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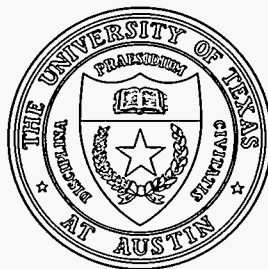
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Nonlinear Theory of Kinetic Instabilities Near Threshold

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Abstract

A new nonlinear equation has been derived and solved for the evolution of an unstable collective mode in a kinetic system close to the threshold of linear instability. The resonant particle response produces the dominant nonlinearity, which can be calculated iteratively in the near-threshold regime as long as the mode does not trap resonant particles. With sources and classical relaxation processes included, the theory describes both soft nonlinear regimes, where the mode saturation level is proportional to an increment above threshold, and explosive nonlinear regimes, where the mode grows to a level that is independent of the closeness to threshold. The explosive solutions exhibit mode frequency shifting. For modes that exist in the absence of energetic particles, the frequency shift is both upward and downward. For modes that require energetic particles for their existence, there is a preferred direction of the frequency shift. The frequency shift continues even after the mode traps resonant particles.

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I. Introduction

There is a variety of plasma problems where instability arises as a result of the free energy available in a particle distribution with an inverted population. When the instability drive γ_L from this inverted distribution function exceeds the damping γ_d from background dissipation mechanisms, the waves grow to a nonlinear level. Recently¹ the near-threshold nonlinear regime has been described for a single-mode electrostatic bump-on-tail problem where $\gamma_L - \gamma_d \ll \gamma_L$. It was shown that the nonlinear development depends on the presence of an energetic particle source and on the rate ν_{eff} of collisional relaxation of the particles interacting with the wave. Incrementally above the instability threshold the mode saturation level is always proportional to the closeness to the threshold when collisionality is taken into account. However, if $\gamma_L - \gamma_d$ is greater than ν_{eff} the mode exhibits an explosive nonlinear evolution, growing faster than exponential to levels that are independent of the closeness to the threshold.¹

The purpose of the present work is to show how these bump-on-tail results can be generalized to any kinetic system where the unperturbed particle orbits are integrable. The theory is extended in several ways. The initial work¹ is limited to the analysis of waves that exist in absence of hot particles. The kinetic component was treated as a perturbation to the mode. In this paper we extend the theory to so-called nonperturbative waves, the waves whose very existence requires the kinetic component. We present a detailed derivation of the results outlined in Ref. 2. The key assumption needed for this generalization is that the system is close to the instability threshold, which still allows the linear properties of the mode to depend on the distribution of hot particles.

A distinctive feature of the near-threshold kinetic problem is that the nonlinear response of resonant particles is the dominant nonlinear response. We employ an action-angle

formalism³ to present the response of the resonant particles in a universal form. We also use a reduced Fokker-Planck collision operator⁴ to describe collisions of resonant particles in a realistic way (in the previous work, collisions were schematically described by a Krook operator). The nonresonant particles can be treated in a linear approximation, and other nonlinear effects, such as mode coupling, are higher order. In this work we use linear theory (including collisions) to lowest order, and then iterate to third order in the field amplitude. This approach is valid as long as resonant particles do not complete a bounce cycle in the field of the wave (because they either scatter out of resonance or the time interval is too short to allow large deflections). We then find that the theory and the ensuing explosive solutions are accurate if the particle nonlinear bounce frequency ω_b is less than γ_L . This is a larger range of applicability than was presumed in Ref. 1. At breakdown the explosive solution shows that the wave frequency shifts by an amount comparable to γ_L . Depending on details the frequency can shift up, down or simultaneously in both directions. After the breakdown of the near-threshold theory a different type of analysis has been used and some of the results are discussed in the text. We will see that the dissipation and drive remain in balance and the mode frequency can change by large amounts compared to γ_L .

The theory presented in this paper has a broad range of applications in describing plasma phenomena. It can be used to interpret the onset of shear Alfvén instability in a tokamak⁵ (more precisely, the Toroidal Alfvén Eigenmode, TAE, driven unstable by energetic particles), or for describing nonperturbative waves, such as the fishbone,^{6,7} an internal kink mode driven unstable by energetic particles. This theory can also be applicable to some collective instabilities in storage rings^{8,9} and there are other potential applications that need development.¹⁰ Ironically, the theory does not resolve a classic problem posed by Simon and Rosenbluth^{11,12} of how to determine the mode saturation level near threshold of a double humped distribution function that just barely satisfies the Penrose criterion¹³ for linear instability. In this case we find that the nonlinearity only becomes important at too large of

a field amplitude where the iterative approach fails.

The structure of the paper is as follows: In Sec. II we extend the derivation of Ref. 1 to obtain a universal nonlinear equation for a weakly unstable mode driven by resonant particles, that is applicable even to nonperturbative modes. In Sec. III we present various nonlinear scenarios described by the universal equation. In Sec. IV we discuss some nonlinear regimes that go beyond the assumptions of the near-threshold theory. An overall summary is given in Sec. V.

II. Basic Equations

Near the threshold of linear instability, the evolution of the unstable mode can generally be analyzed within the assumption of a weak nonlinearity. In this limit, the perturbed current \mathbf{J} that enters Maxwell's equations is a sum of \mathbf{J}_L , a part that is a linear functional of the mode electric field \mathbf{E} , and \mathbf{J}_{NL} , a nonlinear current whose functional dependence on \mathbf{E} is calculated with a perturbation technique. Further, in this paper we assume that \mathbf{J}_{NL} arises solely from resonant particles, and we neglect the other contributions to \mathbf{J}_{NL} which are smaller in the range of validity of our calculations. When we use the Fourier transformed Maxwell's equations, we find

$$\int d\mathbf{r}' \mathbf{g}(\mathbf{r}, \mathbf{r}', \omega, \alpha) \cdot \mathbf{E}(\mathbf{r}', \omega) = \mathbf{J}_{NL} \quad (1)$$

where the matrix $\mathbf{g}(\mathbf{r}, \mathbf{r}', \omega, \alpha)$ includes the contribution from \mathbf{J}_L and α is a parameter that measures closeness to the instability threshold. The linear theory yields the homogeneous equation

$$\int d\mathbf{r}' \mathbf{g}(\mathbf{r}, \mathbf{r}', \omega, \alpha) \cdot \mathbf{e}(\mathbf{r}', \omega) = 0.$$

At the threshold, $\alpha \equiv \alpha_{cr}$, this equation has a real eigenvalue $\omega = \omega_0$ and a nontrivial eigenvector $\mathbf{e}(\mathbf{r}, \omega_0)$ which is determined up to an arbitrary constant. Also, there exists an

adjoint vector $\mathbf{e}^\dagger(\mathbf{r}, \omega_0)$, which is a solution to the equation

$$\int d\mathbf{r}' \mathbf{e}^\dagger(\mathbf{r}', \omega_0) \cdot \mathbf{g}(\mathbf{r}', \mathbf{r}, \omega_0, \alpha_{cr}) = 0.$$

