A Universal Operator Theoretic Framework for Quantum Fault Tolerance

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Recent Papers

"Reliable Final Computational Results from Faulty Quantum Computation"

G Gilbert, M Hamrick, YS Weinstein (MITRE)
submitted to New Journal of Physics

"A Universal Operator Theoretic Framework for Quantum Fault Tolerance"

G Gilbert, M Hamrick, YS Weinstein (MITRE)
V Aggarwal, A Robert Calderbank (Princeton)
submitted to Physical Review A

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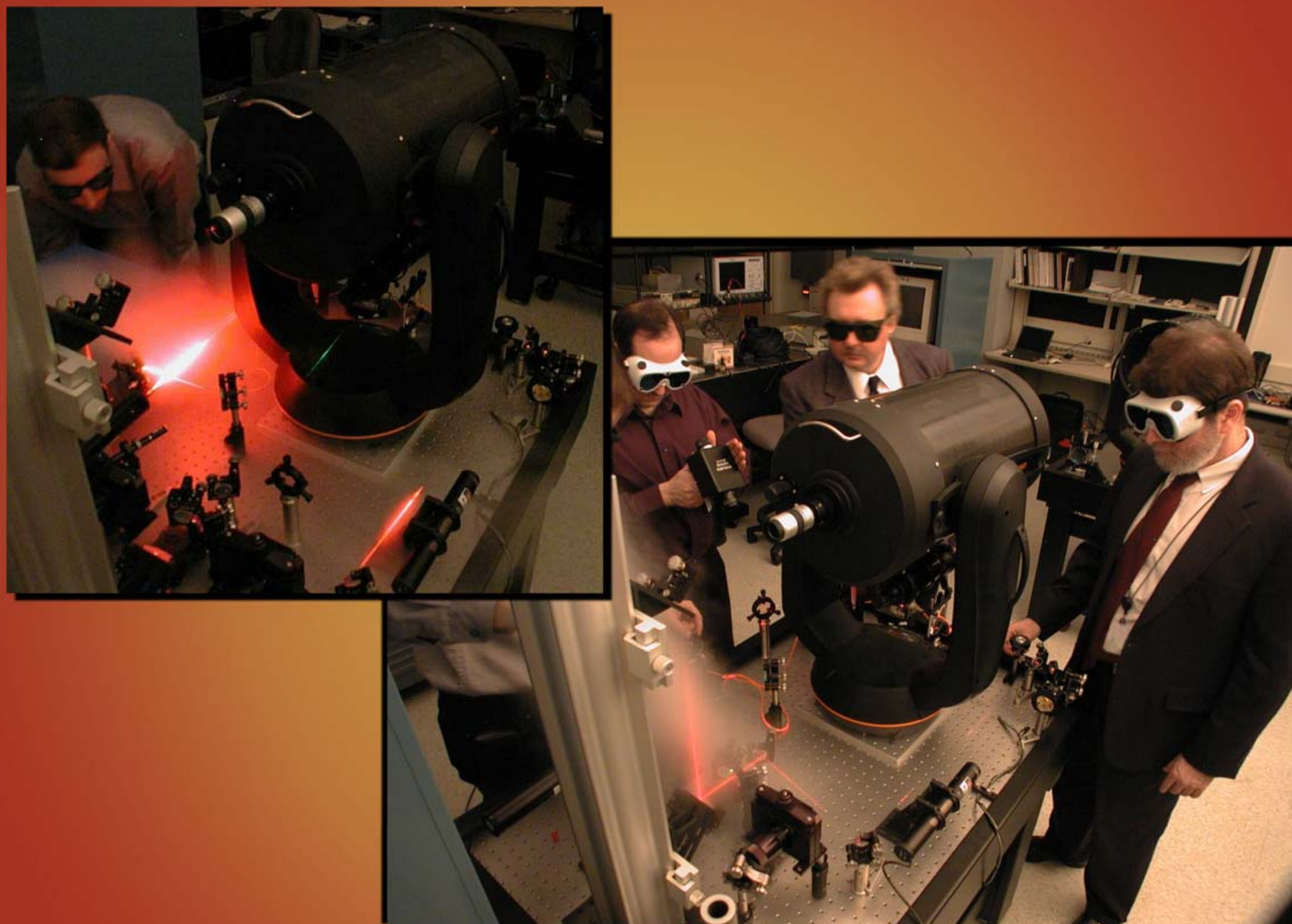
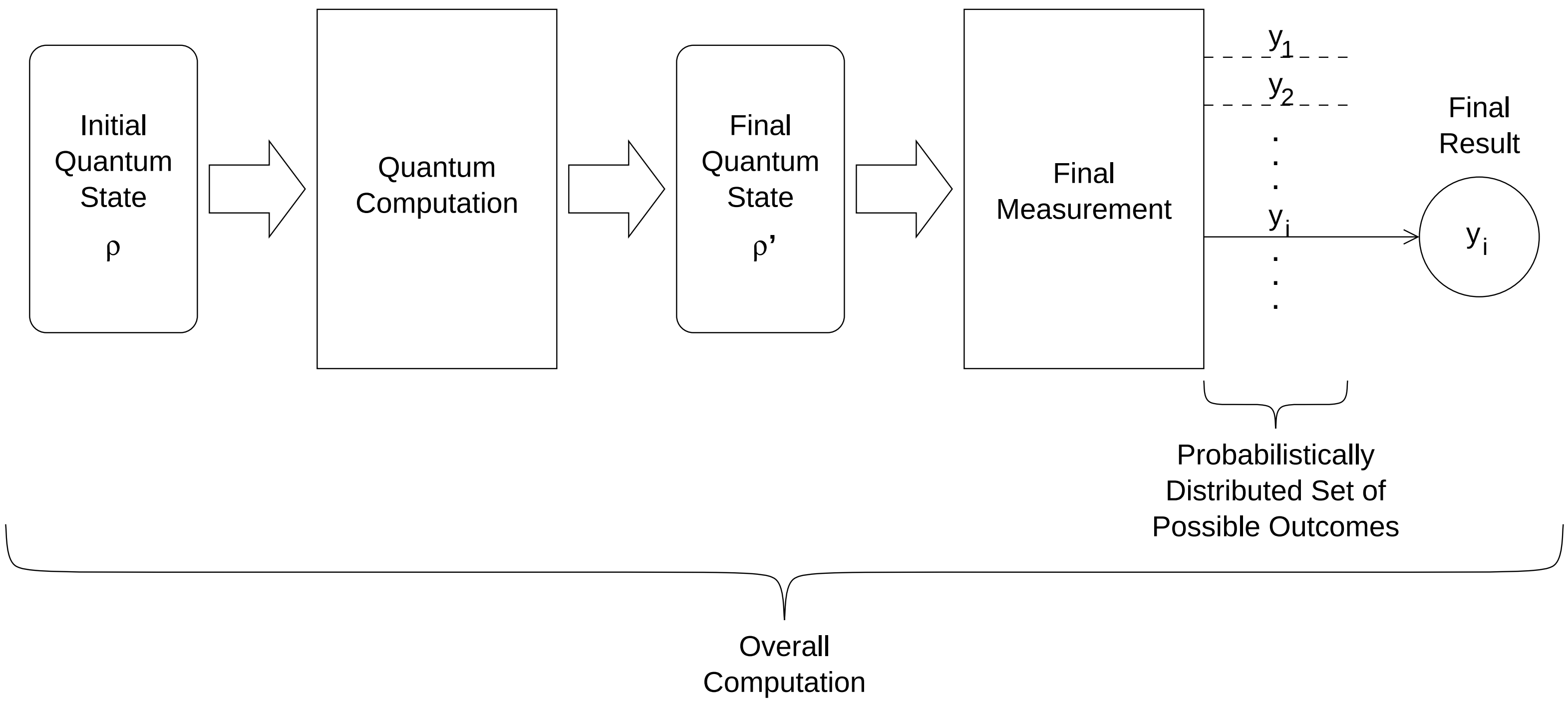


