Enhancement of Particle–Wave Energy Exchange by Resonance Sweeping

When the resonance condition of the particle–wave interaction is varied adiabatically, the particles trapped in a wave are found to form phase space holes or clumps that enhance the particle–wave energy exchange. This mechanism can cause increased saturation levels of instabilities and even allow the free energy associated with instability to be tapped in a system in which background dissipation suppresses linear instability.

Key Words: *resonance, sweeping, hole, clump, adiabatic, free energy, wave momentum*

I. INTRODUCTION

There are many cases in plasma physics where the resonant interactions of particles and waves determine interesting physical phenomena. A common example arises when a coherent mode, a perturbation of the electromagnetic field and the majority of plasma particles, is destabilized through a particle–wave resonance with a minority component that is supplying "free energy" to feed the wave. Examples vary from the basic bump-on-tail instability[1,2] to applications that arise in coherent radiation generators such as free electron lasers.[3,4] Tokamak theory is rich in applications that include current drive,[5] energy "channelling,"[6] and instability arising from fusion produced alpha particles that excite Alfvén waves.[7–10] A more complicated example in tokamaks is the so-called fish-bone instability,[11–13] where the minority species is needed to both form and stabilize the wave.

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In this Comment we wish to emphasize a mechanism whereby the transfer of energy from particles to waves can be enhanced from the prediction of linear theory. Particles trapped by a coherent wave can be adiabatically transported in phase space as the resonance location is continuously swept. However, most of the passing particles do not penetrate the trapping region inside the separatrix. As a result large phase space gradients in the distribution functions build up at the interface between the passing and trapped particles. This gradient allows for enhanced energy transfer. A similar mechanism has been applied to extracting energy in a free electron laser with a variable wiggler,[14] and has been suggested as a means of obtaining enhanced dissipation of high power radio frequency waves in a tokamak.[15] A theory using particle drag[16] which is mathematically identical to resonance sweeping also exhibits enhanced energy transfer. Other studies of such effects have been discussed by Mynick and Pomphrey,[17] where they primarily addressed how particles can be extracted by frequency sweeping and they noted that energy can also be extracted by this method, and by Hsu *et al.*,[18] who noted that frequency chirping can lead to particle loss. Here we show how to determine the field amplitude and energy conversion level as a result of sweeping. We shall show that the large phase space gradients, which depending on the direction of the frequency scan, form either phase space "holes"[19] (where the trapped distribution is depressed from neighboring regions) or phase space "clumps"[20] (where the trapped distribution is enhanced from neighboring regions) and they enhance the energy transfer rate of the resonant particles to the wave compared to linear theory. This phenomenon can cause much higher mode saturation levels than otherwise would be expected in an unstable system. In a system linearly stabilized by background dissipation, the enhanced energy exchange associated with this mechanism allows the wave to grow from an imposed low level seed perturbation, which can facilitate energy channelling, as background dissipation no longer sets a stringent bound for tapping the particle free energy.

II. POWER TRANSFER BY ADIABATIC RESONANCE SWEEPING

In order to tap the free energy reservoir of weak instabilities it is normally necessary for the instability drive to overcome dissipation from mechanisms related to the background plasma. The total power that pumps the wave and determines the rate of change of wave energy, W, which is proportional to the square of the wave amplitude, is equal to the power extracted from the linear drive, minus the power absorbed by the background plasma through dissipation. In linear theory, all the terms in this power transfer equation are proportional to W, and can be written as

$$\frac{\partial W}{\partial t} = 2\gamma_L W - 2\gamma_d W. \tag{1}$$

Related to the wave amplitude is the nonlinear bounce frequency, ω_B, for particles deeply trapped in the wave. It turns out that the response of any physical system with weak instability is universal in terms of ω_B. Taking into account that ω_B is proportional to the square root of the wave amplitude, we rewrite Eq. (1) as

$$\frac{\partial}{\partial t}\overline{\omega}_B^4 = 2\gamma_L \overline{\omega}_B^4 - 2\gamma_d \overline{\omega}_B^4 \tag{2}$$

with $\overline{\omega}_B$ an appropriate average of ω_B.

It has been shown[21–25] that under many conditions a single mode saturates at a level where $\overline{\omega}_B \sim \gamma = \gamma_L - \gamma_d$. The energy released by particles to the wave, ΔW, is then proportional to γ^4. For the bump-on-tail instability one can show that

$$\Delta W \simeq W_F\left(\frac{\gamma}{\omega}\right)^3 \frac{\gamma}{\gamma_L} \tag{3}$$

where W_F is the free energy that in principle can be transferred to the waves. Note that ΔW is a small fraction of W_F; the latter is generally comparable to the kinetic energy of the fast particles. This is a rather low level of conversion of free energy to wave energy.

