |c_t + k_t \alpha| \, dt \right] < \infty
\]

We say an admissible contract \(C\) is incentive compatible if

\[(c, 0) \in \arg \max_{(\bar{c}, a) \in S(C)} U^{\bar{c},a}_0\]

Let \(\mathbb{IC}\) be the set of incentive compatible contracts. For an initial utility \(u_0\) for the agent, an incentive compatible contract is optimal if it minimizes the cost of delivering initial utility \(u_0\) to the agent

\[
v_0 = \min_{(c,k)} J_0
\]

st: \(U^{c,0}_0 \geq u_0\)

\((c, k) \in \mathbb{IC}\)

By changing \(u_0\) we can trace the Pareto frontier for this problem. To make the problem

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\(^5\)Lemma 18 establishes this in the more general setting of Section 5, with both aggregate risk and hidden investment.

\(^6\)This assumption plays the role of a no-Ponzi condition, making sure the principal’s objective function is well defined. It rules out exploding strategies where the present value of both consumption and capital is infinity.
well defined and avoid infinite profits/utility, we assume throughout that \( \rho > r(1 - \gamma) \), and

\[
\alpha \leq \tilde{\alpha} \equiv \frac{\phi \beta \gamma \sqrt{2}}{\sqrt{1 + \gamma}} \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma}}
\]

3 Solving the model

We solve the model as follows. We first derive necessary first-order incentive-compatibility conditions for the agent’s effort and savings choice, using two appropriate state variables: the agent’s continuation utility and consumption level. This allows us to formulate the principal’s \textit{relaxed} problem, minimizing the cost subject to only first-order conditions, as a control problem. We use the HJB equation to solve this problem.

We then derive a sufficient condition for global incentive compatibility (against all deviations, not just local), which uses the same two state variables. The condition is on-path, i.e. for a particular \textit{recommended} strategy of the agent, but it is sufficient because it allows us to bound the agent’s payoff off-path after arbitrary deviations. We show that the solution to the relaxed problem satisfies the sufficient condition, thereby proving it is the optimal contract. More generally, the sufficient condition identifies a whole class of globally incentive compatible contracts, and is useful in a broader context as we show in the next section.

Incentive compatibility

We use the continuation utility of the agent as a state variable for the contract

\[
U_{t}^{c,0} = \mathbb{E}_{t}\left[ \int_{t}^{\infty} e^{-\rho(s-t)} \frac{c_{s}^{1-\gamma}}{1-\gamma} ds \right]
\]

First we obtain the law of motion for the agent’s continuation utility.

\textbf{Lemma 1.} \textit{For any admissible contract} \( C = (c, k) \), \textit{the agent’s continuation utility} \( U_{t}^{c,0} \) \textit{satisfies}

\[
dU_{t}^{c,0} = \left( \rho U_{t}^{c,0} - \frac{c_{t}^{1-\gamma}}{1-\gamma} \right) dt + \Delta_{t} \left( dR_{t} - (\alpha + r) dt \right)
\]

\textit{for some stochastic process} \( \Delta \).
Faced with this contract, the agent might consider stealing and immediately consuming the proceeds, i.e. following a strategy \((c + \phi k_a, a)\) for some \(a\), which results in savings \(h = 0\). The agent adds \(\phi k_t a_t\) to his consumption, but reduces the observed returns \(d R_t\), and therefore his continuation utility \(U_t^{c, 0}\) by \(\Delta_t a_t\). Incentive compatibility therefore requires

\[
0 \in \arg \max_{a \geq 0} \frac{(c_t + \phi k_t a_t)^{1-\gamma}}{1-\gamma} - \Delta_t a_t
\]

Taking FOC yields

\[
\Delta_t \geq c_t^{-\gamma}\phi k_t
\]

which is positive. We need to give the agent some “skin in game”, which exposes him to risk. This is costly because the principal is risk-neutral with respect to \(Z\) so he would like to provide full insurance to the agent.

Notice how the private benefit of the hidden action depends on the marginal utility of consumption \(c_t^{-\gamma}\), so the principal would like to front load the agent’s consumption to relax the risk-sharing constraint. With hidden savings this is not possible. If the principal tries to front load the agent’s consumption, he will secretly save and consume when his marginal utility is higher. The optimal contract must therefore respect the agent’s Euler equation: the discounted marginal utility \(e^{(r - \rho)\gamma} c_t^{-\gamma}\) must be a supermartingale.\(^7\)

The following lemma summarizes all the necessary conditions for incentive compatibility and their implication on the path of the agent’s consumption. We present sufficient conditions in the next subsection.

**Lemma 2.** If \(C = (c, k)\) is an incentive compatible contract, then (6) must hold and the agent’s consumption must satisfy

\[
\frac{dc_t}{c_t} = \left(\frac{r - \rho}{\gamma} + \frac{1 + \gamma (\sigma^c_t)^2}{2}\right) dt + \sigma^c_t dZ_t + dL_t
\]

for some \(\sigma^c\) and a weakly increasing process \(L\).

Equation (7) imposes a lower bound on the growth rate of the agent’s consumption. The first term \(\frac{r - \rho}{\gamma}\) captures the benefit of postponing consumption without risk, given by the risk free rate \(r\), the discount rate \(\rho\), and the elasticity of intertemporal substitution \(1/\gamma\). The

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\(^7\)The agent can expect lower marginal utility in the future, because he can’t borrow. If the agent could have hidden debt, then \(e^{(r - \rho)\gamma} c_t^{-\gamma}\) would have to be a proper martingale. As it turns out, this is the case in the optimal contract.
second term \(\frac{1+\gamma}{2}(\sigma_t^2)^2\) captures the agent’s precautionary motive. A risky consumption profile induces the agent to postpone consumption to self-insure, resulting in a steeper consumption profile.

**State space**

It is convenient to work with the following transformation of the state variables

\[
x_t = \left((1 - \gamma) U_t^{(c,r)}\right)^{\frac{1}{1-\gamma}} > 0
\]

\[
\hat{c}_t = \frac{c_t}{x_t} \geq 0
\]

Variable \(x\) is just a monotone transformation of continuation utility, but it is measured in consumption units (up to a constant). As a result, \(\hat{c}\) measures how front loaded the agent’s consumption is. The state \(\hat{c}\) is related to the agent’s precautionary motive for saving. If the agent faces risk looking forward, he will want to postpone consumption in an attempt to self insure (low \(\hat{c}\)). As a result, while \(x_t\) can take any positive value, \(\hat{c}_t\) has an upper bound.

**Lemma 3.** For any incentive compatible contract \(C\), at all times \(t\), \(\hat{c}_t \in (0, \hat{c}_h]\), where

\[
\hat{c}_h \equiv \left(\frac{\rho - r(1-\gamma)}{\gamma}\right)^{\frac{1}{1-\gamma}} > 0
\]  

(8)

If ever \(\hat{c}_t = \hat{c}_h\), then the continuation contract satisfies \(k_{t+s} = 0\) and \(\hat{c}_{t+s} = \hat{c}_h\) at all future times \(t + s\) and gives the agent a unique deterministic consumption path with growth \((r - \rho)/\gamma\). The contract has cost \(\hat{v}_h x_t\) to the principal, where \(\hat{v}_h \equiv \hat{c}_h^\gamma\).

This upper bound has a simple interpretation. The contract that minimizes the agent’s precautionary motive and maximizes \(\hat{c}\) is the fully safe contract, which gives the agent no capital to manage and lets consumption grow at the deterministic growth rate of \((r - \rho)/\gamma\). This corresponds to \(\hat{c}_h\). A lower \(\hat{c}\) means the agent expects to manage capital and be exposed to risk in the future. The safe contract minimizes the cost of the agent’s consumption, but is very costly because it doesn’t give any capital to the agent and so doesn’t take advantage of the excess return \(\alpha > 0\).
The relaxed problem

Using Ito’s lemma we can obtain laws of motion for $x_t$ and $\hat{c}_t$ from (4) and (7). Using the normalization $\Delta_t \beta / U_t^0 = (1 - \gamma) \sigma_t^x$, we obtain

$$
\frac{dx_t}{x_t} = \left( \frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} + \frac{\gamma (\sigma_t^x)^2}{2} \right) dt + \sigma_t^x dZ_t
$$

(9)

and

$$
\frac{d\hat{c}_t}{\hat{c}_t} = \left( \frac{r - \rho}{\gamma} + \frac{\hat{c}_t^{1-\gamma} - \rho}{1 - \gamma} + \frac{(\sigma_t^x)^2}{2} + \gamma \sigma_t^x \sigma_t^\hat{c} + \frac{1 + \gamma (\sigma_t^\hat{c})^2}{2} \right) dt + \sigma_t^\hat{c} dZ_t + dL_t
$$

(10)

for some $\sigma_t^\hat{c} = \sigma_t^c - \sigma_t^x$. The constraint (6) can be rewritten as

$$
\sigma_t^x \geq \hat{c}_t^{-\gamma} \hat{k}_t \phi \beta,
$$

(11)

where $\hat{k}_t = \frac{k_t}{x_t}$. It will always be binding because conditional on $\sigma_t^x$ it is always better to give the agent more capital to manage. The cost flow for the principal is

$$
c_t - k_t \alpha = x_t \left( \hat{c}_t - \hat{c}_t^{\gamma} \frac{\alpha}{\phi \beta} \sigma_t^x \right)
$$

(12)

We obtain a stochastic control problem with laws of motion of the state variables (9) and (10) with absorption at $\hat{c}_h$, controls $\sigma_t^x$ and $\sigma_t^\hat{c}$, and cost flow (12). This is the relaxed problem.

The principal would like to give capital to the agent because it yields an excess return $\alpha$. However, he must expose the agent to risk to provide incentives. This is costly for two reasons. First, since the agent is risk averse he must be compensated for the exposure to risk with higher utility in the future. This is reflected in the drift of $x_t$ in equation (9). Second, because of hidden savings, exposing the agent to risk induces a precautionary motive - a low $\hat{c}$ - that distorts the optimal intertemporal consumption profile. In fact, these two costs interact because a low $\hat{c}$ further increases the incentives to steal through $\hat{c}^{-\gamma}$ in (11).

Here’s how this works in the recursive formulation. While $x_0$ is determined by the initial utility $u_0$, the principal must choose an initial $\hat{c}_0 \leq \hat{c}_h$. A lower $\hat{c}_t$ means that the
agent expects to be given capital and therefore be exposed to risk in the future, so we can interpret \( \hat{c}_h - \hat{c}_t \) as the principal’s “budget” for risk exposure. If the principal exposes the agent to risk today with \( \sigma_t^x \), he must expose him to less risk in the future. This is reflected in equation (10) where the drift of \( \dot{c} \) is increasing in \( \sigma_t^x \): exposing the agent to risk today eats up the budget for risk exposure in the future. A higher \( \hat{c}_t \) creates distortions because it reduces the principal’s ability to give capital to the agent. Indeed, if \( \hat{c}_t \) ever reached the upper bound \( \hat{c}_h \), the principal would have to give him the perfectly safe contract without any capital. The principal must therefore choose the initial \( \hat{c}_0 \) weighting the benefit of giving capital to the agent against the distortions in intertemporal consumption smoothing and risk sharing. After that, he manages his budget for risk exposure using \( \sigma_t^x \) and \( \sigma_t^\hat{c} \).

In addition to \( \sigma_t^x \) the principal can use a negative \( \sigma_t^\hat{c} \) to relax the agent’s precautionary motive. The agent is concerned the most about risk exposure after bad outcomes, when his utility is low. Thus, the principal can reduce the agent’s precautionary motive by giving him less risk - raising \( \hat{c}_t \) - in the event that \( x_t \) goes down. This is captured by the drift of \( \dot{c}_t \) in equation (10), which can be reduced with a negative \( \sigma_t^\hat{c} \). As it turns out the principal will also prefer to raise \( \hat{c}_t \) after bad outcomes for dynamic hedging reasons: he would rather restrict his ability to give capital to the agent (with a high \( \hat{c}_t \)) when he must deliver less utility \( x_t \) to the agent, and relax it (with a low \( \hat{c}_t \)) when he must deliver more utility \( x_t \).

It turns out that relaxing the agent’s precautionary motive after bad outcomes, \( \sigma_t^\hat{c} \leq 0 \), is also a sufficient condition for the contract to be globally incentive compatible if the necessary conditions of Lemma 2 hold. While our paper is the first to our knowledge to prove this form of a result analytically, the intuition is simple. The marginal value of hidden savings rises in the agent’s precautionary motive, i.e. savings become more valuable when the agent is exposed to risk going forward and therefore \( \hat{c}_t \) is lower. Therefore, whenever a contract reduces the precautionary motive after bad outcomes - these outcomes become more likely when the agent steals - stealing and saving is an unattractive deviation. The agent expects to have hidden savings when they are less valuable to him. This intuition is formalized below in Theorem 3.

The HJB equation

Because preferences are homothetic and the principal’s objective is linear, we know the principal’s cost function takes the form \( v(x, \hat{c}) = \hat{v}(\hat{c})x_t \) with \( \hat{v}(\hat{c}_h) = \hat{v}_h > 0 \). Notice that since we could always raise \( \hat{c}_t \) using \( dL_t \), we know that \( \hat{v}(\hat{c}) \) must be weakly increasing. In
fact, we show below that \( \hat{v}(\hat{c}) \) increases strictly over the interval in which \( \hat{c} \) stays over the course of the optimal contract, so we can drop the term \( dL_t \) from what follows. We will also sometimes write \( \hat{v}_t = \hat{v}(\hat{c}_t) \), and \( \hat{v} \) instead of \( \hat{v}(\hat{c}) \).

The HJB equation associated with this problem is

\[
r \hat{v} x = \min_{\sigma^x, \sigma^c} \left( \dot{c} - \dot{k} \alpha \right) x + \mathbb{E}_t^Q \left[ d(\hat{v}_t x_t) \right]
\]

subject to (9), (10), and (11), and \( \dot{k} \geq 0 \). Using Ito’s lemma and canceling the \( x \) on both sides, we get

\[
r \hat{v} = \min_{\sigma^x, \sigma^c} \dot{c} - \sigma^x \hat{c} \gamma \frac{\alpha}{\phi \beta} + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2} (\sigma^x)^2 \right) + \hat{v}' \left( \frac{\hat{v} - \rho}{\gamma} + \frac{(\sigma^x)^2}{2} + (1 + \gamma) \sigma^x \dot{c} - \frac{1 + \gamma}{2} (\sigma^c)^2 \right) + \hat{v}'' \frac{\sigma^c}{2} (\sigma^c)^2
\]

Even though we have two state variables, \( c \) and \( x \), the HJB equation boils down to a second order ODE in \( \hat{c} \). This is a feature of homothetic preferences and linear technology that makes the problem more tractable. The following lemma characterizes the shape of the principal’s cost function and the range of \( \hat{c}_t \) under the optimal contract.

**Theorem 1.** The principal’s cost function \( \hat{v}(\hat{c}) \) has a flat portion on \([0, \hat{c}_l] \) and a strictly increasing portion on \([\hat{c}_l, \hat{c}_h] \), for some \( \hat{c}_l \in (0, \hat{c}_h) \). The HJB equation (13) holds with equality above \( \hat{c}_l \) and with inequality below \( \hat{c}_l \), i.e.

\[
r \hat{v} < \min_{\sigma^x} \dot{c} - \sigma^x \hat{c} \gamma \frac{\alpha}{\phi \beta} + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2} (\sigma^x)^2 \right) \quad \forall \hat{c} < \hat{c}_l.
\]

Below \( \hat{c}_l \), \( \hat{v}(\hat{c}) = \hat{v}(\hat{c}_l) \), and at \( \hat{c}_l \) the cost function \( \hat{v}(\hat{c}) \) satisfies the smooth-pasting condition \( \hat{v}'(\hat{c}_l) = 0 \), and \( \hat{v}''(\hat{c}_l) > 0 \).

The optimal contract starts at \( \hat{c}_0 = \hat{c}_l \), where \( \sigma^c_0 \) is chosen without taking into account its effect on the agent’s precautionary motive, to maximize

\[
\sigma^c \hat{c} \gamma \frac{\alpha}{\phi \beta} - \hat{v}(\hat{c}_l) \frac{\gamma}{2} (\sigma^x)^2
\]

At \( \hat{c}_l \) we have \( \mu^c(\hat{c}_l) > 0 \) and \( \sigma^c(\hat{c}_l) = 0 \). For all \( t > 0 \), \( \hat{c}_t \in [\hat{c}_l, \hat{c}_h] \), \( \sigma^c_t \leq 0 \) and \( \sigma^x_t \geq 0 \).

Figures 1 and 2 show the cost function and the drift and volatility of state variables \( x \) and \( \hat{c} \) for a numerical solution. To understand why the cost function has a flat portion and
Figure 1: The cost function $\hat{v}(\hat{c})$ solid in blue for the optimal contract, and dashed in red for stationary contracts. The the starting point of the optimal contract is indicated by the blue dot, the optimal stationary contract by the red dot, the optimal portfolio plan with $\hat{\phi} = \phi$ is indicated the black dot, and the optimal contract without hidden savings by the green dot. Parameters: $\rho = r = 5\%$, $\alpha = 1.7\%$, $\gamma = 1/3$, $\phi \beta = 0.2$.

Figure 2: The drift, $\mu^\hat{c}$ and $\mu^x$, and volatility, $\sigma^\hat{c}$ and $\sigma^x$, of the state variables $\hat{c}$ and $x$. The starting point of the optimal contract is indicated by the blue dot, the optimal stationary contract by the red dot, the optimal portfolio plan with $\hat{\phi} = \phi$ is indicated the black dot, and the optimal contract without hidden savings by the green dot. Parameters: $\rho = r = 5\%$, $\alpha = 1.7\%$, $\gamma = 1/3$, $\phi \beta = 0.2$. 
an increasing portion, consider the problem of the optimal choice of \( \hat{c}_0 \). Choosing \( \hat{c}_0 = \hat{c}_h \) is suboptimal, because then the principal cannot give the agent any capital. It is beneficial to give the agent capital and expose him to risk because capital generates an excess return of \( \alpha \). However, risk exposure is costly because the agent is risk averse. In addition, with hidden savings risk exposure also generates a precautionary motive which lowers \( \hat{c} \), distorting the intertemporal consumption profile and further tightening the incentive constraint. Eventually, the costs of risk exposure outweigh the benefits. Under these trade-offs, we denote by \( \hat{c}_l \) the value that minimizes the cost of delivering utility to the agent, indicated with a blue dot in Figure 1. For \( \hat{c} < \hat{c}_l \), the principal has the option to raise \( \hat{c} \) to \( \hat{c}_l \) immediately using the process \( dL_t \). Hence, the cost function is flat over \([0, \hat{c}_l] \). For \( \hat{c} > \hat{c}_l \), he must give the agent an inefficiently low risk exposure to reduce his precautionary motive. As a result the cost function \( \hat{v}(\hat{c}) \) is increasing over \([\hat{c}_l, \hat{c}_h] \), and the optimal contract starts at \( \hat{c}_0 = \hat{c}_l \).

It is conceivable for the function \( \hat{v}(\hat{c}) \) to have flat portions also above the cost-minimizing point, but Theorem 1 rules out this possibility analytically.

The optimal contract starts at the optimal \( \hat{c}_l \) but becomes less risky over time - \( \hat{c} \) drifts up to a steady state, as show in Figure 2. Why does \( \hat{c}_l \) not stay at the cost-minimizing level \( \hat{c}_l \) forever? Because the principal can give more capital to the agent without increasing his precautionary motive if he promises a less risky contract in the future (a higher \( \hat{c} \) in the future). To see this, consider the first-order condition\(^8\) for \( \sigma^x \)

\[
\alpha = \gamma \left( \hat{v}\hat{c}^{-\gamma} \phi \beta \sigma^x \right) + \phi \beta \hat{v}' \hat{c}^{1-\gamma} \left( (1 + \gamma) \sigma \hat{c} + \sigma^x \right)
\]

\[(16)\]

The left hand side and the first term on the right capture the tradeoff between excess return \( \alpha \) and exposure to risk \( \sigma^x \). The second term captures an intertemporal tradeoff. The principal can give more capital to the agent today (exposing him to more risk \( \sigma^x \) today) without a larger precautionary motive (lower \( \hat{c} \)) if he promises a less risky contract in the future (the drift of \( \hat{c} \) is increasing in \( \sigma^x \)). Since less risky contracts require an inefficiently low level of capital - \( \hat{v}(\hat{c}) \) is strictly increasing in \( \hat{c} \) as described above - this is a potentially costly tradeoff for the principal. At the optimal point \( \hat{c}_l \), however, the principal is indifferent about small changes in \( \hat{c} \) because \( \hat{v}'(\hat{c}_l) = 0 \), so the intertemporal trade-off vanishes. The principal gives the agent a large amount of capital, picking \( \sigma^x \) to maximize

\(^8\)This first-order condition characterizes a minimum (rather than a maximum) because \( \hat{v} \gamma + \hat{v}' \hat{c} > 0 \), since \( \hat{v} > 0 \) and \( \hat{v}' \geq 0 \).
(15), and doesn’t care that he is promising less risk and capital in the future. As a result, the drift of \( \dot{c} \) is initially positive. As \( \dot{c} \) moves up from \( \dot{c}_l \), the cost of the contract goes up. The principal would benefit from reducing \( \dot{c} \) - giving the agent a more risky contract - but he cannot do so because he must keep his promise to the agent. The principal then chooses \( \sigma^x \) taking into account this intertemporal tradeoff.

The principal can reduce the cost by setting \( \sigma^\dot{c} < 0 \), as shown in Figure 2, i.e. after bad outcomes the contract becomes less risky for the agent. This has two benefits, as shown by the FOC for \( \sigma^\dot{c} \)

\[
\dot{v}' \left( \gamma \sigma^x + (1 + \gamma) \sigma^\dot{c} \right) + \left( \dot{v}' \sigma^x + \dot{v}'' \dot{c} \sigma^\dot{c} \right) = 0
\]

(17)

The first term says that by setting \( \sigma^\dot{c} < 0 \) the principal can reduce the agent’s precautionary motive. The principal promises the agent less risk - a higher \( \dot{c} \) - after bad outcomes, when it matters most to him. What is going on is that the agent’s consumption \( c_t = \dot{c}_t x_t \) is somewhat insured because \( \dot{c} \) and \( x \) move in opposite directions. As a result, the agent’s precautionary motive is weaker and the drift of \( \dot{c} \) lower, which is attractive to the principal because \( \dot{v}(\dot{c}) \) is increasing. The second term in the FOC captures a hedging motive for the principal: he also prefers to use the relatively costly contracts with high \( \dot{c} \) when he must delivers less utility \( x \), because then he can use the relatively less costly contracts with low \( \dot{c} \) when he must deliver more utility \( x \). Both motives induce a negative \( \sigma^\dot{c} \).

If the agent’s consumption is somewhat insured and he faces less risk after bad outcomes, how exactly is he being punished? The answer is that he faces a lower growth rate in his consumption. After bad outcomes the agent’s access to capital is restricted, and he is given an inefficiently safe contract with low consumption growth. This hurts the agent, but he can’t use hidden savings to get around it. The principal could punish the agent by proportionally scaling down his capital and consumption (keeping \( \dot{c} \) constant), but this would be too costly: it would make the agent’s consumption too risky and give him a precautionary motive for savings that would distort intertemporal consumption smoothing and increase his incentives to steal. Instead, the principal uses the agent’s access to capital to manipulate his precautionary motive.

**Lon-run behavior.** The contract is scale invariant with respect to \( x \) and there is no termination. After a very good history, the agent’s continuation utility \( x \) will be large, and
he will get a large amount of capital and consumption. Conversely, after a bad history his continuation utility $x$ will be small and he will get a small amount of capital and consumption, but he is never completely retired.

The behavior of $\hat{c}$, however, creates non-trivial dynamics. While the contract starts at the lower end of the domain $\hat{c}_0 = \hat{c}_l$, with high growth and risk, in the absence of shocks the drift of $\hat{c}$ takes the contract to a “steady state” $\hat{c}_{ss} \in (\hat{c}_l, \hat{c}_h)$, as shown in Figure 2. But this steady state can be a misleading guide to the long-run behavior of the contract. If the volatility $\sigma_{\hat{c}}$ is high near the steady state, the contract might spend very little time there. Figure 3 shows the stationary distribution of $\hat{c}$. For this numerical solution, the contract spends most of the time near the upper bound $\hat{c}_h$, with low growth and risk, where both the drift and volatility of $\hat{c}$ are small.

Verification theorem and the optimal contract

We know that the principal’s cost function in the relaxed problem satisfies the HJB equation, but how do we know that the equation has no other solutions? And how do we know that we have identified the true (non-relaxed) optimal contract? The following theorem shows that if an appropriate solution has been found, e.g. numerically, then it must be the true cost function, and we can use it to build the optimal contract.

**Theorem 2** (Verification Theorem). Let $\hat{v}(\hat{c}) : [\hat{c}_l, \hat{c}_h] \rightarrow [\hat{v}_l, \hat{v}_h]$ be a strictly increasing
solution to the HJB equation (13) for some \( \hat{c}_l \in (0, \hat{c}_h) \), such that \( \hat{v}_l \equiv \hat{v}(\hat{c}_l) \in (0, \hat{v}_h) \),\( \hat{v}'(\hat{c}_l) = 0 \), \( \hat{v}''(\hat{c}_l) > 0 \) and \( \hat{v}(\hat{c}_h) = \hat{v}_h \). If \( \gamma < \frac{1}{2} \), we also need to check that

\[
1 - \hat{v}_l (\hat{c}_l^{-\gamma} + \hat{c}_l^{2\gamma - 1} \alpha^2 (\phi \beta)^{-2} \hat{v}_l^{-2}) \leq 0
\]  

(18)

Then,

1) For any incentive compatible contract \( C = (c, k) \) that delivers at least utility \( u_0 \) to the agent, we have \( \hat{v}(\hat{c}_l) ((1 - \gamma) u_0)^{\frac{1}{1-\gamma}} \leq J_0(C) \).

2) Let \( C^* \) be a contract generated by the policy functions of the HJB. Specifically, the state variables \( x^* \) and \( \hat{c}^* \) are solutions to (9) and (10) (with potential absorption at \( \hat{c}_h \)), with initial values \( x_0^* = ((1 - \gamma) u_0)^{\frac{1}{1-\gamma}} \) and \( \hat{c}_0^* = \hat{c}_l \). If \( C^* \) is admissible, and \( \sigma^{\hat{c}^*} \) is bounded, then \( C^* \) is an optimal contract, with cost \( J_0(C^*) = \hat{v}(\hat{c}_l) ((1 - \gamma) u_0)^{\frac{1}{1-\gamma}} \).

The HJB equation can be solved as an ODE by plugging in the FOCs. We only need to verify condition (18) in case \( \gamma < \frac{1}{2} \), and that the contract generated by the HJB \( C^* \) is admissible. The following sufficient condition can be useful.

**Lemma 4.** If the candidate contract \( C^* \) constructed in Theorem 2 has \( \mu^{x^*} < r \), then \( C^* \) is admissible and delivers utility \( u_0 \) to the agent.

**Global incentive compatibility.** To finish the section, we provide sufficient conditions for global incentive compatibility of any contract that satisfies the local constraints on savings (10) and effort (11). This result is used in Theorem 2 to verify that the candidate optimal contract \( C^* \) is incentive compatible. However, it is more general than that and can be used to check incentive compatibility of many other contracts of interest (see Section 4).

While (10) and (11) ensure that neither stealing and immediately consuming, nor secretly saving without stealing are attractive on their own, they leave open the possibility that a double deviation (stealing and saving the proceeds for later) could be attractive to the agent. To see how this can happen, notice that since stealing makes bad outcomes more likely, it increases the expected marginal utility of consumption in the future \( E_t^u \left[ e^{(r-\rho)u_c} c_t^{-\gamma} \right] \). Saving the stolen funds for consumption later could therefore be very attractive. However, hidden savings have decreasing marginal value (the first dollar yields \( c_t^{-\gamma} \), the second one less than that), which depends on the agent’s precautionary motive.

