THE THIRD SUMMER SCHOOL ON

PLASMA PHYSICS

15-22 August 1990

Organized by
Association for Plasma Studies of China (APSC)
for
Asian African Association for Plasma Training (AAAPT)

sponsored by
Institute of Physics, Academia Sinica
and
Chinese Society of Physics

at
Tsingdao University
Tsingdao
The People's Republic of China

TOPICS OF MHD THEORY: APPLICATIONS TO FIELD REVERSED CONFIGURATION

R. A. Clemente and P. H. Sakanaka
Departamento de Eletrônica Quântica,
Instituto de Física, UNICAMP,
13081 Campinas, SP, Brazil
Telex: (019)1150, FAX: 55-192-394717,
e-mail:SAKANAKA @ BRUC.ANSP.BITNET

July 20, 1990

Contents

1 RESISTIVE MHD EQUATIONS

A plasma can be described through kinetic equations, but these equations, in spite of describing exactly (at least theoretically) the motion of a statistic system of charged particles, present several troubles when we try to solve them. Moreover, the particle distribution functions are not directly related to macroscopic quantities that can be experimentally measured. In some cases, when the phenomena to be described are not too fast or when the geometry of the problem is complicated, it can be convenient to describe the plasma with macroscopic equation, i.e. through quantities that depend only on $\vec{x}$ (space) and t (time). Obviously this cannot reproduce phenomena conected with the velocity-dependence of the distribution functions, as is the case of Landau damping, but it can help to widen the class of problemas that can be mathematically treated.

From a macroscopical point of view a plasma can be described with fluid variables, ρ (mass density), p (pressure), $\vec{v}$ (local velocity), etc. To the fluid equations we have to add Maxwell's equations and the resulting set will be

-continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{1}$$

- momentum balance equation

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla p + \frac{\vec{j} \times \vec{B}}{c}, \tag{2}$$

- heat balance equation

$$\frac{3}{2} \left(\frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p \right) + \frac{5}{2} p \nabla \cdot \vec{v} = \frac{\vec{j} \cdot \vec{j}}{\sigma}, \tag{3}$$

where thermal conductivity and radiation losses have been neglected, and σ is the plasma conductivity, usually assumed in the Spitzer's form

$$\begin{aligned} \sigma &= \frac{(3kT_e)^{3/2}}{\sqrt{m_e} 4\pi e^2 ln\Lambda}, \\ \Lambda &= 12\pi n_e \lambda_D^3, \\ \lambda_D &= \frac{kT_e}{4\pi e^2 n_e^{1/2}}, \end{aligned}$$

where k is the Boltzmann's constant n_e the electronic density and T_e the electronic temperature. Since $ln\Lambda$ usually is a number comprised between 10 and 20 , the stronger dependence of σ is with $T_e^{3/2}$. In the Maxwell's equation it is usual to neglect the displacement currents and the charge density:

- Gauss' law

$$\nabla \cdot \vec{B} = 0, \tag{4}$$

- Ampère's law

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j}, \tag{5}$$

- Faraday's law

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}. \tag{6}$$

The link between fluid and Maxwell's equation is given by Ohm's law, that we will use in its simpler form:

$$\vec{j} = \sigma \left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right). \tag{7}$$

Many times it is usual to assume an infinite conductivity and the resulting set of equation is know as IDEAL MAGNETOHYDRODYNAMICS (MHD). In this case only equations (3) and (7) modify to

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right) p\rho^{-5/3} = 0 \qquad (adiabatic\ law), \qquad (3bis)$$

$$\vec{E} + \frac{\vec{v}\times\vec{B}}{c} = 0 \qquad (ideal\ Ohm's\ law). \qquad (7bis)$$

2 THE MAGNETIC CONFINEMENT PROBLEM

In general, a laboratory plasma of thermonuclear interest is quite hot and so dilute that any contact with material walls will result in a violent cooling (apart from damages in the wall). In order to have any chance of reaching a controlled thermonuclear device it is important to prevent the contact of the plasma with any material wall. In other words the plasma pressure cannot be balanced by a material container but by some field of force. One way to obtain it is to make currents flowing in the plasma that interacting with the magnetic field, resulting from them and some other external source, can balance the pressure gradient.

From a theoretical point of view the magnetic confinement problem can be divided in three parts:

i) Equilibrium theory: indicates how a plasma can be confined with magnetic fields. It will be seen that infinite equilibrium states are possible but this part of the theory does not predict which are the actual states that can be experimentally obtained. To answer (partially) this las question we need:

ii) Stability theory: determines if a possible equilibrium state is stable or unstable against small perturbations, allowing to find which should be the unstable motions of the plasma. In general several equilibrium can be stable, in this case it is of interest to know which of them, once diffusion effects are taken into account, could last more time. To this aspect some partial answers are offered by:

iii) Transport theory: treats the diffusion phenomena and the decay of the macroscopic variables describing the plasma with the object of estimating how much should last an equilibrium configuration in absence of instabilities.