We now note that we can convert considerably more free energy if we can slowly change the resonance position in time. This problem has already been analyzed in Ref. 16, where it was assumed that the destabilizing particles were slowing down due to drag and that diffusive processes such as pitch angle scattering were unimportant. In fact it turns out that the treatment of the problem is identical whether the resonance function ($\Omega = \omega - kv$ for the electrostatic problem) for a particle changes due to drag (then $dv/dt = -\nu v$) or due to the mode frequency changing in time as $d\omega/dt = \nu\omega$. The analysis shows that if $d\Omega/dt \ll \omega_B^2$, there will be an adiabatic invariant that causes the particles trapped in the wave field to remain trapped as the resonance changes. The particles that are originally trapped at a phase velocity $\omega(0)/k = v_{ph}(0)$, with a distribution weight of $f_0(v_{ph}(0))$, keep their distribution weight as they adiabatically track the resonance. When the phase velocity becomes $\omega(t)/k = v_{ph}(t)$, the weight of the deeply trapped particles will still be $f_0(v_{ph}(0)) \neq f_0(v_{ph}(t))$. Therefore a strong gradient of f develops near the instantaneous phase velocity of the wave which enhances the particle to wave power transfer.

A quantitative expression for the power transfer arises from a straightforward physical picture. We first obtain this expression for the one-dimensional bump-on-tail instability with the resonance function $\Omega = \omega - kv$, where we take ω to be time dependent and k a constant wavenumber. The electrostatic field is taken as $E = \hat{E}\sin(\psi)$, with $\psi = kx - \int_0^t \omega dt$, and we define the quantity $\nu = \dot{\omega}/\omega$.

From basic principles, the energy, ΔE, added to the wave is $\Delta E = \Delta M\, \omega/k$, where ΔM is the momentum that is removed from the particles in a time Δt. This momentum is transferred to the wave. The power transfer, P, is then $P = (\Delta M/\Delta t)\, \omega/k$.

Let us compare the phase space distribution of the particles constituting the free energy reservoir at a time t_0 and a time $t_0 + \Delta t$, where we take $\hat{E}/(d/dt)\hat{E} \gg \Delta t \gg \omega_B/\nu\omega$. The phase space plot is shown in Fig. 1. We assume ν, $\partial f(0)/\partial v > 0$. Note that in this case we have phase space holes where the f-values inside the separatrix at $t = t_0 + \Delta t$ are below the f-values outside the separatrix. As the hole shifts to higher velocity, it is clear that the particle distribution has less energy at the later time, and this missing energy has thus been converted to wave energy.

We now calculate the energy transfer systematically. We assume that $\nu \ll \omega_B^2/\omega$, so that the particle adiabatic invariant J is conserved with

$$J \equiv \frac{1}{k} \oint \frac{d\psi}{2\pi} v \equiv \frac{1}{k} \oint \frac{d\psi}{2\pi} \left[\frac{\omega(k)}{k} + \sqrt{2} \left(\epsilon + \frac{e\hat{E}(t)\cos\psi}{mk} \right)^{1/2} \right] \quad (4)$$

where ϵ is the particle energy per unit mass in the wave frame, which is considered constant in the ψ integration. For passing particles, $\epsilon > |e\hat{E}(t)/mk|$, the loop integral means that ψ varies from $-\pi$ to π. For trapped particles, ψ starts from a turning point $\psi = -\cos^{-1}(e\hat{E}(t)/mk\epsilon) \equiv -\psi_t$, goes to $\psi = \psi_t$ and then back to $-\psi_t$. On the return path, the square

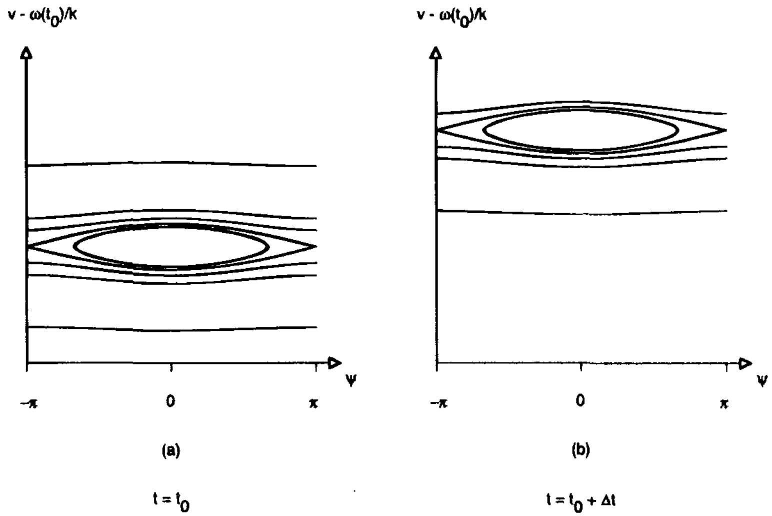

FIGURE 1 Instantaneous phase space contours for the particles in the presence of a wave with variable frequency (a) at time t_0; (b) at time $t_0 + \Delta t$. Contours of constant distribution function are plotted. If the distribution function increases with velocity in the passing region, and there is a hole in the trapped region, the system at time $t = t_0 + \Delta t$ has less kinetic energy than at time t_0, and this can increase the wave energy.

