An interpretation of the translation of drops and bubbles at high Reynolds numbers in terms of the vorticity field

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The steady translation of a drop is reconsidered in the high Reynolds number flow limit, $\mathscr{R} \gg 1$. The standard approach for determining the drag on a spherical drop is to calculate the total energy dissipation in the fluid with the velocity field approximated using the potential flow solution outside the drop and Hill's spherical vortex inside. Kang and Leal [Phys. Fluids **31**, 233 (1988)] provide the first calculation of the drag for a spherical bubble by integrating the normal stresses over the bubble surface. Their detailed calculation shows that the drag coefficient up to $O(\mathscr{R}^{-1})$ depends only on the $O(1)$ vorticity distribution along the bubble surface and is independent of the vorticity distribution in the fluid. Here, this conclusion regarding the role of vorticity is extended to the case of any steady high Reynolds number bubble shape compatible with the steady translational speed; there is no restriction to sphericity. The results are demonstrated without explicit calculations and follow from the representation of the energy dissipation for translating drops in terms of the vorticity field.

Consider the steady translation of a drop or bubble with velocity $\mathbf{U}$ in an infinite bath of otherwise quiescent fluid. The Reynolds number characterizing the fluid motion, $\mathscr{R} = Ua/\nu$, is assumed to be large (here $U = |\mathbf{U}|$, a is the undeformed drop radius, and ν is the kinematic viscosity of the suspending fluid). The interface is assumed to be free of surfactants so that the tangential stress is continuous across the fluid–fluid interface. For a bubble which is commonly supposed to have an inviscid interior, the tangential stress at the surface is set equal to zero.

Early studies of this high Reynolds flow problem focused upon determining the drag force. This is a surprisingly difficult problem with an interesting history of incorrect solutions.[1] Levich[2] determined the drag coefficient [see definition in Eq. (2)] to be $C_D^{\text{bubble}} = 24/\mathscr{R}$ by using an energy dissipation argument with the velocity field estimated using the irrotational flow solution past a sphere. This work was extended by Moore[3] who determined the second term, $O(\mathscr{R}^{-3/2})$, in an expansion for the drag coefficient. Moore used the energy dissipation approach and presented a detailed study of the velocity field everywhere in the fluid. Vorticity is confined to a thin viscous boundary layer along the bubble surface and to a narrow, weak wake downstream of the bubble.

Harper and Moore[4] extended the high Reynolds number flow theory to study the steady translation of spherical drops. The structure of the velocity field is a nearly potential external flow with Hill's spherical vortex characterizing the internal flow. Vorticity is confined to a viscous boundary layer along the external drop surface and a narrow wake downstream of the drop and, because the interior is viscous, there is a boundary layer along the internal drop surface, in addition to an internal boundary layer ("wake") located along the drop axis owing to recirculation of the internal fluid. The assumptions of sphericity and the asymptotic character of the velocity field are restrictive. Nevertheless, Dandy and Leal,[5] using a numerical solution of the Navier–Stokes equations, demonstrate that the basic features of the Harper and Moore theory, both the quantitative prediction of the drag coefficient, as well as the qualitative features of the flow field, are good approximations for Reynolds numbers greater than about 120. The numerical solutions show that for Weber numbers $O(1)$ drops remain nearly spherical.

A direct calculation of the drag coefficient for a spherical bubble, via integration of the surface stresses, is complicated and this calculation was accomplished only recently by Kang and Leal.[6] A major difficulty arises in the calculation since the pressure field in the viscous boundary layer adjacent to the surface is required to obtain even the leading order drag result. As part of their study Kang and Leal demonstrate that the leading order contribution to the drag coefficient for a translating spherical bubble depends on the $O(1)$ vorticity distribution along the bubble surface and is independent of the details of the vorticity distribution in the bulk of the fluid. This observation concerning vorticity and drag provided the stimulus for the work described here.

In this paper we extend the Kang and Leal conclusion regarding the role of vorticity on the motion of spherical bubbles to encompass any steady bubble shape that is compatible with the rise speed. The analysis uses a rearrangement of the energy dissipation argument to obtain an exact expression for the drag coefficient (valid at any Reynolds number), which is dependent solely on the vorticity field. Specific conclusions to be drawn for the high Reynolds number flow limit follow standard boundary layer arguments based upon the asymptotic features of the flow, though are nevertheless limited in that they require that separation or detached wakes, both of which may be observed in applications, do not occur.