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9In other words, even if \( e^{(r-\rho)u_c} c_t^{-\gamma} \) is a martingale under \( P \), it might be a submartingale under \( P^* \).
This observation allows us to derive a sufficient condition to rule out profitable double deviations.

**Theorem 3.** Let $\mathcal{C} = (c,k)$ be an admissible contract with associated processes $x$ and $\hat{c}$ satisfying (9) and (10) and (11), with bounded $\mu^x$, $\mu^{\hat{c}}$, and $\sigma^{\hat{c}}$, and with $\hat{c}$ uniformly bounded away from zero and bounded above by $\hat{c}_h$. Suppose that the contract satisfies the following property

$$\sigma^{\hat{c}}_t \leq 0$$

(19)

Then for any feasible strategy $(\hat{c},a)$, with associated hidden savings $h$, we have the following upper bound on the agent’s utility, after any history

$$U_{t}^{\hat{c},a} \leq \left( 1 + \frac{h_t}{x_t}\hat{c}_t^{-\gamma} \right)^{1-\gamma} U_{t}^{c,0}$$

(20)

In particular, since $h_0 = 0$, for any feasible strategy $U_0^{\hat{c},a} \leq U_0^{c,0}$, and the contract $\mathcal{C}$ is therefore incentive compatible.

Theorem 3 shows that condition (19) is sufficient by providing a closed-form explicit upper bound (20) on the agent’s off-equilibrium payoff for any savings level $h_t \geq 0$. According to (20), if the agent does not have any hidden savings, $h_t = 0$, the most utility he could get is $U_t^{c,0}$, i.e. the utility level he obtains from “good behavior” $(c,0)$. Hence, good behavior is incentive compatible. However, if the agent had somehow accumulated hidden savings in the past, he would want to deviate from $(c,0)$ in the future, at the very least to increase his consumption, and attain a greater utility. Inequality (20) bounds the utility the agent can get, and the bound tightens as $\hat{c}_t$ rises and the agent’s precautionary motive decreases. The bound is consistent with the intuition that the marginal value of hidden savings becomes lower as the agent’s precautionary motive decreases (however, remember that this is just an upper bound on achievable utility).

The sufficient condition $\sigma^{\hat{c}}_t \leq 0$ can be understood as follows. Hidden savings become more valuable when the agent faces more risk, i.e. has a higher precautionary motive. If risk went up after bad outcomes, stealing and saving could be attractive. Stealing makes bad outcomes more likely, so the agent would expect to have a hidden dollar when it is most valuable to him. The condition $\sigma^{\hat{c}}_t \leq 0$ guarantees that risk goes down after bad outcomes, and makes double deviations unprofitable.

As shown above, the optimal contract has the property that $\sigma^{\hat{c}}_t \leq 0$ because the principal
wants to contain the agent’s precautionary motive and the most efficient way to do this is by reducing the agent’s risk exposure after bad outcomes. As it happens it is the same property that is sufficient for global incentive compatibility.

We would like emphasize once more that Theorem 3 identifies \( \sigma_t \leq 0 \) as a general sufficient condition for incentive compatibility, without even assuming that the contract is recursive in variables \( x \) and \( \hat{c} \). These variables are well-defined for an arbitrary contract, and their laws of motion do not need to be Markov for Theorem 3 to apply. Condition \( \sigma_t \leq 0 \) can be used to verify global incentive compatibility of other contracts. For example, it implies that stationary contracts that we discuss below are also all globally incentive compatible.

4 Financial frictions

To understand the role of hidden savings better, it is useful to map the optimal contract into an investment problem with financial frictions. The agent starts with net worth \( n_0 \) and invests in capital \( k \), funding himself with both equity \( e \) and risk free debt \( d \), i.e. \( k_t = n_t + e_t + d_t \). Equity is a risky claim on the return on his capital, which must pay an expected return of \( r \).\(^{10}\) His net worth then follows the dynamic budget constraint

\[
\frac{dn_t}{n_t} = \left( r + \frac{k_t}{n_t} \alpha - \frac{c_t}{n_t} \right) dt + \phi_t \frac{k_t}{n_t} \beta \left( dR_t - (r + \alpha) dt \right)
\]

(21)

where \( \phi_t = \frac{n_t}{n_t + e_t} \) is the fraction of total equity he retains, \( k/n \) is his leverage ratio of assets over his own net worth, and \( c/n \) his payout policy for his consumption. Because the market demands only the risk free return \( r \), the agent appropriates all the excess return \( \alpha \).

The principal may potentially specify the fraction of equity the agent retains \( \phi_t \) (an “equity constraint”), leverage \( k/n \) (a leverage constraint), and the payout policy \( c/n \), all as a function of the history of returns \( R \). This generates a contract \( C = (c, k) \) using (21). Of course, the agent can still divert funds and has access to hidden savings, so it is important that the resulting contract be incentive compatible. If the agent does not misbehave his

\(^{10}\)Recall there is no aggregate risk here. We add it in Section 5.
Figure 4: The implementation of the optimal contract. The starting point of the optimal contract is indicated by the blue dot, the optimal stationary contract by the red dot, the optimal portfolio plan with $\tilde{\phi} = \phi$ is indicated the black dot, and the optimal contract without hidden savings by the green dot. Parameters: $\rho = r = 5\%$, $\alpha = 1.7\%$, $\gamma = 1/3$, $\phi \beta = 0.2$. 
value function takes the form
\[ U^{c,0}_t = \frac{(\omega_t n_t)^{1-\gamma}}{1-\gamma} \]
where \( \omega_t \) captures the agent’s investment opportunities as determined by the financial frictions dictated by the contract. Figure 4 shows the implementation of the optimal contract, as well as other relevant contracts.

**Lemma 5.** The optimal contract can be implemented as a portfolio problem with \( k_t/n_t = \hat{k}_t \hat{v}_t^{-1} \), \( c_t/n_t = \hat{c}_t \hat{v}_t^{-1} \) and
\[ \tilde{\phi}_t \equiv \hat{v}_t \hat{c}_t^{-\gamma} \phi + \hat{v}_t' \hat{c}_t (\beta \hat{k}_t)^{-1} \sigma^2_t < \phi \]
(22)

The agent’s net worth \( n_t = \hat{v}_t x_t \) satisfies (21), and the value function is \( \omega_t = \hat{v}_t^{-1} \). At \( t = 0 \) the leverage constraint is not binding.

We can study the role of hidden savings in terms of the financial frictions they generate. As we will show below, without hidden savings the optimal contract can be implemented with a constant equity constraint \( \tilde{\phi}_n \) and payout policy \( (c/n)_n \), but allowing the agent to choose his leverage \( (k/n)_n \) freely subject to these constraints. The optimal contract is therefore stationary, and the agent’s consumption, capital, and net worth follow geometric Brownian motions.

In contrast, the optimal contract with hidden savings has two important features: 1) a new type of financial friction arises, a leverage constraint on \( k/n \). By restricting the agent’s access to capital, the principal can limit the agent’s exposure to risk and reduce incentives to steal. This allows the principal to relax the equity constraint \( \tilde{\phi} < \phi \) and improve risk sharing. 2) Hidden savings introduce non-trivial dynamics into an otherwise stationary environment. The principal can use access to capital to provide incentives to the agent. If we punished the agent after bad outcomes using only his net worth (therefore reducing his consumption), his consumption would be very risky and he would have large incentives to steal and save to self insure. Instead, the principal can insure the agent’s consumption giving him a lower equity constraint \( \tilde{\phi} \), and punish him by further restricting his access to capital \( k/n \) and therefore his growth \( \mu^n \) after bad outcomes. This hurts the agent, reflected in a lower \( \omega \), but he cannot save his way around it. Furthermore, the principal front-loads the agent’s access to capital. By restricting access to capital in the future, the principal reduces incentives to steal today, but not the other way around. In fact, the optimal contract starts with a non-binding leverage constraint: at \( t = 0 \) the agent
would choose the \( k/n \) indicated by the contract on his own.

The presence of hidden savings therefore generates history dependent financial frictions. The agent starts at \( \hat{c}_0 = \hat{c}_t \), with high leverage, large exposure to risk and very backloaded consumption, but with high growth expectations. Over time, and especially after bad outcomes, his access to capital is reduced (as a multiple of his net worth), he faces less risk and a better consumption profile, but gives up growth (\( \hat{c} \) rises). This is reversed after good outcomes: he is given more access to capital, more risk and growth (\( \hat{c} \) falls). As a result, agents who are initially lucky and have good outcomes will have access to capital and high growth rates, while those that are initially less lucky get starved for capital and languish. To understand the mechanisms more precisely, it is useful to look at some benchmark contracts.

### A simple equity constraint

Perhaps the simplest incentive scheme we can consider is to force the agent to keep a constant fraction \( \tilde{\phi}_t = \phi \) of his equity, but otherwise let him choose his leverage \( k/n \) and consumption \( c/n \). This is a well known portfolio problem, with constant consumption and capital portfolio weights given by

\[
\frac{c}{n}_p = \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\phi \beta \gamma} \right)^2 \tag{23}
\]

\[
\frac{k}{n}_p = \frac{1}{\gamma \phi \beta} \left( \frac{\alpha}{\phi \beta} \right)^2 \tag{24}
\]

and a value function

\[
\omega_p \equiv \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\phi \beta \gamma} \right)^2 \right)^{\frac{\gamma}{1 - \gamma}} \tag{25}
\]

This generates an incentive compatible contract \( C_p = (c_p, k_p) \). First, consumption under the optimal portfolio problem will naturally satisfy the agent’s Euler equation, so the agent won’t find it optimal to postpone consumption. The equity constraint ensures that for every dollar that the agent steals, he loses the same amount in net worth. If the agent immediately consumes the stolen dollar, he gets the marginal utility of consumption, \( c_{p,t}^{-\gamma} \).
On the other hand, a dollar in legitimate net worth has marginal value $\omega_{p}^{1-\gamma}n_{p,t}^{-\gamma}$. In the optimal portfolio problem the marginal value of net worth must equal the marginal utility of consumption, because the agent could always choose to adjust his consumption, i.e.

$$c_{p}^{-\gamma} = \left(\frac{c_{n}}{n}\right)^{-\gamma}n_{p,t}^{-\gamma} = \omega_{p}^{1-\gamma}n_{p,t}^{-\gamma}$$

As a result, exposing the agent to exactly a fraction $\phi$ of his return ensures that he doesn’t find stealing and immediately consuming attractive. In other words, the resulting contract satisfies the necessary conditions (9), (10), and (11). Lemma 3 then ensures it is globally incentive compatible.

**Lemma 6.** The contract generated by the optimal portfolio plan with $\tilde{\phi} = \phi$ is incentive compatible and delivers utility $u_{p} = \frac{(\omega_{p}n_{0})^{1-\gamma}}{1-\gamma}$ to the agent with associated

$$\hat{c}_{p} = \left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1 - \gamma}{2} \left(\frac{\alpha}{\phi \beta \gamma}\right)^{2}\right)^{\frac{1}{1-\gamma}}$$

and cost $\hat{v}_{p} = \hat{c}_{p}^{\gamma}$.

The solution is indicated with a black dot in Figures 1, 2, and 4. It is not optimal, so it has a higher cost for the principal that the optimal contract, $\hat{v}_{p} > \hat{v}_{0}$ (or equivalently $\omega_{p} < \omega_{0}$). But it is useful to understand exactly how the optimal contract improves upon this simple contract. Compared to this contract, the optimal contract has two important features: 1) a leverage constraint on $k/n$ that restricts the agent’s access to capital, and 2) non-trivial dynamics. To isolate the role that each feature plays, we now consider the class of stationary contracts.

**Stationary contracts**

Stationary contracts set $\mu^{\hat{c}} = \sigma^{\hat{c}} = 0$, so that $\hat{c}, \hat{k}, \sigma^{x}$ and $\mu^{x}$ are all constant as well. In order to get $\mu^{\hat{c}} = 0$ we must set $\sigma^{x}$ so that

$$\frac{1}{2}(\sigma^{x})^{2} = \frac{\rho - r}{\gamma} + \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma}$$

(27)

where, using $\hat{c}_{h}^{1-\gamma} = \frac{\rho - r(1-\gamma)}{\gamma} > 0$, we see the right hand side is non-negative for $\hat{c} \leq \hat{c}_{h}$. We can then build a stationary contract for a given $\hat{c}$, using (9) with $x_{0} = (((1 - \gamma)u_{0})^{\frac{1}{1-\gamma}}$. 25
for any initial utility level \( u_0 \) for the agent. Theorem 3 is general enough to ensure global incentive compatibility for this whole class of contracts. The HJB equation (13) yields the cost of the stationary contract.

**Lemma 7.** Take any \( \hat{c} \in (\hat{c}_*, \hat{c}_h] \), where

\[
\hat{c}_* = \left( \frac{2\gamma \rho - r(1 - \gamma)}{1 + \gamma} \right)^{1/\gamma} \in (0, \hat{c}_h)
\]  

(28)

There is an incentive compatible stationary contract for this \( \hat{c} \), where \( \sigma^x(\hat{c}) \) is given by (27) and the cost \( \hat{v}_r(\hat{c})x_0 \) is given by

\[
\hat{v}_r(\hat{c}) = \hat{c} - \frac{\alpha}{\phi \beta} \hat{c}^\gamma \sqrt{2} \frac{\rho - r(1 - \gamma)}{1 - \gamma} \frac{1 + \gamma}{2r - \rho - \frac{1 + \gamma}{1 - \gamma} \rho + \frac{1 - \gamma}{1 - \gamma}(1 + \gamma)}
\]

(29)

For \( \hat{c} \leq \hat{c}_* \) the growth rate \( \mu^x(\hat{c}) > r \) and the corresponding stationary contract violates the No-Ponzi condition (3) and is therefore not admissible. Since stationary contracts are incentive compatible, we have \( \hat{v}(\hat{c}) \leq \hat{v}_r(\hat{c}) \).

**Remark.** Notice that \( \hat{v}_r(\hat{c}_h) = \hat{v}_h = \hat{c}_h^\gamma \), which makes sense because \( \hat{v}_h \) corresponds to a stationary contract with \( \sigma^x = 0 \).

Since stationary contracts give the agent a constant exposure to risk \( \sigma^x \) we can see the tradeoff between risk exposure and the precautionary motive clearly. Equation (27) says that the only way to give the agent a permanently high risk exposure is to accept that he will postpone his consumption for precautionary motives - i.e. low \( \hat{c} \). Figures 1 and 2 show the cost \( \hat{v}_r(\hat{c}) \) and the law of motion of \( x \) for stationary contracts. Giving capital to the agent is attractive at first because capital pays excess return \( \alpha \), so \( \hat{v}_r(\hat{c}) \) falls and \( \sigma^x \) goes up as \( \hat{c} \) moves back from \( \hat{c}_h \). However, at some point the distortions in risk sharing and intertemporal consumption become too large so the cost of stationary contracts goes up below \( \hat{c}_r^{\min} \). The stationary contract with minimum cost is indicated by a red point at \( (\hat{c}_r^{\min}, \hat{v}_r^{\min}) \).

The best stationary contract can also be implemented as an investment problem with financial frictions, setting \( k_t/n_t = \hat{k}_r^{\min}(\hat{v}_r^{\min})^{-1}, c_t/n_t = \hat{c}_r^{\min}(\hat{v}_r^{\min})^{-1} \), and

\[
\hat{\phi}_r^{\min} \equiv \hat{v}_r^{\min}(\hat{c}_r^{\min})^{-\gamma} \phi < \phi
\]

(30)
This yields a value function \( \omega_{r}^{\text{min}} = (\hat{v}_{r}^{\text{min}})^{-1} \).

Since the contract generated by the simple equity constraint considered above is stationary, it is worse than the best stationary contract, as can be seen in Figure 1. How exactly is the best stationary contract improving things? By imposing a leverage constraint on \( k/n \). To see how, recall from the “skin in the game” constraint (11) that if the principal can front-load the agent’s consumption \( \hat{c} > \hat{c}_p \), he can relax the risk sharing problem. This is reflected in a smaller equity constraint (30), which improves risk sharing. However, if the agent is allowed to choose his leverage freely, he would follow the FOC

\[
\frac{k}{n} = \alpha \left( \frac{k}{n} \right)_p > \frac{1}{\gamma(\phi_{r}^{\text{min}} \beta)^2} = \frac{\alpha}{\gamma(\phi \beta)^2}
\]

where the inequality follows because the effective risk of capital for the agent is \( \hat{\phi}_{r}^{\text{min}} \beta < \phi \beta \). This is a problem, because the agent’s exposure to risk \( \sigma^2 = \sigma^c = \frac{k}{n} \hat{\phi}_{r}^{\text{min}} \beta = \frac{\alpha}{\gamma(\phi_{r}^{\text{min}} \beta)^2} \) would then be very high (higher than in the portfolio problem with \( \phi = \phi \)), prompting him to back load consumption \( \hat{c} < \hat{c}_p \) and undoing the whole incentive scheme. The principal therefore must restrict the agent’s access to capital using a leverage constraint.

**Lemma 8.** Compared to the simple equity constraint \( \tilde{\phi} = \phi \), the best stationary contract front loads the agent’s consumption, \( \hat{c}_{r}^{\text{min}} > \hat{c}_p \), relaxes the equity constraint \( \hat{\phi}_{r}^{\text{min}} < \phi \), and imposes a binding leverage constraint \( k/n \leq \hat{k}_{r}^{\text{min}}(\hat{v}_{r}^{\text{min}})^{-1} < \frac{1}{\gamma(\phi_{r}^{\text{min}} \beta)^2} \) and payout policy \( c/n \geq \hat{c}_{r}^{\text{min}}(\hat{v}_{r}^{\text{min}})^{-1} > (\omega_{r}^{\text{min}})^{\gamma-1} \).

It is worth pointing out two things. First, the principal must control the payout policy \( c/n \), but the agent still has access to hidden savings, so he is not forced to consume these payments. Second, hidden savings create a leverage constraint for a completely different reason than models of limited commitment, as in Kiyotaki and Moore (1997), where the agent can “run away” with some resources. Here the leverage constraint keeps the agent from taking too much risk which would make stealing very attractive.

The best stationary contract illustrates how restricting the agent’s access to capital through a leverage constraint can help provide incentives. The optimal contract does even better by adding a dynamic dimension to the financial frictions. The principal front loads the agent’s access to capital and punishes him after bad outcomes by further restricting his access to capital, rather than taking away net worth or consumption (he gives the agent more capital after good outcomes). This improves risk sharing and relaxes the agent’s
precautionary motive, making stealing less attractive, and allows the principal to use the more efficient continuation contracts when he must deliver more utility to the agent (after good outcomes).

Lemma 9. For any $\hat{c} \in [\hat{c}_l, \hat{c}_h]$, the optimal contract has a lower equity constraint than the corresponding stationary contract, i.e. $\phi(\hat{c}) \equiv \hat{v}(\hat{c})\hat{c}^{-\gamma} + \hat{v}'(\hat{c})\hat{c}\beta k(\hat{c}))^{-1}\sigma \hat{c} < \hat{v}_r(\hat{c})\hat{c}^{-\gamma} \phi$.

Of course, if the agent didn’t have access to hidden savings, we could do even better. Let us now turn to the optimal contract without hidden savings.

No hidden savings

Without hidden savings, the optimal contract takes a simple stationary form, with a constant growth rate $\mu^x$, volatility $\sigma^x$, and intertemporal consumption profile $\hat{c}$.\footnote{See Di Tella (2014) for details and extensions to the cases of aggregate shocks to investment opportunities and Epstein-Zin preferences.} The solution is indicated with a green dot in Figures 1, 2, and 4. Without hidden savings, the principal can force the agent to consume in a front loaded way, and can ignore his precautionary motive. This reduces his marginal utility from consumption, and therefore makes stealing and immediately consuming less attractive. By distorting the intertemporal consumption profile, the principal can relax the risk sharing problem. Because it is not necessary to take into account the agent’s precautionary motive, the principal can deliver utility at a lower cost compared to the hidden savings case, and there is no need for leverage constraints or dynamic behavior. The principal always uses the best contract. As a result, the optimal contract can be implemented with a constant equity constraint $\tilde{\phi}_n = \hat{v}_n\hat{c}_n^{-\gamma} \phi$, and a payout policy $(c/n)_n = \hat{c}_n\hat{v}_n^{-1}$, but no leverage constraint. This results in a value function $\omega_n = \hat{v}_n^{-1}$. The payout policy is now very powerful, because the agent must consume everything immediately and cannot save it for later. This result illustrate how the leverage constraint and the dynamic behavior comes exclusively from the presence of hidden savings.

We can characterize the optimal contract without hidden savings using similar tools, ignoring the agent’s Euler equation (7). Because of homothetic preferences, the cost function takes the form $v_n(x) = \hat{v}_nx$, and satisfies an HJB equation

$$r \hat{v}_n = \min \hat{c}, \sigma \frac{\alpha \hat{c} \gamma}{\phi \beta} \sigma x + \hat{v}_n \left( \frac{p - \hat{c}^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2} \sigma x^2 \right)$$

\footnote{See Di Tella (2014) for details and extensions to the cases of aggregate shocks to investment opportunities and Epstein-Zin preferences.}
where notice that $\hat{c}$ is now a choice variable, instead of a state. The FOC for $\sigma^x$ is

$$\alpha = \gamma \hat{v}_n \hat{c}^{-\gamma} \phi \beta \sigma^x$$  \hspace{1cm} (31)

This is similar to equation (16), except that it is missing the last term that comes from the presence of hidden savings. The FOC for $\hat{c}$ is

$$1 = \hat{v}_n \hat{c}^{-\gamma} + \hat{v}_n \gamma^2 (\sigma^x)^2 \hat{c}^{-1}$$  \hspace{1cm} (32)

On the right hand side we have the cost of giving consumption to the agent. On the left hand side we have the benefit. First, if the agent receives more consumption today, the principal must deliver less utility in the future, which reduces the cost. This is the standard tradeoff we would expect to see. But in addition, the principal knows that by front loading the agent he can relax the risk sharing problem. The second term captures this benefit. As a result, the principal has an extra incentive to front load the agent’s consumption. The optimal contract without hidden savings is able to give the agent more capital and better risk sharing, and therefore deliver utility at a lower cost.

**Lemma 10.** The optimal contract without hidden savings has an equity constraint $\tilde{\phi}_n < \phi$ and a binding payout policy $c/n \geq \hat{c}_n \hat{v}_n^{-1} > (\omega_n)^{\frac{\gamma-1}{\gamma}}$, but no leverage constraint.

The difference between the optimal contract with and without hidden savings can be very large. Without hidden savings, if the elasticity of intertemporal substitution of the agent is low, the principal can achieve an arbitrage (the $\hat{k}$ is “infinity” and there is no optimal contract). With EIS less than 2, by front loading the agent’s consumption, the principal can relax the risk sharing problem to the point that in the limit he can get the excess return of capital without exposing the agent to any risk. What is going on is that because the agent has a low EIS, the principal has a lot of power over him and can eliminate the moral hazard problem in the limit. The power of the principal is limited if the elasticity of intertemporal substitution is higher (here we considered EIS = 3 to ensure the optimal contract exists both with and without hidden savings), or if the agent has access to hidden savings, which limit the principal’s ability to manipulate the agent’s consumption.
5 Aggregate risk and hidden investment

The previous results show how the principal can use the agent’s access to capital to manipulate his precautionary motive. This suggests that hidden investment could be an important constraint in this setting. To study the role of hidden investment, it is useful to also introduce aggregate risk, and allow the agent to invest his hidden savings in it. Fortunately, we can easily incorporate hidden investment into our setting and extend our tools to study what role they play. Appendix B has all the formal results. Here we will focus on the main economic insights.

The return on capital is now

$$dR_t = (r + \pi \tilde{\beta} + \alpha - a_t) dt + \beta dZ_t + \tilde{\beta} d\tilde{Z}_t$$

Where $\tilde{Z}$ is an independent Brownian motion that represents aggregate risk, with market price $\pi$. Capital has a loading $\tilde{\beta}$ on aggregate risk, so the excess return on capital for the agent is $\alpha$, as in the baseline. Let $Q$ be the associated martingale measure. Now the agent can invest his hidden savings (besides the risk-free rate). We always allow the agent to invest in aggregate risk, in the same way the principal would be able to. This is really an extension of the notion of hidden savings to an environment with aggregate risk. In addition, the agent may be able to invest his hidden savings in his own private technology. His hidden savings therefore follow the law of motion

$$dh_t = \left( rh_t + z_t h_t (\alpha + \pi \tilde{\beta}) + \tilde{z}_t h_t \pi + c_t - \tilde{c}_t + \phi k_t a_t \right) dt + z_t h_t \left( \beta dZ_t + \tilde{\beta} d\tilde{Z}_t \right) + \tilde{z}_t h_t d\tilde{Z}_t$$

where $z$ is the portfolio weight on his own private technology, and $\tilde{z}$ the weight on aggregate risk. While the agent can choose any position on aggregate risk, $\tilde{z}_t \in \mathbb{R}$, for his hidden private investment we consider two cases: 1) no hidden private investment, $z_t \in H = \{0\}$, and 2) hidden private investment, $z_t \in H = \mathbb{R}^+$. A valid interpretation for hidden investment in the private technology $z_t$ is that the principal can give the agent an amount of capital $k_t$ that he monitors, but the agent can secretly invest more.

A contract $C = (c, k)$ specifies the contractible payments $c$ and capital $k$, contingent not only on returns $R$ but also on the observable aggregate shock $\tilde{Z}$. After signing the contract

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12 We can also study other cases where the agent may not be able to invest in aggregate risk, or only take a positive position, which requires small modifications to the relevant incentive compatibility constraints. We focus on the economically most relevant case, where the agent can always invest his hidden savings in the market in the same way the principal would.
the agent can choose a strategy \((\tilde{c}, a, z, \tilde{z})\) to maximize his utility. The agent’s utility and the principal’s objective function are still given by (1) and (2). As in the baseline setting, it is without loss of generality to look for a contract where the agent does not steal, has no hidden savings, and no hidden investment. A contract is therefore incentive compatible if the agent’s optimal strategy is \((c, 0, 0, 0)\), or \((c, 0)\) for short.

Since there is now aggregate risk that pays a premium, we need to slightly modify the parameter restrictions

\[
\left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^\frac{1}{1-\gamma} > 0 \tag{33}
\]

and

\[
\alpha < \bar{\alpha} \equiv \frac{\phi \beta \gamma \sqrt{2}}{\sqrt{1 + \gamma}} \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1}{2} \left( 1 - \gamma \right) \left( \frac{\pi}{\gamma} \right)^2} \tag{34}
\]

We can check that with \(\pi = 0\) we recover the formulas without aggregate risk.