To treat i) and ii) we will use the ideal MHD equations. Experimentally it has been generally observed that a plasma can be maintained in equilibrium for times quite shorter than the typical macroscopic decaying times. Moreover, the typical wavelengths of macroscopic instabilities are of the order of the plasma size and they develop in very short times, for which the ideal approximation can be considered appropriate (obviously exceptions exist).

Part iii) will be treated using MHD equations with scalar conductivity which we call resistive MHD equations.

In many situations the MHD model can be inadequate, a very strong hypothesis is that of scalar pressure, but for the moment, it is the only one that allows to take into account many of the geometrical characteristics of the plasmas produced in laboratory or existing in several astrophysical problems, by using analytical techniques.

2.1 EQUILIBRIUM THEORY

We can define as equilibrium states the ideal MHD solutions that do not depend on time $\left(\frac{\partial}{\partial t} = 0\right)$ and for which some region with $\nabla p \neq 0$ exists. We can also distinguish two kind of equilibria:

i) Static equilibria: $\frac{\partial}{\partial t} \equiv 0$ and $\vec{v} = 0$.

ii) Stationary equilibria: $\frac{\partial}{\partial t} \equiv 0$ and $\vec{v} \neq 0$.

The ideal MHD equations that describe our equilibrium state are:

$$\nabla\cdot(\rho\vec{v}) = 0, \qquad (8)$$

$$\rho\vec{v}\cdot\nabla\vec{v} = -\nabla p + \frac{\vec{j}\times\vec{B}}{c}, \qquad (9)$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j}, \tag{10}$$

$$\nabla \cdot \vec{B} = 0, \tag{11}$$

$$\nabla \times (\vec{v} \times \vec{B}) = 0, \tag{12}$$

$$\vec{v} \cdot \nabla \left(p\rho^{-5/3}\right) = 0. \tag{13}$$

Using the vectorial identity $\nabla(\vec{A}\cdot\vec{C}) = \vec{A}\times\nabla\times\vec{C}+\vec{C}\times\nabla\times\vec{A}+\vec{A}\cdot\nabla\vec{C}+\vec{C}\cdot\nabla\vec{A}$, from equations (9) and (10) we obtain

$$\rho\vec{v}\cdot\nabla\vec{v} = -\nabla\left(p+\frac{B^2}{8\pi}\right)+\frac{\vec{B}\cdot\nabla\vec{B}}{4\pi}, \tag{14}$$

from which we can distinguish a magnetic pressure $\left[\nabla\frac{B^2}{8\pi}\right]$ and a magnetic tension $\left(\frac{\vec{B}\cdot\nabla\vec{B}}{4\pi}\right)$ due to the curvature of $\vec{B}$ lines. It is usual to define a parameter

$$\beta = \frac{8\pi < p >}{< B^2 >}, \tag{15}$$

where the mean values have to be intended as promedia over the plasma volume (other definitions of β are commonly used, for example, it can also be defined as a local values). β gives a measure of the efficiency of the magnetic field for confining the plasma, typically it varies between zero and unity. If $\beta < 0.1$ we talk about low β configurations, if $\beta \geq 0.1$ we talk of high β configurations.

Equation (14) can give some interesting information, we can see if a plasma can be confined in a limited region of space (out of which $p = \rho = |\vec{v}| = 0$ without magnetic fields due to external conductors. The answer to this question is given by:

2.2 THE VIRIAL THEOREM

Taking the scalar product of equation (14) with a position vector $\vec{x}$, we can obtain (Einstein convention) for each term:

$$\begin{aligned}
\vec{x}\cdot(\rho\vec{v}\cdot\nabla\vec{v}) &= x_j\rho v_i\frac{\partial}{\partial x_i}v_j \\
&= x_j\frac{\partial}{\partial x_i}(\rho v_i v_j) - x_j v_j\frac{\partial}{\partial x_i}\rho v_i \\
&= \nabla(\rho\vec{v}\vec{v}\cdot\vec{x}) - \rho v^2, \\
-\vec{x}\cdot\nabla\left(p+\frac{B^2}{8\pi}\right) &= -\nabla\cdot\vec{x}\left(p+\frac{B^2}{8\pi}\right)+3\left(p+\frac{B^2}{8\pi}\right), \\
\vec{x}\cdot\frac{\vec{B}\cdot\nabla\vec{B}}{4\pi} &= \frac{x_jB_i}{4\pi}\frac{\partial}{\partial x_i}B_j \\
&= \frac{x_j}{4\pi}\frac{\partial}{\partial x_i}(B_iB_j) - \frac{x_jB_j}{4\pi}\frac{\partial}{\partial x_i}B_i \\
&= \nabla\cdot\left(\frac{\vec{B}\cdot\vec{x}\vec{B}}{4\pi}\right) - \frac{B^2}{4\pi}.
\end{aligned}$$

