BRIEF COMMUNICATIONS

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Piston flow in a two-dimensional channel

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A solution using biorthogonal eigenfunctions is presented for viscous flow caused by a piston in a two-dimensional channel. The resulting infinite set of linear equations is solved using Spence's optimal weighting function method [IMA J. Appl. Math. **30**, 107 (1983)]. The solution is compared to that with a shear-free piston surface; in the latter configuration the fluid more rapidly approaches the Poiseuille flow profile established away from the face of the piston. © *2000 American Institute of Physics.* [S1070-6631(00)00805-9]

We present a solution for the viscous, low-Reynolds-number flow generated by a moving piston in a semi-infinite channel. The solution represents the two-dimensional analog of flow in a syringe for the case that the tube length is several times the radius so that a fully developed Poiseuille flow evolves. The analysis uses biorthogonal (Papkovich–Fadle) eigenfunctions that arise in solutions to the biharmonic equation for strip problems in two-dimensional or axisymmetric geometries. This solution has not, to our knowledge, been applied to the familiar syringe configuration.

Given the linearity of the Stokes equations applicable here and the simple geometry of the syringe, it is surprising that an analytical solution to this flow has not been presented (see the interesting discussion in Faber[1]). The axisymmetric rigid piston problem has been treated using a finite-difference method.[2] In fact, the analytical solution of the biharmonic equation (Stokes flow) is quite complicated in finite domains. Meleshko[3] and Krasnopolskaya *et al.*[4] used a superposition method involving dual Fourier series to solve the biharmonic equation in finite rectangular and annular domains. Their solution exhibited good convergence properties, but seems applicable only to finite strips. The method of Joseph and co-workers (e.g., Ref. 5) is apparently suitable but ignores the convergence difficulty indicated by Spence.[6]

The flow near the piston can, in principle, be treated analytically using eigenfunctions developed for cylindrical coordinates.[7] Here we treat the less cumbersome two-dimensional problem that captures the important physical features. Also, in many applications the plunger of the syringe drives fluid into a small tube and flow near this outlet may in some circumstances be treated analytically (e.g., Ref. 8).

We first describe the solution for a translating rigid piston and then compare with results for a shear-free piston surface. The latter models the case of a bubble translating in a narrow channel (Hele-Shaw cell). In particular, we are interested in the detailed flow field as well as the transition length between the uniform flow near the piston and the fully developed parabolic flow. Since the solution method is well established, we tersely summarize the intermediate mathematical steps.

Consider flow in a semi-infinite channel $x \geq 0$, $-H < y < H$ (Fig. 1). The boundary at $x=0$ moves at constant velocity $u=U$, such that the Reynolds number $\mathcal{R}=\rho UH/\mu \ll 1$, where ρ and μ are, respectively, the density and viscosity of the fluid. As $x \to \infty$, a steady Poiseuille flow is established, so that $u \to \frac{3}{2}U[1-(y/H)^2]$.

We present a solution using the method of biorthogonal eigenfunctions.[5] We scale velocities by the piston speed U, lengths by the channel half-width H, and pressures by $\mu U/H$. The velocity field (u,v) can be constructed from a streamfunction $\psi(x,y)$, where $(u,v)=(\partial\psi/\partial y, -\partial\psi/\partial x)$, and hence we consider

$$\nabla^4 \psi(x,y) = 0, \tag{1a}$$

$$\psi(x,\pm 1) = \pm 1, \quad \psi_y(x,\pm 1) = 0, \tag{1b}$$

$$\psi_x(0,y) = 0, \tag{1c}$$

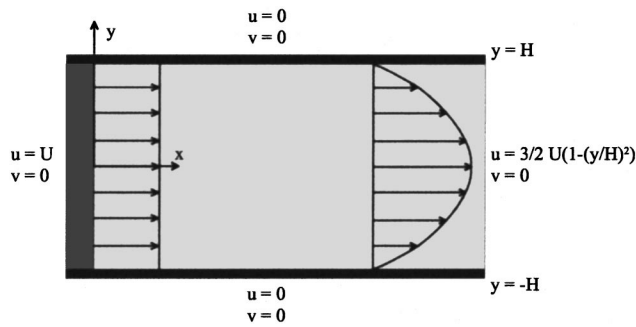[a]Corresponding author. Telephone: (617) 495-3599; fax: (617) 495-9837; electronic mail: has@stokes.deas.harvard.edu

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FIG. 1. Schematic of flow induced by a moving piston.

$$\psi_y(0,y)=1, \quad \text{i.e.,} \quad \psi(0,y)=y, \tag{1d}$$

$$\psi(\infty,y)\to\tfrac{1}{2}(3y-y^3), \tag{1e}$$

in which derivatives are denoted by subscripts.