Since the contract can depend on the history of aggregate shocks \(\tilde{Z}\), so can his continuation utility \(U^{c,0}\) and his consumption \(c\). However, because the agent is not responsible for aggregate shocks, incentive compatibility does not place any constraints on his exposure to aggregate risk. On the other hand, since the agent can invest his hidden savings in aggregate risk, his Euler equation needs to be modified appropriately. His discounted marginal utility must be a supermartingale under any valid trading strategy. Otherwise, the agent could save a dollar, invest it in aggregate risk or his private technology, and consume it later. As a result, the laws of motion for the state variables \(x\) and \(\hat{c}\) need to be modified to

\[
\frac{dx_t}{x_t} = \left( \frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2} (\sigma_t^x)^2 + \frac{\gamma}{2} (\sigma_t^\hat{x})^2 \right) dt + \sigma_t^x dZ_t + \sigma_t^\hat{x} d\tilde{Z}_t \tag{35}
\]

\[
\frac{d\hat{c}_t}{\hat{c}_t} = \left( \frac{r - \rho}{\gamma} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} + \frac{(\sigma_t^{\hat{c}})^2}{2} + \frac{1 + \gamma (\sigma_t^{\hat{c}})^2}{2} \right) dt + \sigma_t^{\hat{c}} dZ_t + \sigma_t^{\hat{c}} d\tilde{Z}_t + dL_t
\] \tag{36}

The “skin in the game” IC constraint is unchanged

\[
\sigma_t^x = \hat{c}_t^{\gamma} \gamma \phi \hat{k}_t \beta \tag{37}
\]
Since the agent can invest his hidden savings in aggregate risk (both long and short) we have a new IC constraint
\[ \tilde{\sigma}^c_i + \tilde{\sigma}^x_i = \frac{\pi}{\gamma} \quad (38) \]
If the agent can also invest his hidden savings in his private technology, \( H = \mathbb{R}_+ \), we also have another IC constraint
\[ \sigma^x_i + \sigma^c_i \geq \frac{\alpha}{\beta \gamma} \quad (39) \]
The interpretation for (38) and (39) is simple: for the agent, aggregate risk has a premium \( \pi \) and his idiosyncratic risk \( \frac{\alpha}{\beta \gamma} \). The principal must allow him to invest in these sources of risk through the contract, or he will do it on his own. This restricts the principal’s ability to provide incentives. In particular, the principal cannot promise to give the agent a perfectly deterministic consumption at some point in the future. If he tried to do this, the agent would just take risk on his own. This is reflected in a lower upper bound \( \hat{c}_h \), which is costly because the principal would like to use a promise of future safety to relax the agent’s precautionary motive. If the agent cannot invest in his private technology, we have
\[ \hat{c}_h = \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{\sigma^x_i}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1-\gamma}}. \]
This is the \( \hat{c} \) corresponding to the optimal portfolio strategy when the agent can invest only in aggregate risk. Notice that if we take \( \pi = 0 \) we recover expression (8) in the baseline setting, corresponding to the optimal portfolio if the agent can only invest in a risk free asset. If the agent can also invest in his private technology, then
\[ \hat{c}_h = \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{\sigma^x_i}{2} \left( \frac{\pi}{\gamma} \right)^2 - \frac{1}{1-\gamma} \left( \frac{\sigma^x_i}{\gamma} \right)^2 \right)^{\frac{1}{1-\gamma}}, \]
which is even lower and corresponds to the \( \hat{c} \) in an optimal portfolio strategy when the agent can invest in both aggregate risk and in his private technology on his own (so he can’t get any risk sharing).\(^{13}\)

We can now characterize the relaxed problem with an HJB equation and appropriate constraints:
\[
0 = \min_{\sigma^x, \sigma^c, \tilde{\sigma}^c, \tilde{\sigma}^x} \left( \hat{c} - r \hat{v} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi \beta} + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma (\sigma^x)^2}{2} + \frac{\gamma (\tilde{\sigma}^c)^2}{2} - \pi \tilde{\sigma}^x \right) \right) + \hat{c}' \left( \frac{r - \rho}{\gamma} - \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{(\sigma^x)^2}{2} + (1 + \gamma) \sigma^x \sigma^c \hat{c} + \frac{1 + \gamma}{2} (\sigma^c)^2 + \frac{(\tilde{\sigma}^c)^2}{2} \right) \quad (40)
\]
\(^{13}\)The parameter restrictions (33) and (34) ensure \( \hat{c}_h \) is positive in both cases.
\[(1 + \gamma)\dot{\sigma}^x \dot{\sigma}^c + \frac{1 + \gamma}{2} (\dot{\sigma}^c)^2 - \dot{\sigma}^c \pi) + \frac{\gamma''}{2} \dot{c}^2 \left( (\sigma^c)^2 + (\dot{\sigma}^c)^2 \right) \]

subject to \(\sigma^x \geq 0\) and (38) and (39).

Figure 5 shows the optimal contract with hidden investment. The optimal contract has the same features as in the baseline setting. Hidden investment, however, restricts the principal’s ability to manipulate the agent’s precautionary motive by promising less risk in the future and after bad outcomes, and thus leads to a higher cost. The contract starts at some \(\hat{c}_l\) and then moves immediately into the interior of the domain \([\hat{c}_l, \hat{c}_h]\). Because the principal cannot promise a completely deterministic contract in the future (\(\hat{c}_h\) is lower with hidden investment), the agent’s precautionary motive is stronger, and the contract starts at a lower \(\hat{c}_l\).

Turn now to the contract’s dynamic behavior. First, using (38) to eliminate \(\dot{\sigma}^c\), and taking FOC for \(\dot{\sigma}^x\), we obtain

\[\dot{\sigma}^x = \frac{\pi}{\gamma}, \quad \dot{\sigma}^c = 0\]

This is the first best exposure to aggregate risk. The principal and the agent don’t have any conflict about aggregate risk, and the principal cannot use it to relax the moral hazard problem, so they implement the first best aggregate risk sharing. The presence of this investment opportunity still affects the optimal contract, however, by limiting the principal’s ability to manipulate the agent’s precautionary motive (it lowers \(\hat{c}_h\)).

In contrast, the FOC for \(\sigma^x\) and \(\sigma^c\) depend on whether the agent can invest his hidden savings in his private technology, as shown in Figure 6. Without hidden investment, the FOCs are the same as in the baseline, (16) and (17), and capture the intertemporal and hedging motives described in Section 3. With hidden investment, the IC constraint (39) can be binding in some region of the state space. This is the case in Figure 6 near the upper bound of the hidden investment contract \(\hat{c}_h\). It is still true, however, that \(\mu \dot{c}^c(\hat{c}_l) > 0\) and \(\sigma^c \leq 0\) always, so the contract dynamics are the same as in the baseline: the agent gets less capital and less risk over time (up to a “steady state”) and after bad outcomes.

In terms of financial frictions, the optimal contract can still be implemented as a portfolio problem with an equity constraint \(\dot{\phi}_l \leq \phi\), a leverage constraint on \(k/n\), and payout policy \(c/n\). How come the principal is able to impose a leverage constraint if the agent can invest on his own with his hidden savings? If the agent invests on his own he is the residual

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14 If the agent didn’t have access to hidden investment in aggregate risk, and the agent’s private technology is exposed to aggregate risk \(\hat{\beta} \neq 0\), then the principal could potentially use the agent’s exposure to aggregate risk to relax the moral hazard problem.
Figure 5: The cost function $\hat{v}(\hat{c})$ solid in blue without hidden investment, green with hidden investment, and dashed in red for stationary contracts. The the starting point without hidden investment is indicated by the blue dot, with hidden investment by the green dot, the optimal stationary contract by the red dot, the optimal portfolio plan with $\hat{\phi} = \phi$ by the black dot. The black dashed line is the locus $A(\hat{c}, \hat{v}) = 0$. Parameters: $\rho = r = 5\%$, $\alpha = 6\%$, $\gamma = 3$, $\phi\beta = 0.2$, $\pi = 4\%$.

Figure 6: The drift, $\mu_{\hat{c}}$ and $\mu_{\hat{x}}$, and volatility, $\sigma_{\hat{c}}$ and $\sigma_{\hat{x}}$, of the state variables $\hat{c}$ and $\hat{x}$, without hidden investment (blue) and with hidden investment (green). The black dashed lines indicate $\hat{c}_l$ and $\hat{c}_h$ with hidden investment, and $\hat{c}_l$ without hidden investment. Parameters: $\rho = r = 5\%$, $\alpha = 6\%$, $\gamma = 3$, $\phi\beta = 0.2$, $\pi = 4\%$. 
claimant and must bear the whole risk, while if he invests through the principal he gets to share some risk and keep only a fraction $\tilde{\phi}_t < 1$. The leverage constraint really limits the amount of capital the principal will agree to share risk on. As in the setting without hidden investment, the optimal contract starts with a non-binding leverage constraint. This implies that the IC constraint on hidden investment is also not binding initially: if the agent doesn’t want to invest more capital while keeping a fraction $\tilde{\phi} < 1$ of the risk, he certainly does not want to invest if he must keep all the risk. At the other extreme, it is worth pointing out that although $\hat{c}_h$ corresponds to the consumption profile under autarky, there are still gains from trade if the agent is able to invest in his private technology, so that the cost for the principal at this point is lower $\hat{v}(\hat{c}_h) < \hat{c}_h$. To see why, notice that the principal can give the agent the same consumption process that he would get in autarky, but more capital. Since $\tilde{\phi} \leq \phi < 1$, the agent can manage more capital than under autarky and yet have the same exposure to risk. Since he keeps the total exposure to risk corresponding to autarky, which is relatively low given that the agent must bear the whole risk, there is a binding leverage constraint here. In other words, if the agent could invest more capital and keep only a fraction $\tilde{\phi} < 1$ he would like to invest more, but not if he has to keep all the risk.

It is useful to ask under what conditions the gains from trade are completely exhausted, and the optimal contract corresponds to letting the agent invest on his own without any risk sharing. This is true in the special (but salient) case with $\phi = 1$, where we have a pure misreporting problem. Without hidden investment, the are still gains from trade, and the optimal contract has the same qualitative features described in Section 3. While the agent can save on his own, the principal can still control his access to capital to provide incentives, and can therefore provide some risk sharing. However, with hidden investment, the optimal contract coincides with autarky. Intuitively, the agent can both save and invest on his own, so the principal cannot provide any risk sharing in an incentive compatible way. We can see this case as a limit in Figure 5. If we let $\phi \rightarrow 1$, while also adjusting $\beta$ so that $\phi \beta$ is constant, all the curves corresponding to case with no hidden investment remain unchanged. The optimal contract with hidden investment, however, becomes progressively worse, because the agent finds investing on his own more attractive (the green curve shifts up). This is reflected in a falling $\hat{c}_h \searrow \check{c}_p = \left(\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left(\frac{\alpha}{\gamma \phi \beta}\right)^2 - \frac{1 - \gamma}{2} \left(\frac{\pi}{2}\right)^2\right)$, where $\check{c}_p$ corresponds to the optimal portfolio with an equity constraint $\tilde{\phi} = \phi$ (which in the limit is 1).
Verifying global incentive compatibility with hidden investment

The agent’s ability to invest his hidden savings makes the verification of global incentive compatibility potentially more difficult. The agent could find it attractive to steal and save the proceeds for later, while investing them in aggregate risk or even his own private technology. However, we can extend the results of Theorem 3 to deal with hidden investment.

**Theorem 7.** Let \( C = (c, k) \) be an admissible contract with associated processes \( x \) and \( \hat{c} \) satisfying (35) and (36), and (37), (38), and in the case of hidden investment (39), with bounded \( \mu^x, \mu^{\hat{c}}, \sigma^{\hat{c}}, \) and \( \hat{\sigma}^{\hat{c}}, \) and with \( \hat{c} \) uniformly bounded away for zero and bounded above by \( \hat{c}_h. \) Suppose that the contract satisfies the following property

\[
\sigma^{\hat{c}}_t \leq 0
\]

Then for any feasible strategy \((\hat{c}, a, z, \hat{z})\), with associated hidden savings \( h \), we have the following upper bound on the agent’s utility, after any history

\[
U_{t}^{\hat{c}, a, z, \hat{z}} \leq \left( 1 + \frac{h_t}{x_t^{\hat{c}} - \gamma} \right)^{1 - \gamma} U_{t}^{c, 0}
\]

In particular, since \( h_0 = 0 \), for any feasible strategy \( U_{0}^{\hat{c}, a, z, \hat{z}} \leq U_{0}^{c, 0} \), and the contract \( C \) is therefore incentive compatible.

### 6 Renegotiation

The optimal contract requires commitment. The principal can relax the agent’s precautionary motive by promising an inefficiently low amount of capital and risk over time and after bad outcomes. This suggests that the agent and principal could be tempted to renegotiate the contract and “start over”, undoing the whole incentive scheme. Here we formalize a notion of renegotiation, and characterize the optimal renegotiation-proof contract. As it turns out, the optimal renegotiation-proof contract is the best stationary contract, so even without commitment we are able to do better than letting the agent invest while keeping a fraction \( \phi \) of his risk.

After signing an incentive compatible contract \( C = (c, k) \), the principal can at any time
offer a new continuation contract that leaves the agent at least as well off (the offer is “take it or leave it”). The question is what kind of contracts can he offer - what is a valid “challenger” to the original contract? Here we explore a notion of internal consistency. If $C$ is renegotiation proof, then surely appropriately scaled parts of it should be a valid challenger. This means that at any stopping time $\tau$, the principal can renegotiate and get a continuation cost $x_\tau \times \inf \hat{v}(\omega, t)$. With this in mind, we say that an incentive compatible contract is renegotiation-proof (RP) if

$$\infty = \arg \min \mathbb{E}_Q \left[ \int_0^\tau e^{-rt}(c - k_t \alpha)dt + e^{-r\tau}x_\tau \inf \hat{v} \right]$$

The optimal contract with hidden savings is not renegotiation proof, because after any history $\hat{v}_t > \hat{v}_l = \inf \hat{v}(\omega, s)$, so the principal is always tempted to “start over”. In fact, it is easy to see that RP contracts must have a constant $\hat{v}_t$. The converse it also true.

**Lemma 11.** An incentive compatible contract $C$ is renegotiation proof if and only if the continuation cost $\hat{v}$ is constant.

Stationary contracts have a constant $\hat{v}$, because $\hat{c}$ is constant. However, those contracts were built using $dL_t = 0$. There are other contracts with a constant $\hat{c}$ that use $dL_t > 0$, i.e. the drift of $\hat{c}$ would be negative without $dL_t$. In addition, there could be non-stationary contracts with a constant cost $\hat{v}(\hat{c})$ for all $\hat{c}$ in the domain. The next Lemma shows they are all worse than the best stationary contract, with cost $\hat{v}_r^{min} = \min_{\hat{c} \in [\hat{c}_*, \hat{c}_h]} \hat{v}_r(\hat{c})$.

**Theorem 4.** The optimal renegotiation-proof contract is the optimal stationary contract with cost $\hat{v}_r^{min}$.

**Remark.** It is possible that $\hat{c}_r^{min} = \hat{c}_h$ if the agent can invest his hidden savings and $\phi$ is close enough to 1, since in this case $\hat{c}_h$ can get arbitrarily close to $\hat{c}_p$.

This result shows that even without full commitment, hidden savings generate a leverage constraint. The principal can do better than just giving the agent an equity constraint $\hat{\phi} = \phi$ by imposing a leverage constraint on $k/n$.

### 7 Conclusions

Hidden savings impose realistic constraints on financial contracts, and can generate commonly observed financial frictions and dynamics in otherwise stationary environments. We
study a classical investment setting with a fund diversion problem. Without hidden savings, the agent must keep an equity stake to provide incentives, but he is otherwise free to lever up. With hidden savings, the principal must use the agent’s access to capital to manipulate his precautionary motive. By reducing access to capital over time, and especially after bad outcomes, the principal reduces incentives to divert funds and secretly save, reducing the cost of providing incentives. After bad outcomes the agent’s consumption is somewhat insured, and he is punished instead with less access to capital and a lower growth rate of consumption. This is true even if the agent is able to secretly invest in capital, because the principal can provide some risk sharing. As a result, with hidden savings the optimal contract requires a leverage constraint in addition to the equity constraint. With renegotiation, the optimal contract takes a simple stationary form, with constant equity and leverage constraints.

We provide a sufficient analytical condition for the validity of the first order approach. If the agent’s precautionary motive is weaker after bad outcomes, the contract is globally incentive compatible. This condition holds in the optimal contract and in a wider class of contracts beyond the optimal one. In fact, the sufficient conditions does not even require a recursive structure, and it is valid even with aggregate risk and hidden investment.
References


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Appendix A - Omitted Proofs

Lemma 1

Consider

\[ Y_t = \mathbb{E}_t \left[ \int_0^\infty e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right] = \int_0^t e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds + e^{-\rho t} U_t^{c,0} \]

Since \( Y \) is an \( \mathbb{F} \)-adapted \( P \)-martingale, and \( \mathbb{F} \) is generated by \( Z \), we can apply a martingale representation theorem to obtain

\[ dY_t = e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho t} dU_t^{c,0} - \rho e^{-\rho t} U_t^{c,0} dt = e^{-\rho t} \Delta_t \beta \, dZ_t \]

for some stochastic process \( \Delta \) also adapted to \( \mathbb{F} \). Dividing by \( e^{-\rho t} \) and rearranging we get (4).

Lemma 2

First, take the strategy \((c + \phi \kappa, a)\). Here \( a \) is zero where (5) is satisfied and \( a = a^* \wedge \bar{a} \) where \( a^* \) achieves the maximum in (5) and \( \bar{a} > 0 \) is an arbitrary bound. Because the objective in (5) is concave and it’s \( \frac{1}{1-\gamma} \) for \( a = 0 \), then for \( a^* \wedge \bar{a} \) it is strictly greater when (5) fails. Notice \( \bar{h}_t = 0 \) by construction, and we can follow strategy \((\hat{c}^n, a^n)\) until some stopping time \( \tau^n \) and then revert to good behavior, where the stopping time \( \tau^n \to \infty \) a.s., ensures it’s feasible and reduces the stochastic integral. We can compare the utility from this strategy \( U^{c^n, a^n} \) with the utility from good behavior \( U^{c,0} \):

\[ U^{c^n, a^n} - U_t^{c,0} = \mathbb{E}_0 \left[ \int_0^{\tau^n} e^{-\rho t} \left( \frac{(c_t + \phi \kappa a_t)^{1-\gamma}}{1-\gamma} - \frac{c_t^{1-\gamma}}{1-\gamma} - \Delta_t a_t \right) dt \right] \]

(41)

where we have used \( Z^n_t = Z_t + \int_0^t \frac{a_s}{\beta} \, ds \) to express \( U_t^{c,0} \) as an expected integral under \( P^a \), and also the fact that \( U_t^{c^n, a^n} = U_t^{c^n, a^n} \). Taking \( n \to \infty \) and using the monotone convergence theorem (the integrand is always positive), if (5) fails we get \( U_0^{c^n, a^n} > U_0^{c,0} \) for some large \( n \), and \( \bar{C} \) is not incentive compatible. (6) is simply the FOC necessary condition for (5) (and sufficient because of concavity).

For (7) this is the result of \( e^{-(\mu-\rho)\gamma t} c_t \) being a supermartingale, which is a standard necessary condition in a savings/consumption problem. Note this doesn’t involve stealing \( a_t \), just hidden consumption \( c \). We can then write it \( e^{-(\mu-\rho)\gamma t} c_t \gamma = M_t - A_t \), where \( M_t = \int_0^t \sigma_s \, dZ_t \) is a local martingale, and \( A \) a weakly increasing process. Using Ito’s lemma, we get the desired expression.

Lemma 3

For the bound, since both \( c_t \geq 0 \) and \( x_t \geq 0 \), we only need to show that \( \hat{c}_t \leq \hat{c}_0 \). Marginal utility of consumption is \( m_t = c_t \gamma \) and the utility flow \( \frac{c_t^{1-\gamma}}{1-\gamma} = \frac{1}{1-\gamma} m_t^{\gamma-1} \). This is a convex and decreasing function of \( m_t \). From Lemma 2, we have by (7) that \( \mathbb{E}_t [m_{t+u}] \leq e^{(\rho-\mu)u} m_t \). Given any \( m_0 \) we have by Jensen’s inequality

\[ \mathbb{E} \left[ \frac{c_t^{1-\gamma}}{1-\gamma} \right] \geq \frac{1}{1-\gamma} \mathbb{E} [m_t]^{\gamma-1} \geq \frac{1}{1-\gamma} m_0^{\gamma-1} e^{(\rho-\mu)u} \frac{2}{1-\gamma} \]
with equality only if \( c \) is deterministic. So

\[
U^{c,0}_t = \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(u-t)} \frac{c_{1-\gamma}^u - \gamma}{1 - \gamma} du \right]
\]

\[
\geq \int_t^\infty e^{-\rho(u-t)} \frac{c_{1-\gamma}^u}{1 - \gamma} e^{(\rho - r)(u-t)} ds = \frac{c_{1-\gamma}^t}{1 - \gamma} \frac{\gamma}{\rho - r(1 - \gamma)}
\]

where the second equality uses the fact that the termination contract must deliver \( \hat{U} \) to the agent. This bound implies

\[
x_t = ((1 - \gamma)U^{c,0}_t)^{\frac{1}{1-\gamma}} \geq c_t \left( \frac{\gamma}{\rho - r(1 - \gamma)} \right)^{\frac{1}{1-\gamma}}
\]

So \( \hat{c}_t = c_t/x_t \) has an upper bound

\[
\hat{c}_t \leq \left( \frac{\rho - r(1 - \gamma)}{\gamma} \right)^{\frac{1}{1-\gamma}} = \hat{c}_h
\]

In addition, since the upper bound can only be achieved with a deterministic consumption, \( U^{c,0} \) is deterministic too. This implies that both \( c_t \) and \( x_t \) grow at rate \( \frac{\gamma}{1 - \gamma} \), so \( \hat{c}_h \) is an absorbing state. In light of (6), we must have \( k_{t+u} = 0 \) in the continuation contract, so we have the “retirement” contract with cost \( \hat{v}_h x_t \). This completes the proof.

**Theorem 1**

The proof is split into parts.

1) The cost function must be bounded above by \( \hat{v}_h \) since we can always just give consumption to the agent without any capital, and obtain cost \( \hat{v}_h \). It must be strictly positive because if \( \hat{v}(\hat{c}) = 0 \) for any \( \hat{c} \in [0, \hat{c}_h] \), then we can scale up the contract and give infinite utility to the agent at zero cost, or else achieve infinite profits.

2) Because we can always move \( \hat{c} \) up using \( dL_t \), we know that \( \hat{c} \) must be weakly increasing. It is useful to define

\[
A(\hat{c}, \hat{v}) \equiv \hat{c} - r \hat{v} - \frac{1}{2} \left( \frac{uv}{\gamma \phi \beta} \right)^2 + \hat{v} \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} \ (42)
\]

which is the HJB equation when \( \hat{v}(\hat{c}) \) is flat. The region where the HJB equation holds cannot have any flat parts, because this would mean that \( A(\hat{c}, \hat{v}(\hat{c})) = 0 \). We know however from Lemma 12 (with \( \pi = 0 \)) that this function can have at most two roots, so \( \hat{v} \) must be strictly increasing in the region where the HJB holds.

The only way the HJB could not hold is if the optimal contract never spends any time there, i.e. if we ever found ourselves there, it would be optimal to immediately jump out using \( dL_t \). The value in that region then must be constant and equal to the value at the destination point (the upper end of the flat region). The HJB should hold as an inequality with \( A(\hat{c}, \hat{v}(\hat{c})) \geq 0 \), since otherwise we could improve by lingering in the flat region for a while before jumping. Since \( \hat{v}(\hat{c}) \) is strictly increasing when the HJB equation holds, the contract will start with \( \hat{c}_0 \) at the upper end of a flat region. It cannot be that \( \hat{c}_0 = 0 \) because of Inada
conditions, so we must have at least one flat interval \([0, \hat{c}_i]\).

3) We would like to show that this is the only flat interval, and the HJB equation holds as an equality in the strictly increasing region \([\hat{c}_i, \hat{c}_h]\), and both regions are connected with smooth pasting, i.e. \(\hat{v}'(\hat{c}_i) = 0\). Suppose then that there is a region \([\hat{c}_1, \hat{c}_2] \subset (0, \hat{c}_i)\) where the cost function is flat, and it’s strictly increasing immediately above it (possibly, \(\hat{c}_1 = 0\)). Let’s show that as \(\hat{c} \searrow \hat{c}_2\), \(A(\hat{c}, \hat{v}(\hat{c})) \to 0\) and \(\hat{v}'(\hat{c}) \to 0\) (i.e. we have smooth pasting). Towards contradiction, imagine there is a kink at \(\hat{c}_2\), i.e. the right-derivative of \(\hat{v}(\hat{c})\) is strictly positive. This can only happen if \(\sigma^\epsilon(\hat{c} + \epsilon) \to 0\) and \(\mu^\epsilon(\hat{c}_2 + \epsilon) \geq 0\) as \(\epsilon \to 0\), since otherwise we would cross into the flat region where the HJB doesn’t hold (with \(\hat{v}'(\hat{c}) > 0\) we must have \(dL = 0\)). First consider the case with \(\liminf \mu^\epsilon(\hat{c}_2 + \epsilon) > 0\). We can contemplate the following deviation: start at \(\hat{c}_2 - \delta\), with the same \(\sigma^x\) and \(\sigma^\epsilon = 0\): by continuity \(\mu^\epsilon > 0\) along this plan, so after some time we will end up at \(\hat{c}_2\) and we can go back to the optimal contract and obtain continuation cost \(\hat{v}(\hat{c}_2)\). The value of this strategy for \(\hat{c} < \hat{c}_2\) extends the cost function \(\hat{v}(\hat{c})\) below \(\hat{c}_2\) and satisfies the HJB equation (with \(\sigma^\epsilon = 0\), so it’s a first order ODE with boundary condition given by \(\hat{v}(\hat{c}_2)\) at \(\hat{c}_2\). However, because \(\hat{v}'(\hat{c}_2) > 0\), we obtain a lower cost at \(\hat{c}_2 - \delta\). This is therefore an attractive deviation, which is a contradiction, so \(\mu^\epsilon(\hat{c}_2 + \epsilon) \to 0\) is the only option left. In this case, since we also have \(\sigma^\epsilon(\hat{c} + \epsilon) \to 0\), it means we have a stationary contract, and because we are at a local minimum of the cost function, we must also be at a local minimum of the cost of stationary contracts \(\hat{v}_r(\hat{c})\) given by (29), implying \(\hat{v}'(\hat{c}_2) = 0\). However, because \(\hat{v}'(\hat{c}_2) > 0\), we get \(\hat{v}(\hat{c} + \epsilon) > \hat{v}_r(\hat{c} + \epsilon)\), which cannot be. We conclude that \(\hat{v}'(\hat{c}_2) = 0\).

We still need to show that \(A(\hat{c}, \hat{v}(\hat{c})) \to 0\) as \(\hat{c} \searrow \hat{c}_2\). To see this it’s useful to use the FOC for \(\sigma^x\) conditional on \(\sigma^\epsilon\) to obtain
\[
\sigma^x = \frac{\alpha \gamma}{\nu} - \hat{v}'(1 + \gamma) \frac{\sigma^\epsilon}{\hat{v} + \hat{v}' \hat{c}}
\]
and plug it into the HJB equation. We can then re-write the HJB
\[
0 = \min_{\sigma^\epsilon} \hat{A} + \hat{B} \sigma^\epsilon + \frac{1}{2} \hat{C} \sigma^\epsilon^2
\]
with
\[
\hat{A} = \hat{c} - r \hat{v} - \frac{1}{2} \left(\frac{\alpha \gamma}{\nu} \right)^2 + \hat{v} \left( \frac{\rho - c^{1-\gamma}}{\gamma} \right) - \hat{v}' \hat{c} \left( \frac{\rho - c^{1-\gamma}}{\gamma} \right)
\]
\[
\hat{B} = \hat{v}'(1 + \gamma) \frac{\alpha \gamma}{\nu} \frac{\sigma^\epsilon}{\hat{v} + \hat{v}' \hat{c}} \geq 0
\]
\[
\hat{C} = \gamma \hat{v}'(1 + \gamma) \frac{\hat{v} - \hat{v}' \hat{c}}{\gamma} + \nu \sigma^\epsilon^2
\]
For the HJB to have a minimum, it must be that \(\hat{C} \geq 0\). If \(\hat{C} = 0\) we can only have a minimum if \(\hat{B} = 0\) as well, in which case \(\hat{A} = 0\), and with \(\hat{v}'(\hat{c}_2) = 0\) we get \(A(\hat{c}, \hat{v}(\hat{c})) \to 0\) as \(\hat{c} \searrow \hat{c}_2\) as desired. If instead \(\hat{C} > 0\), then
\[
\sigma^\epsilon = -\frac{\hat{B}}{\hat{C}} \leq 0
\]
With \(\hat{v}'(\hat{c}_2) = 0\) we must have \(\hat{v}''(\hat{c} + \epsilon) \geq 0\) for small \(\epsilon\) (or else \(\hat{v}'(\hat{c} + \epsilon) < 0\)). We can then show that \(\frac{\hat{C}}{\hat{C}^2} \to \infty\) as \(\hat{c} \searrow \hat{c}_2\), which implies \(\frac{\hat{B}^2}{\hat{C}^2} \to 0\) and therefore \(\hat{A} \to 0\), and therefore \(A(\hat{c}, \hat{v}(\hat{c})) \to 0\) as \(\hat{c} \searrow \hat{c}_2\), as desired. Since this is true in particular when \(\hat{c}_1 = 0\) and \(\hat{c}_2 = \hat{c}_i\), we have proven the smooth pasting
condition.