If we integrate over a volume V $[\vec{x}\cdot(14)]$ we obtain:

$$\begin{aligned}
\int_V d\vec{x}\left(\rho v^2+3p+\frac{B^2}{8\pi}\right) &= \int_{S(V)} d\vec{s}\cdot\left[-\frac{\vec{B}(\vec{B}\cdot\vec{x})}{4\pi}\right. \\
&+ \left.\rho\vec{v}(\vec{v}\cdot\vec{x})+\vec{x}\left(p+\frac{B^2}{8\pi}\right)\right].
\end{aligned} \tag{16}$$

If no external conductors are present, we can extend V to ∞ and we can see that while the left member of (16) is $\neq 0$, the right member vanishes if the plasma occupies a finite region of space (in this case $\vec{B} \sim 1/|\vec{x}|^3$, like a dipole field). This result implies that the surface integral must be necessarily $\neq 0$. This means that external conductors carrying currents $\neq 0$ must exist. The integration volume cannot be extended to the interior of such conductors, this would imply a discontinuity of $\rho v^2 + 3p + \frac{B^2}{8\pi}$ in crossing their surfaces and the Gauss theorem ceases to be valid. In general, the external conductor are assumed perfect, so a discontinuity in $\vec{B}$ will exist ($\vec{B}\cdot\vec{s} = 0$, $\vec{B} \neq 0$ outside and $\vec{B} = 0$ inside the conductor) owing to superficial currents.

Therefore, we can say: "in order to a plasma be confined in a finite volume external magnetic fields produced by external conductors are necessary". Obviously if the plasma volume is infinite (at least in one direction) the theorem ceases to be valid.

2.3 STATIC EQUILIBRIA

Static equilibria ($\vec{v} = 0$) are described by the following equation

$$\nabla p = \frac{\vec{j} \times \vec{B}}{c}, \tag{17}$$

from it we can obtain

$$\vec{j} \cdot \nabla p = 0\,;\ \vec{B} \cdot \nabla p = 0.$$

This means that current lines and $\vec{B}$ lines lie on constant pressure surfaces. Constant pressure surfaces are also named magnetic surfaces. If we assume that over a $p = constant$-surface $\vec{B} \neq 0$ always, taking into account that a $\vec{B}$ line cannot cross itself and also that $\vec{j}$ must be tangent to the surface, it results that, if the surfaces are closed (in a three-dimensional space) it cannot be a simply connected surface (like a spheroid) but it must be multiply connected [for a rigorous demonstration see Alexandroff [1]]. In order to visualize it, think about winding a string on a sphere, it will be impossible without crossing itself, but it will be possible to wind it on a torus. A toroid is the simplest multiply connected surface, and in practice it is what we use to confine a plasma in a finite volume. In general the magnetic surfaces look like, figure (1).

The pressure will have a maximum (p_0) on a toroid which degenerates in a closed curve, in general this curve coincides with a $\vec{B}$-line and it is named magnetic axis.

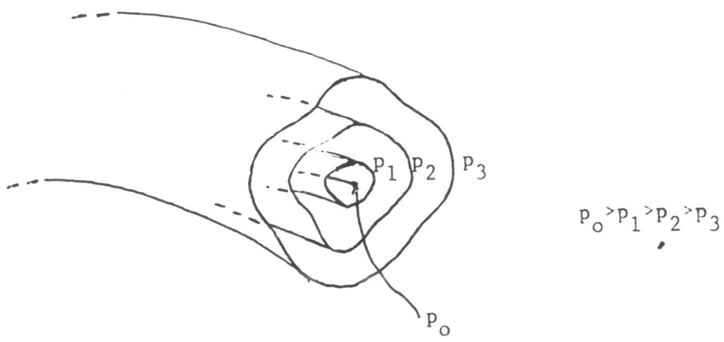


Figure 1:

The magnetic confinement systems can be divided in two wide classes:

i) Systems in which the required magnetic field geometry is due essentially to external conductors. These systems are known as magnetic traps and can be divided in two sub-classes:

a) Open systems: $\vec{B}$-lines extends up to ∞ , in this case we have not a three-dimensional confinement since the plasma can escape along the $\vec{B}$-lines . Examples of this sub-class are magnetic mirrors and cusps, figure (2)

b) Closed systems: $\vec{B}$-lines are confined in a bounded volume. An example. is the stellarator, figure (3)

These are very low β configurations in general, $\vec{B}$ is practically irrotational and the confinement is essentially studied on the basis of the motion of a charged particle in a magnetic field. For example, in a mirror, particles with *velocity* $\parallel$ *to* $\vec{B}$ $>>$ *velocity* $\perp$ $\vec{B}$ can leave the system, in the opposite case, owing to the canonical angular momentum conservation they will be reflected at the ends of the mirror.

