EUROPHYSICS LETTERS 1 February 2004

Europhys. Lett., **65** (3), pp. 365–371 (2004)
DOI: 10.1209/epl/i2003-10096-0

Three-dimensional lubrication model of a contact line corner singularity

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(received 26 June 2003; accepted in final form 24 November 2003)

PACS. 68.08.Bc – Wetting.
PACS. 47.15.Gf – Low-Reynolds-number (creeping) flows.
PACS. 68.15.+e – Liquid thin films.

Abstract. – Drops sliding along inclined planes in situations of partial wetting develop a "corner" at the back of the drop, similar to contact line shapes observed on plates withdrawn from a bath. We discuss, in the framework of lubrication theory, the 3D structure of this corner singularity: similarity solutions exist such that the interface adopts the shape of a cone, with well-defined relationships linking its two characteristic angles. Near both contact lines forming the corner, the solution reduces to the well-known lubrication solution for a single contact line, with an effective capillary number involving the normal component of the contact line velocity.

Coating flows, wetting, and dewetting occur in an enormous variety of situations and pose many challenging scientific problems [1, 2]. Modern research in this area focuses on the influence of rheology of the coating liquid [3], the effect of surfactants, and dynamical influences of high speed among other topics [4]. In many problems involving such thin films, there is a contact line and often the shape and motion of the contact line is fundamental to the dynamics of the process [1,5]. Despite the prevalence of this problem, most of the studies of contact line dynamics, experimental and theoretical, treat two-dimensional situations. We provide here a theoretical description of experiments involving a flow with three-dimensional structure in the neighborhood of a contact line, which provides more insight into the manner in which hydrodynamics in the fluid bulk control the shape of a three-dimensional fluid wedge, and dictate the onset of an instability from a two-dimensional to a three-dimensional configuration.

Recently, Podgorski *et al.* reported experiments of drops sliding down a plane in situations of partial wetting [6]. As suggested in fig. 1a, when the drop velocity lies inside a well-defined window, its rear boundary develops a "corner" of angle ϕ, composed of two contact lines inclined with respect to the drop velocity $\boldsymbol{V}$. This angle is related to the capillary number $Ca = \eta V/\gamma$ (η and γ are, respectively, the viscosity and surface tension), by a scaling

$\sin\phi \propto 1/Ca$. These observations are analogous to earlier ones [5] for plates withdrawn from a bath (or plunging into a bath), where the contact lines separating the wet and dry domains of the plates exhibit sawtooth shapes. In refs. [5, 6], this ϕ-Ca relation was qualitatively explained as follows. The macroscopic contact angle at the back of the drop $\theta_{\rm d} = F(Ca)$, decreases with increasing capillary number Ca until $\theta_{\rm d}$ vanishes (or reaches a critical value) at a critical capillary number $Ca_{\rm c}$. Instead of leaving a thin film on the solid, the contact line inclines to maintain the effective capillary number $Ca\sin\phi$ (*i.e.* that defined in a direction normal to the contact line) equal to $Ca_{\rm c}$ which implies the scaling $\sin\phi \propto 1/Ca$.

To the best of our knowledge, this qualitative argument has never received any justification starting from the basic equations of hydrodynamics, though Podgorski *et al.*'s experiment has received attention in several theoretical works [7–9]. Even the possibility of a corner structure, and the related flow field remain to be established analytically. In this letter, we discuss this problem in the framework of lubrication theory [10], *i.e.* creeping flow with negligible inertia and small slope of the liquid-air interface with respect to the solid plane. We first revisit the "classical" description in 2D of the flow near a single contact line, then adapt it to provide a simplified approach for the corner (also available in ref. [8]), and finally present the exact three-dimensional treatment. As we shall see, similarity solutions of the flow equation delimited by two cornered contact lines exist in which the interface adopts a conical shape, such as that suggested in fig. 1b. A unique relation links the two angles ϕ and Ω of this figure and is in close agreement with recent visualisations of the drop shapes [11, 12]. These similarity solutions generalize that applied recently by Betelù and Diez [13] to a dry line inside a film.

This basic contact line problem is important for two reasons. First, contact line dynamics is still a disputed subject, with the hydrodynamic model [14, 15] coexisting with other descriptions [16]. In spite of many studies, the understanding of the precise manner in which a contact line can move with the line singularity implied by the no-slip condition on a solid surface remains unresolved. Also, this "corner" structure arises in many coating configurations [1,5], especially in the context of air entrainment [1]. As stated in the introduction, most approaches to wetting treat flows directed normal to the contact line, while here the flow is inclined with respect to the normal. Our discussion in this letter is one of the first to account for the possibility of a flow tangent and normal to a moving contact line.