root in Eq. (4) changes sign. Thus, for trapped particles the $\omega(k)/k$ terms cancel over the complete path, and the trapped particle adiabatic invariant, J_t, is given by

$$J_t \doteq \frac{2\sqrt{2}}{2\pi k} \int_{-\psi_t}^{\psi_t} \left[\epsilon + \frac{e\hat{E}(t)\cos\psi}{mk} \right]^{1/2} d\psi. \tag{5}$$

Note that Eqs. (4) and (5) can in principle be inverted to obtain $\epsilon = \epsilon(J)$, and we can define the quantity $\delta v(\psi, J)$ as

$$\delta v(J,\psi) = v - \frac{\omega(t)}{k} = \sqrt{\frac{2}{m}} \left| \epsilon(J) + \frac{e\hat{E}(t)\cos\psi}{mk} \right|^{1/2}. \tag{6}$$

If the mode phase velocity changes faster than the mode amplitude, then a passing particle typically remains passing as it goes around the trapping region from, say, above the separatrix to below the separatrix, by slipping through the x-point.[16] The J-value, J_p, for a passing particle near the separatrix for which $\epsilon \doteq e\hat{E}(t)/mk$ is found from Eq. (4) to be $J_p \equiv \omega(t)/k^2 + (4/\pi k^2)\omega_B(t)$, with $\omega_B(t) \equiv [ek\hat{E}(t)/m]^{1/2}$. Further, the distribution weight $f(J_p)$ remains conserved as the passing particles move away from the separatrix. The trapped particles near the separatrix have J_t values given by $J_t = 8\omega_B(t)/\pi k^2 \equiv J_{\text{sep}}(t)$, and all other trapped particles satisfy $J_t < J_{\text{sep}}(t)$. As the resonance is swept, the trapped particles move with the separatrix and maintain their J_t-invariance and their distribution weight $F(J_t)$.

If $\hat{E}(t)$ increases with time, a few of the passing particles will be constrained in the trapping region inside the separatrix at an f-value that is characteristic of the passing particle distribution. Thus, for trapped particles near the separatrix at time t',

$$f\left[J_t(t') = \frac{8}{\pi k^2}\omega_B(t')\right] \doteq f_0\left[\frac{\omega(t')}{k}\right] \tag{7}$$

where f_0 is the original unperturbed distribution function. Equation (7) is valid for $0 < t' \leq t$. A few particles near the bottom of the well have an f-value that depends on the initial conditions at $t = 0$. For our model, we will assume the distribution is flat in this region, with $f_t(J_t) = f_0(\omega(0)/k)$, for $0 < J_t \leq (8/\pi k^2)\omega_B(0)$. We now delete the subscripts t and p.

Now the difference of momentum (averaged over a spatial period) of the free energy reservoir at $t_0 + \Delta t$ and t_0 is given by

$$\begin{aligned}\Delta M &= \frac{k}{2\pi}\int_{-\pi/k}^{\pi/k} dx \int_{-\infty}^{\infty} dv\, mv\left[f(t_0+\Delta t) - f(t_0)\right] \\ &= m\int_{-\pi}^{\pi}\frac{d\psi}{2\pi}\left[\int_{(\omega/k)(t_0+\Delta t)+\delta v(J,\psi)}^{(\omega/k)(t_0+\Delta t)+\delta v(J,\psi)} dv\, v\,\big(f(t_0+\Delta t) - f_0\big) - \right. \\ &\left.\int_{(\omega/k)(t_0)-\delta v(J,\psi)}^{(\omega/k)(t_0)+\delta v(J,\psi)} dv\, v\,\big(f(t_0+\Delta t) - f_0\big)\right].\end{aligned} \tag{8}$$

At time $t_0 + \Delta t$, we have $v = \frac{\omega(t_0+\Delta t)}{k} + \delta v \doteq \frac{\omega(t_0)}{k} + \nu\Delta t\,\frac{\omega(t_0)}{k} + \delta v$, and at time t_0 we have $v = \omega(t_0)/k + \delta v$. Note that $f(t_0 + \Delta t) - f_0 = f(t_0) - f_0$, because $f(t_0) = f(J)$ is constant and there are negligible changes in the passing particle distributions due to the small difference of phase velocity, $(v\Delta t\,\omega/k)$. We then find

$$\Delta M = m\omega v\Delta t \int_{0}^{(J_{\text{sep}})(t_0)} dJ\,(f(J) - f_0)\,. \tag{9}$$

The power (per unit volume) transferred to the wave is $P = -(\omega/k)\Delta M/\Delta t$.

If we assume $\omega(t)/k - \omega(0)/k \ll f_0/(\partial f/\partial v)$, we can somewhat simplify Eq. (9). We use the approximation

$$f(J) \doteq f_0\left(\frac{\omega(0)}{k}\right) + \left(\frac{\omega(t'(J))}{k} - \frac{\omega(0)}{k}\right)\frac{\partial f_0(\omega(0)/k)}{\partial v}. \tag{10}$$

Then we substitute $f(J)$ into Eq. (8), use $J(t') = 8\omega_B(t')/\pi k^2$, convert the J to a t' integral, integrate by parts using $d\omega/dt = v\omega$ and use the above relation between P and ΔM to find

$$P = \frac{8m}{\pi}\frac{v^2\omega^3}{k^4}\frac{\partial f(\omega(t)/k)}{\partial v}\int_0^t dt'\omega_B(t'). \tag{11}$$

Note that P is independent of the sign of v. If ω/k increases in time, holes are generated, while if ω/k decreases in time, clumps are generated. In both cases there is less kinetic energy in the distribution function in the final state than in the initial state; the difference is converted to wave energy.

