Introduction to Topological Quantum Computing

Steven H. Simon

Alcatel·Lucent

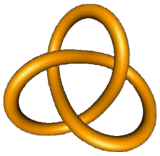
But First: A longwinded introduction on the history of this field

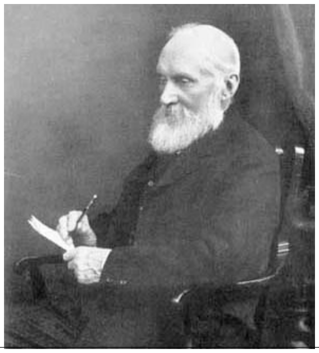
Reference:

Non-Abelian Anyons and Topological Quantum Computation
S. DasSarma , M. Freedman , C. Nayak , S.H. Simon , A. Stern
arXiv:0707.1889, Rev Mod Phys upcoming

The original string theorist (~1867)

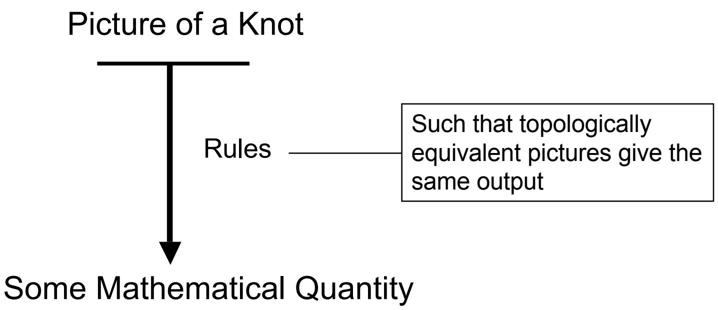
$\overset{?}{=}$





Lord Kelvin

Knot Invariant:



Picture of a Knot

Rules

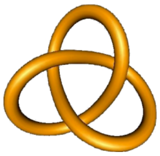
Such that topologically equivalent pictures give the same output

Some Mathematical Quantity

Theory of Knots

?

≠



Jones Polynomial (V. Jones, 1985)

≠

1

$t + t^3 - t^4$

What about this knot?

(Most) Knot Invariants Are “Exponentially Hard” to Calculate

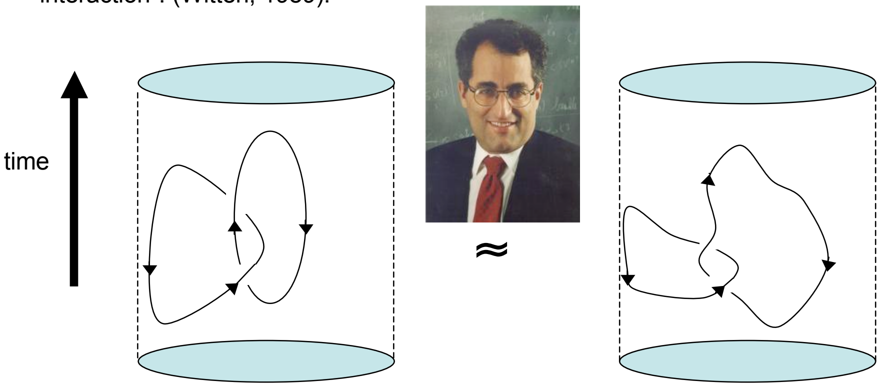
(Jaeger et al 1990)

Seemingly Unrelated: Topological Quantum Field Theory

- TQFT = QFT where amplitudes depend only on the *topology* of the process.
  (Witten, Moore, Seiberg, Froelich, … 1980s)

$$\mathcal{Z} = \int \mathcal{D}\{configs\}\; e^{iS(config)}$$

- For Chern-Simons TQFT, amplitude of a process is given by the Jones Polynomial of the knot. Integrating out Chern-Simons field leaves “topological interaction”. (Witten, 1989).



Proposed TQFT Computer

- If you had a TQFT in your lab, by measuring amplitudes, you could figure out the Jones Polynomial (M. Freedman, 1990s)

Solves a "hard" problem in polynomial time

Flashback to Prehistory of Quantum Computing

Yuri Manin (1980)

Simulating a quantum system (say the Hubbard model) on a classical computer is exponentially hard.

But if you had the physical quantum system in your lab, it could simulate itself easily.

1. What else could you do besides simulate yourself ?
2. What would errors do ?

Proposed TQFT Computer

- If you had a TQFT in your lab, by measuring amplitudes, you could figure out the Jones Polynomial (M. Freedman, 1990s)

1. What else could you do besides simulate yourself ?
2. What would errors do ?
3. Can we build a TQFT ?

Proposed TQFT Computer

The Sweetest Route To Quantum Computing

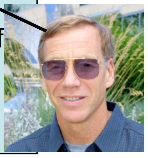
What we now believe:

1. You can do universal quantum computation with certain TQFTs
2. Errors are suppressed in a very nice way
3. Such systems exist (FQHE; more might exist if we look for them)

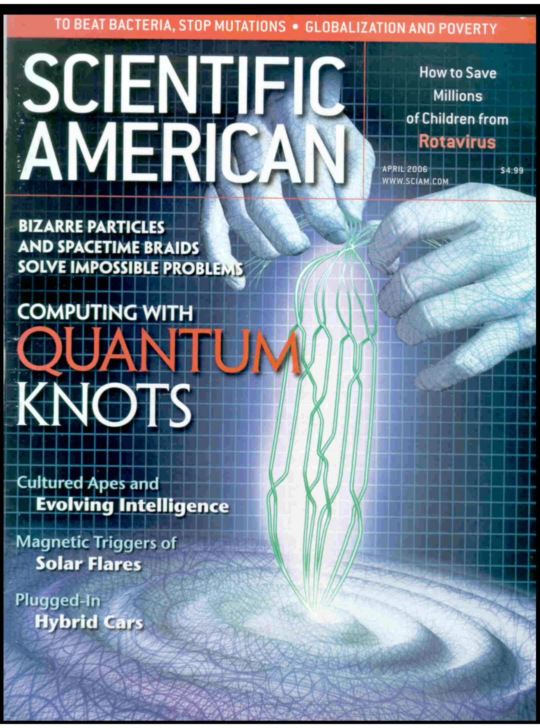
1. What else could you do besides simulat lf
2. What
3. Can w

Microsoft Station Q

A. Kitaev



M. Freedman



Physics Today

Search and Discovery

Devices Based on the Fractional Quantum Hall Effect May Fulfill the Promise of Quantum Computing

The most arcane variant of quantum computing could become the most practicable.

To grasp the potential power of quantum computing, consider its most basic ingredient, the qubit. Unlike the classical binary bit, the qubit can be on or off or in two distinct half-on states. When qubits are combined, their degeneracy balloons to fill a huge Hilbert space in which unitary transformations can change countless states at once.

Quantum mechanics is manifest in small, cold enclaves within the classical macroworld. When heat and other environmental disturbances intrude, they rob a quantum system of its coherence and its ability to compute. Error correction schemes can forestall the loss. But because the schemes work by hiding information among additional qubits, they tax efficiency greedily.

In the face of those limitations, quantum computation based on isolated two-state systems, such as trapped ions, continues to progress. Logic gates and error correction schemes have already been built and run (see, for example, PHYSICS TODAY, February 2005, page 19). Still, computation of any kind relies on executing a long train of operations. When each ion's quantum state in each operation must be protected, the chance of decoherence derailing a calculation is high.

In 1997, Alexei Kitaev of the Landau Institute in Moscow published a revolutionary proposal for fault-tolerant quantum computation.[1] The all-important degeneracy resides not in individual particles but in their shared topology. Just as a rubber ring remains a ring if poked or pulled, Kitaev's particle collective retains its coherence if locally perturbed.

Despite its brilliance, the proposal baffled many physicists. The mathematical notation is unfamiliar and formidably compact; the collective inhabits an artificial two-dimensional lattice; and the Hamiltonian isn't obviously physical. The prospect of ever building a topological quantum computer looked dim.

