BRIEF COMMUNICATIONS

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On the deviatoric normal stress on a slip surface

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A general formula for the deviatoric normal stress on a slip surface for incompressible flows is derived and its application is discussed.
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When studying viscous flow over a deformable surface, knowledge of the deviatoric normal stress on the surface, $2\mu\mathbf{nn}:\mathbf{D}$, is often required in order to determine the shape of the surface. Here $\mathbf{n}$ is the unit normal to the surface, μ is the fluid viscosity, and $\mathbf{D}$ is the rate-of-strain tensor. For a rigid no-slip surface, the deviatoric stress for an incompressible flow is tangential to the rigid surface and the deviatoric normal stress is identically zero (Ref. 1, p. 178). In this note, we derive a formula for the deviatoric normal stress on a deformable slip surface for incompressible flows and we show how this deviatoric normal stress is related to the slip velocity as well as the normal component of the surface velocity. For related ideas applied to surfactant transport at an interface see Ref. 2.

Consider the motion of an incompressible Newtonian fluid with viscosity μ. Let $\mathbf{n}$ and $\mathbf{t}$ be the unit normal and tangent vectors, respectively, on a deformable surface Σ. The gradient operator $\boldsymbol{\nabla}$ on surface Σ can be expressed as

$$\boldsymbol{\nabla}=(\mathbf{I}-\mathbf{nn})\cdot\boldsymbol{\nabla}+\mathbf{nn}\cdot\boldsymbol{\nabla}=\boldsymbol{\nabla}_s+\mathbf{nn}\cdot\boldsymbol{\nabla}, \tag{1}$$

where $\nabla_s=(\mathbf{I}-\mathbf{nn})\cdot\boldsymbol{\nabla}$ is the surface gradient operator. Thus, on surface Σ,

$$\boldsymbol{\nabla}\mathbf{u}=\boldsymbol{\nabla}_s\mathbf{u}+\mathbf{nn}\cdot\boldsymbol{\nabla}\mathbf{u}. \tag{2}$$

The trace of (2) gives

$$\boldsymbol{\nabla}\cdot\mathbf{u}=\boldsymbol{\nabla}_s\cdot\mathbf{u}+\mathbf{nn}:\boldsymbol{\nabla}\mathbf{u}. \tag{3}$$

Since $\mathbf{nn}:\mathbf{D}=\mathbf{nn}:\boldsymbol{\nabla}\mathbf{u}$, then we have

$$\mathbf{nn}:\mathbf{D}=\boldsymbol{\nabla}\cdot\mathbf{u}-\boldsymbol{\nabla}_s\cdot\mathbf{u}. \tag{4}$$

Therefore, for incompressible flows, $\mathbf{nn}:\mathbf{D}=-\boldsymbol{\nabla}_s\cdot\mathbf{u}$, and the deviatoric normal stress on the surface is

$$2\mu\mathbf{nn}:\mathbf{D}=-2\mu\boldsymbol{\nabla}_s\cdot\mathbf{u}. \tag{5}$$

On a surface with fixed shape $(\mathbf{u}\cdot\mathbf{n}=0)$, $\mathbf{u}=u_s\mathbf{t}$, where u_s is the slip velocity, so

$$\mathbf{nn}:\mathbf{D}=-\boldsymbol{\nabla}_s\cdot(u_s\mathbf{t}). \tag{6}$$

For a nonstationary surface, $\mathbf{u}=u_n\mathbf{n}+u_s\mathbf{t}$, where u_n is the local normal velocity of the surface, then

$$\mathbf{nn}:\mathbf{D}=-\boldsymbol{\nabla}_s\cdot(u_n\mathbf{n}+u_s\mathbf{t})=-\boldsymbol{\nabla}_s\cdot(u_s\mathbf{t})-u_n\kappa, \tag{7}$$

where $\kappa=\boldsymbol{\nabla}_s\cdot\mathbf{n}$ is twice the mean curvature of the surface.

Equation (7) is the main result of this note as it relates the deviatoric normal stress to the slip velocity u_s and the normal component of velocity u_n. The shape of the interface clearly is important since variations of $\mathbf{t}$ enter as does the local mean curvature.

Two applications should suffice to illustrate the idea. First, consider the low-Reynolds-number motion of the steady translation at velocity $\mathbf{U}$ of a spherical viscous drop of radius a and viscosity $\lambda\mu$ in a fluid of viscosity μ. As is well known (e.g., Ref. 1, p. 235), the velocity field outside and relative to the drop is

$$\mathbf{u}(\mathbf{r})=-\mathbf{U}+\frac{\lambda a^3}{4(1+\lambda)}\left(\frac{\mathbf{U}}{r^3}-\frac{3\mathbf{U}\cdot\mathbf{rr}}{r^5}\right)+\frac{(2+3\lambda)a}{4(1+\lambda)}\left(\frac{\mathbf{U}}{r}+\frac{\mathbf{U}\cdot\mathbf{rr}}{r^3}\right), \tag{8}$$

where $\mathbf{r}$ is the position vector. So, on the surface $r=a$ we have $u_n=0$ and $u_s=\mathbf{t}\cdot\mathbf{u}=-\mathbf{U}\cdot\mathbf{t}/2(1+\lambda)$. Using (7) and expressing quantities in spherical coordinates $(r,\,\theta,\,\varphi)$, we find

$$2\mu\mathbf{nn}:\mathbf{D}=-2\mu\left(u_s\boldsymbol{\nabla}_s\cdot\mathbf{t}+\frac{\partial u_s}{\partial\theta}\right)=-\frac{2\mu U}{a}\frac{\cos\theta}{1+\lambda}, \tag{9}$$

which can alternatively be evaluated by the more traditional fashion of suitably differentiating (8).

As a second application of (7), consider the familiar Rayleigh–Plesset problem of the radial expansion or oscillation of a bubble of radius $R(t)$ in a viscous fluid of viscosity μ. Here the surface velocity has $\mathbf{u}=u_n\mathbf{n}$, $u_n=dR/dt$, and curvature $\kappa=2/R(t)$. Hence the normal viscous stress that appears in the momentum equation is

$$2\mu \mathbf{nn}:\mathbf{D} = -4\mu \frac{\dot{R}}{R}, \tag{10}$$

as can also be obtained by a detailed calculation of the velocity field.

In conclusion, we have obtained a simple relationship, Eq. (7), for the deviatoric normal stress on an interface, in terms of the instantaneous tangential and normal velocity components and the shape.

[1] G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, 1968).
[2] H. A. Stone, ''A simple derivation of the time-dependent convective-diffusion equation for surfactant transport along a deforming interface,'' Phys. Fluids A **2**, 111 (1990).