When $0 < 1 - \alpha_{cr}/\alpha \ll 1$ and the nonlinear current is sufficiently small, the eigenfunction $\mathbf{e}(\mathbf{r}, \omega)$ must be peaked about $\omega = \omega_0$. We can then expand $\mathbf{g}(\mathbf{r}, \mathbf{r}', \omega, \alpha)$ about $\omega = \omega_0$ and $\alpha = \alpha_{cr}$. To eliminate the lowest order term, we take a dot product of Eq. (1) with $\mathbf{e}^\dagger(\mathbf{r}, \omega_0)$ and integrate over all space. This procedure reduces Eq. (1) to

$$\begin{aligned} \int d\mathbf{r} d\mathbf{r}' \mathbf{e}^\dagger(\mathbf{r}, \omega_0) \cdot [(\omega - \omega_0) \mathbf{g}_\omega(\mathbf{r}, \mathbf{r}', \omega_0, \alpha_{cr}) + (\alpha - \alpha_{cr}) \mathbf{g}_\alpha(\mathbf{r}, \mathbf{r}', \omega_0, \alpha_{cr})] \cdot \mathbf{E}(\mathbf{r}', \omega) \\ = \int d\mathbf{r} \mathbf{e}^\dagger(\mathbf{r}, \omega_0) \cdot \mathbf{J}_{NL}(\mathbf{r}, \omega) \end{aligned}$$

where a subscript indicates a partial derivative. It is now allowable to use the lowest order expression for $\mathbf{E}(\mathbf{r}, \omega)$ in this equation, namely we put

$$\mathbf{E}(\mathbf{r}, \omega) = c(\omega) \mathbf{e}(\mathbf{r}, \omega_0).$$

The factor $c(\omega)$ represents the Fourier components peaked at $\omega = \omega_0$. The real electric field of the mode is

$$\mathbf{E}(\mathbf{r}, t) = C(t) \exp(-i\omega_0 t) \mathbf{e}(\mathbf{r}, \omega_0) + \text{c.c.} \quad (2)$$

where $C(t)$ is a slowly varying mode amplitude. Transformation to the time domain gives the following equation for $C(t)$:

$$iG_\omega \frac{dC}{dt} + (\alpha - \alpha_{cr}) G_\alpha C = e^{i\omega_0 t} \int d\mathbf{r} \mathbf{e}^\dagger(\mathbf{r}, \omega_0) \cdot \mathbf{J}_{NL}(\mathbf{r}, t) \quad (3)$$

where

$$\begin{aligned} G_\omega &\equiv \int d\mathbf{r} d\mathbf{r}' \mathbf{e}^\dagger(\mathbf{r}, \omega_0) \cdot \mathbf{g}_\omega(\mathbf{r}, \mathbf{r}', \omega_0, \alpha_{cr}) \cdot \mathbf{e}(\mathbf{r}', \omega_0) \\ G_\alpha &\equiv \int d\mathbf{r} d\mathbf{r}' \mathbf{e}^\dagger(\mathbf{r}, \omega_0) \cdot \mathbf{g}_\alpha(\mathbf{r}, \mathbf{r}', \omega_0, \alpha_{cr}) \cdot \mathbf{e}(\mathbf{r}', \omega_0). \end{aligned}$$

In order to evaluate the nonlinear term in Eq. (3), we first express $\mathbf{J}_{NL}(\mathbf{r}, t)$ in terms of the particle distribution function, which gives

$$e^{i\omega_0 t} \int d\mathbf{r} \mathbf{e}^\dagger(\mathbf{r}, \omega_0) \cdot \mathbf{J}_{NL}(\mathbf{r}, t) = q e^{i\omega_0 t} \int d\Gamma \mathbf{e}^\dagger(\mathbf{r}, \omega_0) \cdot \mathbf{v}(\mathbf{r}, \mathbf{p}) f_{NL}(\mathbf{r}, \mathbf{p}, t). \quad (4)$$

Here, q is the particle charge, $d\Gamma = d\mathbf{r} d\mathbf{p}$ is the phase space volume element, $\mathbf{v}(\mathbf{r}, \mathbf{p})$ is the particle velocity, and $f_{NL}(\mathbf{r}, \mathbf{p}, t)$ is the nonlinear part of the distribution function.

We now need to find $f_{NL}(\mathbf{r}, \mathbf{p}, t)$ from the kinetic equation

$$\frac{\partial f}{\partial t} + [H, f] = Stf + Q \quad (5)$$

in which the Hamiltonian H splits into $H = H_0 + H_1$ with H_0 being the unperturbed Hamiltonian that determines the equilibrium orbits, and H_1 the perturbation from the mode. The right-hand side of Eq. (5) takes into account particle source Q and collision operator, St . We assume the unperturbed motion described by H_0 to be fully integrable, which allows canonical transformation to action-angle variables. Let I_i with $i = 1, 2, 3$ be the actions and ξ_i be the corresponding canonical angles, so that all physics quantities are periodic functions of ξ_i with the period 2π . Then, the Hamiltonian H can be cast into the form

$$H = H_0(I_1, I_2, I_3) + H_1 \quad (6)$$

with

$$H_1 = 2 \operatorname{Re} C(t) e^{-i\omega_0 t} \sum_{\ell_1, \ell_2, \ell_3} V_{\ell_1, \ell_2, \ell_3}(I_1, I_2, I_3) e^{i\ell_1 \xi_1 + i\ell_2 \xi_2 + i\ell_3 \xi_3} \quad (7)$$

where ℓ_1, ℓ_2 , and ℓ_3 are integers, and $V_{\ell_1, \ell_2, \ell_3}(I_1, I_2, I_3)$ are matrix elements that can be calculated in a standard way, given the mode structure and the unperturbed particle orbits. We have neglected C^2 and other higher order corrections to H as it can be shown that they produce small terms in the final equation, compared to the terms we will generate.

For nearly resonant particles we can relate H_1 to the perturbed electric field given by Eq. (2). The result is

$$H_1 = 2q \operatorname{Re} \left(iC(t) \frac{\mathbf{v} \cdot \mathbf{e}}{\omega_0} e^{-i\omega_0 t} \right).$$

With this expression for H_1 , we find

$$V_{\ell_1, \ell_2, \ell_3}(I_1, I_2, I_3) = \frac{iq}{\omega_0} \int \frac{d\xi_1 d\xi_2 d\xi_3}{(2\pi)^3} \exp[-i(\ell_1 \xi_1 + \ell_2 \xi_2 + \ell_3 \xi_3)] \mathbf{v} \cdot \mathbf{e}.$$

Each term in the perturbed Hamiltonian represents a resonance that can be treated separately if the resonances do not overlap, which we assume here to be the case. For the motion dominated by a single resonance, the summation sign can be dropped in Eq. (7). One can then make a canonical transformation to a new set of action-angle variables so that one of the new angles is $\xi = \ell_1 \xi_1 + \ell_2 \xi_2 + \ell_3 \xi_3$, and I is the corresponding action. The Hamiltonian then reduces to the one-dimensional form

$$H = H_0(I) + 2 \operatorname{Re} C(t) e^{-i\omega_0 t} V(I) \exp(i\xi) \quad (8)$$

where the other two new actions, not shown here, are suppressed as they can be treated as parameters in the new Hamiltonian. The location of the resonance $I = I_r$ is determined by the condition

$$\Omega(I_r) = \omega_0$$

where $\Omega(I) \equiv \frac{\partial H_0}{\partial I} = \ell_1 \frac{\partial H_0}{\partial I_1} + \ell_2 \frac{\partial H_0}{\partial I_2} + \ell_3 \frac{\partial H_0}{\partial I_3}$. In the absence of collisions, the motion of a resonant particle satisfies the pendulum equation

$$\frac{d^2 \xi}{dt^2} + \omega_b^2 \sin(\xi - \omega_0 t - \xi_0) = 0 \quad (9)$$

where

$$\omega_b \equiv |2CV(I_r) \partial \Omega(I_r) / \partial I_r|^{1/2}$$

is the nonlinear bounce frequency of the particle and ξ_0 is a constant phase.

Near resonance, the kinetic equation is

$$\frac{\partial f}{\partial t} + \Omega(I) \frac{\partial f}{\partial \xi} - 2 \operatorname{Re}[iC(t)V \left(\frac{\partial \Omega}{\partial I} \right) \exp(i\xi - i\omega_0 t)] \frac{\partial f}{\partial \Omega} = Stf + Q. \quad (10)$$

In this equation, we have neglected the term $\frac{\partial H_1}{\partial I} \frac{\partial f}{\partial \epsilon}$, which is indeed a justified approximation: this term is small compared to the last term on the left-hand side of Eq. (10) because the perturbed distribution function of the resonant particles has a steeper gradient in I than the perturbed Hamiltonian. For the same reason, we treat the matrix element and $\partial\Omega/\partial I$ as constants evaluated at $I = I_r$ when we solve Eq. (10) for the resonant particles.