That pessimism has shrunk. In April, Sankar Das Sarma of the University of Maryland, Michael Freedman of Microsoft Corp, and Chetan Nayak of UCLA outlined how one might construct a topological logic gate from a familiar material, gallium arsenide.[2]

Turning the outline into a testable device will be demanding. But now two simpler proposals have appeared that, while stopping short of building a gate, would demonstrate the practicability of the underlying physics.[3,4] The race to compute topologically has begun.

Not your high-school physics

Kitaev proved that topological quantum computing was intrinsically robust. However, the notion of exploiting topology to perform calculations was conceived a decade earlier. In 1988, Freedman, then a mathematics professor at the University of California, San Diego, visited Harvard University. There, he learned about two recent advances in quite disparate theoretical fields.

Computer scientists had proved that certain problems, like optimizing a traveling salesman's route through his territory, were as equivalently difficult to solve as calculating the Jones polynomial, an invariant of knots and links in three dimensions. Meanwhile, Edward Witten had proved that a certain quantum field theory (Yang–Mills with a Chern–Simons term) in two spatial dimensions plus time looked mathematically like the Jones polynomial.

Freedman connected the two proofs and had an epiphany: Rather than trying to calculate the Jones polynomial (and, by extension, all those other hard problems), why not simply measure it by manipulating whatever system Witten's quantum field theory applied to?

His friend Cliff Taubes brought him back down to earth. "What Witten thinks of as physics has nothing to do with what you learned in high school," said Taubes. "The stuff probably doesn't exist in the real world."

Deflated, Freedman shelved his idea. Fortunately, Taubes was mistaken. The stuff does exist—in the bizarre, low-temperature physics of the fractional quantum Hall (FQH) effect.

Intuitive wavefunctions

Condensed-matter theorists can write down an appropriate Hamiltonian for a quantum Hall system. But they can't generally solve it to find the ground state and its excitations. Instead, in an approach pioneered by Robert Laughlin, they intuit a wavefunction and then work backward to see if it works.

Laughlin's wavefunction and its subsequent generalizations describe the originally discovered FQH states whose filling factors have odd denominators. But in 1987, Robert Willett and his collaborators found a state with a filling factor of 5/2.

As a string theorist, Greg Moore

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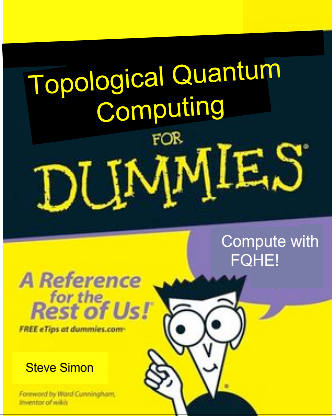
From:
Bonesteel, Hormozi
Simon, Zikos, PRL05

Outline:

- Long-winded introduction ✓
- Dummy’s Guide to Topological Quantum Computing



Dummy’s Guide To Topological Quantum Computing

- Uses 2 Dimensional Systems which are realizations of TQFTs, i.e., have quasiparticles with **NonAbelian Statistics**.
- Quantum Information is encoded in **nonlocal topological degrees of freedom** that do not couple to ***any*** local quantity.
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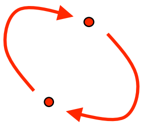
Statistics in Brief:

Statistics:

What happens to a many-particle wavefunction under "exchange" of identical particles.

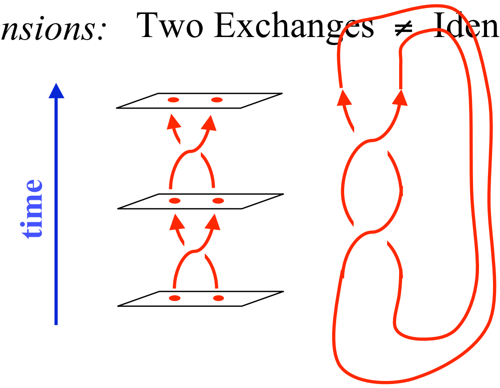


Dogma:

Exchanging twice should be identity

- Bosons $\text{Ø}(r_1,r_2) = \text{Ø}(r_2,r_1)$
- Fermions $\text{Ø}(r_1,r_2) = -\text{Ø}(r_2,r_1)$

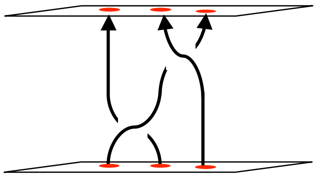
In 2+1 Dimensions: Two Exchanges ≠ Identity



In 3+1 Dimensions: Two Exchanges = Identity

No Knots in World Lines in 3+1 D !

Statistics:



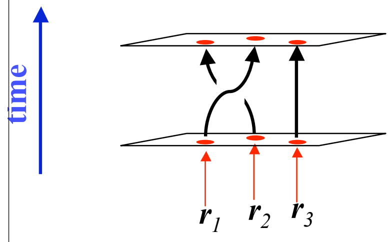
In 3+1 D :

- No Knots in World Lines
- Topologically Different Paths = Different Permutations
- Statistics are Rep of the Permutation Group
- Bosons or Fermions

In 2+1 D :

- Knots in World Lines
- Topologically Different Paths = Different **Braids**
- Statistics are a Rep of the **Braid** Group
- More Possibilities (Anyons + Non-Abelions)

What is NonAbelian Statistics ?



Suppose 2 Degenerate
Orthogonal States $|\psi_1\rangle$, $|\psi_2\rangle$

Vector Represents State

$$\text{Ø}_\mathbf{i} = a_1|\psi_1\rangle + a_2|\psi_2\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

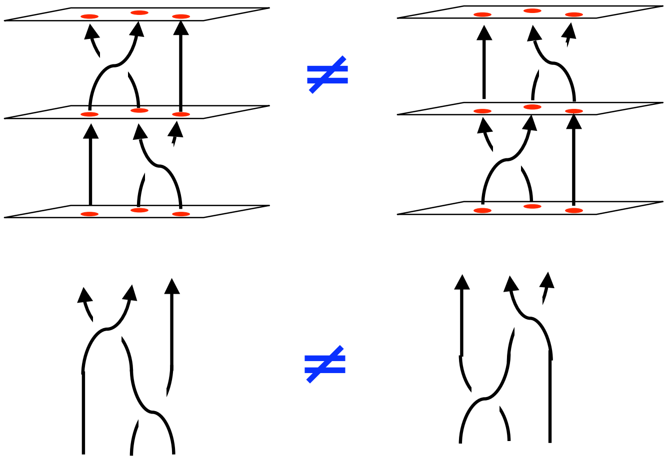
$$\text{Ø}_\mathbf{f} = \tilde{a}_1|\psi_1\rangle + \tilde{a}_2|\psi_2\rangle = \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix}$$

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix} = U \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Unitary Matrix Represents
(Topological) Braid Operation

Matrix Representation
Of The Braid Group
Froelich, Moore, Read, Witten, …

Braid Group is "Non-Abelian" = Non-Commutative



Statistics Are Matrix Representation of Braid Group:
+ Matrices are Non-Abelian → ***NonAbelian Statistics***

Non-Abelian Statistics (Summary) :

- Vector Represents State Within a Degenerate Space

$$|\psi_1\rangle\,,\,|\psi_2\rangle \qquad\qquad \mathbf{\O}_{\mathbf{i}} = a_1|\psi_1\rangle + a_2|\psi_2\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

- Dimension of degenerate space increases exponentially with number of quasiparticles.