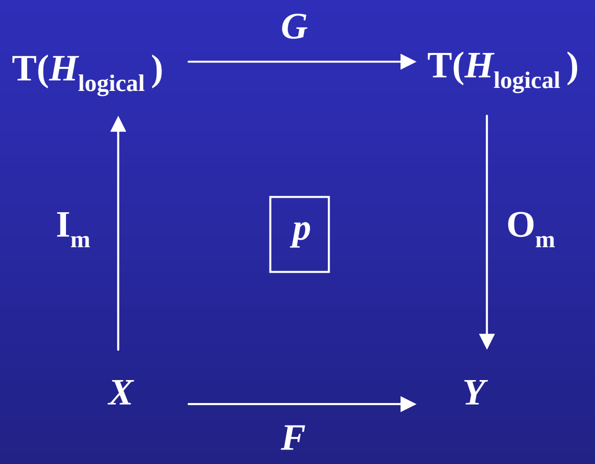
Quantum Computational Path



Accuracy of Quantum Computational Results (1)

Kitaev's Model of Ideal Quantum Computation

Kitaev (1997)



ideal (unitary) quantum computation: $G(\rho) \equiv U \rho U^{+}$

Probability that the ideal quantum computation followed by measurement produces outcome y:

$$\Pr\left[\left(\mathrm{O}_{\mathrm{m}} \circ G \circ \mathrm{I}_{\mathrm{m}}\right)(x) = y\right] = \mathrm{tr}\left(\sqrt{E_y}\, G\left(I_m(x)\right)\sqrt{E_y}\right)$$

Probability that the ideal quantum computation followed by measurement produces the correct result *F*(*x*):

$$\mathrm{tr}\left(\sqrt{E_{F(x)}}\, G\left(I_m(x)\right)\sqrt{E_{F(x)}}\right) > 1 - p$$

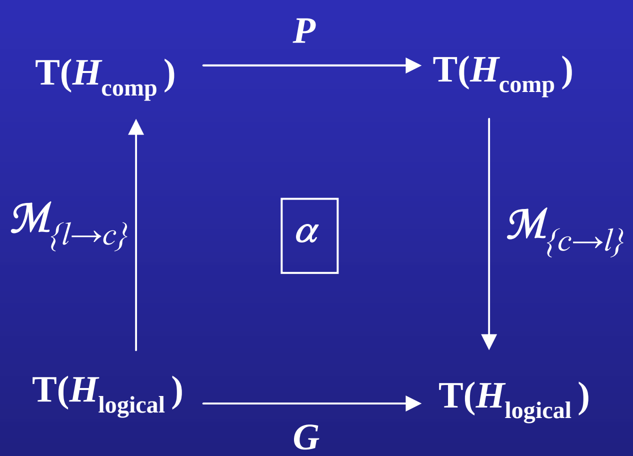
The quantity *p* bounds the probability that the ideal quantum computation followed by measurement fails to produce the correct result

But real quantum computers don't implement *U* exactly.
What's the effect on the failure probability bound of implementation errors?

Accuracy of Quantum Computational Results (2)

The Quantum Computer Condition (QCC)

Gilbert, Hamrick & Weinstein (2007)



Implementation Inaccuracy

$$\left\| \left(\mathcal{M}_{\{c\to l\}} \circ P \circ \mathcal{M}_{\{l\to c\}} \right)(\rho) - U\rho U^{\dagger} \right\|_1 \leq \alpha$$

Actual Computation (P) — **Ideal Computation** ($U\rho U^{\dagger}$)

Non-dynamical maps linking the logical and computational Hilbert spaces ($\mathcal{M}_{\{c\to l\}}$, $\mathcal{M}_{\{l\to c\}}$)

Notational Simplification: $P^{\mathcal{M}} \equiv \mathcal{M}_{\{c\to l\}} \circ P \circ \mathcal{M}_{\{l\to c\}}$

$$\left\| P^{\mathcal{M}} \rho - U\rho U^{\dagger} \right\|_1 \leq \alpha$$

Kitaev model gives failure probability for ideal computation.
QCC describes deviation of the quantum computation from the ideal.
What's the combined effect?

