Numerical Computation of the Optimal Contract. Here we describe the numerical procedure that we use to compute the optimal contract. We use the parameterization of the state space
\[
(\delta - \delta^c, \delta - \delta^c, x) \in [0, \infty) \times [\delta_\ell, \infty) \times [0, \infty),
\]
where \( \delta_\ell = \frac{r(L + R)}{1 - r(R' + L')} - \frac{\nu \sigma}{\sqrt{2r}}. \)

In this space, we solve a system of two PDEs for the functions \( G(x, \delta^c, \delta^c) \), which represents the sum of utilities of the principal and the agent, and \( \xi(x, \delta^c, \delta^c) \), the agent’s information rent. Our initial condition for the system for \((x, \delta^c, 0) \in [0, \infty) \times [\delta_\ell, \infty) \times \{0\} \) is determined by the fixed-liquidation contracts. Specifically, by Proposition 2,
\[
\xi(x, \delta^c, 0) = \xi^c(x + \delta^c,\delta^c) = \frac{\lambda}{r} - \frac{\psi}{r} g(y) \quad \text{and}
\]
\[
G(x, \delta^c, 0) = \frac{y + \delta^c}{r} \left( L(\delta^c) + R(\delta^c) - \frac{\delta^c}{r} \right) g(y), \quad \text{where } g(y) = \exp \left( -\frac{\sqrt{2r}}{\sigma y} \right).
\]

We proceed to describe how we can obtain \( G(\cdot, \cdot, x + dx) \) and \( \xi(\cdot, \cdot, x + dx) \) from \( G(\cdot, \cdot, x) \) and \( \xi(\cdot, \cdot, x) \), hence solving the system of partial differential equations. We use the derivatives \( G_3 \) and \( \xi_3 \) obtained from the partial differential equations,
\[
\begin{align*}
\hat{\delta}(G_3 - G_1) + \hat{\delta}^c(G_2 - G_3) + \frac{1}{2} \nu^2 \sigma^2 G_{11} &= r G - \left( \frac{\delta^c + x + y}{\delta} \right) \\
\hat{\xi}(\xi_3 - \xi_1) + \hat{\xi}^c(\xi_2 - \xi_3) + \frac{1}{2} \nu^2 \sigma^2 \xi_{11} &= r \xi - \lambda
\end{align*}
\]
or, equivalently,
\[
\begin{align*}
(\hat{\delta} - \hat{\delta}^c)(G_3 - G_1) + \hat{\delta}^c(G_2 - G_1) + \frac{1}{2} \nu^2 \sigma^2 G_{11} &= r G - \left( \frac{\delta^c + x + y}{\delta} \right) \\
(\hat{\xi} - \hat{\xi}^c)(\xi_3 - \xi_1) + \hat{\xi}^c(\xi_2 - \xi_1) + \frac{1}{2} \nu^2 \sigma^2 \xi_{11} &= r \xi - \lambda
\end{align*}
\]
where $\dot{\delta}$ and $\dot{\delta} - \dot{\delta}^c$ are obtained as follows. First, notice that we can obtain the agent’s continuation value $W(y, \delta^c, x)$ by integrating

$$W(y, \delta^c, x) = R(\delta^c + x) + \int_y^0 \left( \frac{\dot{\delta}}{\psi} + \xi(y', \delta^c, x) \right) dy'. $$

Second, notice that equations (42) and (44) imply that

$$\dot{\delta}^c = \frac{\psi}{1 - r(R' + L')} \frac{\psi \sigma^2 \nu^2}{1 - r(R' + L')} H_1, \quad \text{where}$$

$$H(y, \delta^c, x) = b_2(y + x + \delta^c, W(y, \delta^c, x), \xi(y, \delta^c, x)).$$

That is, $H_1$ is the derivative of the function $H$ with respect to $y$, i.e. along the volatility trajectory. We can compute the function $H(y, \delta^c, x)$ explicitly as follows. Notice that

$$\left( \frac{d}{d\delta^c} + \frac{d}{d\delta} \right) G(\delta - \delta, \delta^c - \delta^c) = G_2 - G_1 = (b_2(\delta, W, \xi) + 1)(W_2 - W_1) - b_3(\delta, W, \xi) \left( \xi_2 - \xi_1 \right) \frac{\phi = (\delta - \delta_2)(1 - r(L' + R'))}{\psi} \frac{\psi \sigma^2 \nu^2}{1 - r(R' + L')} \frac{\psi \sigma^2 \nu^2}{1 - r(R' + L')} H_1, \quad \text{where}$$

so that

$$H = \frac{G_2 - G_1 + (\delta^c - \delta)}{W_2 - W_1} \frac{1 - r(L' + R')}{\psi} \left( \xi_2 - \xi_1 \right) - 1.$$ 

Finding $\dot{\delta} - \dot{\delta}^c$ is a bit more complicated. Notice that $W(y, \delta^c, x)$ satisfies the PDE

$$(\dot{\delta} - \dot{\delta}^c)(W'_3 - W'_1) + \dot{\delta}^c(W_2 - W_1) + \frac{1}{2} \nu^2 \sigma^2 W_{11} = rW.$$ 

Note also that

$$W_3(y, \delta^c, x) - W_1(y, \delta^c, x) = R'(\delta^c + x) + \int_y^0 \xi_3(y', \delta^c, x) dy' - \xi_3(y, \delta^c, x). \quad \text{(1)}$$

Hence, if we know $\xi_3(y', \delta^c, x)$ for $y' = \{0, dy, 2dy, \ldots (n-1)dy\}$, we can find $W_3(n dy, \delta^c, x)$ by integrating the expression above in order to compute $\dot{\delta} - \dot{\delta}^c$ at $(n dy, \delta^c, x)$ from the PDE for $W(y, \delta^c, x)$. That is, we deduce the rate of change of $\delta$ from the promise-keeping constraint on the agent’s continuation value. We can then compute $\xi_3(n dy, \delta^c, x)$ from the PDE for $\xi(y, \delta^c, x)$, and $G_3(n dy, \delta^c, x)$ from the PDE for $G(y, \delta^c, x)$.

To sum up, we start with the boundary conditions at $\xi(y, \delta^c, 0)$ and $G(y, \delta^c, 0)$, solve for the functions $\xi(y, \delta^c, x)$ and $G(y, \delta^c, x)$ through a system of ODEs in $x$ for values of these functions on the grid of $y$ and $\delta^c$. When we evaluate the derivatives with respect to
x, we do it for each $\delta^c$ in the order of increasing $y$, since we have to perform integration (1) along the way. To jump start integration, we can use the boundary condition

$$\xi(0, \delta^c, x) = \frac{\lambda - \psi}{r},$$

hence $\xi(0, \delta^c, x) = 0$, and we use the boundary condition

$$G(0, \delta^c, x) = L(\delta^c + x) + R(\delta^c + x).$$

Finally, we should note that for the numerical stability of the explicit method, we have to evaluate the derivatives $G_2$, $\xi_2$ and $W_2$ as right derivatives, since the long-run termination threshold is moving up over time.