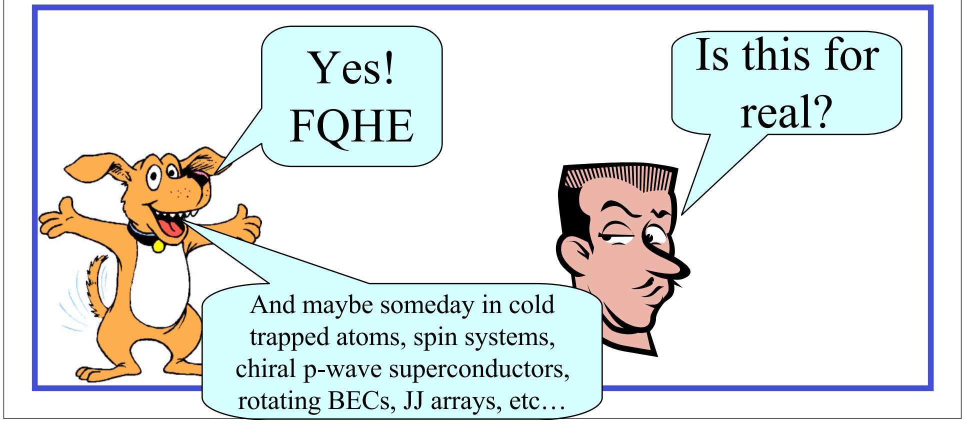
- Unitary (Berry's) Matrix Represents an Adiabatic Braiding Operation

$$\mathbf{\O}_{\mathbf{f}} = U\,\mathbf{\O}_{\mathbf{i}}$$

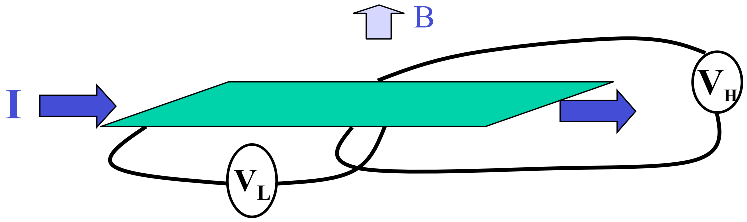
Unitary Matrix Depends Only on The Topology of the Braid = **TQFT**

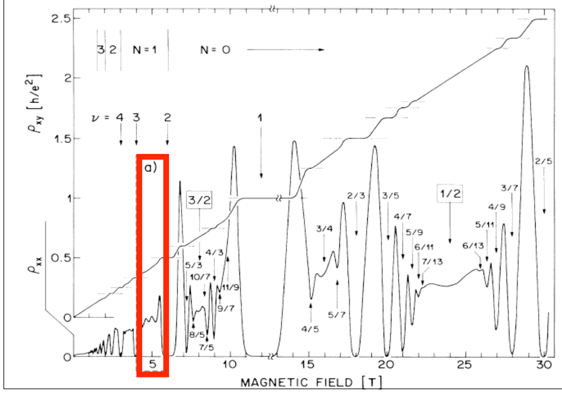
Dummy’s Guide To Topological Quantum Computing

- Uses 2 Dimensional Systems With **“Non-Abelian” Statistics.** (Also known as TQFTs). ✓



FQHE Experiments:

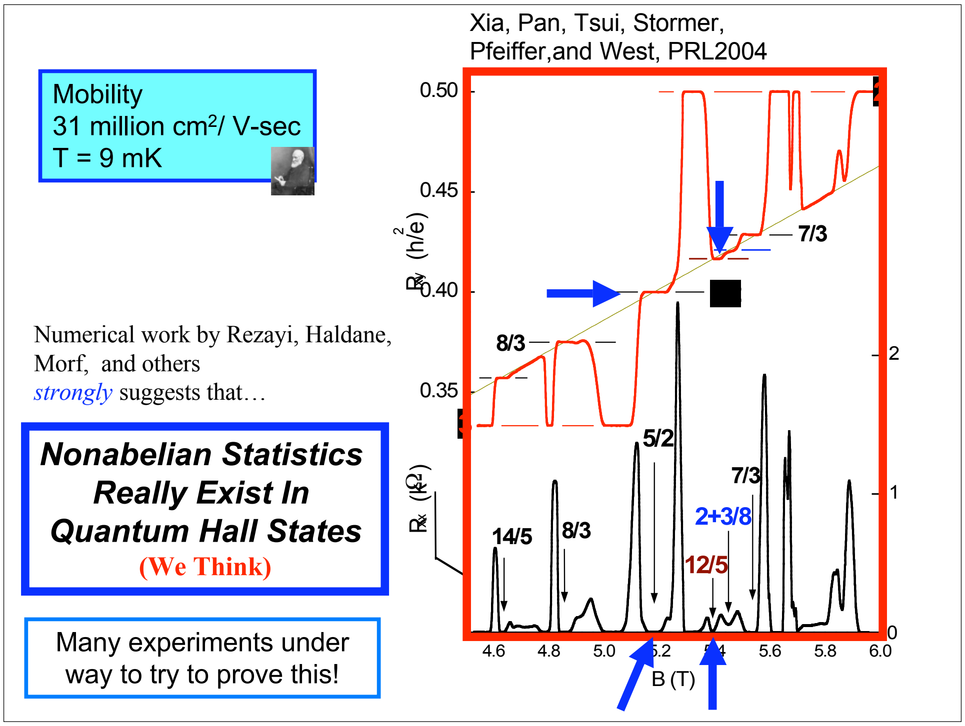




Horst Stormer

Dan Tsui

Bob Laughlin



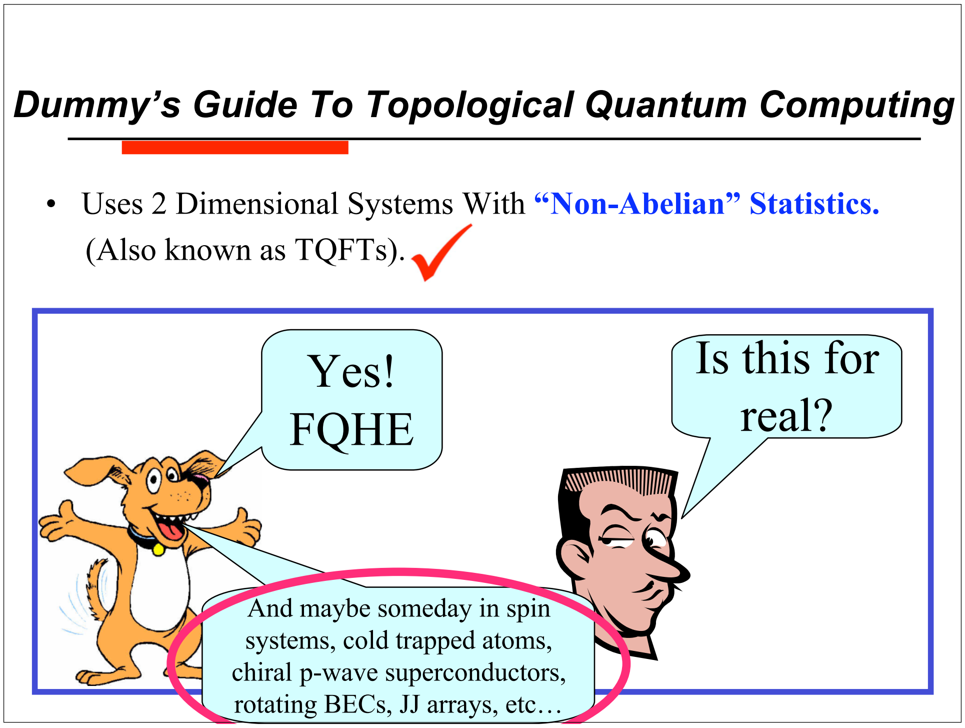
Xia, Pan, Tsui, Stormer,
Pfeiffer,and West, PRL2004

Mobility
31 million cm^2/ V-sec
T = 9 mK

Numerical work by Rezayi, Haldane, Morf, and others *strongly* suggests that…

Nonabelian Statistics Really Exist In Quantum Hall States
(We Think)

Many experiments under way to try to prove this!



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- Uses 2 Dimensional Systems With **"Non-Abelian" Statistics.** (Also known as TQFTs). ✓

And maybe someday in spin systems, cold trapped atoms, chiral p-wave superconductors, rotating BECs, JJ arrays, etc…

Dummy's Guide To Topological Quantum Computing

- Uses 2 Dimer

(Also known

If you could design any Hamiltonian, you could choose one that gives Nonabelian Statistics.

None of these Hamiltonians are particularly simple. None have (yet!) been found in nature.

· **Kitaev, Freedman, Levin, Wen, Nayak, Shtengel, Fendley, Fradkin, …**

And maybe someday in spin systems, cold trapped atoms, chiral p-wave superconductors, rotating BECs, JJ arrays, etc…

Dummy's Guide To Topological Quantum Computing

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(Also known

If you could design any Hamiltonian, you could choose one that gives Nonabelian Statistics.

