Rayleigh-Plateau Instability: Falling Jet

Analysis and Applications

Oren Breslouer
MAE 559
1/08/10
Final Project Report

I) Qualitative Description

A liquid jet, initially of constant radius, is falling vertically under gravity. The liquid length increases and reaches a critical value. At this critical value, the jet loses its cylindrical shape as it decomposes into a stream of droplets. This phenomenon occurs primarily as a result of surface tension.

Joseph Plateau first characterized this instability in 1873 through experimental observation, building on the work of Savart. He noted the instability arose when the liquid column length exceeded the column diameter by a factor of about 3.13 (Plateau, 1873). Lord Rayleigh later corroborated Plateau's work, giving an analytical explanation of this physical observation.

This liquid behavior derives from the existence of small perturbations in any physical system. All real-world flows have some non-negligible external disturbance that will increase exponentially in unstable systems. In general, this deformation of the column, called *varicose perturbations*, is represented as a series of periodic displacement sinusoids, as in Fig. 1 (Rayleigh-Plateau Wikipedia). For certain wavelengths, these perturbation waves will grow larger in time.

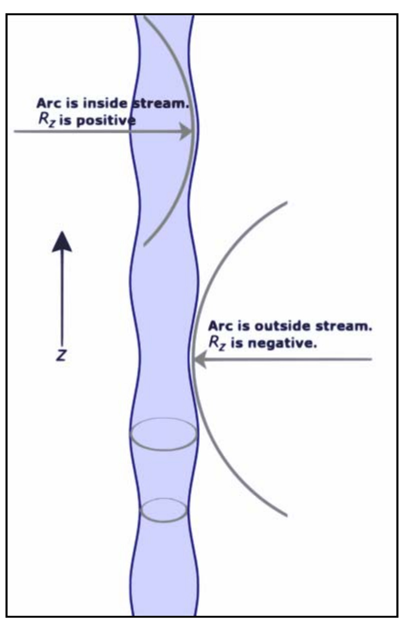
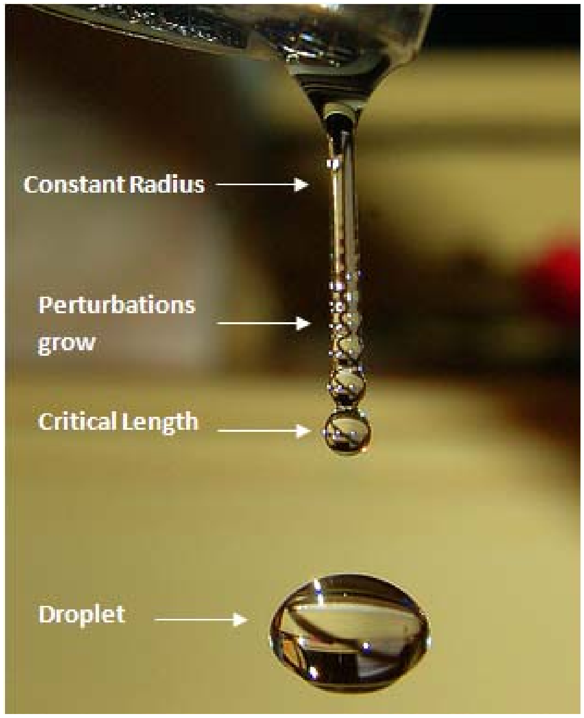


Figure 1: Falling liquid column with periodic perturbations

Note, as the amplitude of the displacement grows, the liquid column will no longer have a constant radius of curvature. Within short times or small lengths, the jet is a cylinder with K_1 equal to $1/R_o$ and K_2 equal to zero. But as shown in Fig. 1 above, the perturbed cylinder now has areas with positive curvature and other areas with negative curvature. From Young-Laplace, the pinched sections have higher pressure ($1/R$ is greater) and the bulging sections have lower pressure, thereby producing a fluid flow due to pressure gradient. This internal flux causes the growth of displacement amplitude which eventually initiates droplet formation. The droplets form when the pinched areas rupture and the bulged areas transform into spherical droplets.

As with all surface tension dominated problems (compressibility and viscous forces are negligible), the specific system geometry depends on energy minimization. A liquid "desires" to be in a minimal energy state. Since surface particles, with only half the neighboring molecules as those in the bulk, have the most energy, the fluid seeks to minimize its surface area. A lower energy state, a result of a total decreased surface area, exists if the fluid breaks into droplets. See Fig. 2 (flickr.com/photos/sqlnerd/204436953/) and Fig. 3 (Hagedorn, 2004) for representational images of the instability.