Energy dissipation argument. When a translating drop has achieved a steady (axisymmetric) shape and rise speed, a mechanical energy balance equates the rate of work done by viscous and pressure forces acting over the drop surface to the rate at which energy is viscously dissipated into thermal energy. In terms of the rise velocity $\mathbf{U}$,

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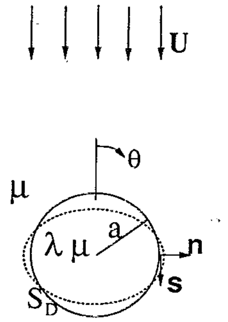

FIG. 1. Steady translation of a drop in a fluid otherwise at rest, as viewed in a reference frame fixed to the drop. Dotted line denotes a typical deformed shape.

the drag force **F**, and the rate-of-strain tensor **E** (a caret denotes variables for the drop fluid), the steady-state energy balance is

$$-\mathbf{F}\cdot\mathbf{U}=2\mu\int_V \mathbf{E}:\mathbf{E}\,dV+2\lambda\mu\int_{\hat{V}}\hat{\mathbf{E}}:\hat{\mathbf{E}}\,dV, \qquad (1)$$

where V and $\hat{V}$ are the fluid volumes external and internal to the drop, respectively. The external and internal fluid viscosities are denoted μ and $\lambda\mu$, respectively (see Fig. 1). For later use S_D indicates the drop surface and S_∞ indicates an external boundary at a large distance. For the axisymmetric shapes of interest here **F** and **U** are antiparallel. A drag coefficient C_D may be introduced as ($F=|\mathbf{F}|, U=|\mathbf{U}|$)

$$C_D=\frac{F}{\frac{1}{2}\rho U^2\pi a^2}=\frac{4}{\pi\mathscr{R}}\int_V \mathbf{E}:\mathbf{E}\,dV+\frac{4\lambda}{\pi\mathscr{R}}\int_{\hat{V}}\hat{\mathbf{E}}:\hat{\mathbf{E}}\,dV, \qquad (2)$$

where all velocities are nondimensionalized with U and lengths by a. Levich's[2] calculation for a bubble gives the drag coefficient accurate to $O(\mathscr{R}^{-1})$ by estimating $\mathbf{E}(\mathbf{x})$ throughout the fluid using the velocity field $\mathbf{u}(\mathbf{x})$ for the irrotational flow past a sphere. Harper and Moore[4] add to this the dissipation from Hill's spherical vortex inside the drop to obtain the drag coefficient for a drop with viscosity ratio λ.

We now recast Eq. (2) in terms of the vorticity vector $\boldsymbol{\omega}=\nabla\wedge\mathbf{u}$. It is then straightforward to demonstrate several features regarding the role of vorticity upon the drag of steadily translating bubbles and drops.

Recasting in terms of vorticity. In order to interpret the energy dissipation approach for the drag calculation in terms of vorticity, we begin with the vector identity

$$2\mathbf{E}:\mathbf{E}=\omega^2+2(\nabla\mathbf{u}):(\nabla\mathbf{u}). \qquad (3)$$

This equation simply expresses a kinematic identity between the components of the rate-of-strain tensor and the vorticity vector. Since the flow is incompressible, $\nabla\cdot\mathbf{u}=0$ and the last term may be written $\nabla\cdot[(\mathbf{u}\cdot\nabla)\mathbf{u}]$. Hence, the drag coefficient may be expressed as

$$C_D=\frac{2}{\pi\mathscr{R}}\int_V \omega^2\,dV-\frac{4}{\pi\mathscr{R}}\int_{S_D+S_\infty}\mathbf{n}\cdot[(\mathbf{u}\cdot\nabla)\mathbf{u}]dS$$
$$+\frac{2\lambda}{\pi\mathscr{R}}\int_{\hat{V}}\hat{\omega}^2\,dV+\frac{4\lambda}{\pi\mathscr{R}}\int_{S_D}\mathbf{n}\cdot[(\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{u}}]dS, \qquad (4)$$

where we have used the divergence theorem with the unit normal **n** directed outward from the particle. It may be verified that the surface integrals in (4) are properly frame invariant to a uniform translation. Also, for the steady translation of a drop, standard results for velocity disturbances in the far field and the wake[7] demonstrate that for finite Reynolds numbers velocity disturbances decay sufficiently rapidly for the S_∞ integral to be neglected.[8] The total viscous dissipation expressed in terms of integrals involving vorticity, as in Eq. (4), is known as the Bobyleff–Forsythe formula.[9]

It is convenient to choose a reference frame fixed at the particle center of mass, so that a uniform flow at infinity approaches the drop. A steady shape requires $\mathbf{u}\cdot\mathbf{n}=0$ along the drop surface S_D, while there may be a nonvanishing tangential slip velocity, $\mathbf{u}_s=u_s\mathbf{s}$, along the interface; here **s** denotes the unit tangent vector to the interface and $s=|\mathbf{s}|$. It follows that along the interface

$$\mathbf{n}\cdot(\mathbf{u}\cdot\nabla)\mathbf{u}=u_s\left(\frac{\partial\mathbf{u}}{\partial s}\right)\cdot\mathbf{n}=u_s^2\left(\frac{\partial\mathbf{s}}{\partial s}\right)\cdot\mathbf{n}. \qquad (5)$$