In cold atom lattices you can design your Hamiltonian … but …. (T, t)

· Zoller et al; Troyer, Whaley, …

And maybe someday in spin systems, cold trapped atoms, chiral p-wave superconductors, rotating BECs, JJ arrays, etc…

Dummy’s Guide To Topological Quantum Computing

- Uses 2 Dimen

  (Also known

1) Real Superconductors: SrRuO4 ?

(2) Chiral p-wave pairing of cold atoms?
Radzihovsky, Gurarie, et al … but … (T,t)

And maybe someday in spin systems, cold trapped atoms, chiral p-wave superconductors, rotating BECs, JJ arrays, etc…

Dummy’s Guide To Topological Quantum Computing

- Uses 2 Dimen

  (Also known

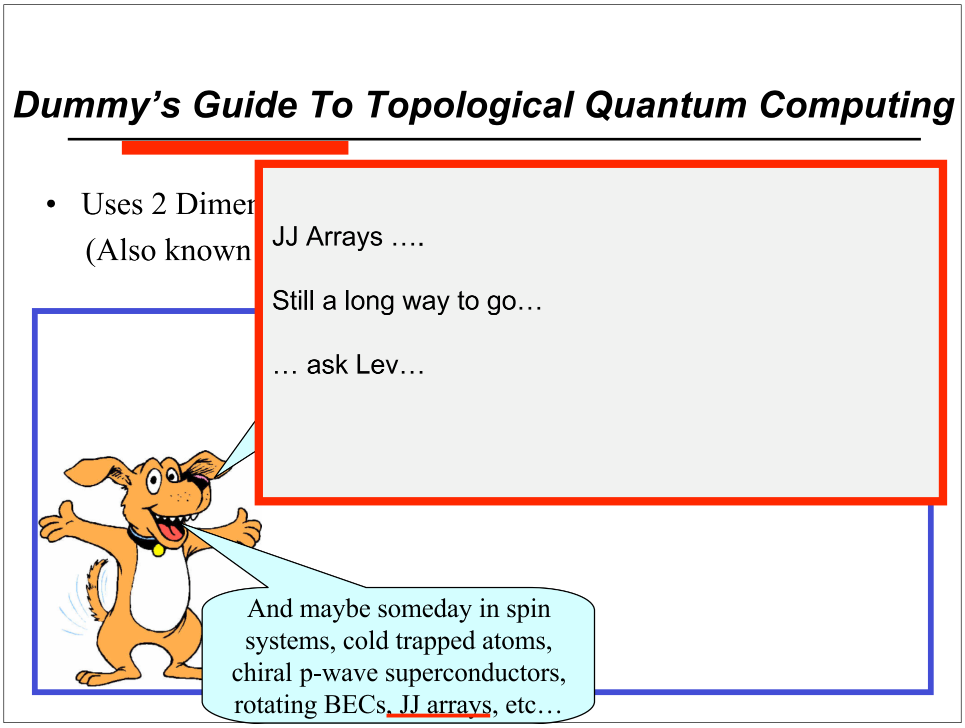
Rotating BEC = Quantum Hall Effect in Disguise

Coriolis Force = Lorentz Force

Ratio of (Rotation Rate / Density) needs to be Increased 1-2 orders of magnitude still. .. and (T, t)

Cooper, Gunn, Wilken, Read, Rezayi, …

And maybe someday in spin systems, cold trapped atoms, chiral p-wave superconductors, rotating BECs, JJ arrays, etc…



Dummy's Guide To Topological Quantum Computing

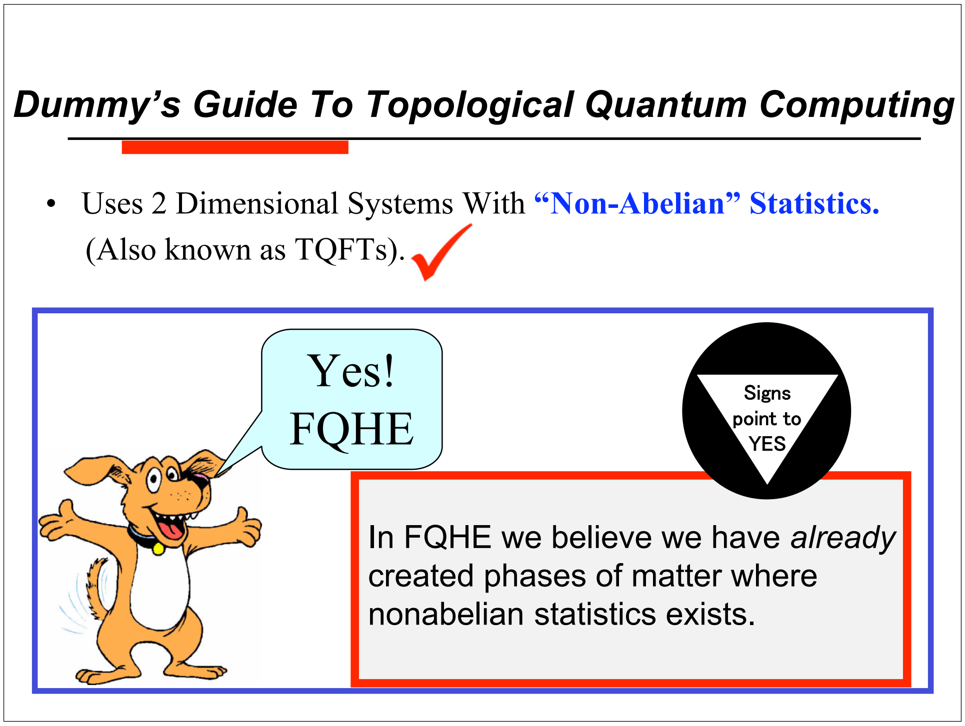
- Uses 2 Dimen

  (Also known

JJ Arrays ….

Still a long way to go…

… ask Lev…



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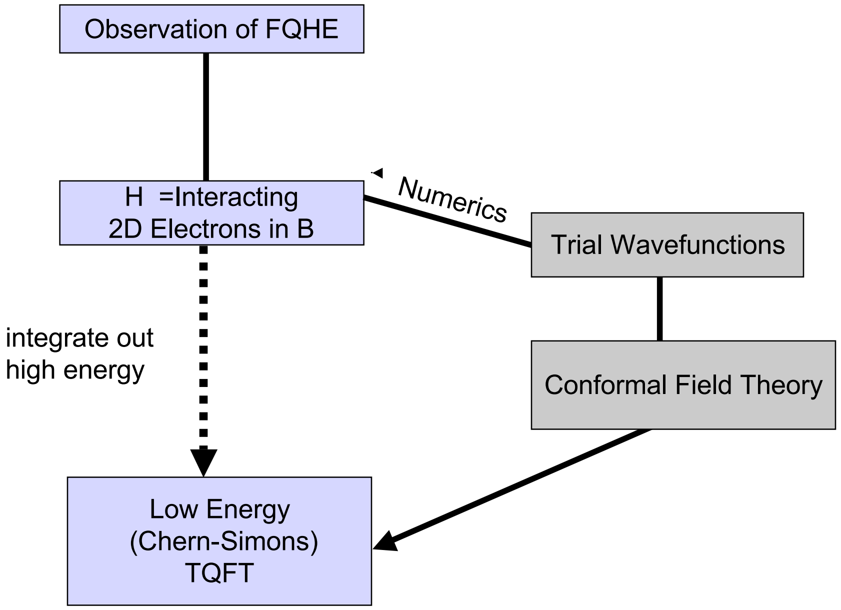
  (Also known as TQFTs). ✓

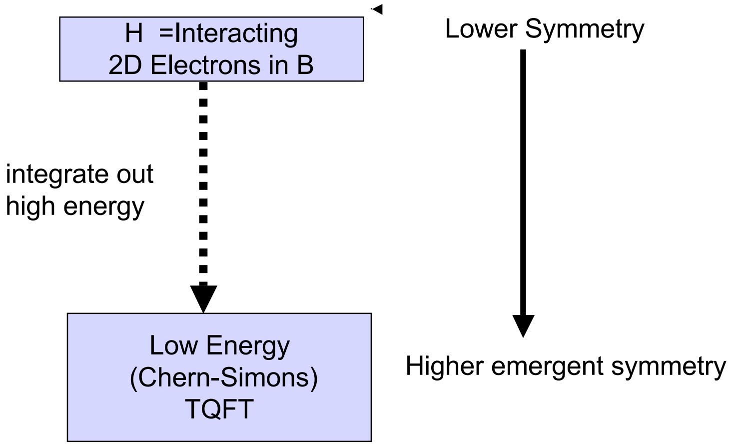
In FQHE we believe we have *already* created phases of matter where nonabelian statistics exists.