.Figure 2: Picture of instability

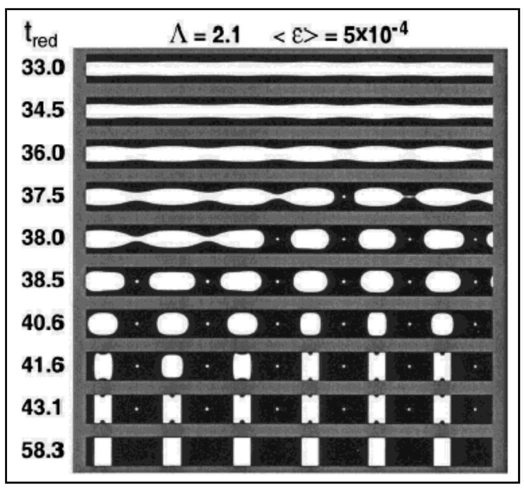


Figure 3: Numerical simulation of instability in horizontal liquid column

As shown later, viscosity and gravity effects (thinning) are neglected per the assumption of insignificant viscous forces (high Re number) and body forces scaling to zero in the governing equations. Rayleigh's treatment of the problem, almost identical to the analysis presented below, included these assumptions. As for the motion of the jet and acceleration under gravity, Rayleigh concludes, "In the cases just considered, the cause of the instability is statical, and the phenomena are independent of the general translatory motion of the jet." (Rayleigh, 1878) The subsequent quantitative analysis considers only liquids for which this is applicable.

II) Motivation and Applications

Lord Rayleigh's initial interest in the problem seems to be wholly academic. He begins his seminal paper, "[m]any, it may even be said, most of the still unexplained phenomena of Acoustics are connected with the instability of jets of fluid". (Rayleigh, 1878) As with Rayleigh, the entirely intellectual exercise of studying this problem is likely fulfilling. The system represents one of many examples of dynamic surface tension fluid flows. The relatively simple experimental results give rise to a rigorous mathematical representation that robustly explains the underlying phenomena. Thus, the problem successfully involves an intimate melding of theory and experiment.

The problem also has analogues to other fluid flow problems. First, similar analysis is appropriate for a thin film coating a cylindrical rod or fiber. This fluid film is inherently unstable and the growth of perturbations mirrors that of the Rayleigh-Plateau instability. In this formulation, viscous effects and the hydrophillicity of the wetted material become important but the general behavior, where a cylindrical column of liquid devolves into a series of droplets, is identical. Second, it is related to the Rayleigh-Taylor instability which occurs between two immiscible and parallel fluids of unequal density. Third, the necking of a falling fluid, such as water disconnecting from a faucet, develops into a droplet much in the same way the liquid column does. Finally, the number of droplets of the crown splash (occurring following a droplet impact into a stationary liquid layer) is determined by the longest wavelength of the Rayleigh-Plateau instability. The splashing rim can be modeled as an unstable cylinder subject to the instability (Deegan, 2008). Elucidation of the Rayleigh-Plateau problem provides insight into these related phenomena.

Another, rather surprising, analogous problem involves the stability of spacetime. A black hole is stretched along some arbitrary dimension into a "black string" and then perturbed along this dimension. It is hypothesized that the black string will breakup into smaller black holes, as in the Rayleigh-Plateau instability. Instead of surface tension and fluid fields such as pressure and velocity, the system is governed by Einstein's general relativity equations, gravity, and other cosmological phenomena (Cardoso, 2006).

In a practical sense, the instability arises in a number of real-world technologies. For applications, I will include the analogous systems similar to the specific problem formulation presented in this paper (a quiescent liquid jet). The most common application is in inkjet printing, where printers use the phenomenon to improve performance. The inkjet stream breaks into extremely small droplets, on the order of 50 microns, that flow at a regular interval in accordance with Rayleigh-Plateau. This small stream of droplets, arising in a predictable manner, is useful for highly precise printing, including increased resolution and accurate coloring. Most inkjet printers initiate the instability with pressure or thermal perturbations behind the ink nozzle, subsequently giving each droplet a charge that determines its deflection onto the paper matrix. The design process matches perturbation magnitude, droplet size, and frequency with printhead structure and ink properties (generally Newtonian fluids). One aspect of the inkjet printer not described by Rayleigh's linear approach is the creation of satellite droplets, small droplets connected to the main droplets. Elimination of this phenomenon is an ongoing problem in the inkjet industry (Martin, et. al, 2008). See Fig. 4 for an image of the general inkjet process (dictionary.zdet.net).