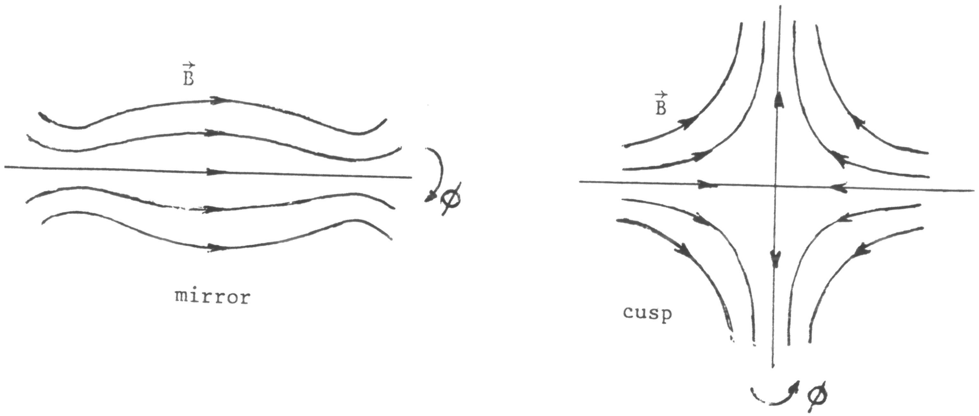


Figure 2:

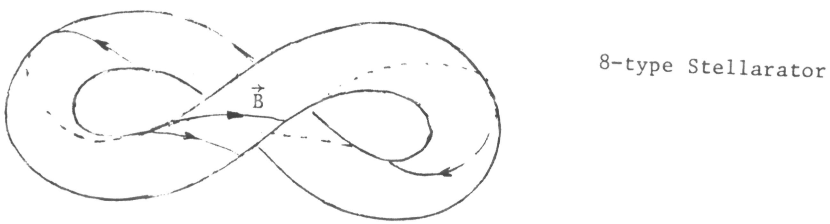


Figure 3:

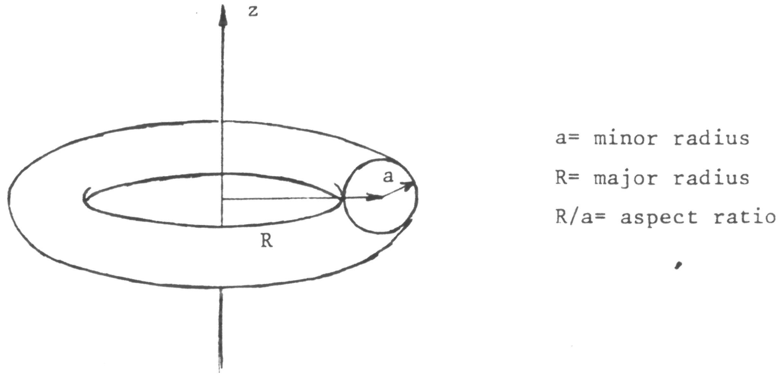


Figure 4:

ii) Systems in which the required magnetic field geometry is due to external conductors and also to currents flowing in the plasma. Also these systems can be divided in open and closed ones with the same criteria used in i). In general they are described using ideal MHD equations. They can be either low or high β configurations. Tokamaks and pinches are examples of this class.

In this note we will restrict our analysis to systems of class ii).

Within open systems we have the longitudinal pinches, they are essentially infinite plasma columns in which longitudinal and/or transversal currents flow. Experimentally the plasma columns are of finite length. In general, their transversal dimensions are much smaller than their length, in such a way that the approximation as infinite column can be considered appropriate. Many times the same approximation is applied to toroidal axisymmetric equilibria, when the minor radius of the torus is much smaller than the major radius, figure (4).($R/a >> 1 \rightarrow$ cylindrical approximation.)

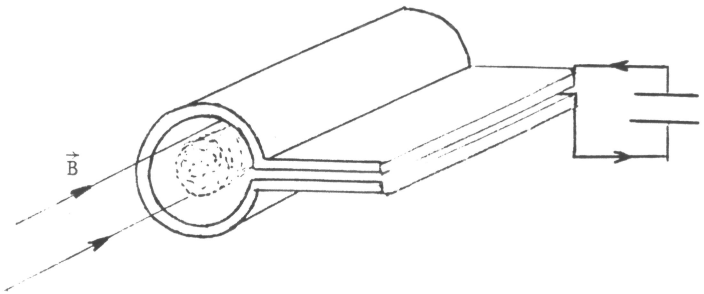


Figure 5:

If $\vec{v} = 0$ and the plasma column is azimuthally symmetric (we use cylindrical coordinates, r, ϕ, z with the z-axis along the plasma column), equation (17) has only radial component and can be written

$$\frac{d}{dr}\left(p + \frac{B_z^2 + B_\phi^2}{8\pi}\right) = -\frac{B_\phi^2}{4\pi r}, \tag{18}$$

these equation can describe 3 cases of interest:

1)$B_\phi = 0$; this corresponds to the ideal θ-pinch in which the plasma current density has only azimuthal component. Experimentally this situation can be produced by discharging a capacitor bank on an elongated coil of circular cross section, on whose inside a low-pressure preionized gas exists, figure (5)
In this case equation (18) reduces to

$$p + \frac{B_z^2}{8\pi} = const, \tag{19}$$

and in principle we can have ∞ solutions for a given external B_z (one for each reasonable profile p(r)).

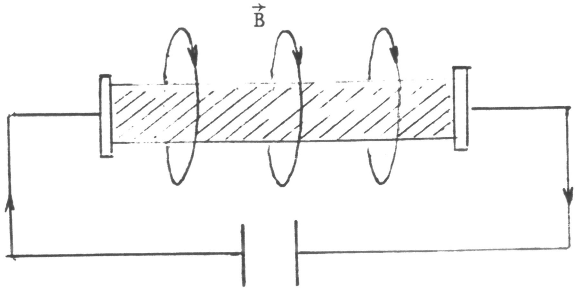


Figure 6:

2) $B_z = 0$, this corresponds to the ideal z-pinch, in which the current density is purely longitudinal. Experimentally they are produced discharging a capacitor bank through two electrodes, figure (6)
In this case equation (18) reduces to:

$$\frac{d}{dr}\left(p + \frac{B_\phi^2}{8\pi}\right) = -\frac{B_\phi^2}{4\pi r}, \tag{20}$$

and we can have ∞ solutions depending on the distribution of j_z.

3)$B_z \neq 0$ and $B_\phi \neq 0$, this case is commonly named stabilized z-pinch . Also in this case it is possible to obtain infinite solutions of equation (18) . Experimentally they are obtained by mixing the θ and z-pinch methods.

If the plasma column is not azimuthally symmetric but has only translational symmetry, equation (18) has two components and we have two coupled differential equations. If $\vec{B} = B_z\vec{e}_z$ we have as in the ideal θ-pinch case

$$\left(p + \frac{B_z^2}{8\pi}\right) = const. \tag{21}$$

this means that we can have θ-pinches of non-circular cross-section. If $\vec{B} = B_x\vec{e}_x + B_y\vec{e}_y + B_z\vec{e}_z$ in order to find equilibria it is convenient to introduce a potential vector $\vec{A} = A(x,y)\vec{e}_z$ such that

$$\vec{B} = B_z(x,y)\vec{e}_z + \nabla A(x,y) \times \vec{e}_z, \tag{22}$$

and equation (18) can be written

$$\nabla\left(p + \frac{B_z^2}{8\pi}\right) = -\left(\frac{\nabla^2 A}{4\pi}\right)\nabla A + \frac{\vec{e}_z}{4\pi}\left(\frac{\partial B_z}{\partial x}\frac{\partial A}{\partial y} - \frac{\partial B_z}{\partial y}\frac{\partial A}{\partial x}\right). \tag{23}$$

Since $\nabla\left(p + \frac{B_z^2}{8\pi}\right)\cdot\vec{e}_z$ must vanish by hypothesis, the z-component of this equation must be zero. This implies $\nabla A \times \nabla B_z = 0$, which is equivalent to the Jacobian of the transformation $x,\ y \rightarrow A,\ B_z$ being zero, it follows $B_z = B_z(A)$. By multiplying (23) $\times\nabla A$ it follows also $p = p(A)$ and equation (23) reduces to:

$$\nabla^2 A = -4\pi\frac{d}{dA}\left(p + \frac{B_z^2}{8\pi}\right). \tag{24}$$

In order to solve it, we have to formulate some hypothesis on the A-dependence of $\left(p + \frac{B_z^2}{8\pi}\right)$. Since in principle, we can make ∞ hypothesis, equation (24) will admit ∞ solutions. Equation (24) is valid inside the plasma, $\nabla^2 A = -\frac{4\pi}{c}j_z$, outside $j_z = 0$ and the equation to solve will be $\nabla^2 A = 0$. In general the equilibrium problem is solved by assigning $\left(p + \frac{B_z^2}{8\pi}\right) = f(A)$ and also the plasma boundary ($p =$ *const.*-surface), once A is known inside the plasma $\nabla^2 A = 0$ is solved outside and matched to the internal solution ($A_{int} = A_{ext}$; $\nabla A_{int} = \nabla A_{ext}$ at the boundary). However, in many applications one forgets the exterior problem and assumes that the plasma boundary coincides with a perfectly rigid and conducting wall, so that the exterior problems is avoided.