Re:Theoretical Evidence



Re:Theoretical Comment

Symmetry Emergence – not symmetry breaking!



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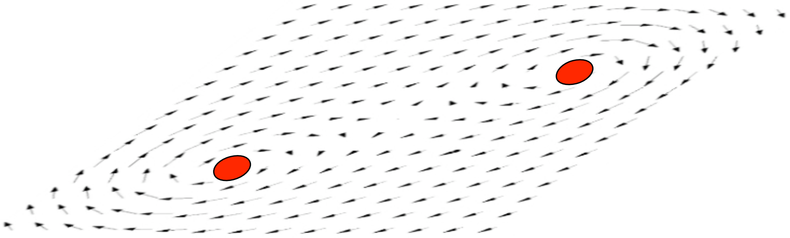
Quasiparticles in Fractional QHE



- Quasiparticles are topological defects
- Charge lives in the core
- topological degrees of freedom live in the “vorticity” (sort of)

The topological degree of freedom can be thought of as the configuration class of the emergent Chern-Simons gauge field.

Energy is independent of this topological degree of freedom.

Topological Quantum Numbers:

- (Usually) An individual quasiparticle exists only in a single state

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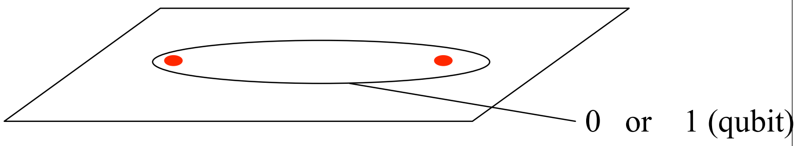
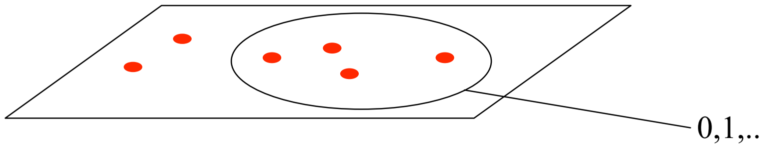
Topological Quantum Numbers:



- Two (or more) quasiparticles can exist in more than one state… described by a quantum number, ex 0 or 1

You cannot determine the quantum number by only measuring one of the quasiparticles

Topological Quantum Numbers:



- Groups of particles similarly have quantum numbers

These quantum numbers can be thought of as describing the global topology of the effective Chern-Simons gauge field

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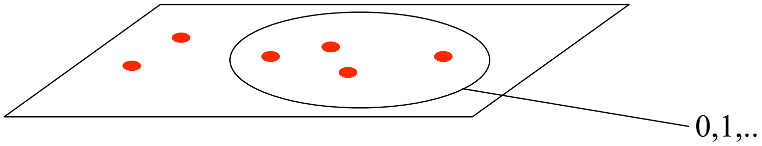
Topological Quantum Numbers:



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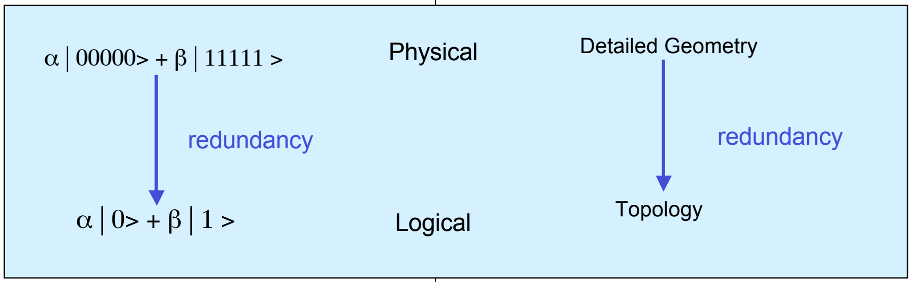
These quantum numbers can be thought of as describing the global topology of the effective Chern-Simons gauge field

No local measurements can determine this quantum number! →
No local operator can couple to this quantum number →
Topological quantum numbers decoupled from "noise"

Comparison with Conventional Error Protection:

Conventional Protection:	Topological Protection:
Many Physical Qubits Code One "Logical" or Computational Qubit	Global topology codes qubits



$\alpha|00000\rangle + \beta|11111\rangle$ — Physical — Detailed Geometry

redundancy — redundancy

$\alpha|0\rangle + \beta|1\rangle$ — Logical — Topology

Software Error Correction	Hardware Error Protection

Midway: Bacon Codes etc

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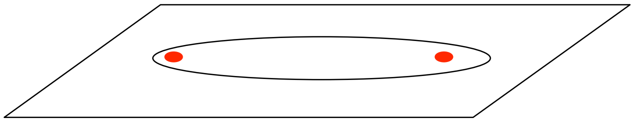
How to Measure ?
How to Manipulate?

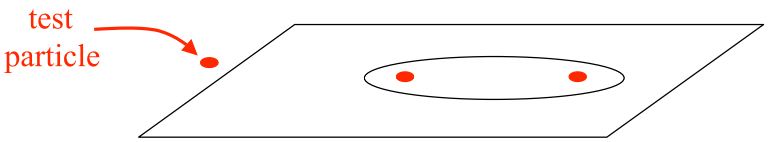
Topological Quantum Numbers:



- To Measure the Quantum Number

(1) Move the particles microscopically close, and measure force

Topological Quantum Numbers:



- To Measure the Quantum Number

(1) Move the particles microscopically close, and measure force

OR

(2) Do an interference experiment surrounding both quasiparticles

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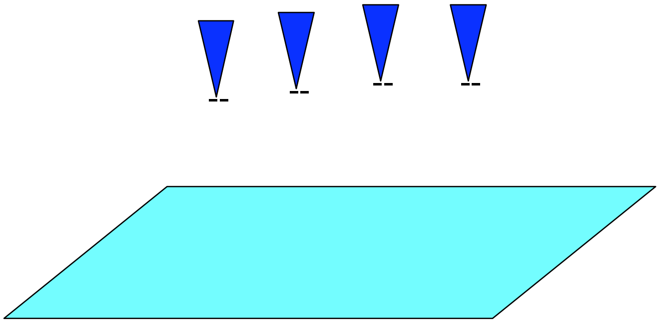
Dragging (Braiding) Particles Adiabatically



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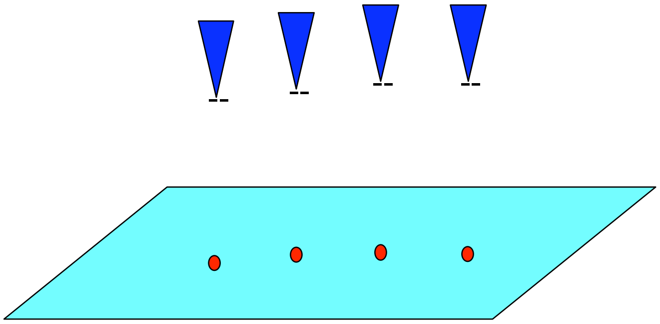