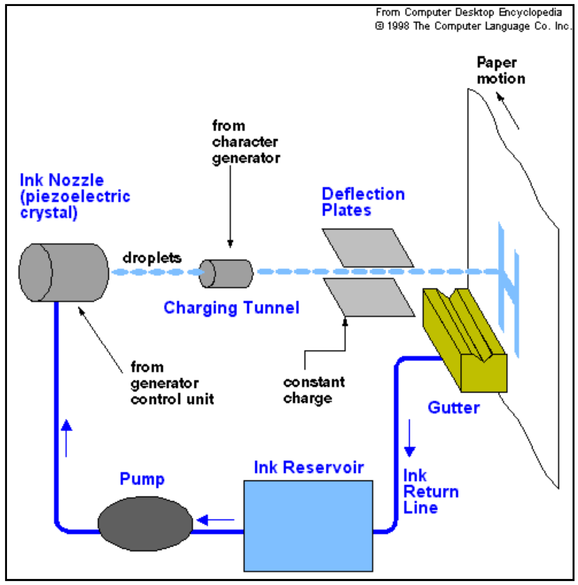


Figure 4: Schematic of inkjet printer

Additionally, the instability occurs in coating optical fibers used in communication systems. The delicate fibers are protected by a urethane acrylate composite material that wets these highly thin glass fibers (Hagedorn, et. al, 2004). The composite is applied prior to UV curing, so the process from wetting to curing must be accomplished on a time scale significantly less than the critical time for Rayleigh-Plateau droplet formation. In Enhanced oil recovery or tertiary recovery, high pressure gas is injected into the oil stratum, thereby forcing oil into the pipe. This represents a significant perturbation to the oil flow, thereby initiating the instability in the confined space of the transferring pipes. Note though, that the oil moves with high velocity and viscous effects would generally temper the onset of instability.

A more recent development involves the so-called "lab on a chip" concept. These millimeter size devices perform laboratory functions relevant to chemistry, biology, and MEMs technology. In these digital microfluidics problems, tiny discrete droplets of liquid, on the order of a picoliter, are transported, mixed, stored, or otherwise manipulated in polymer fluid channels. A process called *flow focusing* employs the Rayleigh-Plateau instability in more efficient mixing processes. Often, the lab on a chip uses a process deemed "droplet on demand" (Bokor). As in inkjet printing, an induced perturbation on a continuous jet can induce droplet breakup. Since a robust solution for the instability exists, the processor can manipulate precisely defined volumes of fluids created at regular intervals.

In a biological context, a 2004 paper by Duclaux et. al hypothesizes that the Rayleigh-Plateau instability may occur within the human lung. The creation of droplets or lenses may impair respiratory performance. Further, the instability pertains to transport in other biological systems, as the arteries, veins, and microtublets of animal cells reduce to tube structures with an imbibing fluid. Finally, a rather obvious and everyday example is the falling of raindrops. Rain does not reach the ground in a continuous column of liquid. Instead, it forms droplets per the Rayleigh-Plateau instability.

These systems are not all explicitly defined by the falling liquid jet, but do involve similar analysis and underlying physical behavior. In general, analysis pertaining to the liquid jet is an important aspect of numerous fluid technologies as well as motivating understanding on analogous systems.

III) Dimensional Analysis

The Rayleigh-Plateau instability is defined by two factors: the growth rate of the perturbations and length scale over which the instability grows. In this system, the time scale is equal to the length scale divided by the jet speed. The length scale, shown in Fig 2, is the length over which the fluid begins to decompose into droplets. The growth rate of the perturbations is important because it characterizes what disturbances initiate the instability. As shown in the next section, the disturbances are a set of sinusoids with varying growth rates and wavenumbers, only a limited number of which are unstable. As discussed previously, many technologies initiate droplet formation as a design parameter. Thus, the exact nature of the disturbance input, perhaps from a piezoelectric element, can be tuned to initiate the instability according to performance specifications (e.g. critical length or droplet size must be less than a prescribed value).

First, I will use dimensional analysis to find a relationship between the critical length, L_{crit}, and fluid properties.

$$L_{crit} = f\left(\rho, R, U_{jet}, \gamma\right) \tag{1}$$

where ρ, R, U_{jet}, γ are density, column radius, jet stream velocity, and surface tension, respectively. One can comprehensively describe the system in terms of energy, with surface tension and momentum flux dominating the system (body forces are negligible). I normalize the critical length, L_{crit}, by the column radius, R. Then, I set the velocity exponent to be one. The resulting equation becomes:

$$\frac{L_{crit}}{R} \sim UR^a\rho^b\gamma^c = U[L]^a\left[\frac{M}{L^3}\right]^b\left[\frac{M}{T^2}\right]^c \tag{2}$$

Solving for a, b, and c, one obtains the following relation for the normalized critical length:

$$\boxed{\boldsymbol{\frac{L_{crit}}{R} \sim U\left(\frac{\rho R}{\gamma}\right)^{\frac{1}{2}}}} \tag{3}$$

There appears a critical time scale, also present in the dispersion relation below.

$$\boxed{\boldsymbol{T_{crit} = \left(\frac{\rho R^3}{\gamma}\right)^{\frac{1}{2}}}} \tag{4}$$

Qualitatively, it is clear that an increase in inertial energy, which depends on $\rho, R,$ and U_{jet}, makes the stream less susceptible to instability generated by external disturbances. For example, consider a firehose versus a garden hose. From common experience, one notes the firehose stream is not as easily diverted as the stream from a garden hose. As for surface tension, one can gauge its importance from the Young-Laplace equation below:

$$\boldsymbol{\Delta p = \gamma \nabla \cdot \underline{n}}$$

As the perturbations grow, the fluid surface has increasingly large gradients mathematically described by the divergence of the normal vector *n*. The more pronounced curvature of the surface shape leads to an increase in pressure gradient, Δp, then greater momentum flux of fluid from high to low pressure, and finally rupture of pinched areas and droplet formation. The jet finally undergoes droplet formation when the pressure gradients induce internal flux that pushes all the fluid from troughs to peaks. The surface tension acts as a gain on this surface shape divergence ($\nabla \cdot \underline{n}$). Thus, the greater the surface tension, the more pronounced the pressure gradient and the faster droplet formation occurs.