Now since at \( \hat{c_2} \) we have \( A(\hat{c}, \hat{v}(\hat{c})) = 0 \), this is a root of \( A \). Below that we have \( A(\hat{c}, \hat{v}(\hat{c})) \geq 0 \). From Lemma 12 we know that this can only be the case if \( \hat{c_2} \) is the first root of \( A(\hat{c}, \hat{v}(\hat{c})) \). We would like to show that it cannot be the case that for \( \hat{c} < \hat{c_2} < \hat{c_2} \) the cost function is lower, i.e. \( \hat{v}(\hat{c_1} - \delta) < \hat{v}(\hat{c_1}) = \hat{v}(\hat{c_2}) \). To see this, imagine the same problem with a smaller \( \alpha' < \alpha \), and cost function \( \hat{v}_{\alpha'}(\hat{c}) \geq \hat{v}(\hat{c}) \). We can pick \( \alpha' \) small enough that \( \hat{v}_{\alpha'}(\hat{c}) = \hat{v}(\hat{c}_2) \), where \( \hat{c}_2 \) is the upper end of the first flat region for the contract with \( \alpha' \) (and it minimizes \( \hat{v}_{\alpha'}(\hat{c}) \)). We can do this because \( \hat{v}_{\alpha'}(\hat{c}) \) is continuously increasing in \( \alpha' \) for any \( \hat{c} \), and has \( \hat{v}_{\alpha'}(\hat{c}) = \hat{v}_h \) for all \( \hat{c} \leq \hat{c}_h \) if \( \alpha' = 0 \). It must be that \( \hat{c}_2 \leq \hat{c}_2 \), because to the right of \( \hat{c}_2 \) \( \hat{v}_{\alpha'}(\hat{c}) \geq \hat{v}(\hat{c}) > \hat{v}(\hat{c}_2) \) for all \( \hat{c} > \hat{c}_2 \). However, looking at \( A(\hat{c}, \hat{v}(\hat{c}_2)) \) we notice it is decreasing in \( \alpha \), so \( A_{\alpha'}(\hat{c}_2, \hat{v}(\hat{c}_2)) > A(\hat{c}_2, \hat{v}(\hat{c}_2)) \geq 0 \). But the previous argument shows that \( A_{\alpha'}(\hat{c}_2, \hat{v}(\hat{c}_2)) = A_{\alpha'}(\hat{c}_2, \hat{v}(\hat{c}_2)) = 0 \) because \( \hat{c}_1 \) is the upper end of the first flat region of the optimal contract for \( \alpha' \).

This is a contradiction, so we cannot have \( \hat{v}(\hat{c}_1 - \delta) < \hat{v}(\hat{c}_1) = \hat{v}(\hat{c}_2) \).

Putting all of this together, we only have one flat region, \([0, \hat{c}_1] \), where the HJB equation holds as an inequality \( A(\hat{c}, \hat{v}(\hat{c})) > 0 \) (the inequality is strict because of Lemma 12 and the fact that \( \hat{c}_1 \) must be the first root), and a strictly increasing region \([\hat{c}_1, \hat{c}_n] \) where the HJB equation holds with equality, and \( \hat{v}(\hat{c}) = \hat{v}(\hat{c}_1) \) for all \( \hat{c} \leq \hat{c}_1 \) and \( \hat{v}'(\hat{c}_1) = 0 \).

4) Now we want to show that \( \hat{v}'(\hat{c}_1) > 0 \). First suppose the \( A(\hat{c}, \hat{v}(\hat{c})) \) is strictly positive below \( \hat{c}_1 \) and strictly negative above it, so that \( A(\hat{c}_1, \hat{v}_1) < 0 \) (we will show that this must indeed be the case below). Consider the first order ODE that results from fixing \( \sigma^e = 0 \) and \( \sigma^e = \frac{\sigma^{\hat{c}}}{\sigma^\gamma} \) in the HJB equation:

\[
\dot{c} - \sigma^e \frac{\partial \bar{A}(\hat{c}, \hat{v})}{\partial \bar{c}} + \hat{v}_f(\hat{c}) \left( \frac{\rho - \hat{c}_1^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} (\sigma^e)^2 \right) + \hat{v}'(\hat{c}) \left( \frac{r - \rho}{\gamma} + \frac{(\sigma^e)^2}{2} - \frac{\rho - \hat{c}_1^{1-\gamma}}{1-\gamma} \right) = 0
\]

and consider the solution with boundary condition \( \hat{v}_{\hat{c}_1}(\hat{c}_1) = \hat{v}(\hat{c}_1) \). Since we already know that \( A(\hat{c}_1, \hat{v}(\hat{c}_1)) = 0 \), we must have \( \hat{v}'(\hat{c}_1) = 0 \) because the term in parenthesis is \( \mu^e \) if \( \sigma^e = 0 \), and Lemma 14 shows this is strictly positive under these conditions. Furthermore, we must have \( \hat{v}'_{\hat{c}_1}(\hat{c}_1) \leq \hat{v}'(\hat{c}_1) \). To see this, if \( \hat{v}'_{\hat{c}_1}(\hat{c}_1) > \hat{v}'(\hat{c}_1) \geq 0 \), then \( \hat{v}'(\hat{c}_1 + \epsilon) > \hat{v}'(\hat{c}_1 + \epsilon) \), while \( \hat{v}_f(\hat{c}_1 + \epsilon) = \hat{v}(\hat{c}_1 + \epsilon) + o(\epsilon) \) (because both have first derivatives equal to zero) for some small \( \epsilon \). By continuity, it will still be the case that \( \mu^e > 0 \) for \( \hat{c}_1 + \epsilon \), so starting at \( \hat{c}_1 + \epsilon - \delta \) we will eventually get to \( \hat{c}_1 + \epsilon \). We can then solve the first order ODE backwards with boundary condition \( \hat{v}_f(\hat{c}_1 + \epsilon) = \hat{v}(\hat{c}_1 + \epsilon) \) and we will obtain a lower cost for \( \hat{c}_1 + \epsilon - \delta \), i.e. \( \hat{v}_f(\hat{c}_1 + \epsilon - \delta) < \hat{v}(\hat{c}_1 + \epsilon - \delta) \), because by continuity \( \hat{v}'(\hat{c}_1 + \epsilon) > \hat{v}'(\hat{c}_1 + \epsilon) \), the solution as well. This cannot be, so we must have \( \hat{v}'(\hat{c}_1) \leq \hat{v}'(\hat{c}_1) \).

Differentiating the first order ODE with respect to \( \hat{c} \) we obtain

\[
0 = \partial_c A(\hat{c}, \hat{v}_1)|_{\hat{c} = \hat{c}_1} + \hat{v}_f(\hat{c}_1) \hat{c}_1 \left( \frac{r - \rho}{\gamma} + \frac{(\sigma^e)^2}{2} - \frac{\rho - \hat{c}_1^{1-\gamma}}{1-\gamma} \right)
\]

where we have used \( \hat{v}_f(\hat{c}_1) = 0 \) and the envelope theorem to compute the derivative \( \partial_c A(\hat{c}, \hat{v}_1)|_{\hat{c} = \hat{c}_1} \). It follows that since \( A(\hat{c}_1, \hat{v}_1) < 0 \), then either \( \hat{v}'(\hat{c}_1) > 0 \) and \( \frac{r - \rho}{\gamma} + \frac{(\sigma^e)^2}{2} - \frac{\rho - \hat{c}_1^{1-\gamma}}{1-\gamma} > 0 \), or both are strictly negative. But we already know that \( \frac{r - \rho}{\gamma} + \frac{(\sigma^e)^2}{2} - \frac{\rho - \hat{c}_1^{1-\gamma}}{1-\gamma} > 0 \), so this leaves only \( \hat{v}'(\hat{c}_1) \geq \hat{v}_f(\hat{c}_1) > 0 \). It only remains to rule out \( A(\hat{c}_1, \hat{v}(\hat{c}_1)) = 0 \) which means \( A(\hat{c}, \hat{v}(\hat{c})) > 0 \) for all \( \hat{c} \neq \hat{c}_1 \) (from Lemma 12).
Since $\hat{A}(\hat{c}, \hat{v}) = 0$ is the HJB equation if $\hat{v}$ is flat, by a martingale verification argument as in Theorem 2 we obtain that the cost of any incentive compatible contract $C$ is strictly larger than $\hat{v}(\hat{c})$, unless it has $\hat{c}_t = \hat{c}_i$ always so that $A(\hat{c}_i, \hat{v}_i) = 0$ always and $\sigma^{x}_t = \frac{\sigma^{\hat{v}}_t}{\hat{v}_t}$. This requires $\mu^{\hat{v}} = \sigma^{\hat{v}} = 0$. But Lemma 14 shows that for $\sigma^{\hat{v}} = 0$ and $\sigma^{x} = \frac{\sigma^{\hat{v}}}{\hat{v}^{\hat{v}}}$. and $A(\hat{c}_i, \hat{v}_i) = 0$ we get $\mu^{\hat{v}} > 0$. Since the optimal contract must be incentive compatible and achieve cost $\hat{v}(\hat{c}_i), A_{c}(\hat{c}_i, v(\hat{c}_i))) = 0$ cannot be. So we have $\hat{v}''(\hat{c}_i) > 0$.

5) $\hat{v}''(\hat{c}_i) > 0$ implies $\sigma^{x}(\hat{c}_i) = 0$ and $\mu^{\hat{v}}(\hat{c}_i) > 0$. This in turn proves that $\hat{c}_t > \hat{c}_0 = \hat{c}_i$. From (44) we get $\sigma^{x}(\hat{c}) \leq 0$, and (43) implies $\sigma^{x}(\hat{c}) \geq 0$. It also implies that at $\hat{c}_i$, since $\hat{v}'(\hat{c}_i) = 0$, the choice of $\sigma^{x}$ maximizes (15).

**Theorem 2**

First let’s extend the function $\hat{v}(\hat{c})$ as described above, with $\hat{v}(\hat{c}) = \hat{v}_i \equiv \hat{v}(\hat{c}_i)$ for all $\hat{c} < \hat{c}_i$ (we always have $\hat{c} \in [0, \hat{c}_i]$). The HJB holds as an equality for $\hat{c} \geq \hat{c}_i$, but we need to check that it holds as an inequality for $\hat{c} < \hat{c}_i$. Using the definition of $A(\hat{c}, \hat{v})$ in (42), notice $A(\hat{c}_i, \hat{v}_i) = 0$, so $\hat{c}_i$ is a root of $A(\hat{c}; \hat{v}_i)$. If $\gamma \geq \frac{1}{2}$, Lemma 12 (with $\pi = 0$) says that it can have at most one root, and it’s positive for small $\hat{c}$, so $A(\hat{c}; \hat{v}_i) \geq 0$ for all $\hat{c} < \hat{c}_i$. For $\gamma < \frac{1}{2}$, it’s convex and can have at most two roots. Condition (18) guarantees that the derivative is negative, so $\hat{c}_i$ is the smaller root, and we also have $A(\hat{c}; \hat{v}_i) \geq 0$ for all $\hat{c} < \hat{c}_i$. Notice that we only need to check (18) if $\gamma < \frac{1}{2}$.

Consider any incentive compatible contract $C = (c, k)$ that delivers utility of at least $u_0$ to the agent, with associated state variables $x$ and $c$. Because $\hat{v}'(\hat{c}_i) = 0$ we can use Ito’s lemma\(^{15}\) and the HJB equation to obtain

$$e^{-rt} \hat{v}(\hat{c}_i t)x_{t} \geq \hat{v}(\hat{c}_0) x_0 - \int_0^t e^{-r(t-s)} \left( \hat{c}_i - k_i \alpha \right) x_s dt + \int_0^t e^{-r(t-s)} \hat{v}(\hat{c}_i) x_s \left( \frac{\hat{v}'(\hat{c}_i)}{\hat{v}(\hat{c}_i)} \hat{c}_i \sigma_i^c + \sigma_i^x \right) dZ_s$$

for the localizing sequence of stopping times $\{\tau^n\}_{n \in \mathbb{N}}$:

$$\tau^n = \inf \left\{ T \geq 0 : \int_0^T e^{-rt} \left( \frac{\hat{v}'(\hat{c}_i)}{\hat{v}(\hat{c}_i)} \hat{c}_i \sigma_i^c + \sigma_i^x \right)^2 dt \geq n \right\}$$

The stopped stochastic integrals are therefore martingales, so take expectations under $Q$ to obtain

$$\mathbb{E}_0^Q \left[ e^{-r\tau^n} \hat{v}(\hat{c}_i t)x_{t} \right] \geq \hat{v}(\hat{c}_0) x_0 - \mathbb{E}_0^Q \left[ \int_0^{\tau^n} e^{-r(t-s)} (c_t - k_t \alpha) dt \right]$$

(45)

Now we would like to take the limit $n \to \infty$, but we need to use the dominated convergence theorem. First,

$$\left| \int_0^{\tau^n} e^{-r(t-s)} (c_t - k_t \alpha) dt \right| \leq \int_0^{\tau^n} e^{-r(t-s)} |c_t - k_t \alpha| dt \leq \int_0^{\tau^n} e^{-r(t-s)} (|c_t| + |k_t \alpha|) dt$$

\(^{15}\)Notice $\hat{v}''$ is discontinuous at $\hat{c}_i$, but this doesn’t change Ito’s formula.
which is integrable because the contract is admissible. Second, for an admissible contract

\[ 0 \leq \lim_{n \to \infty} E_Q^0 \left[ e^{-\rho x^n} \hat{v}(\hat{c}_n) \right] \leq \lim_{n \to \infty} E_Q^0 \left[ e^{-\rho x^n} \hat{v}_h x_{x^n} \right] = 0 \]

To see why the last equality holds, notice that since \( \hat{v}_h x \) is the cheapest way of delivering utility to the agent without capital, the cost of consumption on the contract is

\[ \infty > E^Q \left[ \int_0^\infty e^{-\rho t} c_n dt \right] \geq E^Q \left[ \int_0^\infty e^{-\rho t} c_n dt + e^{-\rho x^n} \hat{v}_h x_{x^n} \right] \]

Taking the limit \( n \to \infty \) and using the monotone convergence theorem, we obtain \( 0 \leq \lim_{n \to \infty} E_Q^0 \left[ e^{-\rho x^n} \hat{v}_h x_{x^n} \right] \leq 0 \).

Upon taking the limit \( n \to \infty \) in (45), we obtain \( \tau \to \tau \) a.s. and therefore

\[ E_Q^0 \left[ \int_0^\infty e^{-\rho t} (c_t - k t \alpha) dt \right] \geq \hat{v}(\hat{c}_0) x_0 \]

Using \( \hat{v}(\hat{c}_0) \leq \hat{v}(\hat{c}_0) \) and \( x_0 \geq (1-\gamma)u_0 \), we obtain the first result.

For the second part, first let’s show that \( C^* \) is incentive compatible. We already know that for the HJB to have a solution with well defined policy functions it must be the case that \( \hat{B} \geq 0 \) and \( \hat{C} > 0 \), and therefore \( \sigma^{\hat{c}^*} \leq 0 \) and \( \sigma^{\hat{c}^*} \geq 0 \). If \( \hat{c}_t \in [\hat{c}_1, \hat{c}_h] \), because \( \sigma^{\hat{c}^*} \) is bounded, so is \( \sigma^{\hat{c}^*} \) and therefore so is \( \mu^{\hat{c}^*} \) and \( \mu^{\hat{c}^*} \). We can then use Lemma 3 and the fact that \( h_0 = 0 \) to show that

\[ U_0^{\hat{c}, a} \leq U_0^{\hat{c}, 0} = u_0 \]

for any feasible strategy \((\hat{c}, a)\), and therefore \( C^* \) is indeed incentive compatible. To show that the cost of the contract is \( \hat{v}_h x_0^* \), we can use the HJB. If \( \hat{c}_t \in [\hat{c}_1, \hat{c}_h] \) always, where the HJB holds, then the same argument as in the first part shows the desired result. Notice that with \( \hat{v}'(\hat{c}_t) = 0 \) and \( \hat{v}''(\hat{c}_t) > 0 \) we get that as \( \hat{c} \downarrow \hat{c}_1 \), \( \hat{B} \to 0 \) and \( \hat{C} \to \hat{C}_1 > 0 \), so \( \sigma^{\hat{c}^*} \to 0 \), \( \sigma^{\hat{c}^*} \to \frac{\hat{c}_1}{\beta \gamma} \), and \( A(\hat{c}_t; \hat{v}_t) = 0 \). From the first part we know \( \hat{v}(\hat{c}_t) \) is weakly below the true cost function, which is also bounded above by \( \hat{v}_h \), so there is a finite cost function and \( \hat{v}_t \) is weakly below it, and therefore \( \hat{v}_t \leq \hat{v}_{\hat{p}} \). Lemma 14 shows that under these conditions \( \mu^{\hat{c}_t}(\hat{c}_t) > 0 \) and therefore \( \hat{c}_t \in [\hat{c}_1, \hat{c}_h] \). Notice that the candidate contract does indeed deliver utility \( u_0 \) to the agent. To see this let \( U^* = \left( \frac{\hat{c}}{\hat{c}} \right)^{1-\gamma} \), so using the law of motion of \( x^* \), (9), we get

\[ U_0^{\hat{c}, a} = E \left[ \int_0^{\tau^n} e^{-\rho t} \frac{c_{t-1}^{1-\gamma}}{1-\gamma} dt + e^{-\rho x^n} U_0^{\hat{c}, a} \right] \]

with some sequence \( \tau^n \to \infty \) a.s. Use the monotone convergence theorem and notice that

\[ \lim_{n \to \infty} E \left[ e^{-\rho x^n} U_0^{\hat{c}, a} \right] = 0 \]

because \( \rho - (1-\gamma)(\mu^{\hat{c}^*} - \frac{\hat{c}_1}{2}(\sigma^{\hat{c}^*})^2) = \hat{c}_1^{1-\gamma} \geq \min \{ \hat{c}_1^{1-\gamma}, \hat{c}_h^{1-\gamma} \} > 0 \). We then get that \( U_0^{\hat{c}, a} = U_0^{\hat{c}, 0} = u_0 \). This completes the proof.
Lemma 4

We know from Lemma 14 that \( \hat{c}^*_0 \in [\hat{c}_i, \hat{c}_h] \) and recall that \( \hat{c}_i > 0 \). Then an upper bounded \( \mu^{x^*} < r \) implies a bounded \( 0 \leq \sigma^{x^*} \leq \sigma_X \). Then

\[
\mathbb{E}^Q \left[ \int_0^\infty e^{-rt} \left( |c^*_t| + |k^*_t| \right) dt \right] \leq 2 \max \left\{ \hat{c}_h, \frac{\sigma X \hat{c}^*_0}{\phi \beta} \right\} \mathbb{E}^Q \left[ \int_0^\infty e^{-rt} x^*_t dt \right] = \infty
\]

where the last inequality follows from \( \mu^{x^*} < r \). Let \( U^*_0 = \left( \frac{c^*_t}{\sigma} \right)^{\frac{1}{1-\gamma}} \), so using the law of motion of \( x^* \), (9), we get

\[
U^*_0 = \mathbb{E} \left[ \int_0^{\tau_n} e^{-\mu t} \frac{c^*_{t-\gamma}}{1-\gamma} dt + e^{-\rho \tau_n} U^*_\infty \right]
\]

with \( \tau_n \to \infty \) a.s. Use the monotone convergence theorem and notice that

\[
\lim_{n \to \infty} \mathbb{E} \left[ e^{-\rho \tau_n} U^*_\infty \right] = 0
\]

because \( \rho - (1-\gamma)(\mu^{x^*} - \frac{2}{\gamma}(\sigma^{x^*})^2) = c^{1-\gamma} \geq \min \{ \hat{c}^{1-\gamma}, c_h^{1-\gamma} \} > 0 \). We then get that \( U^*_0 = U^*_0 = u_0 \).

We conclude that the contract is indeed admissible.

**Lemma 12.** Define the function

\[
A(\hat{c}, \hat{v}) \equiv \hat{c} - r \hat{v} - \frac{1}{2} \left( \frac{\hat{v} \alpha}{\phi \beta} \right)^2 + \hat{v} \left( \rho - \frac{c^{1-\gamma}}{1-\gamma} - \frac{\pi^2}{2} \right)
\]

For any \( \hat{v} \in (0, \hat{v}_h) \), we have \( A(\hat{c}; \hat{v}) > 0 \) for \( \hat{c} \) near 0. In addition, if \( \gamma \geq \frac{1}{2} \) then \( A(\hat{c}; \hat{v}) \) has at most one root in \([0, \hat{c}_h]\), where \( \hat{c}_h = \left( \rho - \frac{c^{1-\gamma}}{1-\gamma} - \frac{\pi^2}{2} \right)^{\frac{1}{1-\gamma}} \). If instead \( \gamma < \frac{1}{2} \), \( A(\hat{c}; \hat{v}) \) is convex and has at most two roots.

**Proof.** First, for \( \gamma < 1 \) \( \lim_{\hat{v} \to 0} A(\hat{c}; \hat{v}) = \hat{v} \frac{\rho - c^{1-\gamma} - \frac{\pi^2}{2}}{1-\gamma} \geq 0 \). For \( \gamma > 1 \), \( \lim_{\hat{v} \to 0} A(\hat{c}; \hat{v}) = \infty \).

For \( \gamma \geq 1/2 \), to show that \( A(\hat{c}; \hat{v}) \) has at most one root in \([0, \hat{c}_h]\) for any \( \hat{c} \in (0, \hat{c}_h) \), we will show that \( A'(\hat{c}; \hat{v}) = 0 \implies A(\hat{c}; \hat{v}) > 0 \) for all \( \hat{c} \leq \hat{c}_h \). Compute the derivative (dropping the arguments to avoid clutter)

\[
A'_c = 1 - \hat{c} \hat{e}^{\hat{c}^*} - \hat{e}^{2\hat{c}^{1-\gamma}} \left( \frac{\alpha}{\phi \beta} \right)^2 \hat{v}
\]

So

\[
A'_c = 0 \implies \hat{c} - \hat{c} \hat{e}^{\hat{c}^*} = \hat{e}^{2\hat{c}^{1-\gamma}} \left( \frac{\alpha}{\phi \beta} \right)^2 \hat{v}
\]

Plug this into the formula for \( A \) to get

\[
A = \hat{c} - r \hat{v} + \hat{v} \rho - \frac{c^{1-\gamma}}{1-\gamma} - \frac{\hat{e}^{2\hat{c}^{1-\gamma}}}{2 \hat{v} \gamma} \left( \frac{\alpha}{\phi \beta} \right)^2 - \hat{v} \frac{\pi^2}{2} \gamma
\]

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and the inequality is strict if

\[
B(\hat{c}, \hat{v}) = \frac{2\gamma - 1}{2\gamma} \hat{c} + \frac{1 - 3\gamma}{2\gamma} \hat{v}^{1-\gamma} + \hat{v}^{\rho - r(1 - \gamma)} - \hat{v} \frac{\pi^2}{2 \gamma} \equiv B(\hat{c}, \hat{v})
\]

is convex in \(\hat{c}\) because \(1 - 3\gamma < 0\) for \(\gamma \geq \frac{1}{2}\), so it’s minimized in \(\hat{c}\) when \(B'_{\hat{c}} = 0\):

\[
\frac{2\gamma - 1}{3\gamma - 1} = \hat{v}^{\hat{c} - \gamma}
\]

and it is strictly decreasing before this point. Now we have two possible cases:

CASE 1: The minimum of \(B\) is achieved for \(\hat{c} \geq \hat{c}_m\), so in the relevant range, it is minimized at \(\hat{c}_m\). So let’s plug in \(\hat{c}_m\) into \(B(\hat{c}, \hat{v})\):

\[
2\gamma B(\hat{c}_m, \hat{v}) = (2\gamma - 1) \hat{c}_m + \frac{\hat{v}}{1 - \gamma} ((\rho - r(1 - \gamma))2\gamma + (1 - 3\gamma)\hat{c}_m^{1 - \gamma}) - \hat{v} \pi^2
\]

\[
= (2\gamma - 1) \hat{c}_m + \frac{\hat{v}}{1 - \gamma} \frac{(\rho - r(1 - \gamma))}{\gamma} (2\gamma^2 + (1 - 3\gamma)) - \frac{\hat{v}}{1 - \gamma} (1 - \gamma) \frac{1}{2}(1 - \gamma) \left( \frac{\pi^2}{2} - \hat{v} \pi^2 \right)
\]

\[
= (2\gamma - 1) \hat{c}_m + \hat{v} \left( \frac{(\rho - r(1 - \gamma))}{\gamma} \right) (1 - 2\gamma) - \frac{1}{2} (1 - \gamma) \left( \frac{\pi^2}{2} \hat{v}(1 - 2\gamma) \right)
\]

\[
(2\gamma - 1) \left( \hat{c}_m - \hat{v} \left( \frac{(\rho - r(1 - \gamma))}{\gamma} \right) \right) \geq 0
\]

and the inequality is strict if \(\hat{v} < \hat{v}_m\). So \(A(\hat{c}, \hat{v}) = B(\hat{c}, \hat{v}) > B(\hat{c}_m, \hat{v}) \geq 0\) for any \(\hat{c} < \hat{c}_m\).