In the familiar case of a contact line receding at constant speed V on a solid surface, with the fluid velocity remaining everywhere normal to the line, the flow reduces to a 2D problem suggested in fig. 1c, in which $h(x)$ denotes the film thickness. The problem is conveniently viewed in a reference frame moving with the contact line, in which the plate is moving along Ox with a velocity $-V$. Very near the contact line, in the well-known framework of lubrication theory applied to wetting [1, 2], the flow reduces to the superposition of a translation imposed by the plate and of a back-flow of Poiseuille type induced by the gradient of capillary pressure:

$$u \approx -V + \frac{z(2h-z)}{2\eta}\gamma h_{xxx}, \tag{1}$$

where u denotes the x component of fluid velocity and the subscript x indicates the derivative of h with respect to x. Mass conservation imposes that the local flux of liquid along Ox $j_x = \int_0^{h(x)} u\,\mathrm{d}z$ vanishes, which can also be understood as a balance between two terms: a flux $j_{x1} = -Vh(x)$ generated by the plate motion and a flux driven by the capillary pressure gradient $j_{x2} = -(h^3/3\eta)(-\gamma h_{xxx})$. The balance $O = j_{x1} + j_{x2}$, yields the equation

$$3\frac{Ca}{h^2} = h_{xxx}. \tag{2}$$

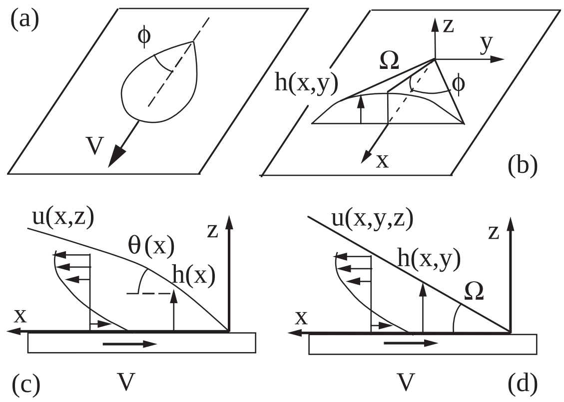

Fig. 1

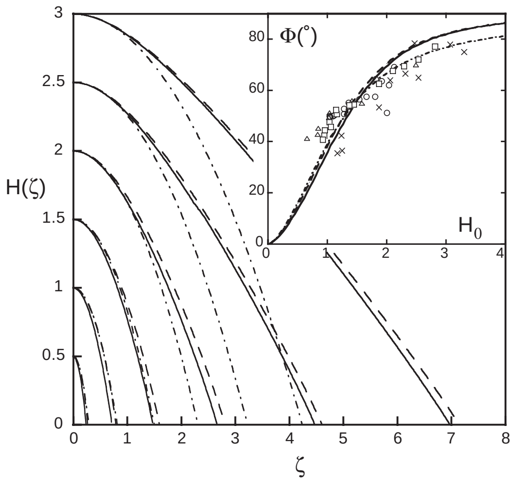

Fig. 2

Fig. 1 – (a) Typical contact line shapes observed on drops sliding along inclined planes in situation of partial wetting. (b) Sketch of the geometry involved near the corner. (c) Usual hydrodynamical description of the flow near a contact line, where the solid plate moves at constant speed with respect to the line. (d) Similar flow structure assumed near the corner.

Fig. 2 – Profiles of the cone cross-section involved in fig. 1b, the free surface being given by $h = Ca^{1/3} x H(\zeta = y/x)$. Insert: relationship linking the opening corner angle ϕ to $H_0 = Ca^{-1/3} \tan\Omega$, Ω being the cone angle viewed from the side on fig. 1b. Continuous lines correspond to the full 3D solution of eq. (9), dashed lines to the planar-flow approximation (2D model, eq. (5)) and dash-dotted lines to the parabolic approximation for the cross-sectional shape. The $\times$ symbols are experimental values deduced from ref. [11] (silicon oil drops sliding on fluoropolymers, $\gamma = 20\,\mathrm{mN\,m^{-1}}$, $\eta = 20\,\mathrm{cP}$) and the open symbols are more recent measurements [12]: (◯) $\eta = 10\,\mathrm{cP}$, (△) 100 cP, and (□) 1000 cP, with again $\gamma = 20\,\mathrm{mN\,m^{-1}}$.