Alternatively, one can view the critical time T_{crit} as a metric of the underlying physics. In systems with large surface tensions relative to inertial energy (small T_{crit}), the total energy of the system will primarily be a function of surface tension. In systems with a large critical time, the low surface tension must do relatively more work to overcome the inertial energy in changing the jet's shape to reduce surface area. Thus, the process takes longer, reflected in an increased critical time scale. Note that this energy-based interpretation applies only to systems without a dissipative effect like viscosity. In these viscous systems, the Ohnesorge number $\left(\frac{\mu}{\sqrt{\rho\gamma L}}\right)$ is most reflective of the dominant effects due to the presence of viscosity.

The presence of a non-dimensional group providing a metric to compare the system's underlying physics parallels non-dimensional groups pertinent to other fluid systems, such as the Reynolds number for flat plate flow. For application purposes where breakup is desirable, such as for inkjet printing, a high surface energy, perhaps from a warming process, is most appropriate. For applications where droplet formation is undesirable, such as for oil excavation, a high jet speed or large cross section area is best.

The second important component of system behavior involves the growth of perturbations, a factor in how quickly the instability arises. As shown in the next section, the displacement due to disturbance takes the form of a wave: $\tilde{R}(x,t) \sim e^{\omega t + ikx}$, with ω equal to the growth rate and k equal to the wavenumber. With the variables defined as before:

$$kR = f(\rho, \omega\ , \gamma) \tag{6}$$

Notice the first non-dimensional group is essentially the radius, *R*, normalized by the wavelength. To obtain the other non-dimensional group, I solve the following equation:

$$kR = f(\omega^a \rho^b \gamma^c) = f\left(\left[\frac{1}{T}\right]^a \left[\frac{M}{L^3}\right]^b \left[\frac{M}{T^2}\right]^c\right) \tag{7}$$

Solving for *a*, *b*, and *c*, one obtains the following relationship between growth rate and wavenumber:

$$\boxed{\boldsymbol{\frac{\rho R^3 \omega^2}{\gamma} = \omega^2 {T_{crit}}^2 = f(kR)}} \tag{8}$$

The specific function, called the dispersion relation, is derived in the next section. Once again, the nature of the instability formation depends on fluid properties, including surface tension, and the cross sectional area. It is also apparent that the growth rate of a disturbance has a dependence on wavelength. This relationship will be elucidated in the next section with a mathematical approach.

IV) Quantitative Description of Problem

The initial step is to approximate the perturbations, assuming the displacements induced by disturbances are much less than the cross sectional radius R_o (Mei, 2004). A wave dependent on both time and spatial direction is chosen as below, with $\varepsilon \ll R_o$. Similar forms are chosen for velocity and pressure perturbations:

$$\begin{gathered} R(x,t) = R_o + \varepsilon(e^{\omega t\,+\,ikz}) \\ u_r = R(r)(e^{\omega t\,+\,ikz}) \\ u_z = Z(r)(e^{\omega t\,+\,ikz}) \\ p = P(r)(e^{\omega t\,+\,ikz}) \end{gathered} \tag{9}$$

Assuming axisymmetry, velocity in only the radial and streamwise direction, and negligible viscous and body forces, the cylindrical Navier-Stokes are as follows:

$$r: \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad (10)$$

$$\theta: (all\ components\ equal\ to\ 0)$$

$$z: \frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Since ε is assumed small, a linearization is appropriate. Neglecting the order ε^2 terms on the left hand side, the Navier-Stokes equations become:

$$r: \frac{\partial u_r}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad (11)$$

$$z: \frac{\partial u_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Using the same procedure for mass conservation, the linearized cylindrical continuity equation is as follows:

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + u_z = 0 \quad (12)$$

Substituting the functions of (9) into the Navier-Stokes and continuity equations gives (13):

$$r: \omega R = -\frac{1}{\rho}\frac{dP}{dr} \quad (13)$$

$$z: \omega Z = -\frac{ik}{\rho}P$$

$$m: \frac{\partial R}{\partial r} + \frac{R}{r} + ikZ = 0$$

One can eliminate *P* and *Z* and obtain a differential equation for *R*. To do so, take the derivative of (13: *r*) and (13: *m*) with respect to *r*, then solve (13: *r*) to get a relation between *Z* and *R* which is substituted into (13: *m*). The resulting differential equation for *R* is:

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} - (1 + (kR)^2)R = 0 \quad (14)$$