We will consider two different descriptions of collisions in Eq. (10): a simplified Krook model and a more realistic diffusive model. For the Krook model, we take

$$Stf + Q = -\nu_r(f - F) \quad (11)$$

where F is the equilibrium distribution function with a nearly constant nonzero slope near the resonance, and ν_r is the relaxation rate. The diffusive collisional operator takes the form

$$Stf + Q = \nu_{\text{eff}}^3 \frac{\partial^2}{\partial \Omega^2} (f - F) \quad (12)$$

where F is again the equilibrium distribution. Equation (12) and the expression for ν_{eff}^3 can be consistently derived from a specific Fokker-Planck collision operator with an appropriate orbit averaging procedure developed in Ref. 4. The specific form for ν_{eff}^3 is given in section (c) of the Appendix. Note that only second derivative term with the perturbed distribution function needs to be retained in the collision operator as this is the dominant term near the resonance where the perturbed distribution is strongly peaked. Equations (11) and (12) lead to similar results if one takes $\nu_r \sim \nu_{\text{eff}}$. The ν -parameter in both equations describes the rate particles decorrelate from resonance when $C(t)$ is sufficiently small.

We explicitly solve Eq. (10) with the diffusive collisional operator and we will also present the results for the Krook model (without derivation). Both solutions are based on a perturbation technique that assumes that either the time interval is short compared with the characteristic bounce period, $2\pi/\omega_b$, or the collisional relaxation rate is much greater than

ω_b . This assumption allows us to seek f in the form of a truncated Fourier series

$$f = F + f_0 + [f_1 \exp(i\xi - i\omega_0 t) + f_2 \exp(2i\xi - 2i\omega_0 t) + \text{c.c.}] \quad (13)$$

where the Fourier coefficients f_0, f_1 , and f_2 are functions of t and I or, equivalently, Ω . Although the second harmonic generally needs to be included in the calculations of the nonlinear response, it turns out that f_2 does not affect the resulting equation for the mode amplitude. Therefore, we ignore f_2 from the very beginning, a procedure that can be verified in a straightforward way. With this simplification, Eqs. (10) and (12) reduce to

$$\frac{\partial f_1}{\partial t} - i[\omega_0 - \Omega]f_1 = iC(t)V \frac{\partial \Omega}{\partial I} \frac{\partial}{\partial \Omega} (F + f_0) + \nu_{\text{eff}}^3 \frac{\partial^2 f_1}{\partial \Omega^2} \quad (14)$$

$$\frac{\partial f_0}{\partial t} = iC(t)V \frac{\partial \Omega}{\partial I} \frac{\partial f_1^*}{\partial \Omega} - iC^*(t)V^* \frac{\partial \Omega}{\partial I} \frac{\partial f_1}{\partial \Omega} + \nu_{\text{eff}}^3 \frac{\partial^2 f_0}{\partial \Omega^2}. \quad (15)$$

We integrate Eqs. (14), and (15) iteratively taking into account that $F \gg f_1 \gg f_0$ and assuming zero initial values for f_1 and f_0 . It is convenient to take the Fourier transformation of Eq. (14) in Ω , which converts it to a first order partial differential equation that can be integrated by the method of characteristics. We first neglect f_0 on the right-hand side of Eq. (14) and find f_{1L} , the part of f_1 , linear in C :

$$f_{1L} = i \int_0^t d\tau C(\tau) V \frac{\partial \Omega}{\partial I} \frac{\partial F}{\partial \Omega} e^{i(\omega_0 - \Omega)(t - \tau)} \exp \left[- \int_{\tau}^t \nu_{\text{eff}}^3(\tau_1)(\tau - \tau_1)^2 d\tau_1 \right].$$

This expression applies to resonant particles for which $\frac{\partial F}{\partial \Omega}$ and V can be treated as constants.

Next, we use f_{1L} instead of f_1 in Eq. (15) to find in a similar manner f_0 and $\frac{\partial f_0}{\partial \Omega}$:

$$\begin{aligned} \frac{\partial f_0}{\partial \Omega} = & -2 \text{Re} \int_0^t d\tau \int_0^t d\tau_1 (\tau - \tau_1)^2 C^*(\tau) C(\tau_1) |V|^2 \left(\frac{\partial \Omega}{\partial I} \right)^2 \frac{\partial F}{\partial \Omega} e^{i(\omega_0 - \Omega)(\tau - \tau_1)} \\ & \cdot \exp \left[- \int_{\tau}^t \nu_{\text{eff}}^3(\tau_2)(\tau_1 - \tau)^2 d\tau_2 - \int_{\tau_1}^{\tau} \nu_{\text{eff}}^3(\tau_3)(\tau_1 - \tau_3)^2 d\tau_3 \right]. \end{aligned}$$

We then substitute $\frac{\partial f_0}{\partial \Omega}$ into Eq. (14) and calculate f_{1NL} , the part of f_1 cubic in C . For $\omega_0 t \gg 1$, the dominant contribution to f_{1NL} has the form

$$f_{1NL} = -i \int_0^t d\tau \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 (\tau_1 - \tau_2)^2 V |V|^2 \left(\frac{\partial \Omega}{\partial I} \right)^3 \frac{\partial F}{\partial \Omega} \cdot \exp \left[- \int_\tau^t \nu_{\text{eff}}^3(\tau_3) (\tau - \tau_3)^2 d\tau_3 - \int_{\tau_1}^\tau \nu_{\text{eff}}^3(\tau_3) (\tau_1 - \tau_2)^2 d\tau_3 - \int_{\tau_2}^{\tau_1} \nu_{\text{eff}}^3(\tau_3) (\tau_2 - \tau_3)^2 d\tau_3 \right] \cdot \left[C(\tau) C(\tau_1) C^*(\tau_2) e^{i(\omega_0 - \Omega)(t - \tau - \tau_1 + \tau_2)} + C(\tau) C^*(\tau_1) C(\tau_2) e^{i(\omega_0 - \Omega)(t - \tau + \tau_1 - \tau_2)} \right]. \quad (16)$$

The nonlinear term on the right-hand side of Eq. (3) is a functional of f_{1NL} . The evaluation of this term involves an integration over phase space, including integration over I , or alternatively, over $\Omega(I)$. As a function of Ω , the integrand is a product of a smooth function, which can be treated as constant near the resonance, and the exponential functions in f_{1NL} . Once integrated over Ω , the exponential functions generate two δ -functions: $\delta(t - \tau - \tau_1 + \tau_2)$ and $\delta(t - \tau + \tau_1 - \tau_2)$, of which only the first one falls into the time domain of Eq. (16). This observation leads to the following structure of Eq. (3):

$$iG_\omega \frac{dC}{dt} + (\alpha - \alpha_{cr}) G_\alpha C = \int_0^t d\tau \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \delta(t - \tau - \tau_1 + \tau_2) (\tau_1 - \tau_2)^2 C(\tau) C(\tau_1) C^*(\tau_2) \int d\Gamma \mathcal{K} \cdot \exp \left[- \int_\tau^t \nu_{\text{eff}}^3(\tau_3) (\tau - \tau_3)^2 d\tau_3 - \int_{\tau_1}^\tau \nu_{\text{eff}}^3(\tau_3) (\tau_1 - \tau_2)^2 d\tau_3 - \int_{\tau_2}^{\tau_1} \nu_{\text{eff}}^3(\tau_3) (\tau_2 - \tau_3)^2 d\tau_3 \right] \quad (17)$$

where

$$\mathcal{K} = 2\pi\omega_0 \delta(\omega_0 - \Omega) V^\dagger V |V|^2 \left(\frac{\partial \Omega}{\partial I} \right)^3 \frac{\partial F}{\partial \Omega}, \quad (18)$$

$$V^\dagger = -\frac{iq}{\omega_0} \int \frac{d\xi_1 d\xi_2 d\xi_3}{(2\pi)^3} \mathbf{e}^\dagger \cdot \mathbf{v} \exp(i\xi). \quad (19)$$

In cases where ν_{eff} is independent of phase space position, one can factor the parameter

$$K = \int d\Gamma \mathcal{K}$$

out of Eq. (17) and obtain a somewhat simpler expression (henceforth, unless otherwise stated, this assumption will be made).

