Supplementary material: Thin-film flows over microdecorated surfaces: Polygonal hydraulic jumps

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Here we provide more details about the modeling of radial flow in the thin film formed upon impact of a jet of radius a on a solid substrate. A hydraulic jump is produced when such a film is bounded by a thicker layer of slow moving liquid that exerts a hydrostatic pressure on the flow. The first model to take into account viscous effects on smooth substrates using a boundary layer approach was described by Watson [1] and is summarized in the first part of this document. As presented in the second part of this supplementary document, we modify the original model to take into account the substrate roughness.

Thin film flow over smooth substrates, main results[1]:
The radial flow in the thin film is considered dominated by viscous effects near the substrate, in the boundary layer, and by inertia outside this layer, where the liquid flows at a velocity estimated to be the jet velocity ($U_0 = \frac{Q}{\pi a^2}$). In the following, r and z are cylindrical coordinates, with $z = 0$ on the substrate and $z \geq 0$ upward, and u and w are the corresponding velocities. The velocity and the thickness of the boundary layer can be determined as functions of the radial distance r using the lubrication approximation:

$$\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{\partial w}{\partial z} = 0 \tag{1a}$$

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2} \tag{1b}$$

where ν is the kinematic viscosity. The boundary conditions and flow rate conservation equations are:

$$u = w = 0, \quad \text{for } z = 0 \tag{2a}$$

$$\frac{\partial u}{\partial z} = 0, \quad \text{for } z = h(r) \tag{2b}$$

$$2\pi r \int_0^{h(r)} u dz = Q \tag{2c}$$

The velocity is assumed to be a self similar function, $u = U(r)f(\eta)$ with $\eta = \frac{z}{h(r)}$, and the system of equations previously described becomes

$$U(r) = \frac{27c^2}{8\pi^4}\frac{Q^2}{\nu}\frac{1}{(r^3+\ell^3)} \tag{3a}$$

$$f(\eta) = \sqrt{3}+1-\frac{2\sqrt{3}}{1+cn[3^{\frac{1}{4}}c(1-\eta)]} \tag{3b}$$

$$h(r) = \frac{2\pi^2}{3\sqrt{3}}\frac{\nu}{Q}\frac{\left(r^3+\ell^3\right)}{r} \tag{3c}$$

where cn is the Jacobian elliptic function. Here c is a numerical constant; $c = 1.402$. To determine the thickness of the boundary layer $\delta(r)$ and the constant ℓ, it is assumed that $\eta = \frac{z}{\delta(r)}$. $\delta(r)$ is an increasing

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function of r. For $r = r_0$, the boundary layer invades the entire thickness of the film $h(r)$.

For $r < r_0$, the velocity of the fluid in the thin film is defined by $u = U(r)f(\eta)$, in the boundary layer, between $z = 0$ and $z = \delta(r)$ and by $u = U_0$ between $z = \delta(r)$ and the free surface $z = h(r)$. For $r > r_0$, the fluid velocity is determined by $u = U(r)f(\eta)$ between $z = 0$ and $z = h(r)$.

The position of the jump is determined by a momentum balance, written at the radial distance $r = r_j$. Accounting for gravitational and inertial terms originally introduced by Watson [1] and a surface tension contribution later introduced by Bush et al. [2], we find

$$\frac{1}{2}\rho g d^2 + \frac{\gamma(d-h)}{r_j} = \rho \int_0^h u^2 dz - \rho U_1^2 d \tag{4}$$

with d and U_1 respectively the thickness of the liquid layer and the velocity immediately outside the jump, defined as $U_1 = \frac{Q}{2\pi r_j d}$, γ the surface tension and ρ the fluid density. Assuming that $d \gg h$, we write Eq. (4) as:

$$\frac{r_j d^2 g a^2}{Q^2}\left(1 + \frac{2\gamma}{\rho g r_j d}\right) + \frac{a^2}{2\pi^2 r_j d} = \frac{2 r_j a^2}{Q^2}\int_0^h u^2 dz \tag{5}$$

When the radius of the hydraulic jump is larger than r_0, which is the case in our experiments, the momentum balance (5) reads:

$$\underbrace{r_j\, g\left(\frac{d\,a}{Q}\right)^2\left(1 + \frac{2}{B_o}\right) + \frac{a^{\,2}}{2\pi^2 r_j d}}_{Y_j} = \frac{27\sqrt{3}}{8\pi^6}c^3 \underbrace{\frac{Q\,a^2}{\nu(r_j^3 + l^3)}}_{X_j^{-1}} \tag{6}$$

where $B_o = \frac{\rho g r_j d}{\gamma}$ is the Bond number and $\ell = \frac{\sqrt{3}}{2\pi}\left(\frac{3c}{2}\frac{Qa^2}{\nu}\left(3\sqrt{3}c - \pi\right)\right)^{\frac{1}{3}}$.