Now, I will rescale this final governing equation into non-dimensional form. I normalize both the coordinate r and the wavenumber k by the nominal radius R_o and the function R by the jet speed. I normalize the traveling wave equation at the final step, as it cancels in all intermediate steps. The resulting equation is below:

$$\tilde{r}^2 \frac{d^2\mathcal{R}}{d\tilde{r}^2} + \tilde{r}\frac{d\mathcal{R}}{d\tilde{r}} - (1 + (K\mathcal{R})^2)\mathcal{R} = 0 \quad (15)$$

where $\mathcal{R}$ is the non-dimensionalized function R and $\tilde{r}$ is the non-dimensionalized variable r. Note the non-dimensionalized form of (15) does not contain either of the non-dimensional groups found in the previous section. This is because the pressure function P has been eliminated. The non-dimensional groups involve surface tension and surface tension phenomena are embedded in the Navier-Stokes' equations through their relation with the pressure distribution.

The solution to (15) is a Bessel function of the first kind:

$$\mathcal{R} = CI_1(K\tilde{r}) \quad (16)$$

From the relation between P and R as in (13: r), the **dimensional** form of P follows:

$$P = \frac{-\omega\rho C}{k} I_o(kr) \quad (17)$$

In (17), there appears both growth rate ω and wavenumber k, yet the dimensional analysis did not provide the functional relationship between these two variables. One cannot make use of a dimensional argument in obtaining a non-dimensional form of the pressure field. Instead, one can normalize the pressure field by the base pressure of the fluid in its stable state. Using Young-Laplace, (18) follows:

$$p_o = \frac{\gamma}{R_o} \quad (18)$$

Dividing (17) by this mean pressure gives a new non-dimensional group and a non-dimensional pressure field:

$$\tilde{P} = \frac{-\omega\rho C}{k}\frac{R_o}{\gamma}I_o(kr) \tag{19}$$

with the new non-dimensional group $\left(\frac{-\omega\rho}{k}\frac{R_o}{\gamma}\right)$ relating to the perturbation of pressure. To obtain the full solution, I apply the appropriate boundary conditions. The speed of perturbed column, $\frac{\partial R(x,t)}{\partial t}$, is approximately equal to the normalized fluid velocity, $\frac{u_r}{U_{jet}}$. This gives $C = \frac{\varepsilon\omega}{U_{jet}I_1}$. Next, I apply Young-Laplace at the surface using the form below, with pressure normalized by nominal surface pressure and R_i being a principal radius of curvature.

$$\tilde{p} = \left[\frac{R_o}{\gamma}\right]\left[\gamma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right] \tag{20}$$

In this case, $1/R_2$ is not zero since the column is no longer a perfect cylinder. To approximate the perturbed geometry, assume the radial curvature is equal to $1/R(x,t)$ as defined above. The tangential radius of curvature, previously zero, is a metric of how much the cylinder is perturbed from its base state. Thus, the second radius of curvature, using the second-derivative definition, is $\frac{1}{\varepsilon k^2 e^{\omega t + ikz}}$. Using (20) to solve for pressure field gives the non-dimensional perturbed pressure and perturbed radius:

$$\boxed{\boldsymbol{\tilde{p} = \frac{\varepsilon}{R_o}(1 + (K\mathcal{R})^2)\, e^{\Omega T + iKZ}}} \tag{21}$$

$$\boxed{\boldsymbol{\mathcal{R} = \frac{\varepsilon\omega}{U_{jet}}I_1(K\tilde{r})}} \tag{22}$$

where growth rate and the time are normalized by T_{crit} as defined previously. For an unstable wavelength, the pressure distribution is shown below. Note the pressure difference between troughs and peaks increases with time for this unstable wavelength.

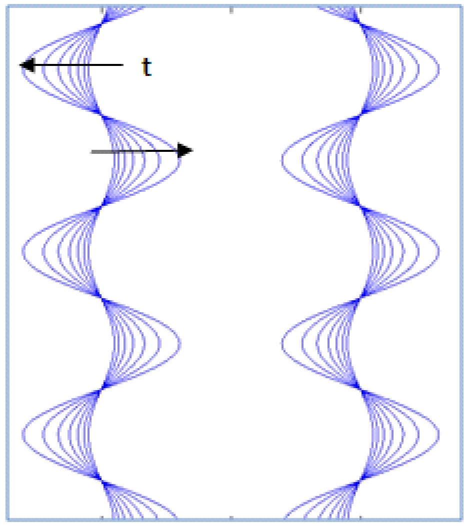


Figure 5: Pressure distribution growth over time

Finally, setting (20) equal to (19) (pressure derived from Young-Laplace equal to pressure derived from Navier-Stokes) yields the dispersion relation:

$$\boxed{\boldsymbol{\omega^2 \frac{\gamma}{\rho R_o{}^3} = \omega^2 T_{crit}{}^2 = kR_o \frac{I_1(kR_o)}{I_0(kR_o)}(1 - (kR_o)^2)}} \quad (23)$$