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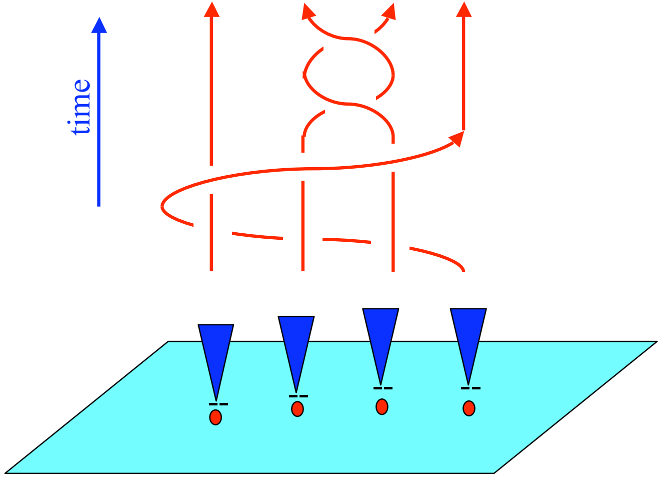
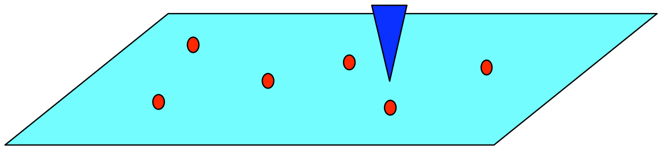


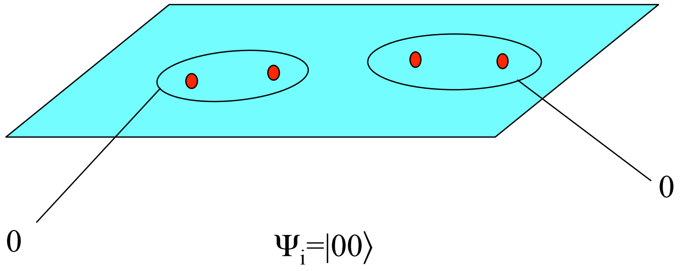
Major Simplification !

Theorem (Simon, Bonesteel, Freedman… PRL05): In any topological quantum computer, all computations can be performed by moving only a single quasiparticle!



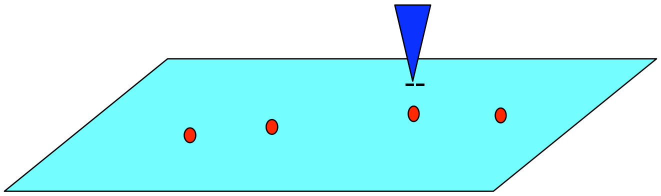
Reduced technological difficulty!

Simple Example: 5/2 state

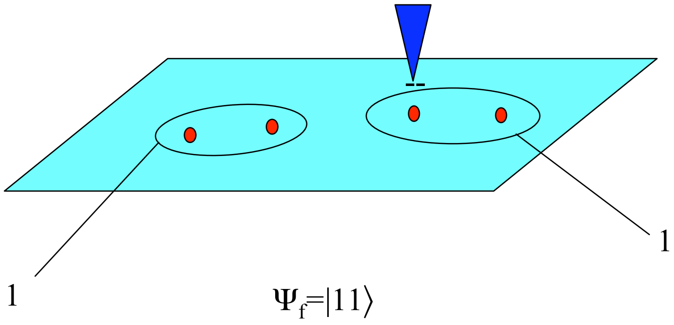


$\Psi_i=|00\rangle$

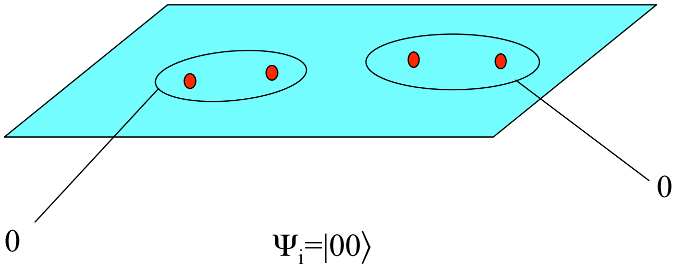
Simple Example: 5/2 state

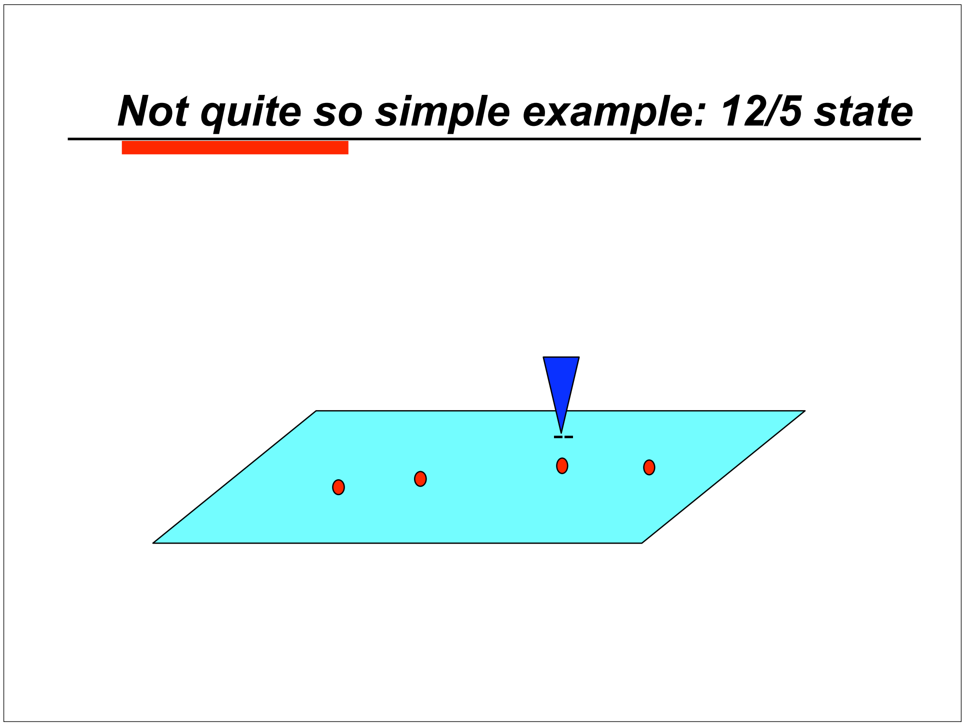


Simple Example: 5/2 state

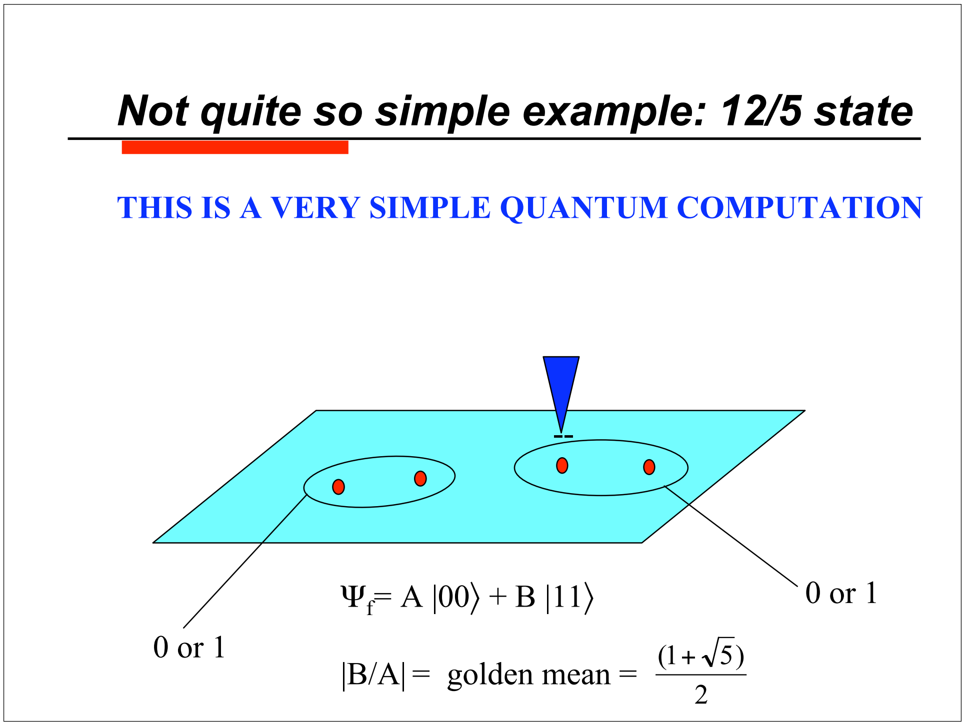


Not quite so simple example: 12/5 state





Not quite so simple example: 12/5 state



Not quite so simple example: 12/5 state

THIS IS A VERY SIMPLE QUANTUM COMPUTATION

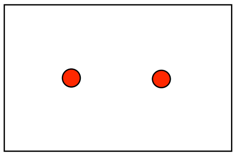
$$\Psi_f = A\,|00\rangle + B\,|11\rangle$$