Now using Eq. (11) and the relation between wave energy and bounce frequency, we find that the change in ω_B due to the adiabatic frequency shift and dissipation is roughly (neglecting numerical factors) given by

$$\frac{\partial \omega_B^4}{\partial t} \approx v^2\omega^2\gamma_L\int_0^t dt'\omega_B(t') - \gamma_d\omega_B^4\,. \tag{12}$$

A rough integration of this equation over a time $t \sim 1/v$ shows that, for a sufficiently low damping,

$$\omega_B \approx (\gamma_L\omega^2)^{1/3}. \tag{13}$$

In terms of the free energy, the energy release, ΔW, is found to be

$$\Delta W = W_F\left(\frac{\gamma_L}{\omega}\right)^{1/3}. \tag{14}$$

When damping is considered, the energy release produced by Eq. (14) is only achieved if ν is fast enough so that the effects of damping in the power transfer can be ignored, a condition that requires $\nu > \gamma_d$. For $\nu < \gamma_d$, the level of the wave energy is somewhat lower, and can be found from Eq. (12) by balancing the terms on the right-hand side. One then finds that, after a time $t \sim 1/\nu$, the level of ω_B is

$$\omega_B \simeq \left(\frac{\omega^2 \gamma_L \nu}{\gamma_d}\right)^{1/3}. \tag{15}$$

Also note that if the background damping determines the level of ω_B, more energy dissipates to the background plasma than instantaneously exists in the wave (this is the original channeling mechanism discussed in Ref. 6). The amount of energy, ΔW_{ch}, that is channelled to the background plasma $\propto \gamma_d \int_0^{\nu^{-1}} dt \omega_B^4$, is found to be

$$\Delta W_{\mathrm{ch}} \approx W_F \left(\frac{\gamma_L}{\omega} \frac{\nu}{\gamma_d}\right)^{1/3}; \quad \nu < \gamma_d. \tag{16}$$

Note that, according to Eq. (14), there is still more energy released when $\nu > \gamma_d$, and hence the optimum strategy to extract free energy is to have $\nu > \gamma_d$. Once this wave energy is "gathered" it can be allowed to be absorbed by the background plasma through dissipative processes, or perhaps even directly converted to a grid using the external antennae circuitry.

III. AMPLIFICATION IN STABLE SYSTEMS

There can be a start-up problem that is especially important in stable systems, where $\gamma_d > \gamma_L$. In this case the transfer of the kinetic energy to wave energy can only be achieved if large phase space gradients are produced so that an enhanced power transfer can occur. The formation of large phase gradients is prevented if the scan rate is too fast, $\nu > \omega_B^2/\omega$, so that trapped particles do not remain trapped, or if collisional effects prevent large gradients from arising.

Let us first neglect collisional effects on the energetic particles. Equation (12) then governs the evolution of ω_B. With γ_d neglected, the solution is

$$\omega_B \approx (\omega^2 \gamma_L)^{1/3} (\nu t)^{2/3}. \tag{17}$$

When $\nu t \approx 1$, this result gives the optimal saturation level, $\omega_B = (\omega^2 \gamma_L)^{1/3} \equiv \omega_{B\max}$, that is estimated in Eq. (13). However, the neglect of the γ_d term in Eq. (12) requires $\gamma_d t < 1$. Thus, if $\gamma_d / \nu > 1$, the upper limit on ω_B is established in Eq. (17) by setting $t = 1/\gamma_d$,

$$\omega_B \simeq \left(\frac{\omega^2 \gamma_L \nu^2}{\gamma_d^2} \right)^{1/3}. \tag{18}$$

Now, in an unstable system we can impose a field that produces a bounce frequency ω_{B0}. Amplification arises if the value of ω_B, estimated from Eq. (18), exceeds ω_{B0} when the maximum value that ν can have (i.e., $\nu = \omega_{B0}^2/\omega$) is used. We thereby obtain the criterion

$$\omega_{B0} > \frac{\gamma_d^2}{\gamma_L}. \tag{19}$$

Note that this result means that for stable systems, the initial bounce frequency of an imposed mode has to be larger than the bounce frequency, $\omega_B \sim \gamma_L$, associated with the natural saturation mechanism of an unstable system. Once the minimum ω_{B0} is imposed, amplification to $\omega_B \approx \omega_{B\max}$ can always be achieved by increasing ν with time, with the optimal choice being $\nu \approx \omega_B^2/\omega$.