CASE 2: If the minimum is achieved for \(\hat{c}_m \in [0, \hat{c}_m]\) it must be that \(\gamma > 1/2\). Then plugging in (46) into \(B\):

\[
B(\hat{c}, \hat{v}) \geq \frac{2\gamma - 1}{2\gamma} \hat{c}_m - \frac{2\gamma - 1}{2\gamma} \hat{c}_m + \hat{v} \frac{\rho - r(1 - \gamma)}{1 - \gamma} - \hat{v} \frac{\pi^2}{2 \gamma}
\]

\[
= \frac{1 - 2\gamma}{2} \hat{c}_m + \hat{v} \frac{\rho - r(1 - \gamma)}{1 - \gamma} - \frac{\hat{v} \pi^2}{2 \gamma}
\]

\[
= \frac{1 - 2\gamma}{2} \hat{c}_m + \frac{2\gamma - 1}{3\gamma - 1} \hat{c}_m \left( \frac{\rho - r(1 - \gamma)}{1 - \gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right)
\]

and dividing throughout by \(2\gamma - 1 > 0\)

\[
= \frac{1}{2} \hat{c}_m + \frac{1}{3\gamma - 1} \left( \frac{\rho - r(1 - \gamma)}{1 - \gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right)
\]

and multiplying by \(\hat{c}_m^{1-\gamma} > 0\) and using \(\frac{\mu_m}{\gamma} < \frac{\hat{c}_m^{1-\gamma}}{\gamma}\):

\[
> \frac{1}{2} \hat{c}_m - \frac{1}{3\gamma - 1} \left( \frac{\rho - r(1 - \gamma)}{1 - \gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right)
\]

\[
= \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1}{2} (1 - \gamma) \left( \frac{\pi^2}{\gamma} \right)^2 \right) \left( \frac{\gamma}{2 \gamma - 1} + \frac{1}{(3\gamma - 1)(1 - \gamma)} \right)
\]

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for a localizing sequence \( \{ \tau_n \}_{n \in \mathbb{N}} \) with \( \tau_n \to \infty \) a.s. We will show that the rhs is non-positive. First write the integral part

\[
E_t^\gamma \left[ \int_t^{\tau_n} e^{-\rho u} \frac{c_0^{1-\gamma}}{1-\gamma} du + \int_t^{\tau_n} d \left( e^{-\rho u} F_u U_{u,0} \right) \right] = E_t^\gamma \left[ \int_t^{\tau_n} e^{-\rho u} (1-\gamma) U_{u,0}^0 Y_u du \right]
\]

with

\[
Y_1 = \frac{\hat{c}_t^{1-\gamma}}{1-\gamma} + \frac{-\rho F_t + \rho F_t - \hat{c}_t^{1-\gamma} F_t}{1-\gamma} + \frac{F_{\hat{h},t} (1-\gamma)^2 \hat{h}_t}{(1-\gamma)^2} + \frac{\gamma (\sigma_t^2 \hat{c}_t^2 F_{\hat{h},t} - 2\sigma_t^2 \sigma_t \hat{c}_t \hat{h}_t + \hat{h}_t^2 F_{\hat{h},t}) + (\sigma_t^2 \hat{c}_t^2 F_{\hat{h},t} - 2\sigma_t^2 \sigma_t \hat{c}_t \hat{h}_t + \hat{h}_t^2 F_{\hat{h},t})}{2(1-\gamma)}
\]

where

\[
F_{\hat{c},t} = \gamma (\gamma - 1) \hat{h}_t \hat{c}_t^{\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{\gamma} \right)^{-\gamma}
\]
\[ F_{h,t} = (1 - \gamma) \hat{c}_t^{\gamma} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} \]
\[ F_{c,t} = -\gamma(\gamma - 1) \hat{h}_t \hat{c}_t^{-\gamma} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} - \gamma^2 (\gamma - 1) \hat{h}_t \hat{c}_t^{-\gamma - 2} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma - 1} \left( 1 - \hat{h}_t (\gamma - 1) \hat{c}_t^{-\gamma} \right) \]
\[ F_{\rho,t} = \gamma(\gamma - 1) \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} - \gamma^2 (\gamma - 1) \hat{h}_t \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma - 1} \]
\[ F_{h,t} = -\gamma \hat{c}_t^{-\gamma} (1 - \gamma) \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma - 1} \]

We know \( e^{-rt}(1 - \gamma)U_t^{c,0} > 0 \), and we will show that \( Y_t \leq 0 \). To do this, we will split the expression into three parts, \( Y_t = A_t + B_t + C_t \), and show each one is non-positive. First take the terms multiplying \( s_t \)

\[ A_t = \frac{1}{(1 - \gamma)} \left( -(1 - \gamma) \sigma_t^{\gamma} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{1 - \gamma} \right) \]
\[ -\gamma(\gamma - 1) \hat{h}_t \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} \sigma_t^{\gamma} \hat{c}_t + (\sigma_t^{\gamma} \hat{h}_t + \hat{k}_t \phi \beta)(1 - \gamma) \hat{c}_t^{-\gamma} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} \]
\[ A_t = \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} \left( -\sigma_t^{\gamma} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right) + \gamma \hat{h}_t \hat{c}_t^{-\gamma - 1} \sigma_t^{\gamma} \hat{c}_t + (\sigma_t^{\gamma} \hat{h}_t + \hat{k}_t \phi \beta) \hat{c}_t^{-\gamma} \right) \]
\[ A_t \leq \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} \gamma \hat{h}_t \hat{c}_t^{-\gamma - 1} \sigma_t^{\gamma} \leq 0 \]

where we have used \( \sigma_t^{\gamma} \geq \hat{k}_t \phi \beta \hat{c}_t^{-\gamma} \) in the first inequality, and \( \hat{h}_t \geq 0 \) and \( \sigma_t^{\gamma} \leq 0 \) in the second. Here is the only place where \( \sigma_t^{\gamma} \leq 0 \) is used.

Second, all the remaining terms that have \( \sigma_t^{\gamma} \) or \( \sigma_t^{\gamma} \) are

\[ B_t = \frac{F_{c,t}}{(1 - \gamma)} \hat{c}_t \left( \frac{1 + \gamma (\sigma_t^{\gamma} + \sigma_t^{\gamma})^2 - \gamma}{2} (\sigma_t^{\gamma})^2 - \sigma_t^{\gamma} \right) + \frac{F_{c,t}}{(1 - \gamma)} (1 - \gamma) \sigma_t^{\gamma} \sigma_t^{\gamma} \hat{c}_t \]
\[ + \frac{F_{h,t}}{(1 - \gamma)} \hat{h}_t \left( -\gamma \left( (\sigma_t^{\gamma})^2 + (\sigma_t^{\gamma})^2 \right) - \frac{F_{h,t}}{(1 - \gamma)} (1 - \gamma) (\sigma_t^{\gamma})^2 \hat{h}_t + \frac{(\sigma_t^{\gamma})^2 \hat{c}_t^2 (\sigma_t^{\gamma})^2 - 2 \sigma_t^{\gamma} \sigma_t^{\gamma} \hat{c}_t \hat{h}_t F_{c,t} + (\sigma_t^{\gamma})^2 \hat{h}_t^2 F_{h,t}}{2(1 - \gamma)} \right) \]
\[ + \frac{F_{c,t}}{(1 - \gamma)} \hat{c}_t \left( \frac{1 + \gamma (\sigma_t^{\gamma} + \sigma_t^{\gamma})^2 - \gamma}{2} (\sigma_t^{\gamma})^2 - \gamma \sigma_t^{\gamma} \sigma_t^{\gamma} \right) + \frac{F_{h,t}}{(1 - \gamma)} \hat{h}_t \left( -\frac{\gamma}{2} (\sigma_t^{\gamma})^2 + \gamma (\sigma_t^{\gamma})^2 \right) \]
\[ + \frac{(\sigma_t^{\gamma})^2 \hat{c}_t^2 F_{c,t} + 2 \sigma_t^{\gamma} \sigma_t^{\gamma} \hat{c}_t \hat{h}_t F_{c,t} + (\sigma_t^{\gamma})^2 \hat{h}_t^2 F_{h,t}}{2(1 - \gamma)} \]
\[ B_t = \frac{(\sigma_t^{\gamma})^2}{2} \left( \frac{F_{c,t}}{1 - \gamma} \hat{c}_t + \frac{F_{h,t}}{1 - \gamma} \hat{h}_t \gamma + \frac{F_{h,h,t}}{1 - \gamma} \hat{h}_t^2 \right) + \frac{(\sigma_t^{\gamma})^2}{2} \left( \frac{F_{c,t}}{1 - \gamma} (1 + \gamma) \hat{c}_t + \frac{F_{h,t}}{1 - \gamma} \hat{c}_t^2 \right) \]
\[ - \sigma_t^{\gamma} \sigma_t^{\gamma} \hat{c}_t \left( \frac{F_{c,t}}{1 - \gamma} \hat{c}_t + \frac{F_{h,t}}{1 - \gamma} \hat{h}_t \right) \]

which plugging in the \( F \)’s gets us

\[ B_t = \frac{(\sigma_t^{\gamma})^2}{2} \left( \frac{\gamma (\gamma - 1) \hat{h}_t \hat{c}_t^{-\gamma - 1} (1 + \hat{h}_t \hat{c}_t^{-\gamma})^{-\gamma}}{1 - \gamma} \right) \hat{c}_t \]

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\[
\begin{align*}
+ \frac{(1 - \gamma) c_t^{-\gamma} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma}}{1 - \gamma} \frac{-\gamma \hat{c}_t^{-2\gamma} (1 - \gamma) \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma - 1}}{1 - \gamma} \hat{h}_t^2 & \\
+ \frac{(\sigma_t^2)^2}{2} \left(\frac{\gamma(\gamma - 1) \hat{h}_t \hat{c}_t^{-\gamma - 1} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma}}{1 - \gamma} \right) (1 + \gamma) \hat{c}_t + \\
- \frac{-\gamma(\gamma - 1) \hat{h}_t \hat{c}_t^{-2\gamma} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma} - \gamma^2 (\gamma - 1) \hat{h}_t \hat{c}_t^{-2\gamma - 2} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma - 1} \left(1 - \hat{h}_t (\gamma - 1) \hat{c}_t^{-\gamma}\right)}{1 - \gamma} \hat{c}_t \hat{h}_t & \\
- \frac{-\sigma_t^2 \sigma_t^{\gamma}}{2} \left(\frac{-\gamma(\gamma - 1) \hat{h}_t \hat{c}_t^{-\gamma - 1} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma}}{1 - \gamma} \right) \hat{c}_t + \\
- \frac{\gamma(\gamma - 1) \hat{h}_t \hat{c}_t^{-\gamma - 1} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma} - \gamma^2 (\gamma - 1) \hat{h}_t \hat{c}_t^{-2\gamma - 1} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma - 1}}{1 - \gamma} \hat{c}_t \hat{h}_t
\end{align*}
\]

which simplifies to:

\[
B_t = \frac{(\sigma_t^2)^2}{2} \left(\frac{-\gamma \hat{h}_t \hat{c}_t^{-\gamma - 1} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma} \hat{c}_t + \hat{c}_t^{-\gamma} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma} \hat{h}_t \gamma - \gamma \hat{c}_t^{-2\gamma} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma - 1} \hat{h}_t^2}{1 - \gamma} \right)
\]

\[
+ \frac{(\sigma_t^2)^2}{2} \left(\frac{-\gamma \hat{h}_t \hat{c}_t^{-\gamma - 1} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma} (1 + \gamma) \hat{c}_t + \gamma \hat{h}_t \hat{c}_t^{-2\gamma} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma} \hat{c}_t^2}{1 - \gamma} \right)
\]

\[
+ \gamma^2 \hat{h}_t \hat{c}_t^{-\gamma - 2} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma - 1} \left(1 - \hat{h}_t (\gamma - 1) \hat{c}_t^{-\gamma}\right) \hat{c}_t^2
\]

\[
- \frac{-\sigma_t^2 \sigma_t^{\gamma}}{2} \left(\frac{\gamma \hat{h}_t \hat{c}_t^{-\gamma - 1} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma} \hat{c}_t - \gamma \hat{c}_t^{-\gamma - 1} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma} \hat{c}_t \hat{h}_t + \gamma^2 \hat{h}_t \hat{c}_t^{-2\gamma - 1} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma - 1} \hat{c}_t \hat{h}_t}{1 - \gamma} \right)
\]

and

\[
B_t = \frac{-\left(\sigma_t^2\right)^2}{2} \left(\frac{\left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma - 1} \hat{c}_t^{-\gamma - 1} \gamma \hat{c}_t^{-\gamma - 1} \hat{h}_t^2}{1 - \gamma} \right)
\]

\[
- \frac{-\left(\sigma_t^2\right)^2}{2} \left(\frac{\left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma - 1} \hat{c}_t^{-\gamma - 1} \hat{h}_t^2}{1 - \gamma} \right) \hat{c}_t \hat{c}_t^{\gamma - 1} \hat{c}_t^{\gamma - 1} \hat{h}_t^2
\]

\[
- \frac{-\sigma_t^2 \sigma_t^{\gamma}}{2} \left(\frac{\left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma - 1} \hat{c}_t^{-\gamma - 1} \hat{h}_t^2}{1 - \gamma} \right) \hat{c}_t \hat{c}_t^{\gamma - 1} \hat{h}_t^2
\]

\[
B_t = \frac{-\left(\sigma_t^2 + \sigma_t^{\gamma}\right)^2}{2} \gamma \hat{h}_t \hat{c}_t^{-\gamma} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma - 1} \hat{c}_t^{-\gamma - 1} \leq 0
\]

And finally all the remaining terms that don't involve \(\sigma_t^2\) or \(\sigma_t^{\gamma}\) are

\[
C_t = \frac{\hat{c}_t^{-\gamma}}{1 - \gamma} - \frac{\hat{c}_t^{-\gamma} \hat{F}_t}{(1 - \gamma)} + \frac{\gamma(\gamma - 1) \hat{h}_t \hat{c}_t^{-\gamma - 1} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma}}{(1 - \gamma)} \hat{c}_t \left(\frac{r - \rho}{\gamma} - \frac{\rho - \hat{c}_t^{-\gamma}}{1 - \gamma}\right)
\]

\[
+ \frac{(1 - \gamma) \hat{c}_t^{-\gamma} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma}}{(1 - \gamma)} \hat{h}_t \left(\frac{\hat{c}_t - \hat{c}_t}{\hat{h}_t} - \frac{\rho - \hat{c}_t^{-\gamma}}{1 - \gamma}\right)
\]

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which is maximized for \( \hat{c}_t = \hat{c}_t + \hat{h}_t \hat{c}_t^{1-\gamma} \). Plugging this in yields

\[
C_t \leq \frac{\hat{c}_t^{1-\gamma} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{1-\gamma}}{1 - \gamma} - \frac{\hat{c}_t^{1-\gamma}}{(1 - \gamma)} + \frac{\gamma (\gamma - 1) \hat{h}_t \hat{c}_t^{-\gamma-1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma}}{1 - \gamma} \hat{c}_t \left( \frac{r - \rho}{\gamma} \frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} \right)
\]

\[
+ \frac{(1 - \gamma) \hat{c}_t^{-\gamma} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma}}{(1 - \gamma)} \hat{h}_t \left( r + \frac{\hat{c}_t - \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right) \hat{c}_t}{h_t} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} \right)
\]

\[
C_t \leq -\gamma \hat{h}_t \hat{c}_t^{-\gamma} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} \hat{h}_t \left( r + \frac{\hat{c}_t - \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right) \hat{c}_t}{h_t} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} \right)
\]

\[
C_t \leq \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right) \hat{c}_t^{-\gamma} \hat{h}_t \left\{ \rho - r + \frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} \right\} \gamma + r - \hat{c}_t^{1-\gamma} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} \}
\]

\[
C_t \leq \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right) \hat{c}_t^{-\gamma} \hat{h}_t \left\{ \rho - \left( \rho - \hat{c}_t^{1-\gamma} \right) - \hat{c}_t^{1-\gamma} \right\} = 0
\]

Since \( A_u, B_u, C_u \leq 0 \) we conclude that \( Y_t \leq 0 \).

Now for the last term, since the agent’s strategy \((\hat{c}, a)\) is feasible, we have that

\[
\lim_{n \to \infty} \mathbb{E}_t^n \left[ e^{-\rho t} U_{x_n}^c \right] = 0
\]

For \( \gamma < 1 \) we use Fatou’s lemma to show

\[
\lim_{n \to \infty} \mathbb{E}_t^n \left[ e^{-\rho t} F_t U_{x_n}^{c,a} \right] \geq 0
\]

As a result, when we take \( n \to \infty \) we get

\[
\mathbb{E}_t \left[ e^{-\rho t} \left( U_{x,a}^c - F(\hat{h}_t, \hat{c}_t) U_{x,a}^{c,a} \right) \right] \leq 0.
\]

For \( \gamma > 1 \), if \( \lim_{n \to \infty} \mathbb{E}_t^n \left[ e^{-\rho t} F_t U_{x_n}^{c,a} \right] = 0 \) for any feasible strategy \((\hat{c}, a)\), then we are done. To show this, let \( N_t = x_t + \hat{h}_t \hat{c}_t^{-\gamma} \), so that \( F_t U_{x,a}^{c,a} = \frac{N_t^{1-\gamma}}{\gamma} \). The law of motion of \( N \) satisfies

\[
dN_t = (\lambda_1 N_t - \lambda_2 \hat{c}_t) dt + \sigma_t N_t dZ_t^a
\]

where \( \lambda_2 = \hat{c}_t^{-\gamma} > 0 \), and \( \lambda_1 = \hat{\mu} + r + 2 \hat{c}_t^{1-\gamma} + \gamma |\hat{\mu}| + \frac{3}{2} \gamma (1 + \gamma)(\hat{\sigma}^2)^2 \). Here’s where we use the assumption that \( \mu^x, \mu^\hat{c} \), and \( \sigma^\hat{c} \) are bounded and \( \hat{c} \leq \hat{c}_t \) is uniformly bounded away from 0. Notice that stealing only reduces the drift of \( N_t \), since the change in the drift of \( x \) and \( h \) cancel out, and it increases the drift of \( \hat{c} \). Since \( N_t > 0 \) always, Lemma 13 and its corollary ensure the desired limit.

**Lemma 5**

Start with the optimal contract \( C = (c, k) \), with associated processes \( x, \hat{c} \), and \( \hat{k} \). Define \( n_t = \hat{v}_t x_t \), so we have \( k_t / n_t = \hat{k}_t \hat{v}_t^{-1}, c_t / n_t = \hat{c}_t \hat{v}_t^{-1} \). The net worth \( n_t \) is the principal’s continuation cost, so it satisfies
the HJB equation (13). This immediately yields
\[ \mu_t n_t = r n_t + k_t \alpha - c_t = n_t (r + \frac{k_t}{n_t} \alpha - \frac{c_t}{n_t}) \]
We also have from Ito’s lemma
\[ \sigma_t^2 = \sigma_t^2 + \frac{\eta_t^2}{\eta_t} \dot{\epsilon}_t \sigma_t^2 = \frac{\dot{\epsilon}_t}{\eta_t} (\eta_0 \phi \beta) + \frac{\dot{\epsilon}_t}{\eta_t} \sigma_t^2 \]
\[ = \left( \frac{\dot{\epsilon}_t}{\eta_t} (\eta_0 \phi \beta) \right) k_t \beta \]
\[ = \left( \frac{\dot{\epsilon}_t}{\eta_t} (\eta_0 \phi \beta) \right) \frac{k_t}{n_t} \beta \]
and using the definition of \( \tilde{\phi}_t \) in (22), we obtain the dynamic budget constraint (21). The continuation utility of the agent under good behavior is
\[ U_{c, 0} = x_1 - \gamma \]
so
\[ U_{c, 0} = (n_t \eta_t)^{1-\gamma} \]
If instead we specify the \( k_t/n_t, c_t/n_t, \) and \( \tilde{\phi}_t \) as functions of the history of \( R, \) and set \( n_0 = \tilde{c}_0 x_0, \) then (21) is a linear SDE with a unique solution. Setting \( c_t = \frac{c_t}{n_t} \times n_t \) and \( k_t = \frac{k_t}{n_t} \times n_t \) we recover the optimal contract.
The inequality \( \tilde{\phi}_t < \phi \) follows from Lemma 17, which establishes \( \tilde{\epsilon}_t < \tilde{\epsilon}_t \), and \( \tilde{\epsilon}_t \geq 0 \) and \( \sigma_t^2 \leq 0 \). To see that the leverage constraint is not binding at \( t = 0 \), consider the agent’s choice of capital under the dynamic budget constraint (21), conditional on following the optimal contract in the future. Since \( \tilde{\epsilon}_t = 0 \), the optimal choice of \( k/n \) is given by
\[ k/n = \frac{1}{\gamma (\phi \beta)^2} \]
Theorem 1 shows that the optimal contract has at \( t = 0 \)
\[ \sigma^* = \frac{1}{\gamma (\tilde{\phi} \beta)^2} = \frac{1}{\gamma \tilde{\phi} \beta} \]
where we used that \( \tilde{\phi} = \tilde{v} e^{-\gamma} \phi \) because \( \tilde{v}' = 0 \) at \( t = 0 \). Using \( \sigma^* = k \tilde{c}^{-\gamma} \phi \beta = \frac{k}{n} \tilde{v} \phi \beta \) we obtain
\[ k/n = \frac{\sigma^*}{\tilde{v} \phi \beta} = \frac{1}{\gamma (\phi \beta)^2} \]
which coincides with the unconstrained portfolio choice of capital.

**Lemma 6**

The optimal portfolio plan generates a stationary contract. We can compute \( \tilde{c}_p = \frac{c}{n_p}/\omega_p \) from (23) and the utility (25) and the cost \( \tilde{v}_p = \omega_p^{-1} \). Lemma 16 ensures that \( \tilde{c}_p \in [\tilde{c}_*, \tilde{c}_h] \) and therefore from Lemma 7 we know we have an incentive compatible contract that delivers utility \( u_p = \frac{(\omega_p n_0)^{1-\gamma}}{1-\gamma} \) to the agent.
Lemma 7

We use the HJB equation (13) with $\hat{\mu} = \sigma^x = 0$. Lemma 15 ensures that $\hat{v}(\hat{c}) > 0$ for all $\hat{c} \in (\hat{c}_*, \hat{c}_h]$. The same argument as in Theorem 2 shows that $\hat{v}_r(\hat{c})$ from (29) is the cost corresponding to the stationary contract with $\hat{c}$ and $\sigma^x$ given by (27), as long as the contract is indeed admissible and delivers utility $u_0$ to the agent. We can check that $\mu^x < r$ for the stationary contract if and only if $\hat{c} > \hat{c}_*$, where $\hat{c}_*$ is given by (28). In this case, since $\mu^x < r$ arguing as in the proof of Lemma 4 we can show that the stationary contract is admissible and delivers utility $u_0$ to the agent if and only if $\hat{c} > \hat{c}_*$. Since the contract satisfies (9), (10), and (11) by construction, Theorem 3 then ensures that it is incentive compatible. This completes the proof.

Lemma 8

Lemma 16 establishes that $\hat{c}_r^{\text{min}} > \hat{c}_p$ in part 1), and therefore $\hat{v}_r^{\text{min}} < (\hat{c}_r^{\text{min}})^\gamma$ by part 2). We then obtain the inequality in equation (30). To see that the leverage constraint is binding, notice that $\sigma^x \equiv \alpha \gamma \phi^{\text{min}} \beta < \alpha \gamma \tilde{\phi}^{\text{min}} \beta$ where the first inequality uses $\hat{c}_r^{\text{min}} > \hat{c}_p$, and the second one $\tilde{\phi}^{\text{min}} < \phi$. Since $\sigma^x = \alpha \gamma \phi^{\text{min}} \beta$ we get

$$\left(\frac{k}{n}\right)^{\text{min}}_r < \frac{\alpha}{\gamma (\phi^{\text{min}} \beta)^2}$$

which is the $k/n$ the agent would choose subject to an equity constraint $\tilde{\phi}^{\text{min}}$. Finally, to see that the payout policy is binding, use $\hat{v}_r^{\text{min}} < (\hat{c}_r^{\text{min}})^\gamma$ to get $c_r^{\text{min}}(\hat{v}_r^{\text{min}})^{-1} > (\hat{v}_r^{\text{min}})^{1-\gamma} = (\omega_r^{\text{min}})^{1-\gamma}$. This completes the proof.

Lemma 9

For any given $c \in [\hat{c}_l, \hat{c}_h]$, we have $\hat{v}(\hat{c}) \leq \hat{v}_r(\hat{c})$, because stationary contracts are incentive compatible and always available as continuation contracts. Furthermore, $\hat{v}'(\hat{c}) \geq 0$ and $\sigma' \hat{c} \leq 0$. This completes the proof.

Lemma 10

From the FOC for $\hat{c}$ (32) we obtain $\hat{v}_n < \hat{c}^x_n$, and therefore $\hat{c}_n \hat{v}_n^{-1} > \hat{c}_n^{x-1} = (\omega_n)^{x-1}$. This completes the proof.

$$k/n = \frac{\alpha}{\gamma \phi_n \beta^2}$$
which is precisely what the agent would choose on his own. This completes the proof.

**Lemma 13.** Assume there are some constants $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, and $\lambda_4 > 0$ such that for any feasible strategy $(\tilde{c}, a, z, \tilde{z})$ there is a non-negative process $N$ with

\[
dN_t \leq (((\lambda_1 + \lambda_2 \sigma^N_t + \lambda_3 \tilde{\sigma}^N_t)N_t - \lambda_4 \tilde{c}_t)dt + \sigma^N_t N_t d\tilde{Z}_t^a + \tilde{\sigma}^N_t N_t d\tilde{Z}_t
\]

for some processes $\sigma^N$ and $\tilde{\sigma}^N$, which can depend on the strategy. Then there is a constant $\lambda_5 > 0$ such that for any $T > 0$ and any feasible strategy $(\tilde{c}, a, z, \tilde{z})$

\[
E^a \left[ \int_0^T e^{-\rho t} \tilde{c}_t^{1-\gamma} dt \right] \leq \frac{\lambda_5 N_0^{1-\gamma}}{1 - \gamma}
\]

**Proof.** First define $n_t$ as the solution to the SDE

\[
dn_t = \left((\lambda_1 + \lambda_2 \sigma^N_t + \lambda_3 \tilde{\sigma}^N_t)n_t - \tilde{c}_t\right) dt + \sigma^N_t n_t d\tilde{Z}_t^a + \tilde{\sigma}^N_t n_t d\tilde{Z}_t
\]

and $n_0 = \frac{N_0}{\lambda_4}$. It follows that $n_t \geq \frac{N_0}{\lambda_4} \geq 0$. Now define $\zeta$ as

\[
\frac{d\zeta_t}{\zeta_t} = -\lambda_1 dt - \lambda_2 d\tilde{Z}_t^a - \lambda_3 d\tilde{Z}_t, \quad \zeta_0 = 1
\]

and

\[
\tilde{n}_t = \int_0^t \zeta_s \tilde{c}_s ds + \zeta_t n_t
\]

We can check that $\tilde{n}_t$ is a local martingale under $P^a$. Since $\zeta_t > 0$ and $n_t \geq 0$ it follows that

\[
E^a \left[ \int_0^{\tau_m \wedge T} \zeta_s \tilde{c}_s ds \right] \leq E^a \left[ \int_0^{\tau_m \wedge T} \zeta_s \tilde{c}_s ds + \zeta_{\tau_m \wedge T} n_{\tau_m \wedge T} \right] = n_0
\]

where $\{\tau_m\}$ reduces the stochastic integral and has $\lim_{m \to \infty} \tau_m = \infty$ a.s. Taking $m \to \infty$ and using the monotone convergence theorem we obtain

\[
E^a \left[ \int_0^T \zeta_t \tilde{c}_t ds \right] \leq n_0
\]

Now we want to maximize $E^a \left[ \int_0^T e^{-\rho t} \tilde{c}_t^{1-\gamma} dt \right]$ subject to this budget constraint. Notice that $a$ appears both in the budget constraint and objective function, but does not affect the law of motion of $\zeta$ under $P^a$, so we can ignore it since we are choosing $\tilde{c}$. The candidate solution $c$ has

\[
e^{-\rho t} c_t^{-\gamma} = \zeta_t \mu
\]

where $\mu > 0$ is the Lagrange multiplier and is chosen so that the budget constraint holds with equality.
any \( \hat{c} \) that satisfies the budget constraint we have
\[
\mathbb{E}^a \left[ \int_0^T e^{-\rho t} \frac{c_1^{1-\gamma}}{1-\gamma} dt \right] < \mathbb{E}^a \left[ \int_0^T e^{-\rho t} \left( \frac{c_1^{1-\gamma}}{1-\gamma} + c_1^{\gamma} (\hat{c}_t - c_t) \right) dt \right] 
\]
\[
\mathbb{E}^a \left[ \int_0^T e^{-\rho t} \frac{c_1^{1-\gamma}}{1-\gamma} dt \right] + \mu \mathbb{E}^a \left[ \int_0^T \zeta_t (\hat{c}_t - c_t) dt \right] \leq \mathbb{E}^a \left[ \int_0^T e^{-\rho t} \frac{c_1^{1-\gamma}}{1-\gamma} dt \right]
\]
Now since \( c_t = (\zeta_t \mu)^{-\frac{1}{\gamma}} e^{-\frac{\rho t}{\gamma}} \) it follows a geometric brownian motion so \( \mathbb{E}^a \left[ \int_0^T e^{-\rho t} \frac{c_1^{1-\gamma}}{1-\gamma} dt \right] \) is finite.

Because of homothetic preferences, we know that \( \mathbb{E}^a \left[ \int_0^T e^{-\rho t} \frac{c_1^{1-\gamma}}{1-\gamma} dt \right] = \lambda_5 \frac{N_1^{1-\gamma}}{1-\gamma} = \lambda_5 \frac{N_1^{1-\gamma}}{1-\gamma} \) for some \( \lambda_5 > 0 \).

\[\square\]

**Corollary.** For \( \gamma > 1 \), \( \lim_{n \to \infty} \mathbb{E}_t^a \left[ e^{-\rho \tau_n} \frac{N_1^{1-\gamma}}{1-\gamma} \right] = 0 \) for any feasible strategy \( (\hat{c}, a, z, \tilde{z}) \).