If the right hand side of (23) is negative, a perturbation with that growth rate and wavenumber will decay as in damped harmonic motion. If the right hand side is positive, that perturbation will grow exponentially and eventually force the system into droplet formation, i.e. instability. For the Rayleigh-Plateau instability, only a real, positive growth rate is of importance. A graphical representation of the dispersion relation follows:

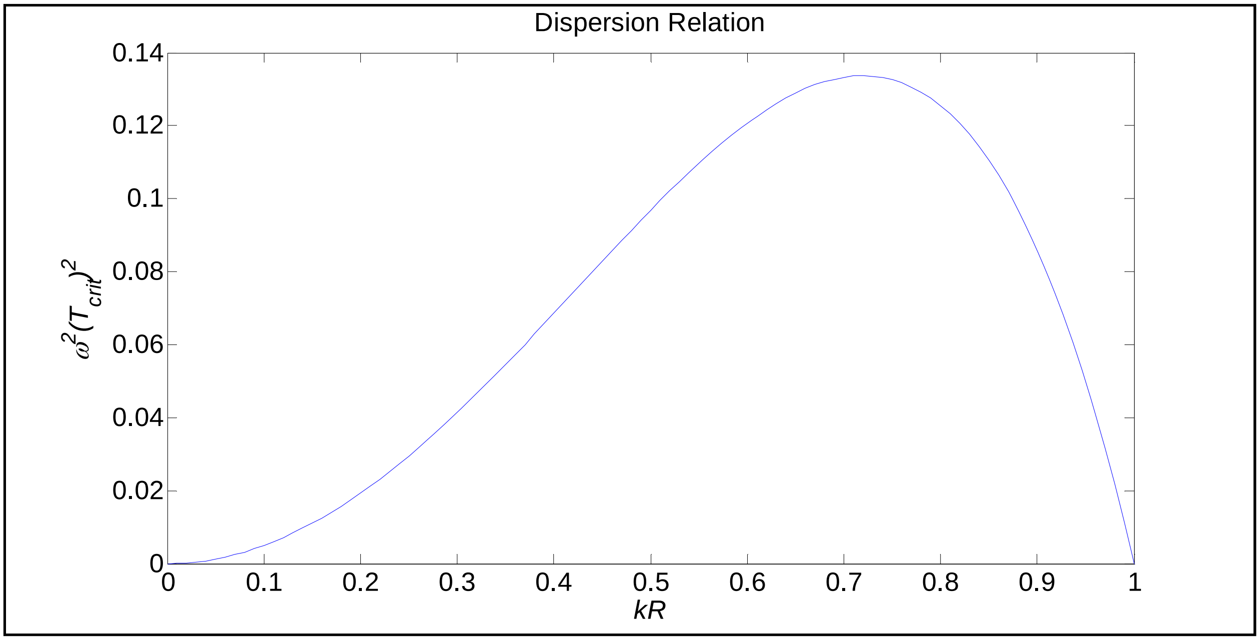


Figure 6: Dispersion Relation

From the above graph, the maximum growth occurs when:

$$\boxed{\omega_{max} \text{ @ } kR_o = 0.697} \qquad (24)$$

The final step is an estimation of the breakup time and distance. Letting the critical time equal to 1, ω_{max} equals 0.1332. Solving, one obtains the critical length scaling constant from dimensional analysis and the critical time:

$$\boxed{\boldsymbol{L_{crit} \cong 2.74\left[U\left(\frac{\rho R^3}{\gamma}\right)^{\frac{1}{2}}\right]}} \qquad (25)$$

$$\boxed{\boldsymbol{T_{crit} \cong 2.74\left[\left(\frac{\rho R^3}{\gamma}\right)^{\frac{1}{2}}\right]}} \qquad (26)$$

Finally, given the fastest growing wavelength and growth rate, droplet formation occurs at frequency $f_{droplet} \sim \lambda U_{jet}$ and the drop radius is approximately $2R_o$. This approximate radius size is obtained by equating the capillary energy of the column with that of the droplet stream. I set the number of droplets, N, equal to the most unstable wavelength multiplied by the critical length, L.

$$E_{col} = E_{drops} : \; 2\pi R_o L\gamma = N\left(4\pi R'^2\gamma\right) \qquad (26)$$

$$= \lambda L\left(4\pi R'^2 \gamma\right) \quad 27)$$

$$\Rightarrow \boxed{\frac{R\prime}{R_o} \approx \left(\frac{\pi}{0.697}\right)^{1/2} \approx 2.1} \quad (28)$$

The results presented in this section categorize both droplet formation and perturbations that cause this instability.