This equation admits an asymptotic solution [17],

$$h(x) \approx (9Ca)^{1/3} x \big(-\operatorname{Log} x \big)^{1/3}, \tag{3}$$

that is singular near the contact line (supposed to be at $x = 0$). Equation (3) is reminiscent of the Cox-Voïnov approximate description of the dynamic contact angle [1, 14, 15] for partially wetting liquids, that is sometimes proposed to relate the apparent macroscopic contact angle θ_d, measured at a macroscopic length scale b, to the equilibrium angle θ_s assumed to apply at a microscopic length scale a: $\theta_\mathrm{d}^3 \approx \theta_\mathrm{s}^3 - 9Ca \operatorname{Log}(b/a)$.

This simplified hydrodynamic approach of contact line motion can be now extended to the case of a corner. In the same spirit, one can construct an approximate model [8] (see figs. 1b and d), where there is a balance of a flux due to the translating plate $-Vh(x)$ with a pressure-driven flux $j_{x2} = -(h^3/3\eta)(\mathrm{d}p/\mathrm{d}x)$, where $p(x,y) = -\gamma \Delta h(x,y)$ and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. This yields an equation similar to (2), in which h depends on both x and y:

$$3\frac{Ca}{h^2} = (\Delta h)_x. \tag{4}$$

As mentioned in [8], this approach, which we will refer to as the "planar-flow approximation" is incorrect because it ignores the nonzero y-component of the flow, but shortly it will help us to understand the full treatment of the problem. Since $h(x, y)$ has dimensions of length, then (4) admits similarity solutions $h(x, y) = Ca^{1/3} x H(\zeta)$ with $\zeta = y/x$ such that

$$K(\zeta) = \frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\zeta\big(1+\zeta^2\big)\frac{\mathrm{d}^2 H}{\mathrm{d}\zeta^2}\right] + \frac{3}{H^2} = 0. \tag{5}$$

This similarity solution corresponds to a conical interface (fig. 1b) that intersects the solid plane forming a corner. Integrating this equation starting from $\zeta = 0$ with $H(0) = H_0$, $H'(0) = 0$ and $H''(0) = -3/H_0^2$ yields the $H(\zeta)$ plots shown in fig. 2 (dashed curves), where each curve corresponds to a different H_0, *i.e.* to a different cone with angles ϕ and Ω defined in fig. 1b, where $\tan\Omega = Ca^{1/3} H_0$. The value ζ_0, where $H(\zeta_0) = 0$, defines the corner angle ϕ via the relationship $\zeta_0 = \tan\phi$, which provides a functional relation between ϕ and $\tan\Omega = Ca^{1/3} H_0$. This constraint is plotted in the inset of fig. 2 with the same convention (dashed curve). An approximate analytical form of this function can be obtained by assuming a parabolic shape of the cone cross-section, *i.e.* in eq. (4) take $h(x, y) \approx H_0 Ca^{1/3} x(1 - y^2/(x^2 \tan^2\phi))$ which is plotted as the dash-dotted curves in fig. 2 and yields

$$\tan^3\Omega \approx \frac{3}{2} Ca \tan^2\phi. \tag{6}$$

This result is also shown in the inset to fig. 2 (dash-dotted curve) and is close to the values obtained by numerical integration of eq. (5), though the curves $H(\zeta)$ are different for large values of ϕ ($\phi > 65°$). The question is now: Do these ideas survive in a three-dimensional description combining liquid motions along y and x?

Referring to the notations of fig. 1b, we balance a flux along x, $j_x = -Vh(x, y) - (h^3/3\eta)\partial p/\partial x$, and a flux along y, $j_y = -(h^3/3\eta)\partial p/\partial y$ with $p(x, y) = -\gamma\Delta h(x, y)$. At steady state $\nabla \cdot j = \partial_x j_x + \partial_y j_y = 0$ leads to

$$3Ca\, h_x = \nabla \cdot \big[h^3 \nabla\Delta h\big], \tag{7}$$

instead of (4), where $\nabla = (\partial_x, \partial_y)$. Again, this nonlinear equation admits similarity solutions $h(x, y) = Ca^{1/3} x H(\zeta)$ with $\zeta = y/x$, which, using (7), gives

$$\big(1+\zeta^2\big)^2\big(H^3 H_{\zeta\zeta\zeta}\big)_\zeta + 3\zeta\big(1+\zeta^2\big)\big(H^3 H_{\zeta\zeta}\big)_\zeta + 2\zeta\big(1+\zeta^2\big)H^3 H_{\zeta\zeta\zeta} + \big(1+3\zeta^2\big)H^3 H_{\zeta\zeta} = 3(H - \zeta H_\zeta). \tag{8}$$