**Proof.** The continuation utility at any stopping time \( \tau_n \) has
\[
U_{\tau_n}^{\hat{c}, a} = \mathbb{E}^a \left[ \int_{\tau_n}^{\tau_n + T} e^{-\rho (t-\tau_n)} \frac{c_1^{1-\gamma}}{1-\gamma} dt + e^{-\rho(T-\tau_n)} U_{\tau_n + T}^{\hat{c}, a} \right]
\]
\[
\leq \mathbb{E}^a \left[ \int_{\tau_n}^{\tau_n + T} e^{-\rho (t-\tau_n)} \frac{c_1^{1-\gamma}}{1-\gamma} dt \right] \leq \lambda_4 \frac{N_1^{1-\gamma}}{1-\gamma}
\]
So at \( t = 0 \) we get
\[
U_0^{\hat{c}, a} = \mathbb{E}^a \left[ \int_0^{\tau_n} e^{-\rho t} \frac{c_1^{1-\gamma}}{1-\gamma} dt + e^{-\rho \tau_n} U_{\tau_n}^{\hat{c}, a} \right] \leq \mathbb{E}^a \left[ \int_0^{\tau_n} e^{-\rho t} \frac{c_1^{1-\gamma}}{1-\gamma} dt + e^{-\rho \tau_n} \lambda_4 \frac{N_1^{1-\gamma}}{1-\gamma} \right]
\]
Take limits \( n \to \infty \) and use the monotone convergence theorem on the first term on the right hand side to get \( 0 \geq \lim_{n \to \infty} \mathbb{E}_t^a \left[ e^{-\rho \tau_n} \frac{N_1^{1-\gamma}}{1-\gamma} \right] \geq 0 \).

\[\square\]

**Lemma 14.** Let \( \hat{c}_t \in (0, \hat{c}_h) \) and \( \hat{v}_l \leq \hat{v}_p \) for any \( \hat{c} \in (0, \hat{c}_h) \). If \( \sigma^\hat{c} = 0 \) and \( \sigma^x = \frac{\alpha}{\beta^x} \frac{\hat{c}_h}{\hat{v}_l} \) and \( A(\hat{c}_t, \hat{v}_l) = 0 \), then
\[
\mu^{\hat{c}} = \frac{r - \rho}{\gamma} - \frac{\rho - \hat{c}_l^{1-\gamma}}{1-\gamma} + \frac{(\sigma^x)^2}{2} > 0
\]

**Proof.** Looking at (10), with \( \sigma^\hat{c} = 0 \) we get for the drift
\[
\mu^{\hat{c}} = \frac{r - \rho}{\gamma} + \frac{1}{2} (\sigma^x)^2 - \frac{\rho - \hat{c}_l^{1-\gamma}}{1-\gamma}
\]
So for any \( \hat{c} \), \( \mu^{\hat{c}} > 0 \) implies
\[
\frac{1}{2} (\sigma^x)^2 > \frac{\rho - r}{\gamma} + \frac{\rho - \hat{c}_l^{1-\gamma}}{1-\gamma}
\]
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Since we also want $A(\hat{c}; \hat{v}) = 0$, we get

$$0 = \hat{c} - r\hat{v} + \hat{v} \left( \frac{\alpha - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2} (\sigma^2)^2 \right)$$

$$< \hat{c} - \hat{v} \hat{c}^{1-\gamma} \equiv M$$

Notice that if $\hat{v} = \hat{c}^\gamma$ we have $M = 0$. If $\hat{v} > \hat{c}^\gamma$ we have $M < 0$ and if $\hat{v} < \hat{c}^\gamma$ we have $M > 0$. So for $A(\hat{c}; \hat{v}) = 0$ and $\mu^\hat{c} > 0$ we need $\hat{v} < \hat{c}^\gamma$. In fact, if $\hat{v} = \hat{c}^\gamma$ and in addition

$$\frac{1}{2} \left( \frac{\alpha}{\phi \beta \gamma} \right)^2 = \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\rho - r}{\gamma}$$

then we have $A = 0$ and $\mu^\hat{c} = 0$. In this case, because we have $\mu^\hat{c} = 0$ we therefore have the value of a stationary contract, i.e. $\hat{v} = \hat{v}_{\hat{r}}(\hat{c})$. This point corresponds to the optimal portfolio with $\hat{v} = \phi$, $(\hat{c}_p, \hat{v}_p)$. We know from Lemma 16 that $\hat{c}_p \in [\hat{c}, \hat{c}_h]$. By assumption, $\hat{v}_l \leq \hat{v}_p$.

First we will show that $\mu^\hat{c} \geq 0$, and then make the inequality strict. Towards contradiction, suppose $\mu^\hat{c} < 0$ at $\hat{c}$. Then it must be the case that $\hat{v} > \hat{c}^\gamma$ because we have $A(\hat{c}, \hat{v}) = 0$. We will show that $A(\hat{c}, \hat{v}) > 0$ and get a contradiction. First take the derivative of $A$:

$$A'(\hat{c}, \hat{v}) = 1 - \hat{v}_l \left( \hat{c}_l^{1-\gamma} + \hat{c}_l^{2\gamma-1} \left( \frac{\alpha}{\phi \beta} \right)^2 \frac{1}{\hat{v}_l^2} \right) < 0$$

where the inequality holds for all $\hat{c} < \hat{v}_l^\frac{1}{\gamma}$. So $A(\hat{c}, \hat{v}) > A(\hat{v}_l^\frac{1}{\gamma}, \hat{v}_l^\frac{1}{\gamma})$. Letting $\hat{c}_m = \hat{v}_l^\frac{1}{\gamma}$ we get

$$A(\hat{c}, \hat{v}) > \hat{c}_m - r\hat{v}_l + \hat{v}_l \left( \frac{\rho - \hat{c}_m^{1-\gamma}}{1-\gamma} - \frac{1}{2} \left( \frac{\alpha}{\phi \beta} \right)^2 \frac{1}{\gamma} \right)$$

$$= \hat{c}_m - r\hat{v}_l + \hat{v}_l \left( \frac{\rho - \hat{c}_m^{1-\gamma}}{1-\gamma} - \gamma \frac{\rho - \hat{c}_m^{1-\gamma}}{1-\gamma} - (\rho - r) \right)$$

$$\Rightarrow A(\hat{c}, \hat{v}) > \hat{c}_m + \hat{v}_l \gamma \frac{\hat{c}_m^{1-\gamma} - \hat{c}_m^{1-\gamma}}{1-\gamma} = \hat{c}_m \gamma \frac{\hat{c}_m^{1-\gamma} - \hat{c}_m^{1-\gamma}}{1-\gamma} \geq 0$$

where the last equality uses $\hat{v}_l = \hat{c}_m^\gamma$ and the last inequality uses $\hat{c}_m = \hat{v}_l^\frac{1}{\gamma} \leq \hat{v}_p^\frac{1}{\gamma} = \hat{c}_p$. This is a contradiction, and therefore it must be the case that $\mu^\hat{c} \geq 0$ at $\hat{c}$.

It’s clear from the previous argument that $\mu^\hat{c}(\hat{c}_l) = 0$ only if $(\hat{c}_l, \hat{v}_l) = (\hat{c}_p, \hat{v}_p)$. We will show this cannot be the case because $\alpha > 0$. First, note that $(\hat{c}_p, \hat{v}_p)$ is a tangency point where $\hat{v}_r(\hat{c})$ touches the locus $\hat{v}_b(\hat{c})$ defined by $A(\hat{c}; \hat{v}_b(\hat{c})) = 0$. If $(\hat{c}_l, \hat{v}_l) = (\hat{c}_p, \hat{v}_p)$ then this must be the minimum point for $\hat{v}_r(\hat{c})$, so the derivative of both $\hat{v}_r(\hat{c})$ and $\hat{v}_b(\hat{c})$ must be zero. This means that $A'(\hat{c}, \hat{v}) = 0$. However,

$$1 - \hat{v}_l \left( \hat{c}_l^{1-\gamma} + \hat{c}_l^{2\gamma-1} \left( \frac{\alpha}{\phi \beta} \right)^2 \frac{1}{\hat{v}_l^2} \right) < 0$$

where the inequality follows from $\hat{v}_l = \hat{v}_p = \hat{c}_l^\gamma$ (note that $\hat{c}_l > 0$ because as Lemma 12 shows $A(\hat{c}, \hat{v})$ is strictly positive for $\hat{c}$ near 0). This can’t be a minimum of $\hat{v}_r(\hat{c})$. Therefore $(\hat{c}_l, \hat{v}_l) \neq (\hat{c}_p, \hat{v}_p)$ and $\mu^\hat{c}(\hat{c}_l) > 0$. This completes the proof.
Lemma 15. The cost function of stationary contracts \( \hat{v}_r(\hat{c}) \) defined by (29) is strictly positive for all \( \hat{c} \in (\hat{c}_*, \hat{c}_h] \) if and only if

\[
\alpha < \bar{\alpha} \equiv \frac{\phi\beta\gamma\sqrt{2}}{1 + \gamma} \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma}}
\]

Proof. We need to check the numerator in (29), since the denominator is positive for all \( \hat{c} \geq \hat{c}_* \):

\[
\hat{c} - \frac{\alpha}{\phi\beta} \hat{c}^{\gamma} \sqrt{\frac{(\rho - r(1 - \gamma))}{1 - \gamma}}
\]

The rest of the proof consists of evaluating this expression at \( \hat{c} = \hat{c}_* \) and showing it is non-positive iff the bound is violated. We get \( \hat{c}_* \) times

\[
1 - \frac{\alpha}{\phi\beta} \sqrt{\frac{2}{1 + \gamma}} \sqrt{\frac{(\rho - r(1 - \gamma))}{\gamma}} - \hat{c}_*^{-1}
\]

So if \( \alpha \geq \bar{\alpha} \) the numerator is non-positive, and if \( \alpha < \bar{\alpha} \) then it’s strictly positive. This completes the proof.

Lemma 11

Proof. If ever \( \hat{v}_t > \inf \hat{v}(\omega, s) \), then renegotiating at that point is better than never renegotiating and obtaining \( \hat{v}_0 \). In the other direction, if \( \hat{v} \) is constant, any stopping time \( \tau \) yields the same value to the principal, so \( \tau = \infty \) is an optimal choice.

Theorem 4

Proof. Since the optimal stationary contract is incentive compatible and has a constant \( \hat{v} \), we only need to show that any incentive compatible contract with constant \( \hat{v} \) has \( \hat{v} \geq \hat{v}_r^{\text{min}} \). This is clearly true for all stationary contracts as defined in Lemma 7, or Lemma 24 with aggregate risk.

There could also be stationary contracts with a constant \( \hat{c} \) but \( dL_t > 0 \). For these contracts the drift \( \mu \hat{c} < 0 \) in the absence of \( dL_t \). Consider the optimization problem

\[
0 = \min_{\sigma^2} \hat{c} - \hat{r} \hat{v} - \sigma^2 \hat{c}^{\gamma} \alpha \frac{\rho - \hat{c}^{1 - \gamma}}{1 - \gamma} + \hat{v} \left( \frac{\rho - \hat{c}^{1 - \gamma}}{1 - \gamma} + \gamma (\sigma^2)^2 - \gamma \left( \pi \gamma \right)^2 \right)
\]
If the constraint is binding, we get the stationary contracts with \( dL_t = 0 \), so \( \hat{v} = \hat{v}_r \). We want to show that it must be binding. Towards contradiction, if the constraint is not binding we have \( \sigma^x = \frac{\mu \cdot \varepsilon}{\beta \cdot \gamma} \), and therefore we have \( A(\hat{c}, \hat{v}) = 0 \), where \( A \) is defined as in Lemma 12. If \( \hat{v} \leq \hat{v}_r^{\text{min}} \) then \( \hat{v} \leq \hat{v}_r \), because the portfolio plan with \( \hat{\phi} = \phi \) is always one of the incentive compatible stationary contracts \( \hat{c}_p \leq \hat{c}_h \) for any valid hidden investment). Then Lemma 26 ensures that \( \frac{r - \rho}{\gamma} - \frac{\rho - \varepsilon^1 - \gamma}{1 - \gamma} + \frac{1}{2} (\sigma^x)^2 + \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 \leq 0 \), which violates the constraint. This means that \( \hat{v} \geq \hat{v}_r^{\text{min}} \).

Finally, if we have a non-stationary contract with a constant \( \hat{v} < \hat{v}_r^{\text{min}} \), the domain of \( \hat{c} \) must have an upper bound \( \hat{c}^* \leq \hat{c}_h \), because otherwise they would have a lower cost than the optimal contract near \( \hat{c}_h \), and this cannot be for an IC contract. For the upper bound \( \hat{c}^* \) we must have \( \sigma^x = 0 \) and \( \mu^x \leq 0 \). But this is the same situation with stationary contracts with \( dL_t > 0 \), and we know their cost is above \( \hat{v}_r^{\text{min}} \).

Lemma 16. Let

\[
\hat{c}_p = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\phi \beta \gamma} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) \frac{1}{1 - \gamma} \in (\hat{c}_*, \hat{c}_h)
\]

\[
\hat{v}_p = \hat{c}_p^\gamma
\]

be the \( \hat{c} \) and \( \hat{v} \) corresponding to the optimal portfolio plan with \( \hat{\phi} = \phi \). We have the following properties

1) \( \hat{c}_p < \hat{c}_r^{\text{min}} \) (the optimal stationary contract)

2) \( \hat{c}^* \) intersects \( \hat{v}_r(\hat{c}) \) only at \( \hat{c}_p \) and \( \hat{c} = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) \frac{1}{1 - \gamma} \) below \( \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) \frac{1}{1 - \gamma} \).

Furthermore, \( \hat{c}^* \geq \hat{v}_r(\hat{c}) \) for all \( \hat{c} \in [\hat{c}_p, \hat{c}_h] \), and \( \hat{c}^* \leq \hat{v}_r(\hat{c}) \) for all \( \hat{c} \in [\hat{c}_*, \hat{c}_p] \).

3) \( A(\hat{c}, \hat{c}^*) = 0 \) only at \( \hat{c} = \hat{c}_p \) and \( \hat{c}_p \). Furthermore, \( A(\hat{c}, \hat{c}^*) \leq 0 \) for all \( \hat{c} \in [\hat{c}_p, \hat{c}_h] \) and \( A(\hat{c}, \hat{c}^*) \geq 0 \) for all \( \hat{c} \in [0, \hat{c}_p] \), and \( \partial_1 A(\hat{c}, \hat{c}^*) < 0 \) for all \( \hat{c} \in (0, \hat{c}_h] \).

Proof. First let’s show that \( \hat{c}_p \in [\hat{c}_*, \hat{c}_h] \). Clearly, \( \hat{c}_p \leq \hat{c}_h \) for any type of valid hidden investment, because \( \phi \leq 1 \). Now write \( \hat{c}_p \):

\[
\hat{c}_p = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\phi \beta \gamma} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) \frac{1}{1 - \gamma} > \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) \frac{1}{1 - \gamma} \left( 1 - \frac{1 - \gamma}{1 + \gamma} \right) \frac{1}{1 - \gamma}
\]

where the inequality comes from \( \alpha < \tilde{\alpha} = \frac{\phi \beta \gamma^2}{\gamma + \gamma} \left( \frac{r - \rho (1 - \gamma)}{\gamma} \right) \frac{1}{1 - \gamma} \). Notice \( 1 - \frac{1 - \gamma}{1 + \gamma} = \frac{2 \gamma}{1 + \gamma} \) and use the definition of \( \hat{c}_* \):

\[
\hat{c}_* = \left( \frac{2 \gamma}{1 + \gamma} \right) \frac{1}{\gamma} \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) \frac{1}{1 - \gamma}
\]
to conclude that \( \hat{c}_* < \hat{c}_p \). The cost of this portfolio contract is \( \hat{c}_p = \hat{c}_p^* \).

Now go to 2). We are looking for roots of \( \hat{c}_r(\hat{c}) = \hat{c}^* \):

\[
\hat{c} - \frac{\alpha}{\phi \beta} \hat{c}^* \sqrt{2} \sqrt{\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 - \hat{c}^1 - \gamma} = \hat{c}^* \left( 2r - \frac{1 + \gamma}{1 - \gamma} \rho + \gamma \left( \frac{\pi}{\gamma} \right)^2 + \hat{c}^1 - \gamma \right)
\]

Divide throughout by \( \hat{c}^* > 0 \) and reorganize the right hand side:

\[
\frac{\hat{c}^1 - \gamma}{1 - \gamma} (1 - \gamma) - \frac{\alpha}{\phi \beta} \sqrt{2} \sqrt{\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 - \hat{c}^1 - \gamma} = -2\gamma \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) + \frac{\hat{c}^1 - \gamma}{1 - \gamma} (1 + \gamma)
\]

\[
\frac{\alpha}{\phi \beta} \sqrt{2} \sqrt{\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 - \hat{c}^1 - \gamma} = 2\gamma \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 - \hat{c}^1 - \gamma \right)
\]

If \( \hat{c} = \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} \) we have a root. If not, then we can write:

\[
\frac{\alpha}{\phi \beta \gamma} = \sqrt{2} \sqrt{\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 - \hat{c}^1 - \gamma}
\]

\[
\hat{c} = \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\phi \beta \gamma} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} = \hat{c}_p < \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}}
\]

We know that at \( \hat{c} = 0, \hat{c}^* = 0 \), while the red curve is always positive above \( \hat{c}_* \) and diverges to infinity as \( \hat{c} \searrow \hat{c}_* \). So we know that \( \hat{c}_p \) is the first time they intersect and therefore \( \hat{c}^* \) intersects the red curve from below. Since they won’t intersect again until \( \hat{c}_h \), we get the other inequality.

For 1), the minimum stationary contract can be implemented with a portfolio strategy with constant
\( c/n = \hat{c}_{\min} \) and \( k/n = \hat{c}_{\min} \), and \( n = \hat{c}_{\min} x \) is exposed to a fraction \( \hat{c}_{\min} \) of the risk in his capital, i.e.
\( \sigma_r(\min) = \sigma_r(\min) = \hat{c}_{\min}^r \). We know it has a lower cost than \( \hat{c}_p \), so this means the agent must keep a smaller share of risk \( \hat{c}_{\min}^r < \phi \). Since \( \sigma_r(\min) = \rho \hat{c}_{\min}^r (\hat{c}_{\min}^r)^{-\gamma} \beta \), we get
\( \hat{c}_{\min}^r = \hat{c}_{\min}^r (\hat{c}_{\min}^r)^{-\gamma} \phi < \phi \)

which implies \( \hat{c}_{\min}^r (\hat{c}_{\min}^r)^{-\gamma} < 1 \) and therefore \( \hat{c}_{\min}^r < (\hat{c}_{\min}^r)^\gamma \). From 2) this means \( \hat{c}_{\min}^r > \hat{c}_p \).

For 3), we are looking for roots of

\[
\hat{c} - r\hat{c}^* - \frac{1}{2} \frac{\alpha \hat{c}^*}{\phi \beta} \hat{c}^* \gamma^2 + \hat{c}^* \left( \frac{\rho - \hat{c}^1 - \gamma}{1 - \gamma} - \frac{1 - \gamma}{2} \frac{\pi^2}{\gamma} \right) = 0
\]

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This works for $c = 0$. Otherwise, divide by $c^\gamma$
\[
\frac{c^{1-\gamma}}{1-\gamma}(1-\gamma) - r - \frac{1}{2} \left(\frac{\alpha}{\phi \beta}\right)^2 - \frac{\rho - c^{1-\gamma}}{1-\gamma} - \frac{1}{2} \gamma \left(\frac{\pi}{\gamma}\right)^2 = 0
\]
\[
\frac{\rho - r(1-\gamma)}{1-\gamma} - \frac{1}{2} \left(\frac{\alpha}{\phi \beta}\gamma\right)^2 - \left(\frac{\pi}{\gamma}\right)^2 = \hat{c}^{1-\gamma}
\]
\[
\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1}{2} \left(\frac{\alpha}{\phi \beta}\gamma\right)^2 - \frac{1}{2} \left(\frac{\pi}{\gamma}\right)^2 = \hat{c}^{1-\gamma}
\]
\[
\hat{c} = \left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1}{2} \left(\frac{\alpha}{\phi \beta}\gamma\right)^2 - \left(\frac{\pi}{\gamma}\right)^2\right)^{\frac{1}{1-\gamma}} = \hat{c}_p
\]
So we have only $\hat{c}_p$ and $\hat{c} = 0$ as roots. This argument also shows that $A(\hat{c}, \hat{c}^\gamma) \leq 0$ for $\hat{c} \in [\hat{c}_p, \hat{c}_h]$, and $A(\hat{c}, \hat{c}^\gamma) \geq 0$ for $\hat{c} \in [0, \hat{c}_p]$. Also, evaluating the derivative $\partial_1 A(\hat{c}, \hat{c}^\gamma)$
\[
\partial_1 A(\hat{c}, \hat{c}^\gamma) = 1 - c^{\gamma-1} - c^{2\gamma-1} \left(\frac{\alpha}{\phi \beta}\right)^2 \frac{1}{\hat{c}^{\gamma}}
\]
\[
\partial_1 A(\hat{c}, \hat{c}^\gamma) = 1 - 1 - \hat{c}^{\gamma-1} \left(\frac{\alpha}{\phi \beta}\right)^2 = -\hat{c}^{\gamma-1} \left(\frac{\alpha}{\phi \beta}\right)^2 < 0
\]
for all $\hat{c} \in (0, \hat{c}_h]$.

\[\square\]

**Lemma 17.** For all $\hat{c} \in [\hat{c}_1, \hat{c}_h]$
\[
\hat{v}(\hat{c}) \leq \hat{c}^\gamma
\]

**Proof.** We already know from the proof of Lemma 26 that at $\hat{c}_1$ we have $\hat{v}(\hat{c}_1) < \hat{c}^\gamma$. We also know that $\hat{v}(\hat{c}) \leq \hat{v}(\hat{c}_1)$, so if ever $\hat{v}(\hat{c}) = \hat{c}^\gamma$ for some $\hat{c} > \hat{c}_1$, it must be with $\hat{c} \leq \hat{c}_op$ and therefore with $A(\hat{c}, \hat{c}^\gamma) = A(\hat{c}, \hat{v}(\hat{c})) \geq 0$ and $\partial_1 A(\hat{c}, \hat{v}(\hat{c})) < 0$ (because $\hat{c} \geq \hat{c}_1 > 0$). From Lemma 12 we know that $A(\hat{c}, \hat{v})$ has at most two roots in $\hat{c}$, and is positive near 0. This means that $A(\hat{c} - \delta, \hat{c}^\gamma) > 0$ for all $\delta \in (0, \hat{c})$.

Now we’ll use the same reasoning as in Theorem 1. We can pick an $\alpha' < \alpha$ so that $\hat{v}(\hat{c}) = \hat{v}(\hat{c}_1')$ and $\hat{c}_1' < \hat{c}$, because $\hat{v}(\hat{c})$ is increasing in $\alpha$. However, $A_{\alpha'}(\hat{c}, \hat{c}^\gamma)$ is decreasing in $\alpha$, so we get $A_{\alpha'}(\hat{c}_1', \hat{c}^\gamma) > A(\hat{c}_1', \hat{c}^\gamma) > 0$, which contradicts $A_{\alpha'}(\hat{c}_1', \hat{v}(\hat{c}_1')) = 0$. So we conclude that $\hat{v}(\hat{c}) \leq \hat{c}^\gamma$ for all $\hat{c} \in [\hat{c}_1, \hat{c}_h]$.

\[\square\]

**Corollary 1.** For all $\hat{c} \in [\hat{c}_1, \hat{c}_h]$
\[
\tilde{\phi}(\hat{c}) = (\hat{v} \hat{c}^{1-\gamma}) \phi + \hat{v}' \hat{c}^{1-\gamma} \beta^{-1} \sigma \hat{c} \leq \phi
\]

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Appendix B: aggregate risk and hidden investment

This appendix formally introduces aggregate risk and hidden investment into the baseline environment. The return on capital is now

\[ dR_t = (r + \pi \tilde{\beta} + \alpha - a_t) \, dt + \beta \, dZ_t + \tilde{\beta} \, d\tilde{Z}_t \]

Where \( \tilde{Z} \) is an independent Brownian motion that represents aggregate risk, with market price \( \pi \). Capital has a loading \( \tilde{\beta} \) on aggregate risk, so the excess return on capital for the agent is \( \alpha \), as in the baseline. Let \( Q \) be the associated martingale measure.

The agent receives cumulative payments \( I \) from the principal and manages capital \( k \) for him. Payments \( I \) can be any semimartingale (it could be decreasing if the agent must pay the principal). This nests the relevant case where the contract gives the agent only what he will consume, i.e. \( dI_t = \eta \, dt \). As in the baseline setting, the agent can steal from the principal at rate \( a \geq 0 \) and decide when to consume \( \tilde{c} \geq 0 \). He can invest his hidden savings in the same way the principal would, not only in a risk-free asset, but also in aggregate risk \( \tilde{Z} \). In addition, the agent may be able to invest his hidden savings in his private technology. His hidden savings follow the law of motion

\[ dh_t = dI_t + \left( rh_t + z_t h_t (\alpha + \pi \tilde{\beta}) + \tilde{z}_t h_t \pi - \tilde{c}_t + \phi k_t a_t \right) \, dt + z_t h_t \left( \beta dZ_t + \tilde{\beta} d\tilde{Z}_t \right) + \tilde{z}_t h_t d\tilde{Z}_t \]

where \( z \) is the portfolio weight on his own private technology, and \( \tilde{z} \) the weight on aggregate risk. While the agent can chose any position on aggregate risk, \( \tilde{z}_t \in \mathbb{R} \), for his hidden private investment we consider two cases: 1) no hidden private investment, \( z_t \in H = \{0\} \), and 2) hidden private investment, \( z_t \in H = \mathbb{R}_+ \).

The cost to the principal is

\[ J_0 = \mathbb{E}^Q \left[ \int_0^\infty e^{-rt} \left( dI_t (R^a) - (\alpha - a_t)k_t(R^a) \right) \, dt \right] \]

where \( R^a \) the observed return that results from the agent’s stealing activity \( a \).

A contract \( C = (I, k, \tilde{c}, a, z, \tilde{z}) \) specifies the contractible payments \( I \) and capital \( k \), and recommends the hidden action \((\tilde{c}, a, z, \tilde{z})\), all contingent not only on returns \( R \) but also

\[ \text{16} \]
on the observable aggregate shock $\tilde{Z}$. After signing the contract the agent can choose a strategy $(\tilde{c}, a, z, \tilde{z})$ to maximize his utility (potentially different from the one recommended by the principal). Given contract $\mathcal{C}$, a strategy is feasible if 1) utility $U^{\tilde{c}, a, z, \tilde{z}}$ is finite, and 2) hidden savings $h_t \geq 0$ always. Since the agent can secretly invest in his private technology, we also impose the No-Ponzi condition on him 3) $\mathbb{E}^Q \left[ \int_0^\infty e^{-rt} (\tilde{c}_t + \alpha z_t h_t) dt \right] < \infty$.

A contract $\mathcal{C} = (I, k, \tilde{c}, a, z, \tilde{z})$ is admissible if 1) $(\tilde{c}, a, z, \tilde{z})$ is feasible given $\mathcal{C}$, and 2) $\mathbb{E}^Q \left[ \int_0^\infty e^{-rt} (dI_t(R^a) + k_t(R^a)adt + a_t k_t(R^a)dt) \right] < \infty$.

An admissible contract $\mathcal{C} = (I, k, \tilde{c}, a, z, \tilde{z})$ is incentive compatible if the agent’s optimal strategy is $(\tilde{c}, a, z, \tilde{z})$ as recommended by the principal. An incentive compatible contract is optimal if it minimizes the principal’s cost

$$v_0 = \min_{\mathcal{C}} J_0(\mathcal{C})$$

subject to $U_0^{\tilde{c}, a, z, \tilde{z}} \geq u_0$

$$\mathcal{C} \in IC$$

To incorporate aggregate risk into the setting, we need to slightly modify the parameter restrictions. We assume throughout that

$$\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 > 0$$

$$\alpha < \bar{\alpha} \equiv \frac{\phi \beta \gamma \sqrt{2}}{\sqrt{1 + \gamma}} \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2}$$

No stealing or hidden savings in the optimal contract

Just like in the baseline setting, it is without loss of generality to look only at contracts which induce no stealing, no hidden savings, and no hidden investment.

**Lemma 18.** It is without loss of generality to look only at contracts that induce no stealing $a = 0$, no hidden savings, $h = 0$, and no hidden investment, $z = \tilde{z} = 0$.