$$|B/A| = \text{ golden mean } = \frac{(1+\sqrt{5})}{2}$$

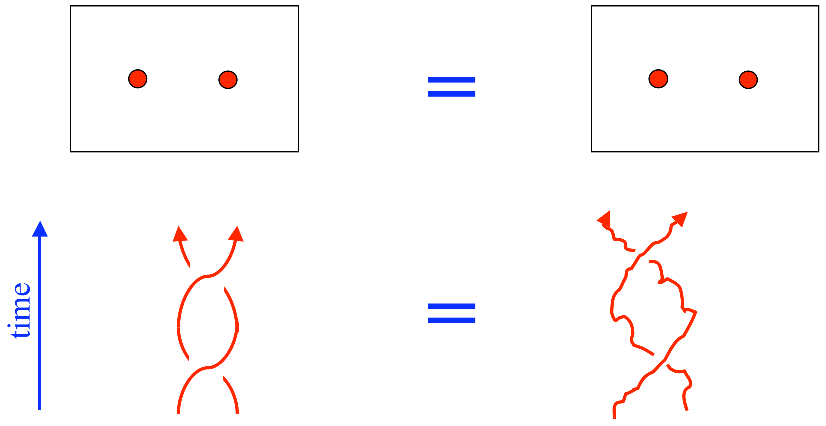
Dummy's Guide To Topological Quantum Computing

- Uses 2 Dimensional Systems which are realizations of TQFTs, i.e., have quasiparticles with **NonAbelian Statistics**. ✓
- Quantum Information is encoded in **nonlocal topological degrees of freedom** that do not couple to ***any*** local quantity. ✓
- States can be manipulated by dragging **(braiding)** quasiparticles around each other. ✓
- The operations (gates) performed on the qubits depends ***only*** on the topology of the braids.

Topological Robustness



Topological Robustness



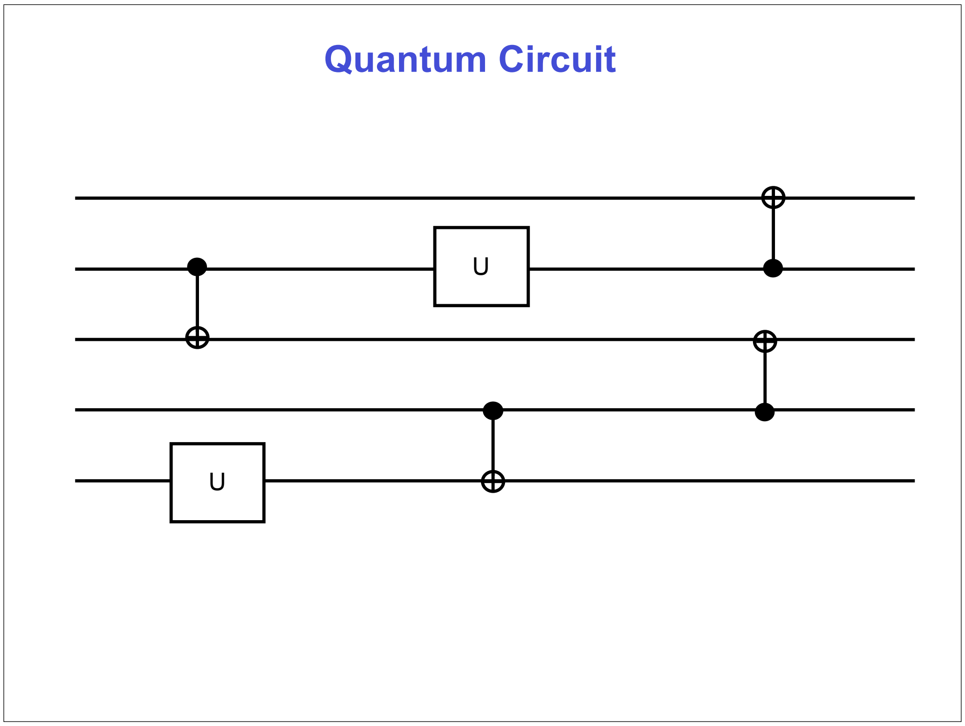
Dummy’s Guide To Topological Quantum Computing

Quantum Information is Topologically
Protected (Isolated) From Decoherence

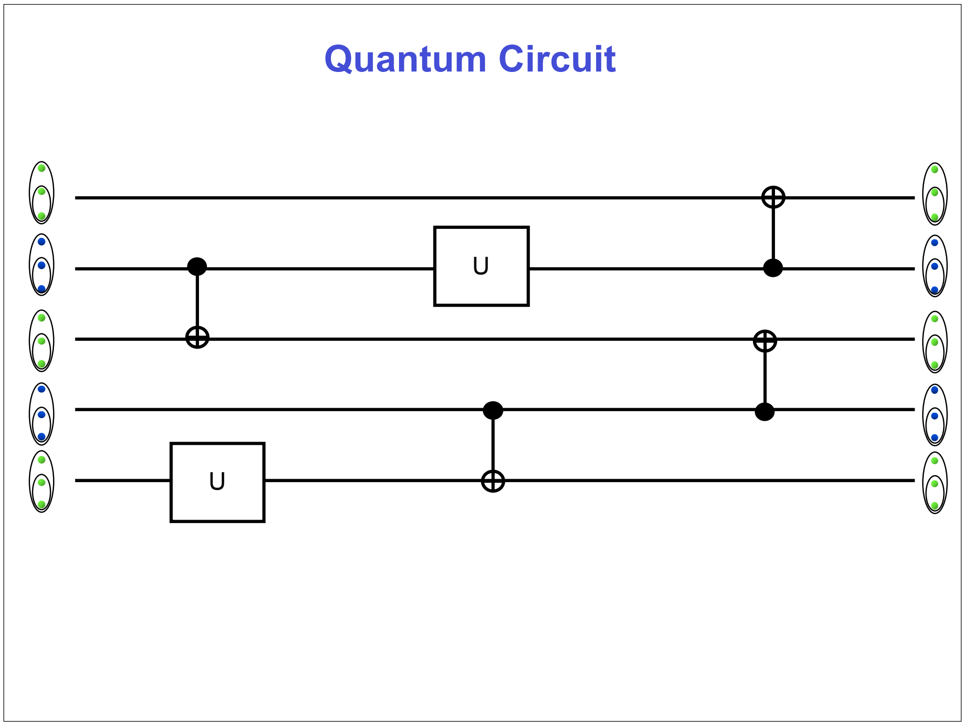
AND

The Operations are Topologically Protected !