The situation is more complicated than for the planar-flow approximation because solutions of eq. (8) are defined by two *independent* parameters $H(0) = H_0$ and $H_{\zeta\zeta}(0) = H_0''$ (owing to symmetry $H_\zeta(0) = H_{\zeta\zeta\zeta}(0) = 0$). For a given value of H_0, eq. (8) allows for an infinite number of solutions, each one having its own value of the central curvature of the cone section. Numerical results for $H(\zeta)$ are shown in fig. 3 where different solutions having the same H_0 but different H_0'' are plotted. Near $\zeta = 0$, these curves are similar to those discussed above, except that, at least for small values of H_0'', they exhibit a minimum instead of a point that touches the axis. When H_0'' is increased, the value of this minimum decreases while the curvature at its minimum increases and even seems to diverge. For a given numerical accuracy, and for large enough H_0'', the curves resemble solutions of the planar approximation with a point nearly touching the axis. The problem of the determination of ϕ (the corner angle) is, however, now complicated by the problem of the selection of H_0''.

At this point, one can try to establish a matching with a microscopic solution valid very close to the contact line (these were the bases of our first attempts), but there is another —macroscopic— argument possible. The numerical solutions with a minimum do not describe a single drop but rather they correspond to a drop in contact with a reservoir of liquid. To be a

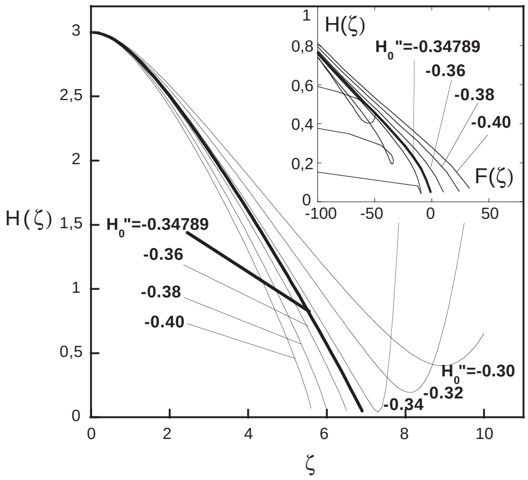

Fig. 3 – Different solutions of eq. (9), for $H(0) = H_0 = 3$ and several values of $H_{\zeta\zeta}(0) = H_0''$. Insert: cross plot combining $H(\zeta)$ and the function $F(\zeta)$ that measures the liquid flux in the x-direction integrated between $y = -\zeta x$ and $y = \zeta x$. Only one curve (bold line) satisfies both $F(\zeta) \to 0$ and $H(\zeta) \to 0$, for a value of ζ that defines $\tan\phi$.

physically reasonable solution of the sliding-drop problem, it is necessary that at steady state the x-flux integrated over the cross-section of the drop remains independent of x and equal to zero (*i.e.* no liquid is exchanged across an effective contact line located at the minimum of H). This argument is expressed mathematically by calculating the flux between $y = -\zeta x$ and $y = \zeta x$,

$$J_x = 2\int_0^{\zeta x} j_x \,\mathrm{d}y = \frac{2}{3}\frac{\gamma}{\eta} Ca^{4/3} x^2 \int_0^{\zeta} K(\zeta')H^3(\zeta')\mathrm{d}\zeta', \tag{9}$$

where $K(\zeta)$ is defined in eq. (5). Now, the desired solution must satisfy $F(\zeta_c) = 0$, where $F(\zeta) = \int_0^{\zeta} K(\zeta')H^3(\zeta')\mathrm{d}\zeta'$, and $\zeta_c = \tan\phi$ is defined as the position of the point touching the plane, if it exists, or at least of the minimum of $H(\zeta)$. The function $F(\zeta)$ is plotted against $H(\zeta)$ in the inset of fig. 3 for different initial values H_0''. As the results in these figures show, there is only one H_0'' value that satisfies the zero-flux condition with $H(\zeta)$ tending to zero. Macroscopically, the corresponding solution is the one relevant to describe the corner singularity. The obtained solution curves $H(\zeta)$ are plotted in fig. 2, and their corresponding prediction for the relationship linking H_0 to ϕ is given in the inset of this figure (continuous curves). The results are very close to the planar-flow approximation, which thus constitutes a reasonable and simple approach to the 3D corner flow, though this was not obvious *a priori*. Further, note that this planar flow naturally satisfies $F(\tan\phi) = 0$ because $K(\zeta) = 0$ in eq. (5). As a check on all of these conclusions, we calculated the transverse flux j_n crossing a line with equation $y = \zeta x$ and, by writing a condition of the form $j_n = 0$ across the contact line, we obtained results that coincide exactly with those deduced from the criterion $F(\zeta) = 0$.