**Remark.** This lemma is also valid for the baseline setting without aggregate risk or hidden investment.
Proof. Imagine the principal is offering contract $C = (I, k, c, a, z, \tilde{z})$ with associated hidden savings $h$. Let $k^h = zh$ and $\tilde{k}^h = \tilde{z}h$ be the agent’s absolute hidden positions in his private technology and aggregate risk respectively. We will show that we can offer a new contract $C' = (I', k', dI', 0, 0, 0)$ under which it is optimal for the agent to choose not to steal, no hidden savings, and no hidden investment, i.e. $\tilde{c}' = dI'$, $a = z = \tilde{z} = 0$. The new contract has $h' = \int_0^t \tilde{c}_i dt$ and $k' = k(R^a) + k^h$.

If the agent now chooses $\tilde{c}' = dI'$, $a = z = \tilde{z} = 0$, he gets hidden savings $h' = 0$ and consumption $\tilde{c}$, so he gets the same utility as under the original contract and this strategy is therefore feasible under the new contract. If instead he chooses a different feasible strategy $(\tilde{c}', a', z', \tilde{z}')$, he gets the utility associated with $\tilde{c}'$. We will show that he could achieve this utility under the original contract by picking consumption $\tilde{c}'$, stealing $dR - dR^a(R^a')$, hidden investment in private technology $k^h(R^a') + (k^h)'$, and hidden investment in aggregate risk $\tilde{k}^h(R^a') + (\tilde{k}^h)'$. Since the strategy $(\tilde{c}', a', z', \tilde{z}')$ is feasible under the new contract $C'$, and $(\tilde{c}, a, z, \tilde{z})$ feasible under the old contract $C$, then in order to ensure the new strategy is feasible under the original contract we only need to show that hidden savings remain non-negative always

$$h_i' = \int_0^t e^{c(t-s)} \left( dI_i(R^a(R^a)) - \tilde{c}_i(R^a) dt + \phi k_i(R^a(R^a)) (dR_t - dR^a_t(R^a)) \right)$$

$$+ (k^h(R^a) + (k^h)'_i) dR_t + (\tilde{k}^h(R^a) + (\tilde{k}^h)'_i)(\pi dt + d\tilde{Z}_t) \geq 0$$

To show this is always non-negative, we will show it’s greater or equal to the sum of two non-negative terms. First, the hidden savings under the original contract, following the feasible strategy $R^a$, had $R^a$ been the true return

$$A_i = \int_0^t e^{c(t-s)} \left( dI_i(R^a(R^a)) - \tilde{c}_i(R^a) dt + \phi k_i(R^a(R^a)) (dR^a_t - dR^a_t(R^a)) \right)$$

$$+ k^h(R^a)dR^a_t + \tilde{k}^h(R^a)(\pi dt + d\tilde{Z}_t) \geq 0$$

Second, hidden savings under the new contract, following the feasible new strategy

$$B_i = \int_0^t e^{c(t-s)} \left( \tilde{c}_i(R^a) dt - \tilde{c}_i'(R^a) dt + \phi (k_i(R^a(R^a)) + k^h(R^a))(dR_t - dR^a_t) \right)$$

$$+ (k^h)'_i dR_t + (\tilde{k}^h)'_i(\pi dt + d\tilde{Z}_t) \geq 0$$

If $\phi = 1$ then $h_i' = A_i + B_i \geq 0$. With $\phi < 1$, we have $h_i' \geq A_i + B_i \geq 0$, because $dR_t - dR^a_t = a'dt \geq 0$ and $k^h(R^a') \geq 0$. This means that $\tilde{c}' = c'$, $a = z = \tilde{z} = 0$ is the agent’s optimal choice under the new contract $C'$, since any other choice delivers an utility that he could have obtained - but chose not to - under the original contract $C$.

We can now compute the principal’s cost under the new contract

$$J_0 = E^Q \left[ \int_0^\infty e^{-rt} \left( \tilde{c}_i - \alpha (k_i(R^a) + k^h) \right) dt + e^{-r\tau} J_{\tau} \right] =$$

$$E^Q \left[ \int_0^\infty e^{-rt} (dI(R^a) - (\alpha - \alpha_t)k_i(R^a)dt) + e^{-r\tau} J_{\tau} \right] - E^Q \left[ \int_0^\infty e^{-rt} \alpha_t k_i(R^a)(1 - \phi)dt \right]$$
\(-E^Q \left[ \int_0^{\tau_n} e^{-rt} \left( dI_t(R^a) - \hat{c} dt + \phi k_t(R^a) a_t dt + k^h_t \alpha dt \right) \right] + E^Q \left[ e^{-r\tau_n} (J'_{\tau_n} - J_{\tau_n}) \right]\)

On the rhs, the first term is the cost under the original contract; the second term the destruction produced by stealing under the original contract, which is non-negative; and the third term is $E^Q [e^{-r\tau_n} h_{\tau_n}] \geq 0$, where $h$ is the agent’s hidden savings under the original contract. To see this, write

\[
\frac{dh_t}{dt} = h_t r dt + dI_t(R^a) - \tilde{c} t dt + \phi k_t(R^a) a_t dt + k^h_t ((\alpha + \pi \tilde{\beta}) dt + \beta dZ_t + \tilde{\beta} d\tilde{Z}_t) + \tilde{k}^h_t (\pi dt + d\tilde{Z}_t)
\]

So

\[
d(e^{-rt} h_t) = e^{-rt} dh_t - re^{-rt} h_t dt = e^{-rt} \left( dI_t(R^a) - \hat{c} dt + \phi k_t(R^a) a_t dt + k^h_t ((\alpha + \pi \tilde{\beta}) dt + \beta dZ_t + \tilde{\beta} d\tilde{Z}_t) + \tilde{k}^h_t (\pi dt + d\tilde{Z}_t) \right)
\]

Now take expectations under $Q$, choosing the localizing process appropriately to get

\[
E^Q \left[ \int_0^{\tau_n} d(e^{-rt} h_t) \right] = E^Q \left[ \int_0^{\tau_n} e^{-rt} \left( dI_t(R^a) - \hat{c} dt + \phi k_t(R^a) a_t dt + k^h_t \alpha dt \right) \right] = E^Q \left[ e^{-r\tau_n} h_{\tau_n} - h_0 \right] \geq 0
\]

Given these inequalities, we can write:

\[
J'_{\tau_n} - J_{\tau_n} \leq E^Q \left[ e^{-r\tau_n} (J'_{\tau_n} - J_{\tau_n}) \right]
\]

Because the original contract was admissible, $\lim_{n \to \infty} E^Q \left[ e^{-r\tau_n} J_{\tau_n} \right] = 0$. Since in addition the agent’s response was feasible, the new contract is also admissible, and we get $\lim_{n \to \infty} E^Q \left[ e^{-r\tau_n} J'_{\tau_n} \right] = 0$ as well. This shows the new contract is admissible, and the cost for the principal is not greater than under the old contract. This completes the proof.

\[
\square
\]

We can then simplify the contract to $C = (c, k)$, and say an admissible contract is incentive compatible if the agent’s optimal strategy is $(c, 0, 0, 0)$, or $(c, 0)$ for short.

**Incentive compatibility**

Since the contract can depend on the history of aggregate shocks $\tilde{Z}$, so can his continuation utility $U^{c,0}$ and his consumption $c$. However, because the agent is not responsible for aggregate shocks, incentive compatibility does not place any constraints on his exposure to aggregate risk. On the other hand, since the agent can invest his hidden savings, his Euler equation needs to be modified appropriately. The discounted marginal utility of a hidden dollar must be a supermartingale under any feasible hidden investment strategy, since otherwise the agent could save a dollar instead of consuming it, invest it in aggregate risk and his private technology, and consume it later when the marginal utility is expected to be higher.
Lemma 19. Take the agent’s hidden investment possibility set \( H \) as given. If \( C = (c, k) \) is an incentive compatible contract, the agent’s continuation utility \( U_{c,0}^{c,0} \) and consumption \( c \) satisfy the laws of motion

\[
dU_{t}^{c,0} = \left( \rho U_{t}^{c,0} - \frac{c_{t}^{1-\gamma}}{1-\gamma} \right) dt + \Delta_{t} dZ_{t} + \bar{\sigma}_{t}^{u} d\bar{Z}_{t} \tag{49}\]

\[
\frac{dc_{t}}{c_{t}} = \left( \frac{r - \rho}{\gamma} + \frac{1 + \gamma}{2} (\sigma_{t}^{c})^{2} + \frac{1 + \gamma}{2} (\bar{\sigma}_{t}^{c})^{2} \right) dt + \sigma_{t}^{c} dZ_{t} + \bar{\sigma}_{t}^{c} d\bar{Z}_{t} + dL_{t} \tag{50}\]

for some \( \Delta, \bar{\sigma}^{c}, \sigma^{c}, \bar{\sigma}^{c} \), and a weakly increasing processes \( L \), such that

\[
\Delta_{t} \geq c_{t}^{-\gamma} \phi_{kt} \tag{51}\]

\[
z(\alpha - \sigma_{t}^{c} \beta \gamma) \leq 0 \quad \forall z \in H \tag{52}\]

\[
\bar{\sigma}^{c} = \frac{\pi}{\gamma} \tag{53}\]

Proof. The proof of (49) and (51) are similar to Lemma 4 and 2, where the \( d\bar{Z} \) term appears because the contract can depend on the history of aggregate shocks. For (50), the proof is analogous to Lemma 2, but now we need the discounted marginal utility

\[
Y_{t} = e^{H_{t}^{c} r - \rho + z_{t}(\alpha + \pi \beta) + \pi_{s} z_{t} - \frac{1}{2} (z_{s} \beta + z_{s})^{2}} ds + \int_{0}^{t} (z_{s} \beta + z_{s}) dZ_{s} + \int_{0}^{t} (z_{s} \beta + z_{s}) d\bar{Z}_{s} c_{t}^{-\gamma} \tag{54}\]

to be a supermartingale for any investment strategy \( z_{t} \in \mathbb{R} \) and \( z_{t} \in H \). Using the Doob-Meyer decomposition, the Martingale Representation theorem, and Ito’s lemma, we can write

\[
\frac{dc_{t}}{c_{t}} = \mu_{t}^{c} dt + \sigma_{t}^{c} dZ_{t} + \bar{\sigma}_{t}^{c} d\bar{Z}_{t} + dL_{t} \tag{55}\]

Since the drift of expression (54) must be weakly negative, we get

\[
\left( r - \rho - \gamma \mu_{t}^{c} + \frac{\gamma}{2} ((1 + \gamma) \sigma_{t}^{c})^{2} + \frac{\gamma}{2} ((1 + \gamma) \bar{\sigma}_{t}^{c})^{2} \right) dt \]

\[
+ \left( z(\alpha + \pi \beta) + \pi \bar{z} - \gamma \sigma_{t}^{c} z_{s} \beta - \gamma \bar{\sigma}_{t}^{c} (z_{s} \beta + \bar{z}) \right) dt - \gamma dL_{t} \leq 0 \tag{55}\]

Taking \( z = \bar{z} = 0 \), which are always allowed, we obtain the expression for \( \mu_{t}^{c} \) in (50), and \( L \) weakly increasing. Once we plug this into (55), and using that \( z_{t} \) can be both positive or negative, we get (53). Condition (52) is therefore necessary to ensure (55) holds.

\[\square\]

The IC constraint (52) depends on whether the agent is allowed to have a hidden investment in his own private technology. If hidden investment in the agent’s private
technology is not allowed, $H = \{0\}$ so condition (52) drops out. If instead hidden investment in the agent’s private technology is allowed, $H = \mathbb{R}_+$, so condition (52) reduces to $\sigma^x \geq \frac{\alpha}{\beta \gamma}$.

**State Space**

We can still use the state variables $x$ and $\hat{c}$. Their laws of motion are

$$\frac{dx_t}{x_t} = \left( \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2}(\sigma^x_t)^2 + \frac{\gamma}{2}(\bar{\sigma}^x_t)^2 \right) dt + \sigma^x_t \, dZ_t + \bar{\sigma}^x_t \, d\bar{Z}_t \quad (56)$$

$$\frac{d\hat{c}_t}{\hat{c}_t} = \left( \frac{r - \rho - \hat{c}^{1-\gamma}}{\gamma} + \frac{(\sigma^x_t)^2}{2} + \gamma \sigma^x_t \hat{c}_t + \frac{1 + \gamma}{2}(\sigma^\hat{c}_t)^2 \right) dt + \sigma^\hat{c}_t \, dZ_t + \bar{\sigma}^\hat{c}_t \, d\bar{Z}_t + dL_t \quad (57)$$

and the incentive compatibility constraints can be written

$$\sigma^x_t = \hat{c}^{-\gamma} \phi \hat{k}_t \beta \quad (58)$$

$$z(\alpha - (\sigma^x_t + \sigma^\hat{c}_t)\beta \gamma) \leq 0 \quad \forall z \in H \quad (59)$$

$$\bar{\sigma}^x_t + \bar{\sigma}^\hat{c}_t = \frac{\pi}{\gamma} \quad (60)$$

As before, $\hat{c}$ has an upper bound $\hat{c}_h$, which must be modified to take into account that it is not incentive compatible to give the agent a perfectly safe consumption stream.

**Lemma 20.** Take the agent’s hidden investment possibility set $H$ as given. For any incentive compatible contract $C$, $\hat{c}_t \in (0, \hat{c}_h]$ at all times, where $\hat{c}_h$ is given by

$$\hat{c}_h \equiv \max_{\sigma^x \geq 0} \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2}(\sigma^x)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} \quad (61)$$

$$st : \quad z(\alpha - \sigma^x_t \beta \gamma) \leq 0 \quad \forall z \in H$$

If ever $\hat{c}_t = \hat{c}_h$, then the continuation contract satisfies $\hat{c}_{t+s} = \hat{c}_h$ and $\hat{k}_t = \frac{\sigma^x \hat{c}_h}{\sigma^\hat{c}_t}$ for all future times $t + s$, and $x_t$ follows the law of motion (56), where $\sigma^x$ is the optimizing choice in (61) and $\bar{\sigma}^x = \frac{\pi}{\gamma}$. The cost of this continuation contract is $\hat{v}_h$. 67
Proof. We know from Lemma 3 that \( \hat{c} \) has an upper bound for any incentive compatible contract. This requires \( \sigma^{\hat{c}} = \tilde{\sigma}^{\hat{c}} = 0 \), \( \mu^{\hat{c}} \leq 0 \), and \( dL_t = 0 \) at that point. Using the law of motion of \( \hat{c} \) we get

\[
\frac{r - \rho}{\gamma} - \rho - \hat{c}^{1-\gamma} + \frac{(\sigma^{x})^2}{2} + \frac{(\tilde{\sigma}^{x})^2}{2} \leq 0
\]

where \( \tilde{\sigma}^{x} = \frac{x}{\gamma} \). If we choose \( \sigma^{x} \) to minimize the rhs of this expression, subject to the IC constraints imposed by hidden investment, we can then solve for the largest \( \hat{c} \) that can be attained by an incentive compatible contract. Since these choices are independent of \( \hat{c} \), this is equivalent to the maximization problem in expression (61). It follows that \( \hat{c}_h \) is an absorbing boundary. Using the IC constraint (58) and the law of motion of \( x \) we obtain the desired result. This completes the proof.

\[
\]

The upper bound \( \hat{c}_h \) restricts the principal’s ability to promise safety in the future. Even if the agent cannot invest his hidden savings in his private technology, \( H = \{0\} \), he can still invest in aggregate risk. In this case the maximizing choice is \( \sigma^{x} = 0 \) and we get \( \hat{c}_h = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{x}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} \). Notice that if \( \pi = 0 \) this boils down to expression (8) in the baseline setting without aggregate risk or hidden investment. If the agent can also invest his hidden savings in his own private technology, \( H = \mathbb{R}_+ \), then the maximizing choice is \( \sigma^{x} = \frac{\alpha}{\beta \gamma} \), and \( \hat{c}_h = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\beta \gamma} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{x}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} \) is lower.

The HJB equation

The optimal contract can be characterized with the same HJB equation as in the case without hidden investment, appropriately extended to incorporate aggregate risk and the new incentive compatibility constraints.

\[
0 = \min_{\sigma^{x}, \sigma^{\hat{c}}, \tilde{\sigma}^{x}, \tilde{\sigma}^{\hat{c}}} \hat{c} - r\hat{v} - \sigma^{x}\hat{c}^{\alpha} + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2}(\sigma^{x})^2 + \frac{\gamma}{2}(\tilde{\sigma}^{x})^2 - \pi \hat{\sigma}^{x} \right)
+ \hat{v}' \hat{c} \left( \frac{r - \rho}{\gamma} - \rho - \hat{c}^{1-\gamma} + \frac{(\sigma^{x})^2}{2} + (1 + \gamma)\sigma^{x}\hat{c}^{\alpha} + \frac{1 + \gamma}{2}(\sigma^{\hat{c}})^2 + \frac{(\tilde{\sigma}^{x})^2}{2} \right)
+ (1 + \gamma)\hat{\sigma}^{x}\hat{c}^{\alpha} + \frac{1 + \gamma}{2}(\hat{c}^{\alpha})^2 - \hat{c}^{\hat{c}^0} \pi + \frac{\hat{v}''}{2} \hat{c}^2 \left( (\sigma^{x})^2 + (\tilde{\sigma}^{x})^2 \right)
\]

subject to \( \sigma^{x} \geq 0 \) and (59) and (60).

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Using (60) to eliminate $\tilde{\sigma}^\hat{c}$, and taking FOC for $\tilde{\sigma}^x$, we obtain

$$\tilde{\sigma}^x = \frac{\pi}{\gamma} \quad \tilde{\sigma}^\hat{c} = 0$$

This is the first best exposure to aggregate risk. The principal and the agent don’t have any conflict about aggregate risk, and the principal cannot use it to relax the moral hazard problem, so they implement the first best aggregate risk sharing.\footnote{If the agent didn’t have access to hidden investment in aggregate risk, and the agent’s private technology is exposed to aggregate risk $\beta \neq 0$, then the principal could potentially use the agent’s exposure to aggregate risk to relax the moral hazard problem.}

The FOC for $\sigma^x$ and $\sigma^\hat{c}$ depend on whether the agent can invest his hidden savings in his private technology. Without hidden investment, the FOCs are the same as in the baseline, (16) and (17). With hidden investment, the IC constraint (59) could be binding in some region of the state space. The shape of the contract, however, is the same as in the baseline without hidden investment.

**Theorem 5.** Take the agent’s hidden investment possibility set $H$ as given. The principal’s cost function $\hat{v}(\hat{c})$ has a flat portion on $[0, \hat{c}_l]$ and a strictly increasing portion on $[\hat{c}_l, \hat{c}_h]$, for some $\hat{c}_l \in (0, \hat{c}_h)$. The HJB equation (62) holds with equality above $\hat{c}_l$ and with inequality below $\hat{c}_l$, i.e.

$$r \hat{v} < \min_{\sigma^x} \hat{v} - \sigma^x \hat{\gamma} \cdot \frac{\alpha}{\phi \beta} + \hat{v} \left( \frac{\rho - \hat{c}_l^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2} (\sigma^x)^2 - \frac{\gamma}{2} (\frac{\pi}{\gamma})^2 \right) \quad \forall \hat{c} < \hat{c}_l. \quad (63)$$

Below $\hat{c}_l$, $\hat{v}(\hat{c}) = \hat{v}(\hat{c}_l)$, and at $\hat{c}_l$ the cost function $\hat{v}(\hat{c})$ satisfies the smooth-pasting condition $\hat{v}'(\hat{c}_l) = 0$, and $\hat{v}''(\hat{c}_l) > 0$.

The optimal contract has $\hat{\sigma}_l^x = \frac{\pi}{\gamma}$ and $\hat{\sigma}_l^\hat{c} = 0$ always. It starts at $\hat{c}_0 = \hat{c}_l$, where $\sigma_0^x$ is chosen without taking into account its effect on the agent’s precautionary motive, to maximize

$$\sigma^x \hat{c}_l^{1-\gamma} \cdot \frac{\alpha}{\phi \beta} - \hat{v}(\hat{c}_l) \frac{\gamma}{2} (\sigma^x)^2 \quad (64)$$

At $\hat{c}_l$ we have $\mu(\hat{c}_l) > 0$ and $\sigma(\hat{c}_l) = 0$. For all $t > 0$, we have $\hat{c}_t \geq \hat{c}_l$, $\sigma_t^x \leq 0$ and $\sigma_t^\hat{c} \geq 0$.

**Proof.** The proof is similar to Theorem 1, except we use the more general definition of $A(\hat{c}, \hat{v})$

$$A(\hat{c}, \hat{v}) = \hat{c} - r \hat{v} - \frac{1}{2} \left( \frac{\hat{v}^\gamma \alpha}{\sigma^\beta} \right) + \hat{v} \left( \frac{\rho - \hat{c}_l^{1-\gamma}}{1 - \gamma} - \frac{1}{2} \frac{\sigma^2}{\gamma} \right)$$
where \( A(\dot{c}, \ddot{v}) = 0 \) is the HJB equation if \( \dot{v} = \ddot{v} = 0 \) and the IC constraint (59) is not binding. Notice we already know from the FOCs that \( \dot{\sigma}^x = \pi/\gamma \) and \( \ddot{\sigma} = 0 \).

**Part 1** goes through without any change. In **part 2**, to show that the region where the HJB holds cannot have flat parts, we notice that with \( \dot{v} = \ddot{v} = 0, \sigma^x \) drops out of the HJB, and can be used to ensure that the IC constraint (59) holds for any choice of \( \sigma^x \). Therefore at a flat part \( A(\dot{c}, \ddot{v}) = 0 \) should hold, which we know cannot be the case.

In **part 3**, the proof that \( \dot{v}'(\dot{c}_1) = 0 \) goes through with small modifications, as well as \( \dot{v}''(\dot{c}_1 + \epsilon) \geq 0 \). Indeed, if at the upper end of a flat region \( \dot{c}_2 \) we have a kink, with the right derivative \( \dot{v}'(\dot{c}_2) > 0 \), we must have \( \sigma^x(\dot{c}_2 + \epsilon) \to 0 \) as \( \epsilon \to 0 \), since otherwise we would cross into the flat region where the HJB doesn’t hold. If the drift is strictly positive there, we can start at \( \dot{c}_2 - \delta \), with the same \( \sigma^x \) that we were using at \( \dot{c}_2 \). The IC constraint (59) does not depend on \( \dot{c} \) so it is still satisfied. Since this extends the solution and \( \dot{v}'(\dot{c}_2) > 0 \), we would get a lower cost, which contradicts the flat region immediately below \( \dot{c}_2 \). The proof for the case with zero drift is unchanged.

With \( \dot{v}'(\dot{c}_2) = 0 \), we must have \( \dot{v}''(\dot{c}_2 + \epsilon) \geq 0 \), or else \( \dot{v}'(\dot{c} + \epsilon) < 0 \) for small \( \epsilon \). At \( \dot{c}_2 \) we must have \( A(\dot{c}_2 - \epsilon, \ddot{v}(\dot{c}_2)) \geq 0 \) because otherwise we could do better by lingering below \( \dot{c}_2 \) before jumping up to \( \dot{c}_2 \). We must also have \( A(\dot{c}_2, \ddot{v}(\dot{c}_2)) \leq 0 \), because at \( \dot{c}_2 \) we have \( \dot{v}' = 0 \), and \( \dot{v}'' \geq 0 \), the RHS of the HJB will be at least as large as \( A(\dot{c}_2, \ddot{v}(\dot{c}_2)) \) (it could be strictly larger if the IC constraint (59) is binding). Since \( A \) is continuous in \( \dot{c} \), we must have \( A(\dot{c}_2, \ddot{v}(\dot{c}_2)) = 0 \). Since this is true in particular with \( \dot{c}_1 = 0 \) and \( \dot{c}_2 = \dot{c}_1 \), we have proven the smooth pasting condition. The rest of part 3 goes through without changes.

**Part 4** goes through with natural modifications. In the case \( A(\dot{c}_1, \ddot{v}_1) < 0 \), we consider setting \( \sigma^x = 0 \), and \( \sigma^x = \frac{\alpha}{\beta} \frac{\dot{c}_1^2}{\gamma} \). This is consistent with IC constraint (59) because \( \dot{c}_1^2 \leq \dot{v}_1 \) from Lemma 17. We then obtain the first order ODE

\[
A(\dot{c}, \ddot{v}_{f_0}) + \dot{v}'_{f_0} \dot{c} \left( \frac{r - \rho}{\gamma} - \frac{\rho - \dot{c}^2}{1 - \gamma} + \frac{\left( \frac{\alpha}{\beta} \frac{\dot{c}_1^2}{\gamma} \right)^2}{2} + \frac{1}{2} \frac{\pi^2}{\gamma^2} \right) = 0
\]

and the rest of the proof goes through, with the only exception that we use the more general Lemma 26 to establish \( \mu^x > 0 \). So we get \( \dot{v}''(\dot{c}_1) > 0 \).

**Part 5** goes through and ignoring the IC constraint (59) we obtain \( \sigma^2(\dot{c}_1) = 0 \) and \( \sigma^x(\dot{c}_1) = \frac{\alpha}{\beta} \frac{\dot{c}_1^2}{\gamma} \), which maximize (64). These satisfy the IC constraint (59) because \( \dot{v}_1 \leq \dot{c}_1^2 \). Lemma 26 implies \( \mu^x(\dot{c}_1) > 0 \). It only remains to show that \( \sigma^x \leq 0 \) always. We already know this is the case when the IC constraint (59) is not binding. If it is binding, then \( \sigma^2 = \frac{\alpha}{\beta} > \sigma^x \), and the FOC for \( \sigma^x \) yields:

\[
\sigma^2 = \frac{\dot{c}_1 \alpha}{\gamma \phi^3} + \dot{v}' c^2 \alpha \frac{\dot{c}_1}{\beta \gamma} \geq \frac{\alpha}{\beta^2 \gamma} \gamma (\gamma - \dot{v}' c^2)
\]

which implies \( \sigma^x \leq 0 \). To see this inequality, multiply both sides by the denominator, which must be positive (second order condition) to get

\[
\dot{c}_1 \frac{\alpha}{\phi^3} + \dot{v}' c^2 \frac{\alpha}{\beta \gamma} \geq \frac{\alpha}{\beta^2 \gamma} (\gamma (\gamma - \dot{v}' c^2)
\]

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We also have a verification theorem for the HJB equation.

**Theorem 6.** Take the agent’s hidden investment possibility set \( H \) as given. Let \( \hat{v}(\hat{c}) : [\hat{c}_l, \hat{c}_h] \to [\hat{v}_1, \hat{v}_h] \) be a strictly increasing \( C^2 \) solution to the HJB equation (62) for some \( \hat{c}_l \in (0, \hat{c}_h) \), such that \( \hat{v}_l \equiv \hat{v}(\hat{c}_l) \in (0, \hat{v}_h] \), \( \hat{v}'(\hat{c}_l) = 0 \), \( \hat{v}''(\hat{c}_l) > 0 \) and \( \hat{v}(\hat{c}_h) = \hat{v}_h \). Assume also that \( \hat{v}(\hat{c}) \leq \hat{c}^\gamma \) for \( \hat{c} \in [\hat{c}_l, \hat{c}_h] \), and, if \( \gamma < \frac{1}{2} \) that

\[
1 - \hat{v}_l \left( \hat{c}^{-\gamma}_l + \hat{c}^{2\gamma - 1}_l \alpha^2 (\phi \beta)^{-2} \hat{v}_l^{-2} \right) \leq 0
\]

Then,

1) For any incentive compatible contract \( C = (c, k) \) that delivers at least utility \( u_0 \) to the agent, we have \( \hat{v}(\hat{c}_l)((1 - \gamma)u_0)^{\frac{1}{1-\gamma}} \leq J_0(C) \).