IV) Experimental Results

The first comprehensive study categorizing the Rayleigh-Plateau instability was published by Belgian physicist Joseph Plateau in *Experimental and Theoretical Statics of Liquids Subject to Molecular Forces Only*. This work led to a more precise theoretical examination of the problem by Lord Rayleigh. In general, the experimental results pertaining to the instability are rather simple. Plateau describes his original experiment as follows (translated from the original French):

> The apparatus of which I made use consists of two vertical discs made out of thin iron, of the same diameter, placed facing each other, and which can gradually be brought closer together or moved apart. I initially placed the mobile disc at 108.40mm from the other, which gave very close to 3.6 for the ratio of the length of the cylinder to its diameter, then I made adhere to the whole of the two discs an oil mass in excess, so that the shape constituted a unduloid rather strongly bulging in the middle. Then I extracted liquid gradually, while observing the shape, and this started to become deformed spontaneously when the sagitta of the bulge above was still approximately 5mm.

Plateau would perturb the fluid (oil in a diluted alcohol to vary the density) thereby creating a bulge and then attempt to revert the shape back to the base cylindrical geometry. He varied the distance between the two discs with circular orifices, noting the diameter/length ratio and perturbation amplitude when the deformation occurred. He performed seven experiments, using diameter/length ratios of 3.6, 3.3, 3.18, 3.14, 3.09, 3.11, and 3.13. He concludes:

> One can thus affirm, aside from any theoretical result, that **the limit of the stability of the cylinder lies between the values 3.13 and 3.18**, which differ between them only by 0.05.

A basic fluid system undergoing the Rayleigh-Plateau instability is (clean) water flowing from a residential sink.

$$T_{crit} \cong 2.74\left[\left(\frac{\rho R^3}{\gamma}\right)^{\frac{1}{2}}\right] = 2.74\left[\left(\frac{1000\,\frac{kg}{m^3}\,(0.005\,m)^3}{70x10^{-3}\,\frac{N}{m}}\right)^{\frac{1}{2}}\right] = 0.12\text{ secs}$$

Additionally, the Ohnesorge number relates viscous forces and inertial and surface tension effects. In addition to the Reynolds number, this provides a justification for the inviscid assumption. The ink properties used below are for ethylene glycol (0.5) + water (0.5). Note the Ohnesorge number is far greater for the inkjet system. The qualitative analysis remains appropriate, but the exact formation of the instability, an important aspect of the design process, is not wholly described by the inviscid jet.

$$Oh_{sink} = \left[\frac{\mu}{\sqrt{\rho\gamma L}}\right] = \left[\frac{1x10^{-3}\text{ Pa}\cdot\text{s}}{\sqrt{1000\,\frac{kg}{m^3}\left(70x10^{-3}\,\frac{N}{m}\right)(0.005\text{ m})}}\right] = 0.0017$$

$$Oh_{ink} = \left[\frac{\mu}{\sqrt{\rho\gamma L}}\right] = \left[\frac{7.61\text{ mPa}\cdot\text{s}}{\sqrt{1094\,\frac{kg}{m^3}\left(46x10^{-3}\,\frac{N}{m}\right)(30\text{ nm})}}\right] = 6.2$$

Table 1 (Lin, 1998) below provides typical values of breakup in jet streams. The jet is defined by *Re*, *We*, *Q*, and *k* being Reynolds Number, Weber number (T_{crit}), ratio of ambient and liquid densities, and wavenumber of largest disturbance, respectively.

Re_L	$10^3/We_L$	$Q \times 10^3$	k_r
2	1.25	1.3	0.1669
102	1.25	1.3	0.5725
4×10^2	2.50	1.3	0.7701
4×10^4	1.25	0.1	0.7088
11016	0.882(−2)	1.3	35.417
36720	0.882(−2)	1.3	40.051
67411	0.882(−2)	1.3	41.580
116122	0.802(−2)	1.3	42.368

Table 1: Common numbers for jet breakup

As noted in the application section, the Rayleigh-Plateau instability also occurs for a thin film coating a wire or fiber and for a cylindrical column not under the influence of gravity (low Bond number). One can observe this behavior by spreading a thin film of oil or syrup across a string. A picture (http://www.treehugger.com/water-on-stem-teaser.jpg) of water droplets on a plant follows:

Figure 7: Water droplets on a plant stem

A 2004 paper by Hagedorn, et. al, studied polymer deformation within a confined space. This system likely has more real-world relevance than the jet described in this report, yet the underlying physics are almost identical. For the experimental set-up, the system was subject to minimal confinement effects and essentially matches the liquid jet. They used a nylon thread of radius 127 μm with a temperature of 503 K to avoid non-Newtonian effects (a basic assumption of the mathematical analysis above). The underlying instability is quite similar to the jet, as shown in Fig. 8 below.

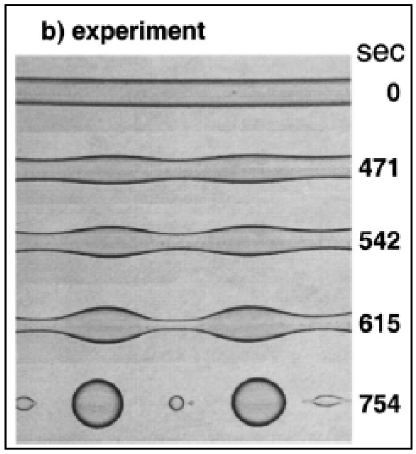


Figure 8: Polymer deformation in confined matrix

Note the order of magnitude for critical time is much higher than for the dynamic jet. This is common for the droplet-on-a-string system with high viscosity. For a string with radius 100 μm, perturbations on the order of 1 μm, and using standard cooking oil, the critical time is generally on the order of an hour (matches well with theory).