The other important physical process is particle scattering from either small angle collisions or velocity space diffusion arising from external heating. This process makes the achievement of an adiabatic scanning process more difficult. Straightforward dimensional arguments show that diffusion processes cause trapped particles to escape from the trapping region at a rate given by $\nu_{\text{eff}} \simeq \nu_s \omega^2/\omega_B^2$, where ν_s is the rate of relaxation of the overall velocity distribution. Then, if we scan the frequency for a time greater than $1/\nu_{\text{eff}}$, the change in the distribution, Δf, in the trapping region is

$$\Delta f \approx \frac{\partial f}{\partial v} \frac{\nu \omega}{k} \frac{1}{\nu_{\text{eff}}}. \tag{20}$$

Thus, if the scan time is greater than $1/\nu_{\text{eff}}$, we need to replace $(\omega(t'(J)) - \omega(0))/k$ by $\nu\omega/(\nu_{\text{eff}} k) = \nu \omega_B^2/(\omega \nu_s k)$. This result has been rigorously obtained in Ref. 16. An interpolation formula for arbitrary ν_{eff} is obtained if in Eq. (11) we make the replacement

$$\int_0^t dt'\omega_B(t') \to \int_0^t dt'\omega_B(t') \exp\left(-\int_{t'}^{t} dt''\nu_{\text{eff}}(t'')\right). \tag{21}$$

Thus, when $\nu_{\text{eff}}\, t > 1$, Eq. (12) changes to

$$\frac{\partial\omega_B^4}{\partial t} = \frac{\nu^2}{\nu_s}\omega_B^3\gamma_L - \gamma_d\omega_B^4. \tag{22}$$

We now observe that if ν_s is too large, $\omega_B = \omega_{B\max}$ cannot be achieved. The value of ν_s for which this transition occurs is determined by the conditions, $\nu_{\text{eff}} \equiv \nu_s\omega^2/\omega_B^2 \approx \nu$, $\nu \le \omega_B^2/\omega$ and $\omega_B \le \omega_{B\max} \equiv (\gamma_L\omega^2)^{1/3}$. From these conditions we find that if

$$\nu_s > \frac{\gamma_L^{4/3}}{\omega^{1/3}}, \tag{23}$$

then $\omega_B < \omega_{B\max}$. If $\nu_s < \gamma_L^{4/3}/\omega^{1/3}$, it is possible to achieve $\omega_B \approx \omega_{B\max}$ if ω_{B0} is sufficiently large. To obtain the relevant criteria, let us integrate Eq. (22) when γ_d is neglected, and we set $\nu = \omega_B^2/\omega$. We find

$$\frac{\omega_B^3}{\omega_{B0}^3} = \frac{1}{(1 - t/T_{\text{sng}})} \tag{24}$$

with $T_{\text{sng}} = \alpha\nu_s\omega^2/\omega_{B0}^3\,\gamma_L$ and α is a constant of order unity. Note that ω_B does not change appreciably until $t/T_{\text{sng}} \approx 1$. At $t/T_{\text{sng}} \simeq 1$, a singularity in ω_B occurs. This singular behavior means that for $t \sim T_{\text{sng}}$, one needs to change the evolution equation for ω_B to Eq. (12), from which one finds that $\omega_B \approx \omega_{B\max}$ is achieved. However, one also needs $T_{\text{sng}} < \omega/\omega_{B0}^2$, or else the frequency scans out of the resonance region before the singular transition occurs. Thus we find the restriction $\omega_{B0} > (\nu_s/\gamma_L)\omega$. An additional restriction on ν_s comes from the requirement that Eq. (22) be in the correct collisionality regime, i.e., $\nu_s\,(\omega^2/\omega_{B0}^2)\,T_{\text{sng}} > 1$, which leads to the condition, $\omega_{B0} < (\nu_s^2\omega^4/\gamma_L)^{1/5}$. The final restriction for our solution is that the damping term in Eq. (22) be unimportant, leading to the relation $\omega_{B0} > (\omega^2\nu_s\gamma_d/\gamma_L)^{1/3}$ (or equivalently $\gamma_d T_{\text{sng}} < 1$). Thus, the conditions that need to be satisfied to achieve $\omega_B \simeq \omega_{B\max}$, when $\nu_s < \gamma_L^{4/5}/\omega^{1/5}$, are

$$\left(\frac{\nu_s^2\omega^4}{\gamma_L}\right)^{1/5} > \omega_{B0} > \left(\frac{\omega^4\nu_s\gamma_d}{\gamma_L}\right)^{1/3},\quad \left(\frac{\nu_s\omega}{\gamma_L}\right). \tag{25}$$

Note that these conditions require $\nu_s > \gamma_d^5/\gamma_L^2\omega^2$, which shows that

$$\omega_{B0} > \left(\frac{\omega^2 \nu_s \gamma_d}{\gamma_L}\right)^{1/3} > \frac{\gamma_d^2}{\gamma_L},$$

and hence with collisions a somewhat larger seed value for ω_{B0} is needed. Further, from the conditions, $\gamma_L^{4/3}/\omega^{1/3} > \nu_s > \gamma_d^5/\gamma_L^2\omega^2$, we find that the optimal $\omega_{B\max}$ can only be achieved if $\gamma_d < \omega^{1/3}\gamma_L^{2/3}$.