2) Let \( C^* \) be a contract generated by the policy functions of the HJB. Specifically, the state variables \( x^* \) and \( \hat{c}^* \) are solutions to (56) and (57) (with potential absorption at \( \hat{c}_h \)), with initial values \( x_0^* = ((1 - \gamma)u_0)^{\frac{1}{1-\gamma}} \) and \( \hat{c}^*_l = \hat{c}_l \). If \( C^* \) is admissible and \( \sigma^{\hat{c}^*} \) bounded, then \( C^* \) is an optimal contract, with cost \( J_0(C^*) = \hat{v}(\hat{c}_l)((1 - \gamma)u_0)^{\frac{1}{1-\gamma}} \).

**Proof.** The proof is very similar to Theorem 2, but we use the definition

\[
A(\hat{c}, \hat{v}) = \hat{c} - r\hat{v} - \frac{\hat{c}^\gamma \alpha}{\hat{v}^\gamma \beta} - \hat{v} \left( \frac{\rho}{1 - \gamma} - \frac{1}{\gamma} \frac{\pi^2}{\gamma} \right)
\]

and use it to show that the HJB holds as an inequality below \( \hat{c}_l \). Notice that the optimal policies in the HJB imply \( \hat{\sigma}^x = \pi/\gamma \) and \( \hat{\sigma}^c = 0 \) throughout. With \( \hat{v}'(\hat{c}_l) = 0 \) and \( \hat{v}''(\hat{c}_l) > 0 \), the IC constraint (59) is not binding at \( \hat{c}_l \), because \( \hat{v}(\hat{c}_l) \leq \hat{c}^\gamma \), so we get \( \hat{\sigma}^x(\hat{c}_l) = \frac{\alpha}{\beta \gamma \phi \hat{v}_l} \) and \( \hat{\sigma}^c(\hat{c}_l) = 0 \) and therefore \( A(\hat{c}_l, \hat{v}_l) = 0 \). Lemma 26 then shows that \( \mu^c(\hat{c}_l) > 0 \). We also want to show that \( \sigma^x \geq 0 \) and \( \sigma^c \leq 0 \) for all \( \hat{c} \in [\hat{c}_l, \hat{c}_h] \). When this is true when the IC constraint (59) is not binding. Using \( \hat{v}(\hat{c}) \leq \hat{c}^\gamma \) we can show this is also the case if the constraint is binding, as in Part 5 of Theorem 5. We can then use Theorem 7 to establish the global incentive compatibility of the candidate optimal contract. The rest of the proof is unchanged.

The following result is useful to verify admissibility.
Lemma 21. If the candidate contract $C^*$ constructed in Theorem 6 has $\mu^{x^*} < r + \pi^2/\gamma$, then $C^*$ is admissible and delivers utility $u_0$ to the agent.

Proof. We know that $\hat{c}_i \in [\hat{c}_l, \hat{c}_h]$ and recall that $\hat{c}_i > 0$. Then an upper bounded $\mu^{x^*} < r + \pi^2/\gamma$ implies a bounded $0 \leq \sigma^{x^*} \leq \bar{\sigma}$, and we also know that $\sigma^{x^*} = \pi/\gamma$. Then

$$E_Q \left[ \int_0^\infty e^{-\rho t} (|x^*_t| + |k^*_t\alpha|) dt \right] \leq 2 \max \left\{ \hat{c}_h, \frac{\sigma X \hat{c}_h^{\alpha}}{\phi \beta} \right\} E_Q \left[ \int_0^\infty e^{-\rho t} x^*_t dt \right] < \infty$$

where the last inequality follows from $\mu^{x^*} < r + \pi^2/\gamma$ (notice the expectations is taken under $Q$). Let $U^*_0 = \frac{(x^*)_t^{1-\gamma}}{1-\gamma}$, so using the law of motion of $x^*$, (9), we get

$$U^*_0 = E \left[ \int_0^{\tau^n} e^{-\mu t} \hat{c}_1^{1-\gamma} dt + e^{-\rho \tau^n} U^*_{\tau^n} \right]$$

with $\tau^n \to \infty$ a.s. Use the monotone convergence theorem and notice that

$$\lim_{n \to \infty} E \left[ e^{-\rho \tau^n} U^*_{\tau^n} \right] = 0$$

because $\rho - (1 - \gamma)(\mu^{x^*} - \frac{3}{2}(\sigma^{x^*})^2 - \frac{3}{2} (\pi^2/\gamma)^2) = \hat{c}_1^{1-\gamma} - \min\{\hat{c}_h^{1-\gamma}, \hat{c}_l^{1-\gamma}\} > 0$. We then get that $U^*_0 = U^*_0 = u_0$. We conclude that the contract is indeed admissible.

\[\square\]

Verifying global incentive compatibility

We can extend Theorem 3 to verify global incentive compatibility.

Theorem 7. Let $C = (c, k)$ be an admissible contract with associated processes $x$ and $\hat{c}$ satisfying (56) and (57), and (58), (59), and (60), with bounded $\mu^x$, $\mu^\hat{c}$, $\sigma^\hat{c}$, and $\bar{\sigma}^\hat{c}$, and with $\hat{c}$ uniformly bounded away for zero and bounded above by $\hat{c}_h$. Suppose that the contract satisfies the following property

$$\sigma^{\hat{c}} \leq 0$$

Then for any feasible strategy $(\hat{c}, a, z, \tilde{z})$, with associated hidden savings $h$, we have the following upper bound on the agent’s utility, after any history

$$U^*_{t}^{\hat{c}, a, z, \tilde{z}} \leq \left( 1 + \frac{h_t}{x_t^{1-\gamma}} \right)^{1-\gamma} U^*_0$$

In particular, since $h_0 = 0$, for any feasible strategy $U^*_{0}^{\hat{c}, a, z, \tilde{z}} \leq U^*_0$, and the contract $C$ is therefore incentive compatible.
Proof. As in the proof of Lemma 3, we get

\[ e^{-\rho t} \left( U_{t}^{c,a,z} - F(\hat{h}_{t}, \hat{c}_{t}) U_{t}^{c,a} \right) = \mathbb{E}_{t}^{\pi} \left[ \int_{t}^{\infty} e^{-\rho s}(1-\gamma)U_{u}^{c,a} Y_{u} d + e^{-\rho s} \left( U_{u}^{c,a,z} - F_{u,}\hat{U}_{u}^{c,a} \right) \right] \]

where \( Y_{t} = A_{t} + B_{t} + \hat{B}_{t} + C_{t} \) must be modified to 1) incorporate hidden investment in term \( B_{t} \), and 2) incorporate aggregate risk in term \( \hat{B}_{t} \). Terms \( A_{t} \) and \( C_{t} \) are not changed, and we already know they are non-positive from the proof of Theorem 3. We only need to show that \( B_{t} + \hat{B}_{t} \leq 0 \) as well.

Start with \( B_{t} \). Because the agent can now invest his hidden savings, we have the following expression:

\[ B_{t} = \frac{F_{b,t}}{(1-\gamma)} \hat{c}_{t} \left( \frac{1}{2} + \frac{\gamma}{2} (\sigma_{t}^{x} + \sigma_{t}^{z})^{2} - \frac{\gamma}{2} (\hat{\sigma}_{t}^{x})^{2} - \hat{\sigma}_{t}^{x} \hat{c}_{t} \right) + \frac{F_{h,t}}{(1-\gamma)} (1-\gamma) \sigma_{t}^{x} \hat{c}_{t} \]

Notice that when the agent invests in his private technology, we get an expected return \( z_{t}(\alpha + \pi \beta) \), but we have only included the term \( z_{t}\alpha \) in the expression for \( B_{t} \). We will include the premium for aggregate risk \( \pi \beta \) in the term \( \hat{B}_{t} \), which takes an analogous form:

\[ \hat{B}_{t} = \frac{F_{b,t}}{(1-\gamma)} \hat{c}_{t} \left( \frac{1}{2} + \frac{\gamma}{2} (\hat{\sigma}_{t}^{x} + \hat{\sigma}_{t}^{z})^{2} - \frac{\gamma}{2} (\hat{\sigma}_{t}^{x})^{2} - \hat{\sigma}_{t}^{x} \hat{c}_{t} \right) + \frac{F_{h,t}}{(1-\gamma)} (1-\gamma) \hat{\sigma}_{t}^{x} \hat{c}_{t} \]

The two terms \( B_{t} \) and \( \hat{B}_{t} \) are very similar. The only difference is that instead of \( z_{t}\beta \) - the exposure to idiosyncratic risk \( Z \) induced by hidden investment in the private technology - we have \( z_{t}\beta + \hat{z}_{t} \) - the exposure to aggregate risk from investment in both private technology and aggregate risk; and instead of \( z_{t}\beta \hat{\sigma}_{t}^{x} \) we have \( (z_{t}\beta + \hat{z}_{t}) \pi \). In addition, all the volatilities are with respect to aggregate risk \( \hat{Z} \) instead of \( Z \).

Now let’s re-write these terms in a more convenient form. Take \( B_{t} \) and simplify to obtain:

\[ B_{t} = \frac{F_{b,t}}{(1-\gamma)} \hat{c}_{t} \left( \frac{1}{2} + \frac{\gamma}{2} (\sigma_{t}^{x} + \sigma_{t}^{z})^{2} - \frac{\gamma}{2} (\sigma_{t}^{x})^{2} - \gamma \sigma_{t}^{x} \sigma_{t}^{z} \right) + \frac{F_{h,t}}{(1-\gamma)} \hat{h}_{t} \left( z_{t}\alpha + \frac{\gamma}{2} (\sigma_{t}^{x})^{2} - \gamma \sigma_{t}^{x} z_{t}\beta \right) \]

\[ + \frac{(\hat{\sigma}_{t}^{x})^{2} \hat{c}_{t}^{2} F_{b,t} + 2\hat{\sigma}_{t}^{x} (z_{t}\beta + \hat{z}_{t}) \hat{c}_{t} \hat{h}_{t} F_{b,t} + (z_{t}\beta + \hat{z}_{t})^{2} \hat{h}_{t}^{2} F_{b,t}^{2}}{2(1-\gamma)} \]

We know from the proof of Theorem 3 that with \( z_{t} = 0 \) this will be

\[ B_{0,t} = -\frac{(\sigma_{t}^{x} + \sigma_{t}^{z})^{2}}{2} \gamma \hat{h}_{t} \hat{c}_{t}^{1-\gamma} \left( 1 + \hat{h}_{t} \hat{c}_{t}^{-\gamma} \right)^{-\gamma-1} \hat{c}_{t}^{\gamma-1} \leq 0 \]
So let’s gather all the terms with \( z_t \) into \( B_{1,t} \) so that \( B_t = B_{0,t} + B_{1,t} \)

\[
B_{1,t} = \frac{F_{h,t}}{(1 - \gamma)} \frac{\hat{h}_t (z_t \alpha - \gamma \sigma_1^2 z_t \beta)}{(1 - \gamma)} + \frac{2 \sigma_t^2 z_t \beta \mu \hat{h}_t F_{h,k,t} + ((z_t \beta)^2 - 2 z_t \beta \sigma_t^2) \hat{h}_t^2 F_{h,h,t}}{2(1 - \gamma)}
\]

Now plug in the formula for \( F_{h}, F_{h,k}, \) and \( F_{h,h} \) to obtain:

\[
B_{1,t} = \hat{c}_t^{-\gamma} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} \hat{h}_t z_t (\alpha - \gamma \beta \sigma_t^2) + \frac{1}{2} \left( 2 \sigma_t^2 z_t \beta \hat{c}_t \hat{h}_t \left( -\gamma \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} + \gamma^2 \hat{h}_t \hat{c}_t^{-2\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma - 1} \right) \right)
\]

Reorganize to get:

\[
B_{1,t} = \hat{c}_t^{-\gamma} \hat{h}_t z_t \left( \alpha - \gamma \beta (\sigma_t^2 + \sigma_t^5) \right) \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} + \frac{1}{2} \left( 2 \sigma_t^2 z_t \beta \hat{h}_t \hat{c}_t^2 \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma - 1} \right)
\]

Adding the two terms back together we get

\[
B_t = B_{0,t} + B_{1,t} = \hat{c}_t^{-\gamma} \hat{h}_t z_t \left( \alpha - \gamma \beta (\sigma_t^2 + \sigma_t^5) \right) \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} + \frac{1}{2} \left( 2 \sigma_t^2 z_t \beta \gamma - ((z_t \beta)^2 - 2 z_t \beta \sigma_t^2) \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma - 1} \right)
\]

The second term is always positive, so we only need to concern ourselves with the first one. Performing the same algebraic steps on \( B_t \), and taking advantage of the structural similarities, we obtain

\[
\bar{B}_t = \hat{c}_t^{-\gamma} \hat{h}_t (z_t \beta + \hat{z}_t) \left( \pi - \gamma (\hat{\sigma}_t^2 + \hat{\sigma}_t^5) \right) \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma}
\]

where the second term is also positive. Finally, we can add the two first terms

\[
\hat{c}_t^{-\gamma} \hat{h}_t \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} \left[ z_t (\alpha - \gamma \beta (\sigma_t^2 + \sigma_t^5)) + (z_t \beta + \hat{z}_t) \left( \pi - \gamma (\hat{\sigma}_t^2 + \hat{\sigma}_t^5) \right) \right]
\]

where the last inequality follows from the incentive compatibility conditions (59) and (60). Notice this argument works for any contract where the discounted marginal utility of consumption is a supermartingale under any valid trading strategy. The rest of the proof dealing with the terminal term follows the same
steps as in Theorem 3.

Financial Frictions

The implementation as a portfolio problem is still valid, but now the agent is also exposed to aggregate risk $\tilde{Z}$

$$
\frac{dn_t}{n_t} = \left( r + \frac{k_t}{n_t} \alpha + \tilde{\sigma}_t^n \pi - \frac{c_t}{n_t} \right) dt + \tilde{\phi}_n \beta dZ_t + \tilde{\sigma}_t^n d\tilde{Z}_t
$$

(65)

where $k_t/n_t$, $c_t/n_t$, and $\tilde{\phi}_t$ satisfy the same expressions, and $\tilde{\sigma}_n = \pi/\gamma$.

Lemma 22. The optimal contract can be implemented as a portfolio problem with $k_t/n_t = \hat{k}_t \hat{v}_t^{-1}$, $c_t/n_t = \hat{c}_t \hat{v}_t^{-1}$, $\tilde{\sigma}_n = \pi/\gamma$ and

$$
\tilde{\phi}_t \equiv \hat{v}_t \hat{c}_t^{-1} \gamma \phi + \hat{v}_t' \hat{c}_t (\beta \hat{k}_t)^{-1} \sigma_t^2 \tilde{\phi}_t < \phi
$$

(66)

The agent's net worth $n_t = \hat{v}_t x_t$ satisfies (21), and the value function is $\omega_t = \hat{v}_t^{-1}$. At $t = 0$ the leverage constraint is not binding.

Proof. The proof is the same as in Lemma 5, but using the HJB equation (62), and with the addition that $\tilde{\sigma}_n = \hat{\sigma}_n = \pi/\gamma$.

Optimal portfolio with $\tilde{\phi} = \phi$. The optimal portfolio with $\tilde{\phi} = \phi$ can be adapted to the presence of aggregate risk

$$
\left( \frac{c}{n} \right)_p = \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\phi \beta \gamma} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2
$$

(67)

$$
\left( \frac{k}{n} \right)_p = \frac{1}{\gamma} \frac{1}{\phi \beta} \frac{\alpha}{\phi \beta} \text{Sharpe ratio}
$$

(68)

$$
\tilde{\sigma}_n = \frac{\pi}{\gamma}
$$

(69)
We can build a contract using (65) with (67), (68), and \( \bar{\phi} = \phi \), with initial \( n_0 \) taken as given. The agent then gets utility

\[
 u_p = \frac{(\omega_p n_0)^{1-\gamma}}{1-\gamma}
\]

where \( \omega_p \equiv \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2} \left( \frac{\alpha}{\phi \beta \gamma} \right)^2 - \frac{1-\gamma}{2} \left( \frac{\pi}{\gamma} \right) \right)^{\frac{\gamma}{\gamma-1}}. \)

**Lemma 23.** The contract generated by the optimal portfolio plan with \( \bar{\phi} = \phi \) is incentive compatible and delivers utility \( u_p = \frac{(\omega_p n_0)^{1-\gamma}}{1-\gamma} \) to the agent with associated

\[
 \hat{c}_p = \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2} \left( \frac{\alpha}{\phi \beta \gamma} \right)^2 - \frac{1-\gamma}{2} \left( \frac{\pi}{\gamma} \right) \right)^{\frac{1}{1-\gamma}}
\]

and cost \( \hat{v}_p = \hat{c}_p^\gamma \).

**Proof.** The proof follows the same lines as that of Lemma 6, but uses Lemma 24 to establish the incentive compatibility of the resulting contract.

\[
 \Box
\]

**Stationary contracts**  Set \( \mu^c = \sigma^c = \tilde{\sigma}^c = 0. \) To do this we need to set

\[
 \frac{1}{2} (\sigma_x^*)^2 = \frac{\rho - \hat{c}_p^{1-\gamma}}{1-\gamma} + \frac{\rho - r}{\gamma} - \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2
\]

and \( \sigma_x^* = \pi/\gamma \) which implies \( \tilde{\sigma}^c = 0. \)

**Lemma 24.** Take any \( \hat{c} \in (\hat{c}_*, \hat{c}_h) \), where

\[
 \hat{c}_* \equiv \left( \frac{2\gamma}{1+\gamma} \right)^{\frac{1}{1-\gamma}} \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1-\gamma}} \in (0, \hat{c}_h)
\]

There is an incentive compatible stationary contract for this \( \hat{c} \), where \( \sigma^*_p(\hat{c}) \) is given by (72), \( \tilde{\sigma}^*_p(\hat{c}) = \pi/\gamma \), and the cost \( \hat{v}_p(\hat{c}) x_0 \) is given by

\[
 \hat{v}_p(\hat{c}) = \frac{\hat{c} - \frac{\alpha}{\phi \beta} \hat{c}^\gamma \sqrt{2} \sqrt{\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2}}{2r - \rho - \frac{1+\gamma}{1-\gamma} \rho + \gamma \left( \frac{\pi}{\gamma} \right)^2 + \frac{\hat{c}^{1-\gamma}}{1-\gamma} (1+\gamma)}
\]

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For $\hat{c} \leq \hat{c}_*$ the growth rate $\mu^c(\hat{c}) > r + \frac{\pi^2}{\gamma}$ and the corresponding stationary contract violates the No-Ponzi condition (3) and is therefore not admissible. Since stationary contracts are incentive compatible, we have $\hat{v}(\hat{c}) \leq \hat{v}_r(\hat{c})$.

Proof. The proof is similar to the case without aggregate shocks in Lemma 7, using the HJB (62). First, using $\alpha < \bar{\alpha}$, we can verify that $0 \leq \hat{c}^* \leq \hat{c}_h$, regardless of whether the agent can invest in his hidden savings. Second, $\hat{v}_r(\hat{c}) > 0$ for all $\hat{c} \in (\hat{c}_r, \hat{c}_b)$ from Lemma 25. We can check that $\mu^c < r + \frac{\pi^2}{\gamma}$ for the stationary contract if and only if $\hat{c} > \hat{c}_*$, where $\hat{c}_*$ is given by (73). Arguing as in the proof of Lemma 21 we can show that the stationary contract is admissible and delivers utility $u_0$ to the agent if and only if $\hat{c} > \hat{c}_*$. Since the contract satisfies (56), (57), and (58), and (60) by construction, we only need to check that (59) holds too. It’s east to see this is the case because $\hat{c} \leq \hat{c}_h$. Lemma 7 then ensures that it is incentive compatible. This completes the proof.

The results of Lemmas 8 and 9 are unchanged.

Additional results

Lemma 25. The cost function of stationary contracts $\hat{v}_r(\hat{c})$ defined by (74) is strictly positive for all $\hat{c} \in (\hat{c}_*, \hat{c}_h)$ if and only if $\alpha < \bar{\alpha}$.

Proof. We need to check the numerator in (74), since the denominator is positive for all $\hat{c} \geq \hat{c}_*$:

$$\hat{c} \left( 1 - \frac{\alpha}{\beta} \sqrt{2} \right) \left( \hat{c} \gamma^{-1} \left( \frac{\mu - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) \hat{c} \gamma^{-1} - 1 \right)$$

The rest of the proof consists of evaluating this expression at $\hat{c} = \hat{c}_*$ and showing it is non-positive iff the bound is violated, since the expression is increasing in $\hat{c}$. We get $\hat{c}$ times

$$1 - \frac{\alpha}{\beta} \sqrt{2} \sqrt{1 + \frac{1}{2\gamma}} \sqrt{\left( \frac{\rho - r(1 - \gamma)}{\gamma} \right) - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2}$$

So if $\alpha \geq \bar{\alpha}$ the numerator is non-positive, and if $\alpha < \bar{\alpha}$ then it’s strictly positive. This completes the proof.

Lemma 26. Let $\hat{c}_l \in (0, \hat{c}_h)$ and $\hat{v}_l \leq \hat{v}_p$. If $\sigma^\hat{c} = \bar{\sigma} = 0$, $\sigma^x = \frac{\alpha}{\beta} \frac{\hat{c}_l}{\gamma \psi}$, and $\bar{\sigma}^x = \pi / \gamma$, and $A(\hat{c}_l, \hat{v}_l) = 0$, where

$$A(\hat{c}, \hat{v}) \equiv \hat{c} - r\hat{v} - \frac{1}{2} \left( \frac{\hat{c} \alpha}{\beta \psi} \right)^2 + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} - \frac{1 \pi^2}{2 \gamma} \right)$$

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\[ \mu^\phi = \frac{r - \rho}{\gamma} - \frac{\rho - c_l^{1-\gamma}}{1 - \gamma} + \frac{1}{2} (\sigma^x)^2 + \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 > 0 \]

**Proof.** Looking at (57), with \( \sigma^\phi = \tilde{\sigma}^\phi = 0 \) we get for the drift
\[ \mu^\phi = \frac{r - \rho}{\gamma} + \frac{1}{2} (\sigma^x)^2 + \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 - \frac{\rho - c_l^{1-\gamma}}{1 - \gamma} \]

So \( \mu^\phi > 0 \) implies
\[ \frac{1}{2} (\sigma^x)^2 + \frac{1}{2} (\sigma^x)^2 > \frac{\rho - r}{\gamma} + \frac{\rho - c_l^{1-\gamma}}{1 - \gamma} \]

Since we also want \( A(\hat{c}; \hat{v}) = 0 \), we get
\[ 0 = \hat{c} - r \hat{v} + \hat{v} \left( \frac{\rho - c_l^{1-\gamma}}{1 - \gamma} - \frac{\gamma}{2} (\sigma^x)^2 - \frac{\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) \]
\[ < \hat{c} - \hat{v} c_l^{1-\gamma} \equiv M \]

Notice that if \( \hat{v} = c_l^{1-\gamma} \) we have \( M = 0 \). If \( \hat{v} > c_l^{1-\gamma} \) we have \( M < 0 \) and if \( \hat{v} < c_l^{1-\gamma} \) we have \( M > 0 \). So for \( A(\hat{c}; \hat{v}) = 0 \) and \( \mu^\phi > 0 \) we need \( \hat{v} < c_l^{1-\gamma} \). In fact, if \( \hat{v} = c_l^{1-\gamma} \) and in addition
\[ \frac{1}{2} \left( \frac{\alpha}{\phi \beta \gamma} \right)^2 + \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 = \frac{\rho - c_l^{1-\gamma}}{1 - \gamma} + \frac{\rho - r}{\gamma} \]

then we have \( A = 0 \) and \( \mu^\phi = 0 \). In this case, because we have \( \mu^\phi = 0 \) we therefore have the value of a stationary contract, i.e. \( \hat{v} = \hat{v}_c(\hat{c}) \) given by (74). This point corresponds to the optimal portfolio with \( \hat{\phi} = \phi, (\hat{c}_p, \hat{v}_p) \). We know from Lemma 16 that \( \hat{c}_p \in [\hat{c}_c, \hat{c}_h] \). By assumption, \( \hat{v}_1 \leq \hat{v}_m \).

First we will show that \( \mu^\phi \geq 0 \), and then make the inequality strict. Towards contradiction, suppose \( \mu^\phi < 0 \) at \( \hat{c}_l \). Then it must be the case that \( \hat{v}_l > \hat{v}_1^{1-\gamma} \) because we have \( A(\hat{c}_l, \hat{v}_l) = 0 \). We will show that \( A(\hat{c}_l, \hat{v}_l) > 0 \) and get a contradiction. First take the derivative of \( A \):
\[ A'_c(\hat{c}_l, \hat{v}_l) = 1 - \hat{v}_l \left( \hat{c}_l^{1-\gamma} + \hat{c}_l^{2\gamma-1} \left( \frac{\alpha}{\phi \beta} \right)^2 \frac{1}{\phi^2} \right) < 0 \]

where the inequality holds for all \( \hat{c} < \hat{v}_l^{\frac{1}{\gamma}} \). So \( A(\hat{c}_l, \hat{v}_l) > A(\hat{v}_l^{\frac{1}{\gamma}}, \hat{v}_l) \). Letting \( \hat{c}_m = \hat{v}_l^{\frac{1}{\gamma}} \) we get
\[ A(\hat{c}_l, \hat{v}_l) > \hat{c}_m - r \hat{v}_l + \hat{v}_l \left( \frac{\rho - \hat{c}_m^{1-\gamma}}{1 - \gamma} - \frac{1}{2} \left( \frac{\alpha}{\phi \beta} \right)^2 \frac{1}{\gamma^2} - \frac{\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) \]
\[ = \hat{c}_m - r \hat{v}_l + \hat{v}_l \left( \frac{\rho - \hat{c}_m^{1-\gamma}}{1 - \gamma} - \rho \frac{\hat{c}_m^{1-\gamma}}{1 - \gamma} - (\rho - r) \right) \]
\[ \Rightarrow A(\hat{c}_l, \hat{v}_l) > \hat{c}_m + \hat{v}_l \gamma \hat{c}_m^{1-\gamma} - \hat{c}_m^{1-\gamma} = \hat{c}_m \gamma \hat{c}_m^{1-\gamma} - \hat{c}_m^{1-\gamma} \geq 0 \]

where the last equality uses \( \hat{v}_l = \hat{c}_m^{1-\gamma} \) and the last inequality uses \( \hat{c}_m = \hat{v}_l^{\frac{1}{\gamma}} \leq \hat{v}_p^{\frac{1}{\gamma}} = \hat{c}_p \). This is a contradiction, and therefore it must be the case that \( \mu^\phi \geq 0 \) at \( \hat{c}_l \).
It’s clear from the previous argument that $\mu^\hat{c}(\hat{c}_l) = 0$ only if $(\hat{c}_l, \hat{v}_l) = (\hat{c}_p, \hat{v}_p)$. We will show this cannot be the case because $\alpha > 0$. First, note that $(\hat{c}_p, \hat{v}_p)$ is a tangency point where $\hat{v}_r(\hat{c})$ touches the locus $\hat{v}_b(\hat{c})$ defined by $A(\hat{c}; \hat{v}_b(\hat{c})) = 0$. If $(\hat{c}_l, \hat{v}_l) = (\hat{c}_p, \hat{v}_p)$ then this must be the minimum point for $\hat{v}_r(\hat{c})$, so the derivative of both $\hat{v}_r(\hat{c})$ and $\hat{v}_b(\hat{c})$ must be zero. This means that $A'_l(\hat{c}_l, \hat{v}_l) = 0$. However,

$$1 - \hat{v}_l \left( \hat{c}_l^{\gamma-1} + \hat{c}_l^{2(\gamma-1)} \left( \frac{\alpha}{\phi_3} \right)^2 \frac{1}{\hat{v}_l^2} \right) < 0$$

where the inequality follows from $\hat{v}_l = \hat{v}_p = \hat{c}_l^\gamma$ (note that $\hat{c}_l > 0$ because as Lemma 12 shows $A(\hat{c}, \hat{v}_l)$ is strictly positive for $\hat{c}$ near 0). This can’t be a minimum of $\hat{v}_r(\hat{c})$. Therefore $(\hat{c}_l, \hat{v}_l) \neq (\hat{c}_p, \hat{v}_p)$ and $\mu^\hat{c}(\hat{c}_l) > 0$. This completes the proof.

□
Appendix C - numerical algorithm

TO BE COMPLETED