The rate of change of the surface tangent vector **s** along S_D yields the curvature along the path. Hence,

$$\frac{\partial\mathbf{s}}{\partial s}=-\kappa_s\mathbf{n}, \qquad (6)$$

where κ_s denotes the curvature measured along the path of the surface flow $\mathbf{u}_s$. Since the tangential velocity is continuous across the interface ($u_s=\hat{u}_s$), substitution of (5) and (6) into Eq. (4) yields

$$C_D=\frac{2}{\pi\mathscr{R}}\int_V \omega^2\,dV+\frac{2\lambda}{\pi\mathscr{R}}\int_{\hat{V}}\hat{\omega}^2\,dV$$
$$+\frac{4(1-\lambda)}{\pi\mathscr{R}}\int_{S_D}u_s^2\kappa_s\,dS. \qquad (7)$$

At steady state, continuity of tangential stress across the interface can be expressed in terms of the surface velocity u_s and the surface vorticity ω_s according to[5]

$$\omega_s-\lambda\hat{\omega}_s=2\kappa_s(1-\lambda)u_s. \qquad (8)$$

In particular, for the special case of an inviscid bubble ($\lambda=0$) the tangential stress vanishes which requires[7] $\omega_s=2\kappa_s u_s$. Physically, at a stress-free surface, vorticity is generated owing to local rotation of fluid elements associated with flow over a curved boundary. Hence, using (7) and (8) the drag coefficient C_D may be expressed completely in terms of the vorticity according to

$$C_D=\frac{2}{\pi\mathscr{R}}\int_V \omega^2\,dV+\frac{2\lambda}{\pi\mathscr{R}}\int_{\hat{V}}\hat{\omega}^2 dV$$
$$+\frac{1}{\pi\mathscr{R}(1-\lambda)}\int_{S_D}\frac{(\omega_s-\lambda\hat{\omega}_s)^2}{\kappa_s}\,dS. \qquad (9)$$

Equation (9) is the principal analytical result of this communication. In particular, (9) demonstrates that the vorticity distribution internal and external to the drop is sufficient to calculate the drag coefficient in steady flows for any drop shape compatible with the steady translation. We next consider several features of drop and bubble motion which follow directly from (9).

Discussion. The characteristic (asymptotic) features of high Reynolds number flow around a bubble ($\lambda=0$) are discussed by Moore,[3] Batchelor,[7] and Leal.[10] In particular, the bulk of the fluid is irrotational. There is a thin viscous boundary layer of thickness $O(\mathscr{R}^{-1/2})$ along the bubble surface, where the vorticity is $O(1)$, which feeds into a narrow stagnation region at the back of the bubble. Also, there is a narrow "inviscid" wake, with thickness $O(\mathscr{R}^{-1/4})$ and approximate length $O(\mathscr{R}^{1/2})$, where the vorticity is $O(\mathscr{R}^{-1/4})$ and beyond which velocity gradients viscously decay. With these estimates it is clear from Eq. (9) that the leading order drag coefficient only depends on the $O(1)$ surface vorticity distribution as appears in the final integral on the right-hand side. The boundary layer, stagnation region, and wake yield $O(\mathscr{R}^{-3/2})$ contributions to the drag. Hence, we see that using Eq. (9) the drag-surface vorticity relationship, noted by Kang and Leal[6] for a spherical bubble, holds for any shape compatible with the steady rise speed.

By way of example, we note that for a spherical bubble, the potential flow solution yields $u_s=u_\theta(r=1,\theta)=\frac{3}{2}\sin\theta$, $\kappa_s=1$, $\omega_s=3\sin\theta$. Hence, we deduce from (9) the well-known result $C_D^{\text{bubble}}=24/\mathscr{R}$. Also, it is possible to demonstrate the relationship between drag on a spherical bubble and the surface vorticity using derivations in standard textbooks in conjunction with Eq. (8). One such approach is outlined in the Appendix.

For the case of a viscous drop, the potential flow outside is coupled to Hill's spherical vortex inside.[4,11] The additional $O(1)$ vorticity contributions from this internal motion gives $C_D^{\text{drop}}=24(1+\frac{3}{2}\lambda)/\mathscr{R}$. Hence, the connection between surface vorticity and drag which exists for a bubble no longer holds owing to viscous dissipation which accompanies internal fluid motion with nonzero vorticity. Experimental observations and numerical simulations demonstrate that a detached eddy typically exists behind the drop. This eddy would make an $O(1)$ contribution to the above estimate of the drag coefficient. Nevertheless, numerical simulations[5] show that provided the Reynolds number is large enough (typically >120), the Harper and Moore theory provides an adequate qualitative and quantitative description of the flow.