IV. NONADIABATIC FREQUENCY SCANNING

We now note that in unstable systems ω_B can be amplified above the natural level, $\omega_B \sim \gamma_L$, when the frequency is scanned nonadiabatically ($\nu > \omega_B^2/\omega$). However, the potential amplification level is still lower than in the adiabatic case.

In the nonadiabatic regime, new particles enter the region of resonance at the rate

$$\frac{d\Delta N}{dt} \simeq \frac{\nu\omega}{\omega_B}\Delta N \tag{26}$$

where ΔN is the number of trapped particles. If $\nu\omega/\omega_B > \omega_B$, the trapped distribution does not have a chance to flatten, and the wave will continue to grow. On the other hand, the change of the frequency is comparable to its original value in a time $t \sim 1/\nu$. Thus, in order to have at least one growth time of amplification in the nonadiabatic regime, we need

$$\gamma_L > \nu > \omega_B^2/\omega. \tag{27}$$

Thus, the level of ω_B that can be reached in a nonadiabatic scan is $\omega_B \lesssim (\omega\gamma_L)^{1/2}$, which is a lower level than is in principle achievable with an adiabatic scan.

We can model these effects quantitatively as follows. We let the conventional growth rate satisfy the relationship

$$\gamma' = \frac{\gamma_L}{\left[1 + \frac{\omega_B}{3\gamma' + \nu\omega/\omega_B + \nu_{\text{eff}}}\right]}. \tag{28}$$

For $\omega_B = 0$ we recover linear theory. If there are no sources ($\nu = 0$, $\nu_{\text{eff}} = 0$), we have $\gamma \rightarrow 0$ when $\omega_B \rightarrow 3\gamma_L$, which is a result previously observed in particle simulations. In Ref. 16, it is shown that $\gamma \simeq \gamma_L \nu_{\mathrm{eff}}/\omega_B$ if $\nu_{\mathrm{eff}}/\omega_B \ll 1$, where ν_{eff} represents the rate at which particles re-enter the flattened resonance region due to diffusion. Here we have conjectured that by scanning the frequency rapidly, we have introduced a similar source as ν_{eff}, which is now given by $\nu\omega/\omega_B$. We can then write a model equation that includes all the processes we have described. In the adiabatic term we include a cut-off when $\nu\omega > \omega_B^2$. The model equation is

$$\frac{d\omega_B^4}{dt} = -\gamma_d\omega_B^4 + \gamma'\omega_B^4$$

$$+\exp\left(-\frac{\nu\omega}{\omega_B^2}\right)(\omega\nu)^2\gamma_L\int_0^t dt'\omega_B\exp\left(-\int_{t''}^{t} dt''\nu_{\mathrm{eff}}\right) + \tilde{\omega}_b^4(t)(\gamma_d - \gamma_L) \quad (29)$$

where $\tilde{\omega}_b(t)$ is included to describe an imposed external perturbation.

V. GENERALIZATION TO TOROIDAL SYSTEMS

The preceding considerations also apply to more complicated systems. For example, one can show that Eqs. (1)–(3) are valid for low-frequency waves in a toroidally symmetric tokamak plasma, where the perturbation varies as $\hat{A}(t)\sin(n\phi - \omega t)$. The concept of particles trapped in a wave at a bounce frequency $\omega_B \propto \hat{A}^{1/2}$ is also valid in this case. The resonance function is $\Omega = \omega - n\varpi_\phi(H, P_\phi, \mu) - \ell\varpi_\theta(H, P_\phi, \mu)$, where ϖ_ϕ and ϖ_θ are the toroidal and poloidal drift frequencies of a particle, n and ℓ are integers, H is the particle energy, P_ϕ is the toroidal angular momentum and μ is the magnetic moment. It follows from general arguments that $H' = H - (\omega/n)P_\phi$ is conserved as a particle interacts with a toroidal wave. Also, μ is conserved in the low-frequency wave. If we consider an adiabatic sweeping of the frequency (i.e., $d\omega/dt \equiv \nu\omega \ll \omega_B^2$), then for a particle trapped in the wave, P_ϕ will change according to the relation

$$\frac{d\Omega}{dt} = \nu\omega - \frac{dP_\phi}{dt}\frac{\partial}{\partial P_\phi}\bigg|_{H';\mu}(n\varpi_\phi + \ell\varpi_\theta) = 0 \quad (30)$$

while H' and μ remain constant. As the resonance moves, most of the passing particles will skim the separatrix and remain passing on the other side of the separatrix, while the trapped particles will track the resonance. One will again generate large phase space gradients. Similar to Eq. (4), the adiabatic invariant can be written as

$$J = \oint \frac{d\psi'}{2\pi} \left\{ \left[\epsilon + \frac{\omega_B^2 \cos\psi'}{\left(\partial\Omega / \partial P_\phi \big|_{H';\mu}\right)^2} \right]^{1/2} \sqrt{2} + P_{\phi r} \right\} \tag{31}$$

where

$$\Omega(P_{\phi r}) = 0, \qquad \epsilon = \frac{(P_\phi - P_{\phi r})^2}{2} - \frac{\omega_B^2 \cos\psi}{(\partial\Omega / \partial P_\phi)^2}. \tag{32}$$