When Eqs. (17)–(19) are applied to specific problems, transformation from I to other variables can be useful. For example, natural transformation of the operator $\frac{\partial}{\partial I}$ for a symmetric torus is given in section (a) of the Appendix.

The linear growth rate γ is given by $\gamma = -(\alpha - \alpha_{cr}) \text{Im}(G_\alpha/G_\omega)$ and it is convenient to rescale time in Eq. (17) in the units of γ^{-1} . In addition, we introduce a new amplitude

$$A = aC \exp(ibt)$$

where $a = |K/G_\omega|^{1/2}/\gamma^{5/2}$ and $b = (\alpha - \alpha_{cr}) \text{Re}(G_\alpha/G_\omega)$ so that Eq. (17) attains a standard form,

$$\frac{dA}{dt} = A - e^{i\phi} \int_0^{t/2} \tau^2 d\tau \int_0^{t-2\tau} d\tau_1 e^{-\nu_d^3 \tau^2 (2\tau/3 + \tau_1)} A(t - \tau) A(t - \tau - \tau_1) A^*(t - 2\tau - \tau_1) \quad (20)$$

where $\nu_d = \nu_{\text{eff}}/\gamma$ is assumed to be time independent, and ϕ is a constant angle defined by the relation

$$e^{i\phi} \equiv iK|G_\omega|/(|K|G_\omega).$$

A similar derivation can be carried out with the Krook collision operator (11). The resulting dimensionless equation has the form

$$\frac{dA}{dt} = A - e^{i\phi} \int_0^{t/2} \tau^2 d\tau \int_0^{t-2\tau} d\tau_1 e^{(-2\nu_a \tau - \nu_a \tau_1)} A(t - \tau) A(t - \tau - \tau_1) A^*(t - 2\tau - \tau_1) \quad (21)$$

where $\nu_a = \nu_r/\gamma$.

In the perturbative case, the matrix g in Eq. (1) is nearly Hermitian, which gives $e^\dagger = e^*$ and $V^\dagger = V^*$. The factor K is real in this case. The quantity G_ω is purely imaginary for any Hermitian matrix g with $\text{Im } G_\omega |C|^2$ being the mode energy. We thus conclude that the value of ϕ can only be 0 or π in the perturbative case. Note that $\phi = 0$ corresponds to a positive energy wave with negative dissipation from resonant particles.

The absolute value of the dimensionless amplitude A in Eqs. (20), (21) measures the square of the typical nonlinear bounce frequency ω_b , namely

$$\omega_b^2 \approx \gamma_L^2 \left(\frac{\alpha - \alpha_{cr}}{\alpha_{cr}} \right)^{5/2} |A|. \quad (22)$$

where for perturbative instabilities γ_L is the growth rate in the absence of background dissipation, and for nonperturbative instabilities γ_L is roughly the growth rate when $\alpha = 2\alpha_{cr}$. It should also be noted that the small parameter $\left(\frac{\alpha - \alpha_{cr}}{\alpha_{cr}} \right)^{5/2}$ in Eq. (22) gives the basis for the neglect of higher than cubic nonlinear terms in Eqs. (20) and (21), as long as $|A| \ll [\alpha_{cr}/(\alpha - \alpha_{cr})]^{5/2}$. Further discussion of the applicability range of Eqs. (20) and (21) is given in section (b) of the Appendix.

III. Steady-State Saturation, Limit Cycle, Explosion

Equation (21) is of the form derived in Ref. 1, except for the additional phase factor $e^{i\phi}$ and the complex conjugate appearing in the nonlinear term. In the limit of large t and when $\cos \phi > 0$, Eq. (21) has a periodic solution with constant amplitude:

$$A = \frac{2\nu_a^2}{(\cos \phi)^{1/2}} \exp(-it \tan \phi), \quad (23)$$

a generalization of the steady state solution found in Ref. 1 to $\phi \neq 0$. A similar solution can be readily obtained for Eq. (20):

$$A = \frac{\nu_d^2}{\left(\cos \phi \int_0^\infty dz \exp(-2z^3/3) \right)^{1/2}} \exp(-it \tan \phi). \quad (24)$$

Note that if $\nu_a/\nu_d = 0.71$ the two models give the same steady state levels. For an unstable system with a negative value of $\cos \phi$, the nonlinearity enhances the drive and always leads to a hard nonlinear scenario where the mode grows to a large amplitude regardless of closeness to the instability threshold. Note that $\cos \phi > 0$ is a necessary but not a sufficient condition

for the mode to saturate at a low level as a hard scenario is possible even when $\cos \phi > 0$. In this case, however, it requires sufficiently low collisionality (see below).

We now address the question of stability for the constant amplitude solutions (23) and (24). In order to make the analysis more compact, we use the transformation

$$A = a(t) \exp(-it \tan \phi), \quad (25)$$

delete the subscripts in ν_a and ν_d , and also combine Eqs. (20) and (21) into

$$\frac{da}{dt} = (1 + i \tan \phi)a - \frac{e^{i\phi}}{\nu^2} \int_0^\infty d\tau \int_0^\infty d\tau_1 Q(\nu\tau, \nu\tau_1) a(t-\tau) a(t-\tau-\tau_1) a^*(t-2\tau-\tau_1) \quad (26)$$

with $Q(x, y) = x^2 e^{-x^2(2x/3+y)}$ for Eq. (20) and $Q(x, y) = x^2 e^{-2x-y}$ for Eq. (21). We have extended the integration limits in Eq. (26) to infinity, reflecting the limit of large t . We now linearize Eq. (26) about the steady state $\alpha = \alpha_0 = \text{const}$ with

$$|a_0|^2 = \nu^4 \left[\cos \phi \int_0^\infty dx \int_0^\infty dy Q(x, y) \right]^{-1} \quad (27)$$

and look for a solution of the form

$$a = a_0 + \delta a_1 \exp(\nu \lambda t) + \delta a_2 \exp(\nu \lambda^* t) \quad (28)$$

with λ an eigenvalue. The solvability condition for the ensuing linear equations for δa_1 and δa_1^* yields the dispersion relation

$$\lambda^2 \nu^2 \cos^2 \phi - 2\lambda \nu Q_1 \cos^2 \phi + Q_1^2 = Q_2^2 \quad (29)$$

where

$$Q_1(\lambda) = \left[\int_0^\infty dx \int_0^\infty dy Q(x, y) \right]^{-1} \int_0^\infty dx \int_0^\infty dy Q(x, y) [1 - \exp(-\lambda x) - \exp(-\lambda x - \lambda y)]$$

$$Q_2(\lambda) = \left[\int_0^\infty dx \int_0^\infty dy Q(x, y) \right]^{-1} \int_0^\infty dx \int_0^\infty dy Q(x, y) \exp(-2\lambda x - \lambda y)$$

For large values of ν ($\nu \gg 1$) all roots of Eq. (29) have $\text{Re } \lambda < 0$, while if ν is sufficiently small unstable roots are found. The critical value of ν at which the first unstable root appears is shown in Fig. 1 as a function of ϕ for both the diffusive and the Krook models. Note that lower stable steady state values of $|A|$ can be achieved in the diffusive model than in the Krook model. For example, for $\phi = 0$ the lowest stable values for $|A|$ are 38 for the Krook model and 6 for the diffusive model.