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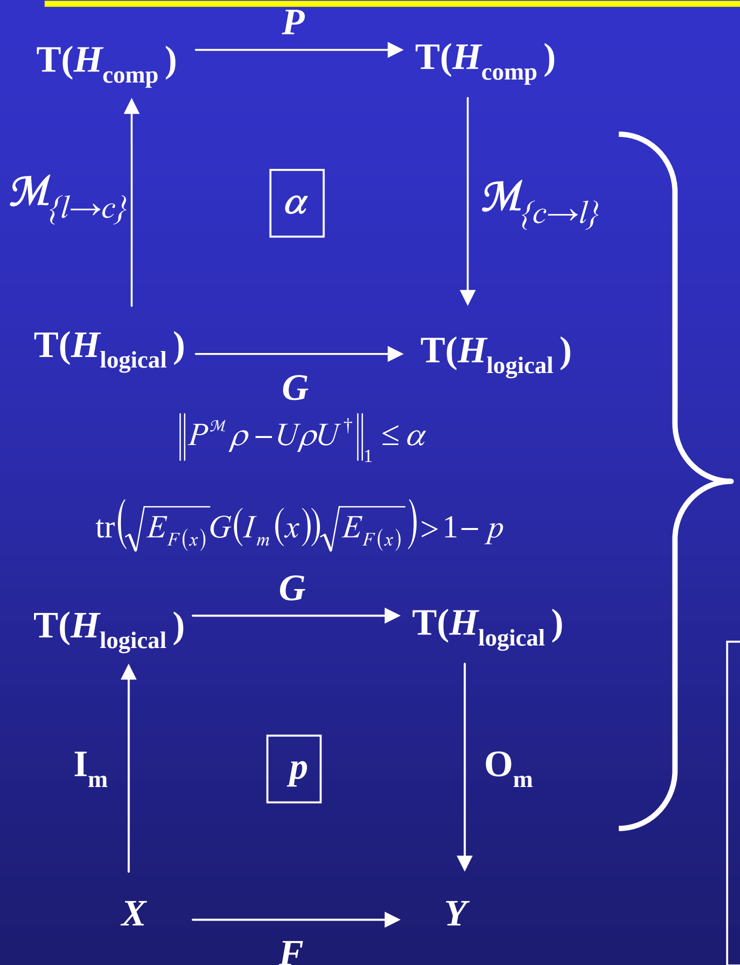
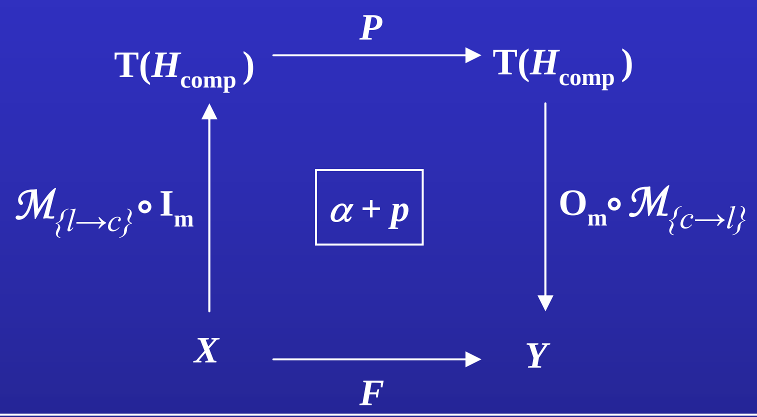
Accuracy of Quantum Computational Results (3)