In the classical representation adopted in Fig. 3, the left hand side of Eq. (6), Y_j, is plotted as a function of $X_j = \frac{\nu(r_j^3 + l^3)}{Q\,a^2}$.

Flow over rough substrates and polygonal hydraulic jumps:

In this Letter, we utilize substrates textured at the micron scale. The origin of the z-axis is now located at the top of the posts. The presence of the roughness is captured by the modeling of two separate effects:

1) A Navier slip boundary condition at $z = 0$:

$$u\,|_{z=0} = \lambda \frac{\partial u}{\partial z}\,|_{z=0} = b\,h(r)\frac{\partial u}{\partial z}\,|_{z=0} \tag{7}$$

Here $b\,h(r)$ is the slip length λ as prescribed for a similarity solution to exist. b is a non-dimensional slip parameter which, for such Reynolds number values, is expected to have the symmetry of the lattice. Thus, we postulate the following constitutive relation: $b(\theta, \xi) = \xi \frac{\ell_o(\theta)}{2\,D}$. $\xi(\epsilon, \kappa)$ a function of the roughness porosity and the aspect ratio of the posts, whose value is determined experimentally. $\frac{\ell_0}{2D}$ is a non-dimensional parameter which captures the anisotropy of the slip length and is defined phenomenologically by the open frontal area (Fig. 4a).

2) The flow rate in the thin film is reduced by a quantity $q_{leak} = \alpha Q$. q_{leak}, the "leakage flow", corresponds to the amount of liquid flowing through the texture and is assumed to be proportional to the total flow rate. Here α is a coefficient that we determine experimentally.

Our approach to determine the position of the jump is similar to other studies on smooth substrates as presented above [1–3]. We search for a self similar solution to satisfy the governing equations (Eq. 1), new boundary conditions (Eq. 2a and 7) and conservation equation ($Q - q_{leak} = 2\pi r \int_0^{h(r)} u dz$). Using the momentum equation (Eq. 1b) and the slip boundary condition (Eq. 7), we obtain a system of equations that defines $c_1(b)$, the analogue of the constant c previously mentioned and $f(\eta = 0, b)$ noted $f(b)$:

$$c_1(b) = c - f(b)\, F_2^1(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, f(b)^3) \tag{8a}$$

$$c_1(b) = \frac{f(b)}{b\sqrt{1 - f(b)^3}} \tag{8b}$$

where F_2^1 is the hypergeometric function.
We define $c_2(b)$ as $c_2(b) = 1 - c_1(b)\; c\; \frac{3\sqrt{3}}{2\pi} \int_0^1 f(\eta, b) d\eta$. The function $f(\eta, b)$ is obtained numerically whereas $U(r)$ and $h(r)$ have analytical expressions:

$$U(r) = \frac{27c^2}{8\pi^4 \nu \left(1 - c_2(b)\right)^2} \frac{(1-\alpha)^2 Q^2}{(r^3 + \ell^3)} \tag{9a}$$

$$h(r) = \frac{2\pi^2}{3c\sqrt{3}} \frac{\nu\, c_1(b)}{(1-\alpha)Q} \left(1 - c_2(b)\right) \frac{\left(r^3 + \ell^3\right)}{r} \tag{9b}$$

where ℓ is now a function of b:

$$\ell(b) = \left(\frac{(1-\alpha)Qa^2}{\nu} \frac{9\sqrt{3}c}{16\pi^3} \right)^{\frac{1}{3}} \left(\frac{2\sqrt{3}c}{\left(1 - c_2(b)\right)^2} - \frac{\pi b\, c_1(b)}{f(0)\left(1 - c_2(b)\right)} + \frac{\sqrt{3}\sqrt{1 - f(0)^3}\; c\; b\; c_1(b)}{f(0)\left(1 - c_2(b)\right)^2} \right)^{\frac{1}{3}} \tag{10}$$

Using the momentum equation (Eq. 1b) for a jump radius larger than r_0, we can define $r_j(b)$:

$$r_j(b)\, g \left(\frac{d\, a}{(1-\alpha)Q} \right)^2 \left(1 + \frac{2}{B_o} \right) + \frac{a^2}{2\pi^2 r_j(b) d} = \frac{27\sqrt{3}}{8\pi^6} \left(\frac{c}{g(b)} \right)^3 \frac{(1-\alpha)Q\, a^2}{\nu\left(r_j(b)^3 + \ell(b)^3\right)} \tag{11}$$

where $g(b) = \frac{1 - c_2(b)}{(1 - f(b)^3)^{\frac{1}{6}}}$

Using this equation, we can determine the radius of the jump r_j as a function of b, i.e. of the azimuthal angle θ for a given lattice geometry. Results for different lattices are presented using an angular representation in Fig. 4c, for both smooth and rough surfaces.

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