When the steady state nonlinear solution is unstable, the mode cannot converge to the steady level. What develops instead when ν is close to the critical value, is a limit cycle of the type discussed in Ref. 1. An example of such a cycle is shown in Fig. 2. As ν goes deeper into the unstable range, bifurcations destroy periodicity of the cycle but the mode amplitude can still be limited in this regime (see Fig. 2). Further, at even smaller values of ν , the mode develops an explosive singularity, evolving into a hard nonlinear regime that runs out of the applicability range of Eqs. (20) or (21).

The corresponding explosive solutions were first found in Ref. 1 (for $\phi = 0$). Here, we present another example of such a solution for Eqs. (20), (21). As in the analysis of Ref. 1, we look for an asymptotic solution that becomes singular at a finite time t_0 , at which stage ν and the linear drive can be neglected. In this limit, there is no difference between Eqs. (20) and (21). The solution has the form

$$A(t) = g[\chi(t)](t_0 - t)^{-5/2} \quad (30)$$

where $g[\chi]$ is a periodic function of $\chi \equiv \ln(t_0 - t)$. This structure of A allows a common time factorization and then we can reduce Eqs. (20) and (21) to

$$e^{-i\phi} \left(\frac{5}{2} g - \frac{dg}{d\chi} \right) = \int_0^\infty d\xi \int_0^\infty d\eta U(\xi, \eta) g[\chi + \ln(1 + \xi)] g[\chi + \ln(1 + \xi + \eta)] g^*[\chi + \ln(1 + 2\xi + \eta)] \quad (31)$$

with

$$U(\xi, \eta) = \frac{\xi^2}{(1 + \xi)^{5/2} (1 + \xi + \eta)^{5/2} (1 + 2\xi + \eta)^{5/2}}.$$

We now observe that

$$g(\chi) = \rho \exp(i\sigma\chi) = \rho \exp[i\sigma \ln(t_0 - t)] \quad (32)$$

is an exact solution to Eq. (31) if the constants ρ and σ , with σ real, satisfy the complex relation

$$e^{-i\phi} \left(\frac{5}{2} - i\sigma \right) = |\rho|^2 \int_0^\infty d\xi \int_0^\infty d\eta U(\xi, \eta) \exp \{ i\sigma \ln[(1 + \xi)(1 + \xi + \eta)/(1 + 2\xi + \eta)] \}$$

that can also be rewritten as

$$e^{-i\phi} \left(\frac{5}{2} - i\sigma \right) = |\rho|^2 \int_0^\infty dz F(z) \exp(i\sigma z) \quad (33)$$

with

$$F(z) = e^{-7z/2} \int_0^1 \frac{s^3 ds}{[1 + s(1 - e^{-z})^{1/2}]^4 [1 + s(1 - e^{-z})^{-1/2}]^2}.$$

Thus, in order for σ to be real, we require

$$\frac{\sigma \cos \phi + \frac{5}{2} \sin \phi}{\sigma \sin \phi - \frac{5}{2} \cos \phi} = \frac{\int_0^\infty dz F(z) \sin(\sigma z)}{\int_0^\infty dz F(z) \cos(\sigma z)}. \quad (34)$$

Note that $\sigma > 0$ corresponds to a nonlinearly increasing frequency of the mode, while $\sigma < 0$ corresponds to a decreasing frequency. A plot of σ vs. ϕ for two roots of Eq. (34) is shown in Fig. 3. The roots are related by symmetry: if $\sigma(\phi)$ is a root of Eq. (34), then $\sigma(-\phi) = -\sigma(\phi)$ is also a root. When the kinetic response is nonperturbative, the frequency shift in the explosive regime can reach a substantial fraction of the mode's initial frequency before solution (30) breaks due to higher nonlinearities.

The previous analytic solution for $\phi = 0$, found in Ref. 1 has a special symmetry that gives both upshifted and downshifted frequency components with equal amplitudes. However, with a nonzero ϕ , one can show that a two-component explosive solution does not even exist and only a one component asymptotic solution presented in this paper has been found. We

interpret this observation as meaning that ultimately a specific direction to the frequency shift is selected in the explosive regime. Nonetheless, the numerical results indicate for sufficiently small ϕ that a nearly symmetric solution (like that found in Ref. 1) is appropriate for transient behavior. For $\phi > \pi/8$, the numerical solutions are close to the asymptotic solutions found in this paper, where there is a definite direction to the frequency shift. This direction corresponds to the root with the lower absolute value of σ (see Fig. 3).

It should be noted that the oscillations of the mode amplitude described by Eqs. (30)–(32) are not directly due to particle trapping (indeed, particle trapping would only occur when the explosive solution is beyond its range of validity). The qualitative explanation for these oscillations is that when the slope of the particle distribution function decreases nonlinearly at the location of the original resonance, steeper slopes build up on both sides of the resonance next to it. In the symmetric case ($\phi = 0$) discussed in Ref. 1, the mode frequency splits into two sidebands that tend to grow faster than the original mode. Hence, we obtain an explosive overall growth of the amplitude with the oscillations at the beat frequency that increases as the sidebands move apart. This process continues until the mode traps resonant particles where the presented theory fails. The corresponding peak amplitude of the mode is unrelated to the closeness to the instability threshold. For perturbative instabilities, like the bump-on-tail instability, this peak amplitude can be estimated from the condition $\omega_b \approx \gamma_L$ where γ_L is the instability growth rate without the background damping. This is a much higher level than the underestimated value $\omega_b \approx \gamma$ presented in Ref. 1. For those instabilities that have $\gamma \approx \omega$ far above the threshold, ω_b can grow up to $\omega_b \approx \omega$.

The explosive solutions we have discussed can also be initiated with a large enough fluctuation when $\nu > \nu_{cr}$ (see Fig. 1) and even in a linearly stable system. A straightforward argument (confirmed by numerical testing) shows that the fluctuation level needed to induce an explosion satisfies the condition $(\gamma_L/|\gamma|)^{5/2} > |A| > (1 + \nu)^{5/2}$.

IV. Extensions of Theory

A. Steady solutions

The theory presented in Secs. II and III is limited to the near-threshold regime. We have seen that if the collisionality is large enough, $\nu_{\text{eff}} \gtrsim \gamma_L - \gamma_d$, a steady state solution arises where the mode saturation level is proportional to an increment above threshold, viz. $\omega_b \approx (1 - \gamma_d/\gamma_L)^{1/4} \nu_{\text{eff}}$. As $1 - \gamma_d/\gamma_L$ increases, the iterative method eventually fails and we then need a different technique to solve the problem. For a steady response of a perturbative mode we need to balance the power being dissipated by the wave, with the power emitted by resonant particles. This relation takes the form

$$-2\gamma_d \text{Im } G_\omega |C|^2 = \int d\Gamma \frac{\partial H_1}{\partial t} f \quad (35)$$

where f is a stationary (in the wave frame) solution to Eq. (10). In addition to the solution in the limit $1 - \gamma_d/\gamma_L \ll 1$ presented in this paper, an analytic solution has previously been found¹⁴ in the limit $\gamma_d/\gamma_L \ll 1$. The two solutions can be combined (accounting for the phase space dependence of ν_{eff}) to give the following interpolation formula for the steady state with the diffusive model of collisions:

$$\frac{\gamma_d}{\gamma_L} = \frac{\int d\Gamma G / [1 + 0.57u/(1 + 1.45/u)^{1/3}]}{\int d\Gamma G} \quad (36)$$

where

$$G(\Gamma) = |V|^2 \frac{\partial F}{\partial I} \delta(\Omega - \omega_0),$$

$$u(\Gamma) = \left(\omega_b(\Gamma) / \nu_{\text{eff}}(\Gamma) \right)^3$$

and $\omega_b(\Gamma)$ is given by Eq. (9).