Composite diagram commutes with probability $>1-(p+\alpha)$



$$\mathrm{tr}\left(\sqrt{E_{F(x)}}\,P^{\mathcal{M}}(I_m(x))\sqrt{E_{F(x)}}\right) > 1-(p+\alpha)$$

Failure probability $p_f \le (p+\alpha)$

- Kitaev model shows <u>ideal</u> quantum computation can produce correct output, within p
- the QCC states that <u>real</u> quantum computation can realize ideal quantum computation, within α
- our result shows that real quantum computation can produce correct output, within $p + \alpha$

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Achieving a Specified Probability of Success

We can now derive a bound on the implementation inaccuracy that will guarantee the correct result with a specified probability

- End user specifies maximum probability of failure — We wish to achieve $p_f \le \hat{p}$
  - $\hat{p} < \frac{1}{2}$ means majority voting can be used
- Our result bounds total probability of failure — $p_f \le p + \alpha$
- Bounds p_b on failure probability, p, for ideal quantum computation are known for algorithms of interest: — $p \le p_b$
  - *e.g.*, in Grover search for 1 out of n items $p_b = 1/n$
- We require that the implementation error tolerance be bounded by $\hat{p} - p_b$ — $\alpha \le \hat{p} - p_b$
- It follows that the probability that the computation fails to return the correct result is bounded, as required, by $\hat{p}$ — $p_f \le p + \alpha \le p_b + \alpha \le \hat{p}$

But the standard theory of fault tolerance yields a bound on the probability that the quantum computation results in the correct quantum state, not the implementation inaccuracy, which is a bound on the error norm of the state. Can we relate the two?

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Fault Tolerance Theory

- Fault tolerance techniques enable correct dynamical evolution of the quantum computation in the presence of noise
  - Circuit is subject to (random) errors
  - Techniques such as concatenated quantum error correcting codes are used to correct errors
  - Fault tolerant circuit operates on encoded states
  - Not all errors are corrected, only the more likely ones
  - There is a residual error probability that the circuit fails to produce the desired quantum state as its output
    - Residual error probability can be estimated from theory
      - Doesn't address quantum uncertainties in the final measurement
    - Need to find connection between residual error probability and probability of obtaining the correct final result

Residual Error Probability ε_N

$$\frac{\varepsilon_N}{\varepsilon_{th}} \cong \left(\frac{\varepsilon_0}{\varepsilon_{th}}\right)^{2^N}$$

(for a single logical gate using simplified model for concatenated error codes)

ε_0 Elementary gate failure probability

ε_{th} Error threshold (function of circuit structure)

N Number of levels of concatenation

Preskill (1998)

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Specifying Fault Tolerance to Guarantee Probability of Obtaining Correct Final Result

- Fault tolerance theory yields a residual error probability $\approx \mathcal{N}_g \varepsilon_N$
  - $\mathcal{N}_g$ is the number of logical gates in the formal description of the algorithm
- We derive from this a bound on implementation inaccuracy given by: $\left\| P^{\mathcal{M}} \cdot \rho - U \rho U^{\dagger} \right\|_1 \leq 2\mathcal{N}_g \varepsilon_N$
- Success criterion is achieved if this bound satisfies $2\mathcal{N}_g \varepsilon_N \leq \hat{p} - p_b$
- For a given circuit and a given probability of elementary gate failure, this gives the number of levels of concatenation required to achieve the success criterion for the overall result

$$N_{\min} \approx \log_2 \frac{\ln \dfrac{2\mathcal{N}_g \varepsilon_{th}}{\hat{p} - p_b}}{\ln \dfrac{\varepsilon_{th}}{\varepsilon_0}}$$