As a simple example let us consider the case in which $\left(p + \frac{B_z^2}{8\pi}\right) \propto A$. In this case equation (24) reduces to $\nabla^2 A = -(4\pi/c)j_0$ (uniform longitudinal current density). A straightforward solution for A is

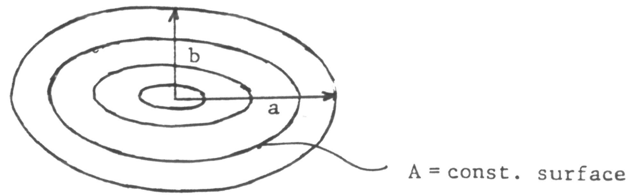


Figure 7:

$$A = \frac{2\pi j_0}{c}\frac{a^2b^2}{a^2+b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \tag{25}$$

which represents a pinch of elliptical cross section. Each $A =$ *const.*-surface can represent the plasma boundary if we choose $A = 0$, the boundary is an ellipse of semi-axes a and b, figure (7)

2.4 TOROIDAL EQUILIBRIA

Toroidal equilibria have the advantage of presenting closed magnetic surfaces thus preventing particle losses along $\vec{B}$-lines. Probably a fusion reactor based on magnetic confinement will be of a toroidal shape. Toroidal equilibria can have either low β (tokamaks) or high β (toroidal pinches, compact tori), anyway both are described by the same equations.

In order to simplify things we will assume that equilibria are axisymmetric and choose the z-axis as the symmetry axis. Moreover, we will consider stationary equilibria for which $\vec{v} = v_\phi(r,z)\vec{e}_\phi$ only. Azimuthal rotation is quite common in reversed field θ-pinches but it has also been observed in tokamaks. In order to obtain an equation describing this kind of equilibria we will follow quite closely the work of Maschke and Perrin [2].

In general, in a plasma the thermal conductivity along $\vec{B}$-lines is greater than

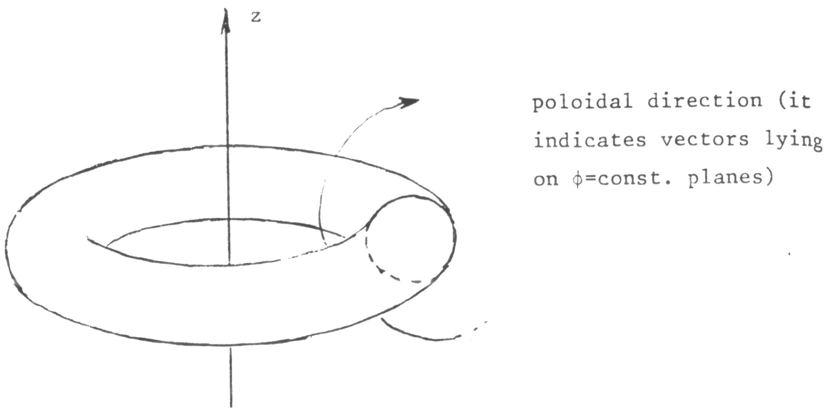


Figure 8:

hat across them, in such a way that it is reasonable to assume that the plasma emperature $T = T_e + T_i$ is constant along B-lines ($\vec{B} \cdot \nabla T = 0$). This assumption implifies considerably the problem of finding rotating toroidal equilibria, but other ssumptions are possible (for example, constant entropy along B-lines).

Owing to axisymmetry and since $\nabla \cdot \vec{B} = 0$ we can express $\vec{B}$ as:

$$\vec{B}(r,z) = \frac{2I(r,z)}{cr}\vec{e}_\phi - \frac{\vec{e}_\phi \times \nabla\psi(r,z)}{2\pi r}, \tag{26}$$

vhere I and ψ have to be determined. We can note that the azimuthal component f the potential vector ($\vec{B} = \nabla \times \vec{A}$) is related to ψ, since, apart from an arbitrary auge, $A_\phi = \psi/2\pi r$, moreover, ψ can be interpreted as the poloidal magnetic flux:

$$\psi(r,z) = 2\pi \int_0^r dr' r' B_z(r',z). \tag{27}$$

Analogously we can identify I with the poloidal current flux. figure(8)

$$I(r,z) = 2\pi \int_0^r dr' r' j_z(r',z). \tag{28}$$

Expressing $v_\phi(r,z) = \Omega(r,z)r$, with Ω angular velocity, Faraday's law can be written

$$\nabla \times \vec{E} = -\nabla \times \left(\frac{\vec{v} \times \vec{B}}{c}\right) = 0 = \nabla \times \left[\frac{\Omega r}{c}\vec{e}_\phi \times \left(\frac{\vec{e}_\phi \times \nabla\psi}{2\pi r}\right)\right], \tag{29}$$

from which it results

$$\nabla\Omega \times \nabla\psi = 0, \tag{30}$$

which is equivalent to make zero the Jacobian of the transformation $r,\ z \to \psi,\ \Omega$. Therefore, in general, $\Omega = \Omega(\psi)$. This means that constant ψ-surfaces (they are nested tori) rotate rigidly around the z-axis. Equation (30) is also known as "Isorotation law", found originally by Ferraro [3] studying magnetic fields in stars.