Using separation of variables, we describe the transition of the flow from the piston surface to the downstream profile (1e) with

$$\psi(x,y)=\frac{1}{2}(3y-y^3)-\sum_{\substack{n=-\infty\\(n\neq0)}}^{\infty}\frac{c_n}{\lambda_n}\phi_n^{(1)}(y)e^{-\lambda_n x}, \tag{2}$$

where the eigenfunctions satisfy (1a), (1b) and vanish as $x\to\infty$. Thus $\mathrm{Re}(\lambda_n)>0$ ($n\neq0$) and the resulting odd functions of y are given by

$$\phi_n^{(1)}(y)=\frac{\sin\lambda_n y}{\sin\lambda_n}-\frac{y\cos\lambda_n y}{\cos\lambda_n} \quad (n\neq0). \tag{3}$$

The complex-valued eigenvalues $\{\lambda_n;n\geq1\}$ are the zeros of $\sin2\lambda_n-2\lambda_n$ in the first quadrant, arranged in order of an ascending real part. The asymptotic form for large n,[9]

$$\lambda_n\sim\left(n+\frac{1}{4}\right)\pi+\frac{i}{2}\ln[(4n+1)\pi]-\frac{\ln[(4n+1)\pi]}{(4n+1)\pi}, \tag{4}$$

is readily established, and very few Newton iterations, even for $n=1$, are needed for sufficient accuracy. Negative values of n correspond to the complex conjugates, and thus $c_{-n}=\bar{c}_n(n\geq1)$, which yields a real-valued ψ. The real part of λ_1 is ~3.73, which sets the length scale of adjustment to the parabolic flow. The corresponding even functions for (1) have a lowest eigenvalue with real part ~2.1, and hence in a purely numerical solution, small errors could trigger a more slowly decaying, incorrect component. It is worth keeping this point in mind when considering direct numerical solutions.

The complex-valued coefficients $\{c_n\}$ are determined from the end conditions (1c), (1d) that yield $(0<y<1)$

$$\sum_{n\neq0}c_n\phi_n^{(1)}(y)=0, \quad \sum_{n\neq0}c_n\Phi_n^{(1)}(y)=\frac{1}{2}(1-3y^2), \tag{5}$$

where

$$\Phi_n^{(1)}=\lambda_n^{-1}\frac{d\phi_n^{(1)}}{dy}=\frac{y\sin\lambda_n y-\tan\lambda_n\cos\lambda_n y}{\cos\lambda_n} \quad (n\neq0). \tag{6}$$

Unfortunately, the above sets of functions (3), (6) do not appear in the same biorthogonality relation and the consequent difficulty distinguishes the rigid piston from the stress-free one. We define two new functions:

$$\phi_n^{(2)}=\lambda_n^{-2}\frac{d^2\phi_n^{(1)}}{dy^2}+\phi_n^{(1)}=\frac{2\sin\lambda_n y}{\lambda_n\cos\lambda_n}, \tag{7a}$$

$$\Phi_n^{(2)}=\lambda_n^{-2}\frac{d^2\Phi_n^{(1)}}{dy^2}+\Phi_n^{(1)}=\frac{2\cos\lambda_n y}{\lambda_n\cos\lambda_n}. \tag{7b}$$

Then, the matrix construction given by Ref. 7 shows that the biorthogonality relations of use for (5) are

$$\int_0^1[\phi_m^{(2)}(\phi_n^{(1)}-\phi_n^{(2)})+\phi_m^{(1)}\phi_n^{(2)}]dy=-\frac{2\sin^2\lambda_n}{\lambda_n^2\cos^2\lambda_n}\delta_{mn}, \tag{8a}$$

$$\int_0^1[\Phi_m^{(2)}\Phi_n^{(1)}+\Phi_m^{(1)}\Phi_n^{(2)}]dy=-\frac{2\sin^2\lambda_n}{\lambda_n^2\cos^2\lambda_n}\delta_{mn}, \tag{8b}$$

for non-zero m,n. The first of these relations identifies the pair $[\phi_m^{(2)},\phi_m^{(1)}]$ as the adjoint of $[\phi_n^{(1)},\phi_n^{(2)}]$ with respect to the matrix $\begin{bmatrix}1&-1\\0&1\end{bmatrix}$.