This interpolation formula has been tested with a precise calculation for a physical system where $|V|^2$, ν_{eff}^3 and $\partial\Omega/\partial I$ are independent of Γ . A comparison of the calculated value of

$\omega_b^3 \gamma_d / \nu_{\text{eff}}^3 \gamma_L$, as a function of γ_d / γ_L , with the interpolation formula, is shown in Fig. 4, and good agreement is observed.

B. Long time evolution of explosive solution

Let us now consider a case where collisionality is small ($\nu_{\text{eff}} \ll \gamma_L$) and let us imagine that the increment above marginal stability increases slowly with time. As long as $\gamma_L - \gamma_d$ is sufficiently small compared with ν_{eff} , the steady saturated level discussed in Sec. IVA is appropriate. However, when $(\gamma_L - \gamma_d)$ becomes greater than ν_{eff} , explosive solution emerges. When this solution reaches a level $\omega_b \approx \gamma_L$, the iteratively obtained Eq. (17) fails. At the point of failure we also observe that the solution has a frequency shift, $\delta\omega$, that is comparable to γ_L .

The evolution of perturbative instabilities beyond the point of the explosive solution breakdown has recently been analyzed.¹⁵ The results indeed confirm that ω_b saturates at the level $\omega_b \approx \gamma_L$. However, the surprise is that the two sideband frequencies continue to shift from the frequency of the original mode by an amount $\delta\omega$ that is much greater than γ_L and the waves do not readily damp. We find that the two frequencies arise because a “hole” and “clump” form in the particle trapping region. When $\delta\omega \gg \gamma_L$, the trapped particles carry their phase space density with them adiabatically as the sideband frequencies continue to slowly change compared to a particle bounce time. This creates a disparity between the trapped particle distribution and the value of the ambient passing particle distribution, which is necessary to support two BGK-type modes that emerge from the sidebands. The energy losses from the background dissipation are compensated by the energy released by passing particles. The passing particle flux to the separatrix skims around the separatrix and energy is released as the particles go from one side of the separatrix to the other.

Figure 5 illustrates our explanation. We plot the spatially averaged particle distribution and the mode spectrum as a function of $\Omega - \omega_0$ and time. We see that the initial distribution

function has a plateau-like region at the original resonance position, but as time evolves, a depression in the distribution propagates to larger values of $\Omega - \omega_0$ and bump in the distribution propagates to lower values of $\Omega - \omega_0$. The BGK-modes last until collisions become important. Then the trapped and passing particle distribution mix on a time scale $T \sim \gamma_L^2 / \nu_{\text{eff}}^3$, the rate of frequency shift diminishes, and the waves damp. The total frequency shift is $\delta\omega \sim \gamma_L (\gamma_L / \nu_{\text{eff}})^{3/2}$ if this value is smaller than the width of the distribution function.

The evolution of the nonperturbative instability beyond the explosive phase is somewhat different from that of the perturbative instability. We have seen that there is a preferred direction to the frequency shift in the nonperturbative case. Therefore, either a hole or a clump, but not both, will form. Technically, the problem of the long time evolution is easier in the perturbative case since a fully nonlinear treatment is only needed for the resonant particles while the rest of the system response can still be treated linearly. In the nonperturbative case, the explosive singularity leads to a complete breakdown of all aspects of the linear problem, and new techniques of analysis are required.

Large frequency shifts have been observed in experiments for the fishbone mode⁶ and for the hot electron interchange mode¹⁰; both of which are nonperturbative modes. The pattern of the early stage of the fishbone instability is consistent with our explosive scenario.² It remains a challenge to consistently describe the mode and particle evolution beyond the explosive phase.

V. Summary

In this paper we have shown how a nonlinear single-mode near-threshold theory, first discussed for the bump-on-tail instability in Ref. 1, generalizes to a wide range of kinetic problems. This generalization leads to new results which include:

The extension of the nonlinear theory to arbitrary geometry and to nonperturbative

modes (i.e. modes whose very existence requires the kinetic response of the particles). Now the theory applies to any mode and any device where the equilibrium orbits are integrable.

An analytic description for the onset of a frequency shift in a definite direction in the nonlinear evolution of nonperturbative instabilities.

Presentation of the nonlinear equation for the mode with a realistic diffusive collision operator as opposed to the idealized Krook model for collisions.

A new explosive solution that applies to the nonlinear equation for the complex wave amplitude. We find that the applicability range of the explosive solutions extends considerably beyond the limit estimated in Ref. 1.

Finally we have given a brief discussion of later developments of our theory that are reported in more detail elsewhere.

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It is a pleasure to dedicate this paper to the memory of Michail Solomonovich Ioffe whose adroit ability to convert theoretical visions to experimental reality has been an inspiration to our field.

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Appendix: Technical Details

a. Operator $\partial/\partial I$ in a symmetric torus

For the guiding center motion in a toroidally symmetric magnetic field, the operator $\partial/\partial I$ can be expressed in terms of the three conserved quantities in the nonperturbed field: the particle energy E , the canonical toroidal angular momentum P_ϕ , and the magnetic moment μ . One can readily establish that

$$\partial/\partial I = \ell_1 \partial/\partial I_1 + \ell_2 \partial/\partial I_2 + \ell_3 \partial/\partial I_3.$$

It can also be shown that it is always allowable to take $I_2 = P_\phi$ and $I_3 = \mu mc/q$; here q is the particle charge. The particle energy E as a function of I_1 , I_2 and I_3 is the Hamiltonian of the unperturbed system. We now choose I_1 to be the action for the poloidal motion, so that the quantities $\omega_1 \equiv \partial E/\partial I_1$, $\omega_2 \equiv \partial E/\partial I_2$ and $\omega_3 \equiv \partial E/\partial I_3$ are the frequencies of the poloidal, toroidal and gyromotion, respectively. We can then rewrite the operator $\partial/\partial I$ in the form

$$\partial/\partial I = (\ell_1 \omega_1 + \ell_2 \omega_2 + \ell_3 \omega_3) \partial/\partial E + \ell_2 \partial/\partial I_2 + \ell_3 \partial/\partial I_3.$$

At the resonance, the sum $\ell_1 \omega_1 + \ell_2 \omega_2 + \ell_3 \omega_3$ equals the mode frequency ω . In addition, ℓ_3 must be taken zero for the low-frequency modes, and ℓ_2 is nothing else than the toroidal mode number n . Hence, we find

$$\frac{\partial}{\partial I} = \omega \frac{\partial}{\partial E} + n \frac{\partial}{\partial P_\phi} = \omega \frac{\partial}{\partial E} \Big|_{P'_\phi} = n \frac{\partial}{\partial P_\phi} \Big|_{E'}$$

with $P'_\phi = P_\phi - nE/\omega$ and $E' = E - \omega P_\phi/n$.

b. Validity limit of cubic integral equation and explosive solution

It is clear from Eq. (9) that the particle motion can be described perturbatively for short enough time scales, satisfying the condition

$$\int_0^t \omega_b dt \ll 1.$$

With collisions present, the time of validity of perturbation theory can be indefinitely long if the decorrelation time τ_c , which is $1/\nu_{\text{eff}}$ or $1/\nu_r$ depending on description of collisions, is less than ω_b^{-1} . Hence, a perturbative treatment is expected to be applicable if

$$\min \left(\int_0^t \omega_b dt; \omega_b \tau_c \right) \ll 1. \quad (\text{A-1})$$

The explicit evaluation of the next (fifth) order nonlinear terms in Eq. (17) shows that those terms are indeed smaller than the cubic term when condition (A-1) is satisfied.