Also similar to the one-dimensional electrostatic problem is the expression for the power converted to the wave

$$P = \frac{\Delta E}{\Delta t} = \frac{\omega}{n} \frac{\Delta M_\phi}{\Delta t}, \tag{33}$$

where ΔM_ϕ is the toroidal angular momentum that is lost by the resonant particles during the time interval Δt as the frequency is swept. We then find that

$$\begin{aligned} P &= \frac{\omega^2 \nu}{n} \int \frac{d^3p d^3r}{\left|\frac{\partial\Omega}{\partial P_\phi}\right|_{H';\mu}} \delta\left[P_\phi - P_{\phi r}(t)\right] \int_0^{J_{\rm sep}(t)} dJ \left[f_0\left(P_{\phi r}(t)\right) - f(J) \right] \\ &= \frac{8\nu^2 \omega^2}{n\pi} \int \frac{d^3p d^3r}{\left|\frac{\partial\Omega}{\partial P_\phi}\right|^2_{H';\mu}} \delta\left[P_\phi - P_{\phi r}(t)\right] \frac{\partial f_0(P_\phi)}{\partial P_\phi}\bigg|_{H';\mu} \int_{t_0}^{t} dt' \omega_B(t', P_\phi, H') \end{aligned} \tag{34}$$

where we used the relation

$$f(J) = f_0\left(P_{\phi r}(t)\right) + \left[P_{\phi r}\left(t'(J)\right) - P_{\phi r}(t)\right] \frac{\partial f_0}{\partial P_\phi}\bigg|_{H';\mu}.$$

This expression is a generalization of the expression obtained in Ref. 16 where the calculation was done for a plasma slab when the mode frequency is much less than the diamagnetic frequency of the kinetic component. Observe that in terms of an average bounce frequency, ϖ_B, nearly all conclusions of the bump-on-tail problem apply to the more complicated problem. The main difference is that $\bar{\omega}_B$ is now a suitable average over particles in resonance, whereas in the bump-on-tail problem there is only one point resonant in phase space.

VI. DISCUSSION

In summary we discuss our results and their implications.

(1) If the frequency changes there can be an enhanced saturation level of an unstable wave. Without a frequency change, the saturation level for the initial value problem is given by $\omega_B \approx \gamma_L$. This level can be enhanced to $\omega_B \approx (\gamma_L \omega^2)^{1/3} \min(1, \nu/\gamma_d)$ by an adiabatic scan of the frequency, where the frequency scan rate, ν, satisfies $\nu < \omega_B^2/\omega$. In a single scan, the fraction of free energy that can be converted to wave energy is $(\gamma_L/\omega)^{1/3}$. High efficiency of energy conversion of free energy to wave energy may be possible with multiple scans. When $\nu > \omega_B^2/\omega$, the response is nonadiabatic, and the level of ω_B that can be achieved in this case is limited to $\omega_B \sim (\omega\gamma_L)^{1/2}$, which is lower than that for an adiabatic scan.

Normally, there is no reason to expect the frequency to vary during an instability of a stationary system. However, instabilities often arise in a plasma where the background properties vary with time. In the fishbone experiment, the plasma is being heated and is reaching parameters that approach marginal stability for MHD waves. In this case the frequency of the fishbone is sensitive to the closeness to marginal stability. It may be that the approach of the background plasma to marginal stability allows the frequency to change. In any event, if there exists a mechanism that allows the frequency to shift, the processes described here allow large wave amplitudes to grow and perhaps give rise to the kind of energetic particle transport that has been observed experimentally[9–11] and discussed in computer simulations.[18,26]

(2) For a stable system, the adiabatic response to a frequency change can allow waves to spontaneously grow from a "seed" perturbation. Our analysis shows that growth is feasible if either the initial perturbation is large enough or the frequency scan rate is made to increase as the mode grows. Sufficiently rapid diffusion in velocity space sets limits to obtaining large amplifications of ω_B. Once a wave is created, its energy can be extracted by terminating the frequency scan, thereby allowing the wave to damp because of dissipation to the background plasma (this is energy channelling) or by transferring the wave energy back to a grid through

appropriate phasing with an external antenna. This latter procedure is a form of direct conversion, and a further study is needed to assess the feasibility of this method. If feasible, this form of direct conversion may be applicable to a D–He^3 fusion system, where 15 MeV protons are a principal fusion product.

(3) We have indicated a mechanism that allows plasma to amplify a low level signal to a level that one would not expect on the basis of linear theory. This arises because it is possible to use particle adiabaticity to create phase space holes or clumps. The scaling given here is generic to kinetic systems. The increased amplitudes achievable with frequency sweeping may then allow mode overlap to occur, thereby producing global plasma transport.