In some cases, the boundary data are such that one biorthogonality condition [e.g., (8a)] suffices to solve explicitly for the coefficients $\{c_n\}$ in (2). Such "canonical" problems include the case where the normal velocity and shear stress are applied at $x=0$, as these conditions are related by a second derivative. For "noncanonical" problems, as occurs here where the two components of velocity are prescribed at $x=0$, an infinite set of linear equations must be solved numerically to determine the $\{c_n\}$.

However, a naive application of the biorthogonality closure procedures leads to convergence difficulties. As discussed by Spence (Ref. 6, Sec. 3.1), the convergence criterion $\sup_m\Sigma_n|F_{mn}|\leq1-\epsilon(\epsilon>0)$, for the solution of the truncated system of equations, $c_m=\Sigma F_{mn}c_n+d_m$ [see (10) below], cannot be negotiated by redefining $\{c_n\}$. Spence[6] introduced an "optimal weighting" procedure to make the linear system diagonally dominated, which ensures that the matrix to be inverted is well conditioned. Spence also demonstrated that the eigenvalues of this matrix have a limit point equal to 1, which leads to minimal truncation errors.

Following Spence,[6] we apply Galerkin weighting functions $[A_m\phi_m^{(1)}+B_m\phi_m^{(2)}]$ and $[C_m\Phi_m^{(1)}+D_m\Phi_m^{(2)}]$ to the respective equations (5) to obtain

$$\sum_{n\neq0}c_n\int_0^1\{\phi_n^{(1)}[A_m\phi_m^{(1)}+B_m\phi_m^{(2)}]$$

$$+\Phi_n^{(1)}[C_m\Phi_m^{(1)}+D_m\Phi_m^{(2)}]\}dy$$

$$=\frac{1}{2}\int_0^1(1-3y^2)[C_m\Phi_m^{(1)}+D_m\Phi_m^{(2)}]dy \quad (m\neq0). \tag{9}$$

The coefficient ratios are chosen to eliminate inverse powers of $(\lambda_n-\lambda_m)$ that appear in the coefficient matrix when $n\neq m$ and are $A_m:B_m:C_m:D_m=2:-1:-2:1$, i.e., independent of m. After much algebra, we find that the coefficients $\{c_n\}$ are determined by the linear system

$$c_m + \sum_{n=-\infty, n\neq 0, m}^{\infty} c_n \frac{2\lambda_m}{\tan^2 \lambda_m} \frac{(\lambda_m \tan \lambda_n - \lambda_n \tan \lambda_m)}{(\lambda_n + \lambda_m)^3}$$

$$= \frac{3}{\lambda_m} + \cot \lambda_m \quad (m \neq 0). \tag{10}$$

This system of equations is well conditioned and, after truncation ($-N \leq n \leq N$), is solved using the IMSL routine DLSACG. Though there are oscillations in matching of values to the boundary conditions at $x=0$, convergence occurs rapidly with increasing N; e.g., for streamline plots, results are essentially unchanged with $N \geq 20$.

It is interesting to compare these results with the flow obtained when the boundary conditions at $x=0$ are changed to represent a (flat) shear-free interface. This case is a model for the fluid motion when a liquid is displaced by air, as when a bubble translates in a narrow channel or tube (of course, the real interface would not be flat). In this case the boundary conditions at $x=0$ are $\partial v / \partial x = 0$ or $\psi_{xx} = 0$ and (1d). We note that though the x velocity imposed on the boundary is the same as before, there can now be a y velocity along the boundary.

Bhattacharji and Savic[10] presented a solution (in the moving frame) in terms of a Fourier sine transform,

$$\psi(x,y) = \frac{2}{\pi} \int_0^\infty \frac{\cosh k \sinh ky - y \sinh k \cosh ky}{k(\cosh k \sinh k - k)} \sin kx \, dk. \tag{11}$$

An approximate (though unfortunately incorrect) expression for this integral was found[10] by expanding the hyperbolic functions into power series and using an integral identity to obtain $\psi \approx \frac{1}{2}(y - y^3)(1 - e^{-\sqrt{5}x})$ in the moving frame, which purports to show the tendency to Poiseuille flow for large x.

However, the correct procedure here is to rewrite the integral in terms of residues at the zeros of the denominator and retain those with the slowest decay as $x \to \infty$. This rearranged form can also be obtained by the eigenfunction expansion method since the problem is now canonical. A correct evaluation occurs after it is observed that, due to the velocity discontinuity at the corners, one must set $\psi_{yy} = -\delta(y - 1 + \epsilon)$ at $x = 0 (0 \leq y < 1)$, for arbitrarily small $\epsilon > 0$, where δ is the delta function.