Condition (A-1) sets the limit on ω_b for which the explosive solution is valid. For the explosive solution, the dC/dt term in Eq. (17) equals the nonlinear term with $\nu_r = 0$. This relation gives the following estimate:

$$\frac{1}{C} \frac{dC}{dt} \approx \gamma_L \left(\int_0^t \omega_b dt \right)^4 \ll \gamma_L \quad (\text{A-2})$$

where γ_L is the instability growth rate far above the threshold (at $\alpha \sim 2\alpha_{cr}$). The breakdown occurs when

$$\frac{1}{C} \frac{dC}{dt} \approx \gamma_L,$$

which determines the characteristic time scale near the singularity: $\Delta t \approx 1/\gamma_L$. We now find from Eq. (A-2) that the corresponding limit for ω_b is $\omega_b \approx 1/\Delta t \approx \gamma_L$.

c. Form of ν_{eff}

Suppose the Fokker-Planck operator is of the form

$$St \equiv \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D} \cdot \frac{\partial}{\partial \mathbf{v}},$$

where $\partial/\partial\mathbf{v}$ is a velocity space derivative with the spatial position \mathbf{r} fixed, and \mathbf{D} a dyadic describing velocity space diffusion. The distribution function, f , is in general a function of $I(\Omega), \xi$ and two additional action variables, but only the derivative with respect to I is large near the resonance. Hence, the dominant part of the collisional term is

$$Stf = \nu_{\text{eff}}^3 \frac{\partial^2 f}{\partial \Omega^2}$$

with

$$\nu_{\text{eff}}^3 = \overline{\frac{\partial I}{\partial \mathbf{v}} \cdot \mathbf{D} \cdot \frac{\partial I}{\partial \mathbf{v}}} \left(\frac{\partial \Omega}{\partial I} \right)^2,$$

where the bar denotes bounce average over the nonperturbed orbit.

Using the result of section (a) of this Appendix we can take $I = P_\phi/n$ at constant E' .

Then we find

$$\nu_{\text{eff}}^3 = \overline{\frac{\partial P_\phi}{\partial \mathbf{v}} \cdot \mathbf{D} \cdot \frac{\partial P_\phi}{\partial \mathbf{v}}} \left(\frac{\partial \Omega}{\partial P_\phi} \bigg|_{E'} \right)^2.$$

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FIGURE CAPTIONS

FIG. 1. Stability boundaries for the steady state nonlinear solution. These curves plot the value of ν_{cr} vs. ϕ with the dotted curve for the Krook collisional model and the solid curve for the diffusive collisional model. $\nu < \nu_{cr}$ corresponds to instability of the steady state.

FIG. 2. Transition from steady state saturation to the explosive nonlinear regime as ν decreases. Plots of the value of the normalized amplitude, A , vs. normalized time t for the diffusive case with $\phi = 0$.

FIG. 3. Nonlinear eigenvalues $\sigma(\phi)$ for the explosive solution.

FIG. 4. Steady state saturated level of a single mode. The curve is the interpolation formula while the diamonds and triangles correspond to steady solutions of Eqs.(3) and (10) for $(\nu_{eff}/\gamma_L)^3 = 5$ and $(\nu_{eff}/\gamma_L)^3 = 10$ respectively.

FIG. 5. Spontaneous formation of coherent modes with time dependent frequencies in the near-threshold regime of the bump-on-tail instability. (a) The spatial average of the distribution $\langle f \rangle = F + f_0$ as a function of time and $\Omega - \omega_0$; (b) the evolution of the mode spectral intensity $|c(\omega)|^2$ as a function of time and frequency shift.