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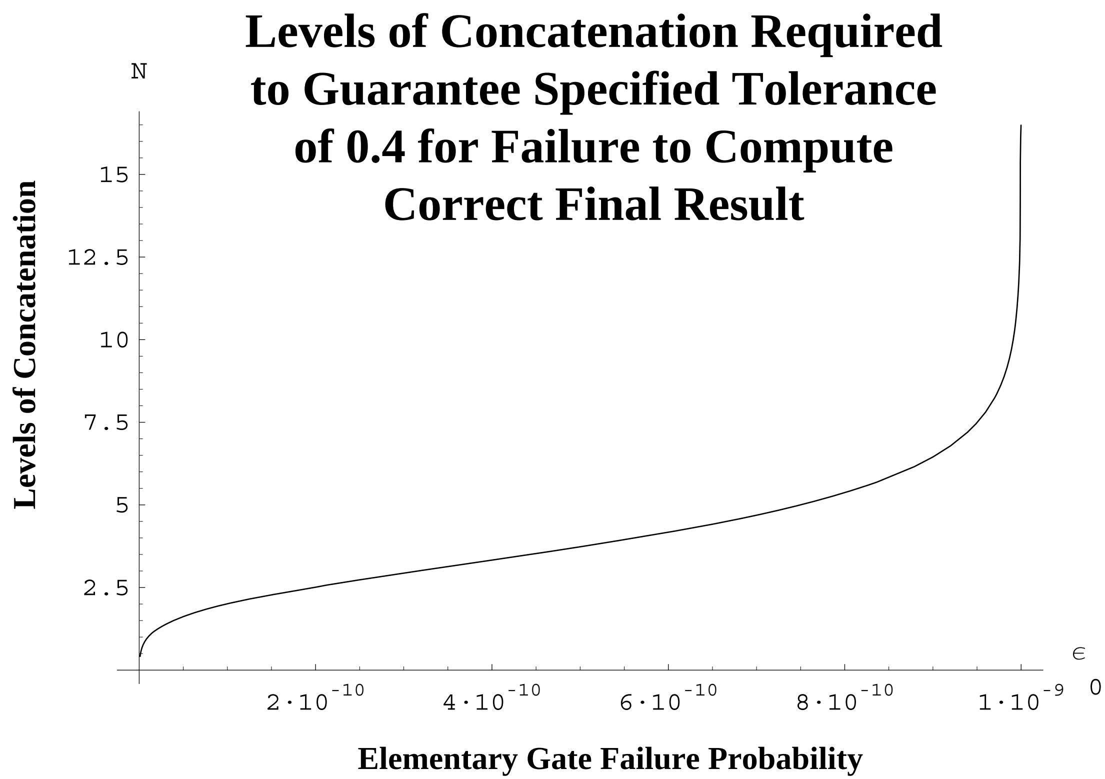
Example



Failure probability tolerance for final result	$\hat{p} = 0.4$
Number of gates in circuit	$\mathcal{N}_g = 10^{12}$
Measurement failure bound for quantum algorithm	$p_b = 0.2$
Fault tolerant error threshold	$\varepsilon_{th} = 10^{-9}$

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Quantum Error Correction (1)

QEC $$V_{dec} R \varepsilon V_{enc} \rho = \rho$$

OQEC $$V_{dec} Tr_B F_{AB} R \varepsilon W_{\rho_B} V_{enc} \rho = \rho$$

EAQEC $$\mathcal{V}_{dec} \mathcal{R} \mathcal{E} \mathcal{V}_{enc} \rho = \rho$$

EAOQEC $$\mathcal{V}_{dec} Tr_B \mathcal{F}_{AB} \mathcal{R} \mathcal{E} \mathcal{W}_{\rho_B} \mathcal{V}_{enc} \rho = \rho$$

V_{enc}, V_{dec}	encoding, decoding
$\mathcal{V}_{enc}, \mathcal{V}_{dec}$	encoding, decoding w/ebits
W_{ρ_B}	$: W_{\rho_B} \equiv \rho_A \otimes \rho_B \oplus 0_K$
$\mathcal{W}_{\rho_B}$	$: \mathcal{W}_{\rho_B} \equiv \rho_A \otimes \rho_B \oplus 0_K$ w/ebits
ε	error dynamics
R	measurement and recovery
$\mathcal{R}$	measurement and recovery w/ebits
$F_{A,B}$	projects H onto $A \otimes B$
$\mathcal{F}_{A,B}$	projects H onto $A \otimes B$ w/ebits