We first write the conditions $\psi_{xx} = 0$ and (1d) as

$$\sum_{n\neq 0} \lambda_n c_n \phi_n^{(1)}(y) = 0, \tag{12a}$$

$$\sum_{n\neq 0} \lambda_n c_n (\phi_n^{(2)}(y) - \phi_n^{(1)}(y)) = -3y + \delta(y - 1 + \epsilon). \tag{12b}$$

However, to apply the appropriate biorthogonality relation to determine the c_n, we add these two boundary conditions to arrive at the system

$$\begin{pmatrix} 0 \\ -3y + \delta(y - 1 + \epsilon) \end{pmatrix} = \sum_{n\neq 0} \lambda_n c_n \begin{pmatrix} \phi_n^{(1)} \\ \phi_n^{(2)} \end{pmatrix}, \tag{13}$$

since this form of the boundary conditions corresponds to one of the two canonical forms described by Spence.[6] The
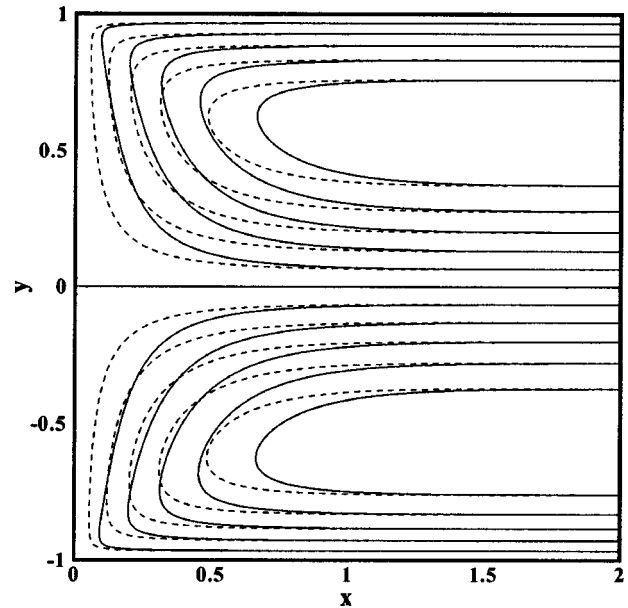

FIG. 2. Contours of the streamfunction ($N=50$) shown with respect to the fixed piston for the rigid-piston (solid) and shear-free (dashed) cases. $\psi = 0$ at all walls and along the centerline, and increases (decreases) by increments of about 0.032 away from the walls in the upper (lower) half.

solution for the coefficients may be shown to be $c_n = \cot \lambda_n$, so that $\psi(x,y)$ follows from (2) and (3).

Streamlines for the rigid-piston and stress-free cases are given in Fig. 2, in the moving reference frame. There are no corner eddies and the local flow problem near the corner is simply the "G.I. Taylor scraper." Note that the decay of the velocity toward a Poiseuille flow is set by the real part of the largest eigenvalue, which is the same for the two cases. In the shear-free case, however, there is flow in the y direction on $x=0$, which allows more rapid adjustment to the Poiseuille profile. This adjustment is controlled by the value of the first coefficient c_1, which is smaller for the shear-free case. Along the centerline, $u = 0.99 u_{\max}$ at $x \approx 1.25$ for the case of the solid piston, whereas for the shear-free interface this occurs at $x \approx 1.0$. Similar numerical results were obtained when the method of Joseph and co-workers was implemented [this required taking a derivative of (1d)], with the most noticeable deviations between the two solution methods occurring close to the boundary at $x=0$.

In the corresponding axisymmetric flow with the pipe wall at $r=1$, (1a) is replaced by $L_{-1}^2 \psi = 0$, where L_{-1}^2 is the axisymmetric operator obtained from the Stokes equations, and (3) by

$$\psi_n^{(1)}(r) = \frac{r J_1(\lambda_n r)}{J_1(\lambda_n)} - \frac{r^2 J_2(\lambda_n r)}{J_2(\lambda_n)} \quad (n \neq 0), \tag{14}$$

where $\{\lambda_n; n \geq 1\}$ are the zeros in the first quadrant of $J_1^2(s) - J_0(s) J_2(s) = 0$. Thus

$$\lambda_1 = 4.463 + 1.468i, \quad \lambda_2 = 7.693 + 1.727i, \tag{15}$$

$$\lambda_n \sim \left(n + \frac{1}{2}\right)\pi + \frac{i}{2} \ln[(4n + 2)\pi], \quad \text{as } n \to \infty. \tag{16}$$

The ensuing adjustments to the above calculation can be inferred by reference to Ref. 11. Some related problems in elasticity were discussed by Duncan Fama.[12]

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