V) Generalizations and New Questions

As noted throughout this report, the physics describing droplet formation from a flowing dynamic jet can be applied to a number of other systems. These include a thin film on a fiber or string, a thin film inside a cylindrical tube, and tube-shaped polymers. The systems have slight differences which arise in the governing equations, but the general qualitative structure of the problems is similar. In each case, a stable liquid geometry goes unstable due to the presence of external disturbances. This occurs for the same reason in all systems: the fluid system reaches its minimal energy state by minimizing its surface area through the formation of droplets. However, unlike the dynamic jet, viscous effects are often a non-negligible component of system behavior, such as the manner by which the film adheres to the string. Additionally, another variable, the string thickness or tube radius, especially in relation to the perturbation amplitude, enters the analysis.

The treatment presented in this report is a specific formulation of a more general problem defined by two parameters: scale of Reynolds number and inner fluid properties. The Reynolds number provides a metric for comparing viscous and inertial effects. Further, one can analyze a system with an inner thread of liquid surrounded by air or an inner thread of air surrounded by a liquid using the same approach.

As for questions related to future research, the exact nature of the Rayleigh-Plateau instability has been understood for quite some time. Current research relates to either the inhibition or inducement of the phenomenon, much of which pertains to microfluidics devices. An interesting question involves the control problem of stabilizing a capillary bridge. Absent of a control system, the length of these bridges are restricted by the Plateau limit. Control systems are designed in order to detect the amplitude of the perturbations (ε in the quantitative section) and apply active control to dampen deviations from the base state.

In a more pedagogical sense, little research has been done for nonlinear geometries, such as polygonal strings or curved jet streams. These problems have an innate curvature which would likely affect the pressure field. Also, does the instability occur at all length scales? At exceedingly small length scales, such as on the order of nanometers, do molecular forces begin to affect the behavior, possibly acting as a stabilizing or destabilizing mechanism? Further, systems with a very large characteristic length, such as flows within a gaseous planet or star, traverse very large distances with variations in temperature along its axial direction. Since temperature has a significant effect on surface tension, how will this affect the onset of the instability? Also, how do a non-axisymmetric perturbation and a rotating velocity field affect the instability growth? And finally, how does the instability manifest with multiple jets of varying densities and speeds coalescing?

VI) References

Arkin, William, *Trends in lasers and electro-optics research*, Nova Science Publishers 2005

Aslanov, S., *Theory of Breakup of a liquid jet into droplets*, Technical Physics, Volume 4, Number 11, 1999

Bokor, Jeffrey, *Droplet on Demand Technology*, Research homepage: orange.eecs.berkeley.edu/research.html

Cardoso, Vitor, Ó. Dias, *Black Holes and Strings in the Water Tap*, Phys. Rev. Lett. 96, 181601 (2006)

Deegan, Robert D., et. al., *Rayleigh-Plateau Instability Causes the Crown Splash*, http://www-personal.umich.edu/~rddeegan/PDF/CrownSplashPRL.pdf, 2008

dictionary.zdnet.com/definition/inkjet+printer.html

Duclaux, Virginie; Quéré, David; Clanet, Christophe**,** *Does Plateau-Rayleigh instability occur in our lungs?*, American Physical Society, 57th Meeting of the Division of Fluid Dynamics, 2004

flickr.com/photos/sqlnerd/204436953/

Hagedorn, John, et. al, *Breakup of a fluid thread in a confined geometry: droplet-plug transition, perturbation sensitivity, and kinetic stabilization with confinement*, Phys. Rev. E 69, 056312 (2004)

Lin, S. P., *Drop and Spray Formation from a Liquid Jet*, Annu. Rev. Fluid Mech. Vol. 30 1998

Martin, Graham, Hoath, Stephen, *Inkjet printing - the physics of manipulating liquid jets and drops*, *J. Phys.: Conf. Ser.* 105, 2008

Mei, C.C., web.mit.edu/1.63/www/Lec-notes/Surfacetension/Lecture5.pdf, , 2004

Plateau, Joseph, *Experimental and Theoretical Statics of Liquids Subject to Molecular Forces Only*, 1873

Tabeling, P., *Introduction to Microfluidics*, pages 118-119, Oxford University Press, 2006

http://www.treehugger.com/water-on-stem-teaser.jpg

Rayleigh, Lord, *On the Instability of Jets*, 1878

wikipedia.org/wiki/Navier-Stokes_equations#Cylindrical_coordinates

wikipedia.org/wiki/Enhanced_oil_recovery

wikipedia.org/wiki/Plateau%E2%80%93Rayleigh_instability