As a final note, it may be possible to use the linearized vorticity equation, described by Kang and Leal,[6] in conjunction with Eq. (9) to study the bubble or drop problem further. This approach may provide an alternative for calculating the first correction to the drag.[3] Admittedly, though, an attempt by the author leads to no simplifications from Moore's original development. Lastly, during the preparation of this note the author noticed that Ryskin[12] developed similar equations starting with the Bobyleff–Forsythe equation in order to study the effective viscosity of suspensions of spherical bubbles and drops.

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APPENDIX: A SPHERICAL BUBBLE

For a spherical bubble, it is possible to demonstrate the drag-surface vorticity relationship using derivations found in textbooks. For example, Batchelor[7] (Eq. 5.14.8) shows that the drag force, expressed in spherical coordinates (r,θ,ϕ), is

$$FU=-4\pi\mu aU^2\int_0^\pi u_\theta\frac{\partial u_\theta}{\partial r}\sin\theta\,d\theta, \tag{A1}$$

where all variables under the integral sign are dimensionless. The (azimuthal) vorticity ω_ϕ evaluated along the surface $r=1$ satisfies $\partial u_\theta/\partial r=u_\theta-\omega_\phi$. Hence,

$$FU=-4\pi\mu aU^2\int_0^\pi u_\theta(u_\theta-\omega_\phi)\sin\theta\,d\theta. \tag{A2}$$

Since the tangential stress is zero along the bubble surface, then $2u_\theta=\omega_\phi$ [Eq. (8)]. Therefore, we have

$$FU=\pi\mu aU^2\int_0^\pi \omega_\phi^2\sin\theta\,d\theta, \tag{A3}$$

which shows that for a spherical bubble the drag coefficient only depends on the surface vorticity distribution.

[1] J. F. Harper, "The motion of bubbles and drops through liquids," Adv. Appl. Mech. **12**, 59 (1972).

[2] V. G. Levich, "Motion of gaseous bubbles with high Reynolds numbers," (in Russian) Zh. Eksp. Teor. Fiz. **19**, 18 (1949).

[3] D. W. Moore, "The boundary layer on a spherical gas bubble," J. Fluid Mech. **16**, 161 (1963); see also, D. W. Moore, "The velocity of rise of distorted gas bubbles in a liquid of small viscosity," J. Fluid Mech. **23**, 749 (1965).

[4] J. F. Harper and D. W. Moore, "The motion of a spherical liquid drop at high Reynolds number," J. Fluid Mech. **32**, 367 (1968).

[5] D. S. Dandy and L. G. Leal, "Buoyancy-driven motion of a deformable drop through a quiescent liquid at intermediate Reynolds numbers," J. Fluid Mech. **208**, 161 (1989).

[6] I. S. Kang and L. G. Leal, "The drag coefficient for a spherical bubble in a uniform streaming flow," Phys. Fluids **31**, 233 (1988).

[7] G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, 1967), pp. 364–370.

[8] In the potential far field, $u_{\text{dist}}\sim 1/r^3$ so that contributions from this part of S_∞ are negligible. Also, the narrow wake region has a cross-sectional area which grows proportionately to the downstream distance z, while the velocity disturbance created by the bubble decays with downstream distance as $u_{\text{dist}}\sim 1/z$ and so once again the far-field contribution is negligible. Finally, the reader may note that for a viscously dominated flow, $u_{\text{dist}}\sim 1/r$ and again the integral over S_∞ vanishes.

[9] J. Serrin, *Mathematical Principles of Classical Fluid Dynamics*, Handbuch der Physik (Springer-Verlag, New York, 1959), Vol. 8/1, p. 125.

[10] L. G. Leal, *Laminar Flow and Convective Transport Processes* (Butterworth-Heinemann, Boston, 1992), pp. 609–621.

[11] The stream function, $\hat{\psi}(r,\theta)$ expressed in spherical coordinates, for Hill's spherical vortex is $\hat{\psi}(r,\theta)=\frac{3}{4}r^2(1-r^2)\sin^2\theta$ which leads to the internal vorticity $\hat{\omega}(r,\theta)=\frac{15}{2}r\sin\theta$.

[12] G. Ryskin, "The extensional viscosity of a dilute suspension of spherical particles at intermediate microscale Reynolds numbers," J. Fluid Mech. **99**, 513 (1980).