From (26) we can express $\vec{j}$ as

$$\begin{aligned} \vec{j} &= \frac{c}{4\pi}\nabla \times \left(-\frac{\vec{e}_r}{2\pi r}\frac{\partial\psi}{\partial z} + \vec{e}_\phi\frac{2I}{cr} + \frac{\vec{e}_z}{2\pi r}\frac{\partial\psi}{\partial r}\right) \\ &= \frac{c}{4\pi}\left(-\vec{e}_r\frac{2}{cr}\frac{\partial I}{\partial z} - \frac{\vec{e}_\phi}{2\pi r}\nabla^{*}\psi + \vec{e}_z\frac{2}{cr}\frac{\partial I}{\partial r}\right), \end{aligned} \tag{31}$$

where $\nabla^{*} \equiv r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ is known as Grad-Shafranov operator.

Using (31) we obtain for $\rho\vec{v}\cdot\nabla\vec{v} + \nabla p = \nabla \times \vec{B} \times \vec{B}/4\pi$:

$$\left(\frac{\partial p}{\partial r} - \rho\Omega^2 r\right)\vec{e}_r + \frac{\partial p}{\partial z}\vec{e}_z = -\frac{1}{16\pi^3 r^2}(\nabla^{*}\psi)\nabla\psi - \frac{I}{\pi c^2 r^2}\nabla I + \frac{\nabla I \times \nabla\psi}{4\pi r^2 c}. \tag{32}$$

The last term has only ϕ-component and must vanish, so we obtain $I = I(\psi)$, and (32) can be written

$$\left(\nabla^{*}\psi + \frac{8\pi^2}{c^2}\frac{dI^2}{d\psi}\right)\nabla\psi = -16\pi^3 r^2\left(\nabla p - \rho\Omega^2 r\nabla r\right). \tag{33}$$

Let us assume that the plasma is formed by electrons and ions of unitary charge, indicating with n their density we can write $p = nkT$. It is therefore possible to transform the right member of (33) (this is a trick introduced by Maschke and Perrin [2])

$$16\pi^3 r^2 \left(\nabla p - \rho\Omega^2 r \nabla r\right) = 16\pi^3 r^2 \left\{ n\nabla \left[kT \ln\frac{\rho}{\rho_0} - (m_e+m_i)\frac{\Omega^2 r^2}{2}\right] + n\left[k\left(1-\ln\frac{\rho}{\rho_0}\right)\frac{dT}{d\psi} + r^2\Omega\frac{d\Omega}{d\psi}(m_e+m_i)\right]\nabla\psi\right\} \quad (34)$$

where we used the fact that $\vec{B}\cdot\nabla T = 0 = \nabla\psi\times\nabla T$, $m_{e,i}$ are the electronic and ionic masses and ρ_0 a constant. From (33) and (34) we get

$$\left\{\nabla^{*}\psi + \frac{8\pi^2}{c^2}\frac{dI^2}{d\psi} + 16\pi^3 r^2 n\left[k\left(1-\ln\frac{\rho}{\rho_0}\right)\frac{dT}{d\psi} + r^2\Omega\frac{d\Omega}{d\psi}(m_e+m_i)\right]\right\}\nabla\psi = -16\pi^3 r^2 n\nabla\left[kT\ln\frac{\rho}{\rho_0} - (m_e+m_i)\frac{\Omega^2 r^2}{2}\right]. \quad (35)$$

By taking the vectorial product $\nabla\psi\times(35)$ it follows that

$$kT\ln\frac{\rho}{\rho_0} - (m_e+m_i)\frac{\Omega^2 r^2}{2} = \Theta(\psi) \quad (36)$$

and we can express the plasma density as

$$\rho = \rho_0 \exp\left[\frac{\Theta}{kT} + (m_e+m_i)\frac{\Omega^2 r^2}{2kT}\right]. \quad (37)$$