Finally, it is interesting to examine qualitatively the behavior of our 3D solution near each contact line. Let us assume that the function $H(\zeta)$ has a "touching point", *i.e.* that $h(x,y)$ goes to zero near the contact line (*i.e.* very close compared to macroscopic scales). Then, when ζ is close to $\zeta_c = \tan\phi$, the highest derivatives in eq. (9) should dominate, in which case eq. (8) should be equivalent to $(1+\zeta_c^2)^2(H^3H_{\zeta\zeta\zeta})_\zeta = -3\zeta_c H_\zeta$, which after one integration is similar to eq. (2) written for a single contact line. A direct calculation then leads to an approximation for h very near the contact line of equation $y = x \tan \phi$:

$$h \approx (9Ca \sin \phi)^{1/3} \xi \big(-\mathrm{Log}\, \xi \big)^{1/3}, \tag{10}$$

where $\xi = x \sin \phi - y \cos \phi$ is the coordinate normal to contact line supposed to lie at $\xi = 0$. Comparison with eq. (2) shows that the three-dimensional corner model reduces to the classical lubrication description of a single contact line, when examined very close to each contact line forming the corner, where the capillary number Ca is now replaced by $Ca \sin \phi$. From a physical point of view, this result means that the dissipation in the corner is essentially governed by the component of velocity normal to contact line. This result is consistent with the qualitative arguments supporting the scaling $\sin \phi \propto 1/Ca$ found in refs. [5,6]. Obviously, any condition written at the inclined contact line will involve the quantity $Ca \sin \phi$. It is nevertheless important to note that there is still something missing in our model to get this extra condition, and thus, the prefactor of the scaling. One would need to treat explicitly the singular behavior near the contact line, possibly establishing a matching with a microscopic model of wetting [18]. The difficulty is then to keep a similarity solution while introducing a specific microscopic scale, which is outside of the scope of our paper.

Despite this limitation, we believe that our similarity solution is essential for understanding the experiments of refs. [5,6]. To support this intuition, we have plotted in the inset of fig. 2 experimental measurements of ϕ and $H_0 = Ca^{-1/3} \tan \Omega$ deduced from recent visualisations of silicon oil drops ($\eta \approx 10, 20, 100$ and $1000\,\mathrm{cP}$, $\gamma \approx 20\,\mathrm{dyne/cm}$) sliding on a glass coated with fluoropolymers [11,12]. The agreement with our theoretical curves is very good. The applicability of the lubrication approach is here justified because the dynamic receding contact angles are rather low (lower than 20°), and as the Reynolds numbers, at least for the most viscous liquid used ($Re = 10^{-4}$ for $\nu = \eta/\rho = 10\,\mathrm{cm^2 s^{-1}}$, $V = 1\,\mathrm{mm\,s^{-1}}$ and a millimeter drop height). As usual with wetting, provided that one is looking close enough to the contact line, gravity terms are negligible compared to those involved in our analysis.

In summary, the formation of a corner has been analytically related to a hydrodynamic theory. We have shown that similarity solutions of the lubrication equations exist that are compatible with a corner-shaped contact line. These solutions imply a nonzero contact angle, *i.e.* a nonzero cone angle Ω viewed from the side, that is uniquely related to the corner opening angle ϕ (fig. 2b) in good agreement with recent visualisations [11,12]. This result may seem surprising as refs. [5,6] arguments supposed a zero macroscopic receding contact angle, but the recent visualisations [11,12] indeed have shown that Ω does not vanish at threshold. Nevertheless, a thorough understanding of the 3D corner flow is not yet complete, because the selection mechanism of the angle ϕ remains to be explained in detail. We have however been able to show that the relevant capillary number is $Ca \sin \phi$ instead of Ca, which, for the first time, gives a rigorous basis to the scaling $\sin \phi \propto 1/Ca$ found in experiments.

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In its early stages this work benefitted from discussions with J.-M. Flesselles, T. Podgorski and S. K. Wilson. We also thank J. Eggers and M. Fermigier for helpful advice and discussions around the hydrodynamical description of wetting. We are indebted to B. Andreotti, A. Daerr, N. Le Grand and E. Rio for the data reported in fig. 2 and stimulating discussions. One of us (LL) also acknowledge recent exchanges with V. Roy about lubrication approach of a single contact line. HAS thanks the Harvard MRSEC (and so NSF), and LL the Fondation Langlois (Fondation de l'Ouest) for their financial supports.

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