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Quantum Error Correction (2)

$$\lim_{c\to 0} \mathcal{V}_{\substack{enc\\dec}} = V_{\substack{enc\\dec}}$$

$$\lim_{c\to 0} \mathcal{F}_{AB} = F_{AB}$$

$$\lim_{c\to 0} \mathcal{R} = R$$

$$\lim_{\dim B\to 0} Tr_B = 1$$

$$\lim_{\dim B\to 0} W_{\rho_B} = 1$$

OQEC contains QEC

EAQEC contains QEC

EAOQEC contains EAQEC and OQEC

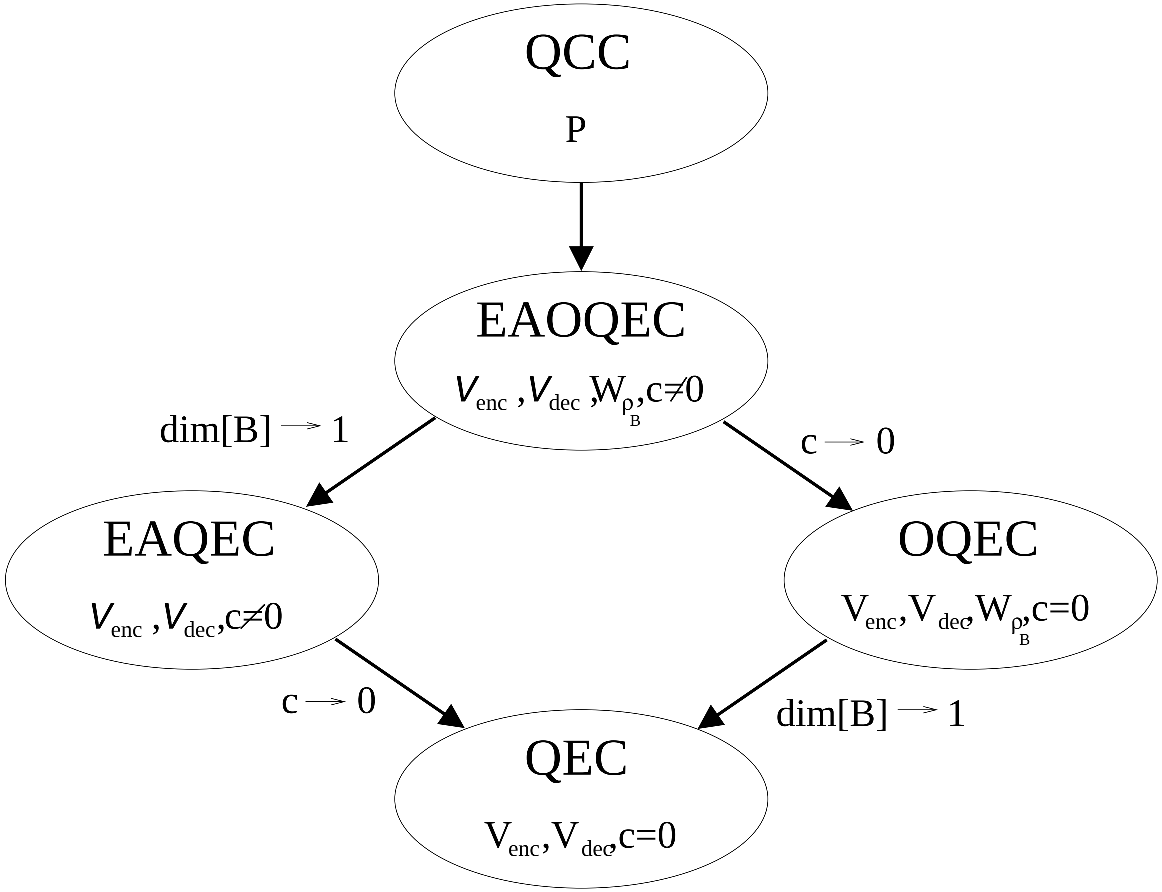
EAOQEC: $\mathcal{V}_{dec} Tr_B \mathcal{F}_{AB} \mathcal{R}\mathcal{E} W_{\rho_B} \mathcal{V}_{enc} \rho = \rho$