We note that varying the frequency can be difficult when an external antenna excites a normal mode for which the frequency is defined by the plasma. In this case it may be necessary to change the mode frequency by changing plasma parameters in time. Another possibility is to use a wavepacket that changes its phase velocity as it propagates in an inhomogeneous plasma. This can have an effect similar to frequency sweeping.

It would be interesting to exhibit the enhancement of the power extraction rate in a laboratory experiment under controlled conditions. Such experiments can be devised on a variety of different plasma systems, from a Q-machine to a tokamak.

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References

1. D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851 (1949).
2. A.I. Akhiezer and Ya. B. Fainberg, Zh. Eksp. Teor. Fiz. **21**, 1262 (1951).
3. B.I. Cohen, R.B. Cohen, W.M. Nevins and T.B. Rognlein, Rev. Mod. Phys. **63**, 949 (1991).

4. M.N. Rosenbluth, H.V. Wong and B.N. Moore, Phys. Fluids B **2**, 1635 (1990).
5. N.J. Fisch, Rev. Mod. Phys. **59**, 175 (1987).
6. N.J. Fisch and J.R. Rax, Phys. Rev. Lett. **69**, 612 (1992).
7. C.Z. Cheng, L. Chen and M.S. Chance, Ann. Phys. **161**, 21 (1985).
8. G.Y. Fu and J.W. Van Dam, Phys. Fluids B **1**, 1949 (1989).
9. K.L. Wong, R.J. Fonck, S.F. Paul, D.R. Roberts, E.D. Fredrickson, R. Nazikian, H.K. Park, M. Bell, N.L. Bretz, R. Budny, S. Cohen, G.W. Hammett, F.C. Jobes, D.M. Meade, S.S. Medley, D. Mueller, Y. Nagayama, D.K. Owens and E.J. Synakowski, Phys. Rev. Lett. **66**, 1874 (1991).
10. W.W. Heidbrink, E.J. Strait, E. Doyle and R. Snider, Nucl. Fusion **31**, 1635 (1991).
11. K. McGuire, R. Goldston, M. Bell, M. Bitter, K. Bol, K. Brau, D. Buchenhauer, T. Crowley, S. Davis, F. Dylla, H. Eubank, H. Fishman, R. Fonck, B. Grek, R. Grimm, R. Hawryluk, H. Hsuan, R. Hulse, R. Izzo, R. Kaita, S. Kaye, H. Kugel, D. Johnson, J. Manickam, D. Manos, D. Mansfield, E. Mazzucato, R. McCann, D. McCune, D. Monticello, R. Motley, D. Mueller, K. Oasa, M. Okabayashi, K. Owens, W. Park, M. Reusch, N. Sauthoff, G. Schmidt, S. Sesnic, J. Strachan, C. Surko, R. Slusher, H. Takahashi, F. Tenny, P. Thomas, H. Towner, J. Valley and R. White, Phys. Rev. Lett. **50**, 891 (1983).
12. L. Chen, R.H. White and M.N. Rosenbluth, Phys. Rev. Lett. **52**, 1122 (1954).
13. B. Coppi and F. Porcelli, Phys. Rev. Lett. **57**, 2272 (1986).
14. N. Kroll, P. Morton and M.N. Rosenbluth, J. Quantum Elect. **OE-17**, 1436 (1981).
15. R.H. Cohen and T.B. Rognlein, Phys. Fluids B **3**, 3406 (1991).
16. H.L. Berk and B.N. Breizman, Phys. Fluids B **2**, 2235 (1990).
17. H.E. Mynick and N. Pomphrey, Nucl. Fusion **34**, 1277 (1994).
18. C.T. Hsu, C.Z. Cheng, P. Helander, D.J. Sigmar and R. B. White, Phys. Rev. Lett. **72**, 2503 (1994).
19. H.L. Berk, C.E. Nielsen and K.V. Roberts, Phys. Fluids **13**, 980 (1974).
20. T. H. Dupree, Phys. Fluids **25**, 277 (1982).
21. B.C. Fried, C.S. Liu, R.W. Means and R.Z. Sagdeev, "Nonlinear Evolution and Saturation of Unstable Electrostatic Wave," Report # PPG-93, University of California, Los Angeles (1971). Copies may be ordered from the National Technical Information Service, Springfield, VA 22161.
22. M.B. Levin, M.G. Lyubarsky, I.N. Onishchenko, V.D. Shapiro and V.I. Shevchenko, Soviet Physics JETP **35**, 898 (1972).
23. H.L. Berk, B.N. Breizman and M. Pekker, in *Physics of High Energy Particles in Toroidal Systems*, AIP Conf. Proc. 311, edited by T. Tajima and M. Okamoto (American Institute of Physics, New York, NY, 1994), p. 18; also Phys. Plasmas **2**, 3007 (1995).
24. G.Y. Fu and W. Park, Phys. Rev. Lett. **74**, 1594 (1995).
25. Y. Wu, R.B. White, Y. Chen and M.N. Rosenbluth, Phys. Plasmas **2**, 4555 (1995).
26. R.B. White, R.J. Goldston, K. McGuire, A.H. Boozer, D.A. Monticello and W. Park, Phys. Fluids **26**, 2958 (1983).