- The operations (gates) performed on the qubits depends ***only*** on the topology of the braids

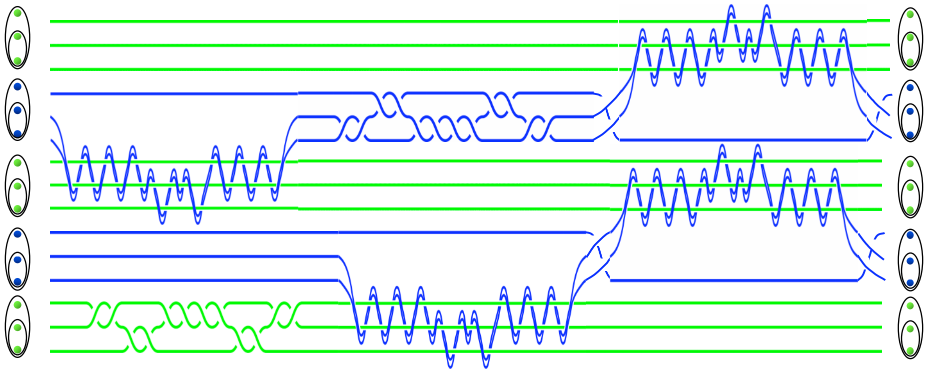


Quantum Circuit



Quantum Circuit

Braid



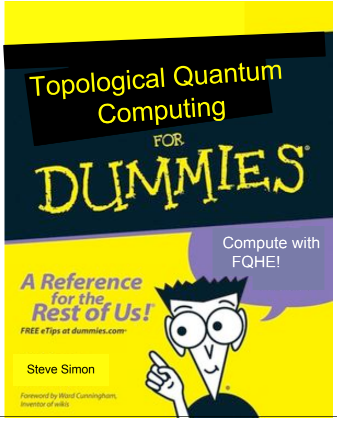
From an interview with Bill Gates in the May, 2004 online edition of Scientific American

```
SA: Well, some of the physical entities [non-Abelian
anyons, a type of sub-atomic particle] need to actually be
discovered before they can move ahead from theory to create
the type of quantum computer that Michael Freedman is
working on at Microsoft Research?
```

BG: There is an approach that doesn't involve an advance in physics; it involves a missing piece in terms of a physical phenomenon that we don't know about. The models are in terms of existing physical phenomena. Now, can you really build that. Does it scale up? It's hard to say.

Outline:

- Long-winded introduction ✓
- Dummy's Guide to Topological Quantum Computing ✓
- FAQ



Frequently Asked Questions

1. *Is there ANY decoherence?*
2. *Why is topological so much better than nontopological q–computing?*
3. *Does it matter if you put the quasiparticles back where they started?*
4. *Can you quantum compute with the 5/2 state?*
5. *What other fractions are interesting?*
6. *How far apart should the particle be, and how fast should you move the particles around.*
7. *Can you show me a CNOT gate? how accurate is it?*
8. *How many braid operations does it take to do a useful computation*
9. *Do you really believe this?*

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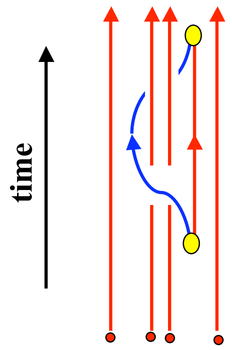
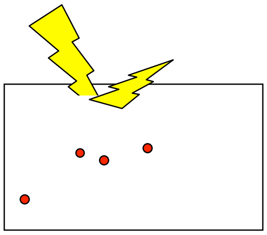
Decoherence?





Topologically Nontrivial Processes Creates Decoherence

These process can be eliminated exponentially at low T!

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Why is topological so good

- Topological Protection of Memory -
  - Decoherence is suppressed *EXPONENTIALLY*
- Operations are *AUTOMATICALLY PRECISE (quantized)*
- *"Beauty is truth, truth beauty,--that is all*
  ye know on earth, and all ye need to know." – Keats

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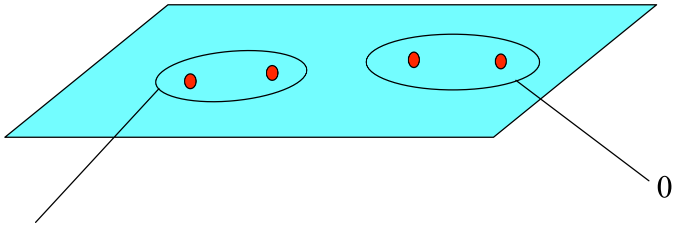
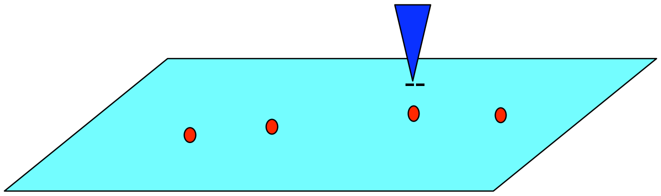
Not quite so simple example: 12/5 state



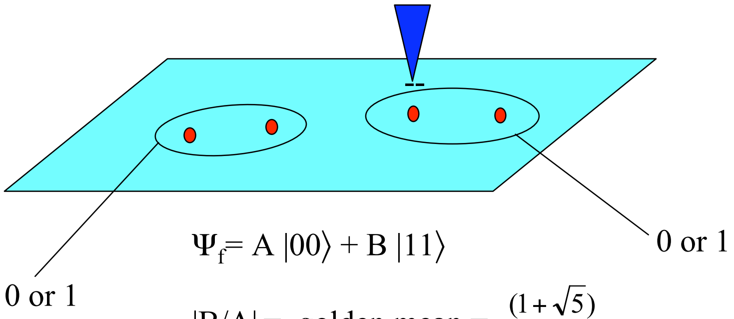
$\Psi_i = |00\rangle$

Not quite so simple example: 12/5 state



Not quite so simple example: 12/5 state

THIS IS A VERY SIMPLE QUANTUM COMPUTATION



$\Psi_f = A\,|00\rangle + B\,|11\rangle$

$|B/A| =$ golden mean $= \dfrac{(1+\sqrt{5})}{2}$

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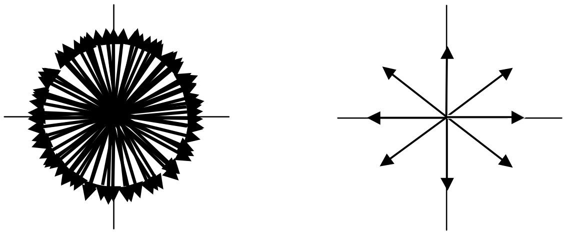
Does 5/2 compute?

Universal Q-Computation ⇔ Approximate any Unitary Transform

The Moore-Read Pfaffian Cannot Do This!

Braid Group Representation is not “Dense”



BUT…. Hybrid schemes might work!

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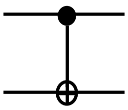
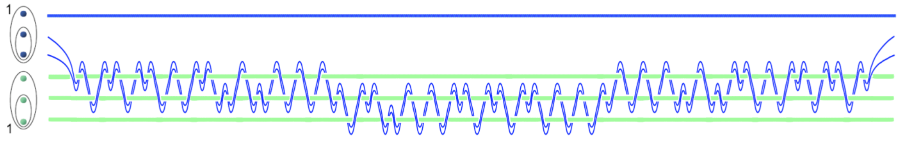
Universal Set of Topologically Robust Gates

(Bonesteel, Hormozi, Simon, 2005, 2006)

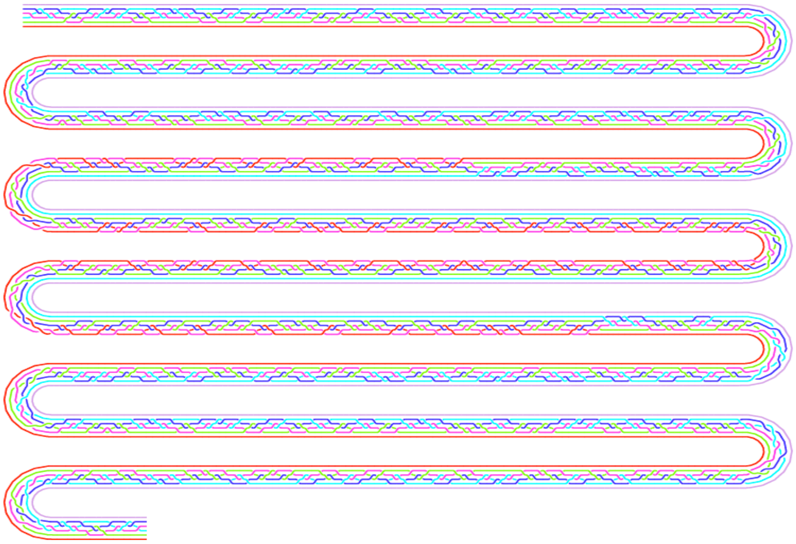
Single qubit rotations: $|\psi\rangle \;-\boxed{U_{\vec{\phi}}}-\; U_{\vec{\phi}}\,|\psi\rangle$

Controlled NOT:





Solovay-Kitaev Improved CNOT



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Introduction to Topological Quantum Computing

Thanks for Listening!

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