$$\Rightarrow \mathcal{V}_{dec} Tr_B \mathcal{F}_{AB} \mathcal{R}\mathcal{E} W_{\rho_B} \mathcal{V}_{enc} \rho - \rho = 0 \qquad \text{(EAOQEC)}$$

$$\Rightarrow \left\| \tilde{P}\rho - U\rho U^{\dagger} \right\|_1 \leq \alpha \qquad \text{(QCC)}$$

with $\tilde{P} \equiv \mathcal{V}_{dec} Tr_B \mathcal{F}_{AB} \mathcal{R}\mathcal{E} W_{\rho_B} \mathcal{V}_{enc}$, $\alpha = 0$ and $U = I$

Quantum Error Correction (3)



all forms of error correction are special cases of the QCC

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Operator Quantum Fault Tolerance (1)

- **Would like "top-down" approach to fault tolerance based on system-level dynamical constraint**

$$\left\| \widetilde{P}\rho - U\rho U^{\dagger} \right\|_1 \le \alpha$$

full specification of system dynamics with success criterion

$$\left\| \widetilde{P}^{(i)}\rho - U\rho U^{\dagger} \right\|_1$$

characterization system dynamics at i^{th} level of concatenation

$$\Rightarrow \quad \frac{\sup_{\rho}\left\| \widetilde{P}^{(i+1)}\rho - U\rho U^{\dagger} \right\|_1}{\sup_{\rho}\left\| \widetilde{P}^{(i)}\rho - U\rho U^{\dagger} \right\|_1} < 1$$

<u>Operator Quantum Fault Tolerance</u> (OQFT)

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Operator Quantum Fault Tolerance (2)

$$\mathcal{M}_{\{c\to l\}}\left[\Upsilon\left(\mathcal{M}_{\{l\to c\}}\rho\right)\Upsilon^{\dagger}\right]=U\rho U^{\dagger}$$

Υ (acting on H_{comp}) faithfully implements U (acting on H_{log})

$$P^{(i)}\left(\mathcal{M}_{\{l\to c\}}\rho\right)=\left(1-\varepsilon_f^{(i)}\right)\Upsilon\left(\mathcal{M}_{\{l\to c\}}\rho\right)\Upsilon^{\dagger}+\varepsilon_f^{(i)}Q_f^{(i)}\left(\mathcal{M}_{\{l\to c\}}\rho\right)$$

model for P that includes local stochastic noise and locally correlated stochastic noise (*i.e.*, standard quantum fault tolerance)

$$\frac{\varepsilon_f^{(i+1)}}{\varepsilon_f^{(i)}}\cdot\frac{\sup_{\rho}\left\Vert\widetilde{Q}^{(i+1)}\rho-U\rho U^{\dagger}\right\Vert_1}{\sup_{\rho}\left\Vert\widetilde{Q}^{(i)}\rho-U\rho U^{\dagger}\right\Vert_1}<1$$ **OQFT success criterion**	$$\frac{\varepsilon_f^{(i+1)}}{\varepsilon_f^{(i)}}<1$$ **standard FT success criterion**

OQFT can produce more accurate (larger) error thresholds than standard FT

Conclusion

- The probability that a quantum computation produces the correct final answer can be simply expressed in terms of the intrinsic failure probability due to quantum uncertainty in measurement and the implementation inaccuracy
- The resulting expression can be used to specify the “amount” of fault tolerance required to achieve a specified success probability that the quantum computer yields the correct final answer
- All forms of error correction and avoidance are special cases of the QCC
- The QCC provides a universal operator-theoretic framework for quantum fault tolerance based on a top-down criterion, leading to improved accuracy for error threshold values

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