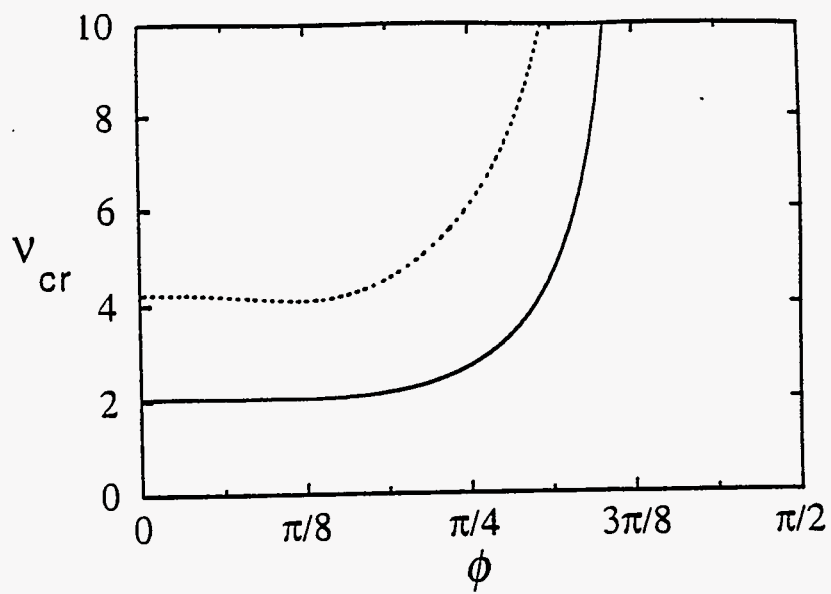


Fig. 1

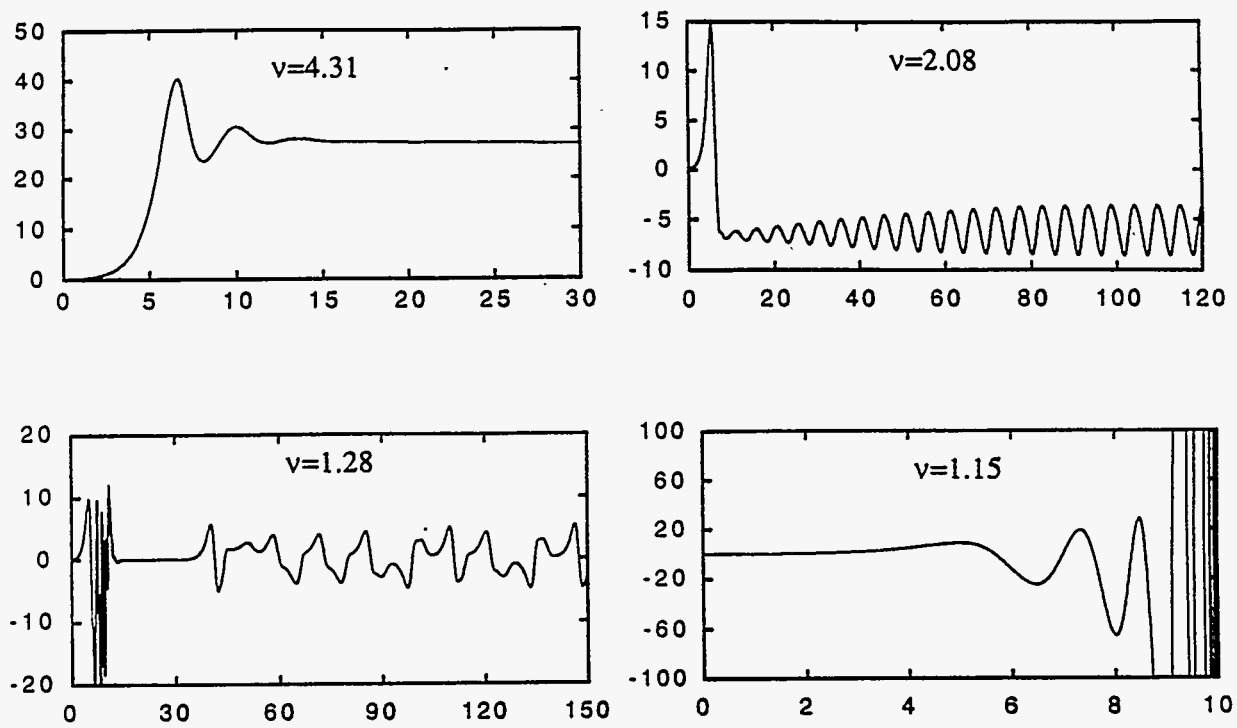


Fig. 2

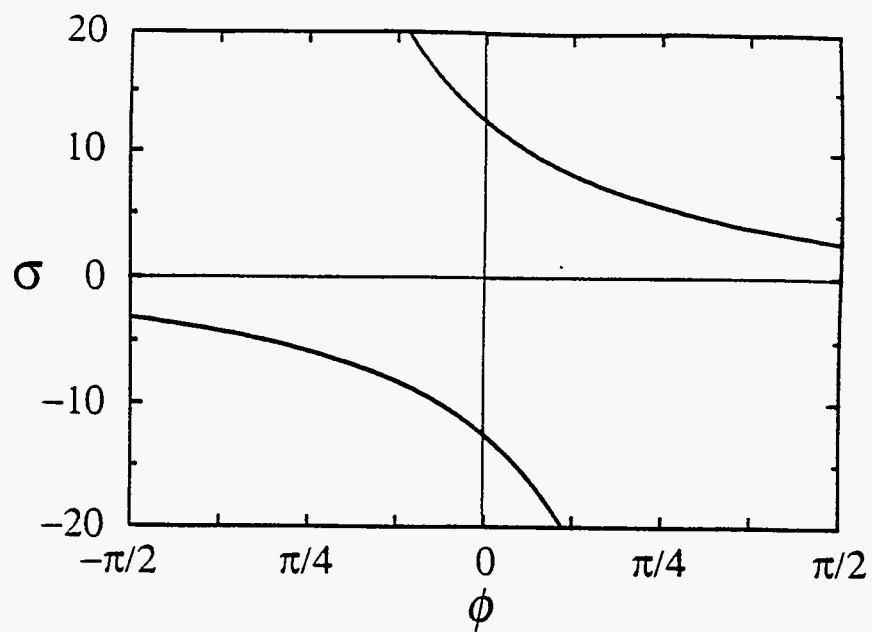


Fig. 3

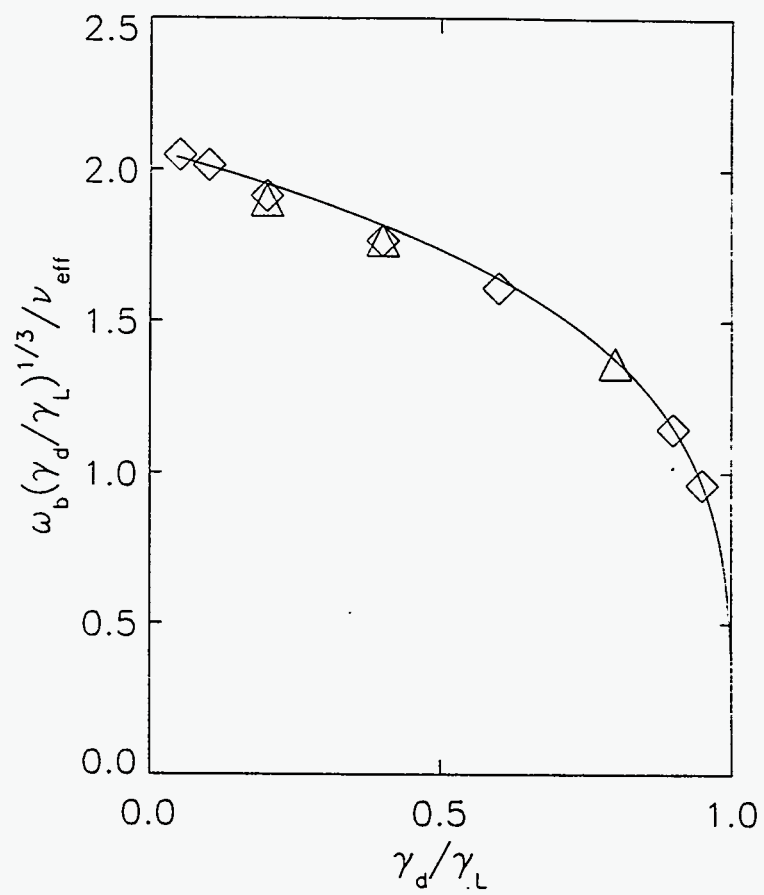


Fig. 4

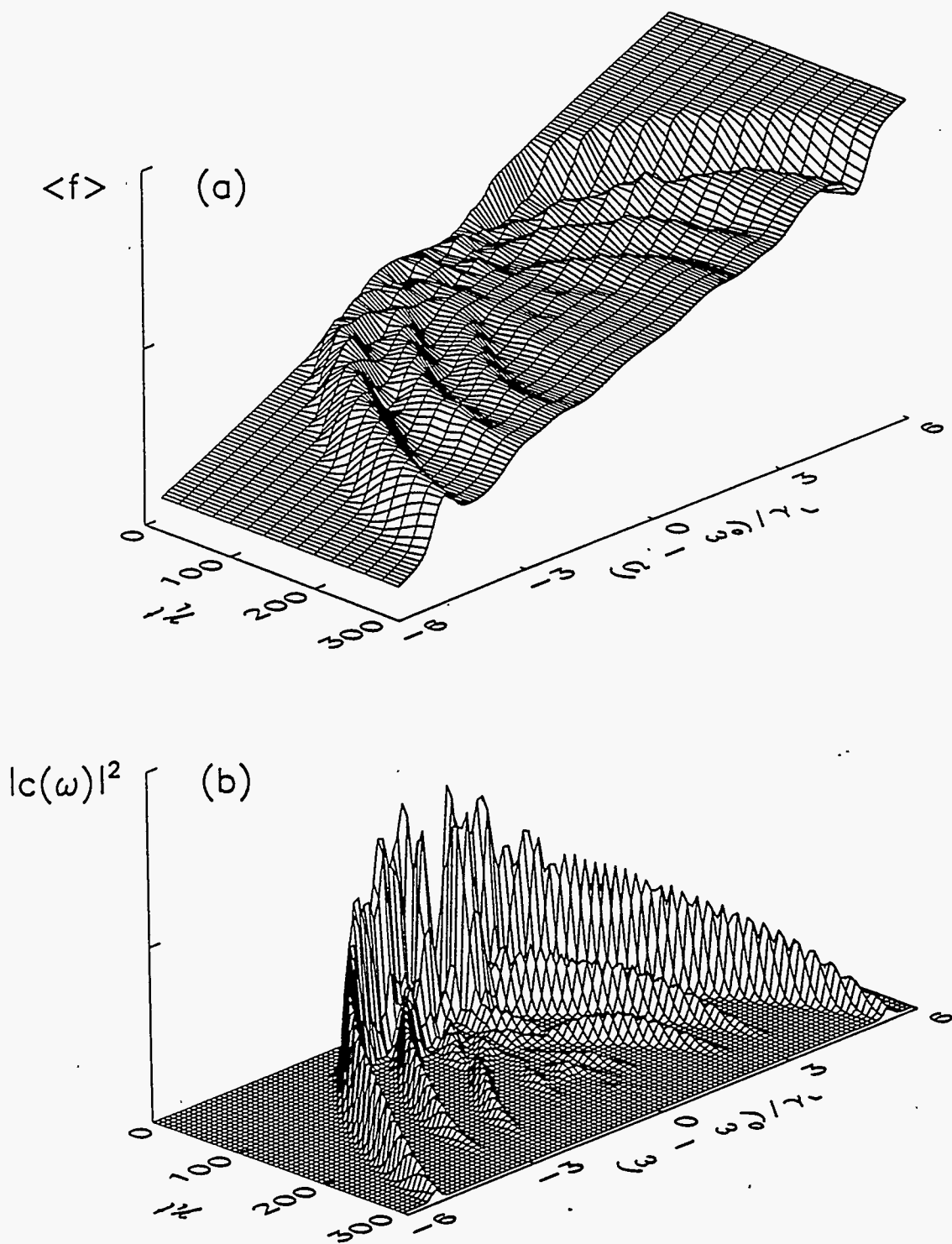


Fig. 5