Introducing the function

$$g = \frac{\rho_0 kT}{(m_e+m_i)} e^{\Theta/kT} = p e^{-(m_e+m_i)\frac{\Omega^2 r^2}{2kT}}, \quad (38)$$

we obtain the final form of the Maschke-Perrin equation

$$\nabla^{*}\psi = -16\pi^3 r^2 \exp(m_e+m_i)\frac{\Omega^2 r^2}{2kT}\left[\frac{dg}{d\psi} + g(m_e+m_i)r^2\frac{d}{d\psi}\frac{\Omega^2}{2kT}\right] - \frac{8\pi^2}{c^2}\frac{dI^2}{d\psi}. \quad (39)$$

The solution of this equation can describe the equilibrium of a rotating toroidal plasma. At the beginning the problem consisted of 8 nonlinear coupled equations for the 8 unknowns ρ, p, $\vec{v}$, $\vec{B}$. The hypothesis of axisymmetry and azimuthal velocity reduced the number of unknowns to 5 (ρ, p, Ω, ψ, I), and the resulting 5 equations have been reduced to 1 equation for ψ plus 4 hypothesis on the other variables (ρ and p are related to g and T).

In order to solve (39) it is necessary to formulate hypopthesis on the functional dependence of ρ, Ω, T, I with ψ. Since in principle an ∞ number of hypothesis can be made we can have ∞ possible equilibrium states.

Equation (39) is valid inside the plasma, if the plasma is not in contact with a perfectly rigid and conducting wall but it is surrounded by a vacuum region, remembering that $\nabla^{*}\psi = -\frac{8\pi^2 r}{c} j_\phi$, we need to solve in that region

$$\nabla^{*}\psi = 0, \quad (40)$$

the inner and outer solutions have to be matched at the plasma boundary, moreover the outer solution must be constant on the surface of eventual external conductors. In general, this is a quite difficult problem that can be solved numerically. In many applications one forgets the exterior problem and assume that the plasma boundary coincides with a $\psi = const$ surface resulting from solutions of (39).

Very little can be done analytically on the solution of equation (39), in the literature essentially 3 solutions have appeared, one due to Maschke and Perrin [2] and the other to Missiato and Pantuso-Sudano [4] and to Clemente and Farengo [5], all corresponding to linear cases of (39) and to the hypothesis that $\Omega^2/kT = const$, which greatly simplifies the task.

As an example of solution let us consider the case of a plasma embedded in a vessel of rectangular cross-section of radii $R-a$ and $R+a$ and height L , figure (9).

We can assume that either p or ψ vanish at the wall. In order to get a linear

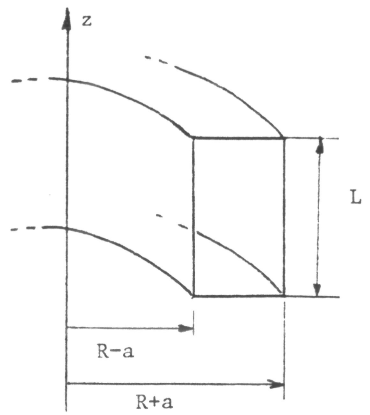


Figure 9:

homogeneous expression for equation (39) we make the following hypothesis:

$$g = g_0\psi^2 \; ; \; I^2 = I_0^2\psi^2 + I_1 \; ; \; \Omega^2/kT = const.$$

It is convenient to use adimensional variables:

$$x = \frac{r}{R} \; ; \; y = \frac{z}{L}.$$

Equation (39) reduces to

$$x\frac{\partial}{\partial x}\frac{1}{x}\frac{\partial}{\partial x}\psi + \frac{R^2}{L^2}\frac{\partial^2\psi}{\partial y^2} = -\left(\alpha^2 x^2 e^{3\epsilon x^2/2} + \delta^2\right)\psi \propto j_\phi \qquad (41)$$

from which we see that j_ϕ also will vanish at the wall if ψ vanishes. The constants appearing in (39) are respectively

$$\alpha^2 = 32\pi^3 R^4 g_0,$$
$$\delta^2 = \frac{16 I_0^2 R^2}{c^2},$$
$$\epsilon = \frac{(m_e + m_i)\Omega^2 R^2}{3kT},$$

where ϵ is a measure of the amount of rotation in the plasma. Taking $y = 0$ as the symmetry plane of the vessel and assuming the same symmetry for ψ and that ψ vanishes at the plasma boundary , we can try to find a solution of the form:

$$\psi = F(x)cos(\pi y),$$

it automatically satisfies $\psi = 0$ at $y = \pm\frac{1}{2}$, and the following equation for F results

$$x\frac{\partial}{\partial x}\frac{1}{x}\frac{\partial}{\partial x}F = -\left(\alpha^2 x^2 e^{3\epsilon x^2/2} + \delta^2 - \frac{\pi^2 R^2}{L^2}\right)F.$$

At this point it is convenient to introduce a new variable $w = \frac{x^2}{2}$ and the equation transforms to

$$\frac{\partial^2 F}{\partial w^2} = -\left(\alpha^2 e^{3\epsilon w} + \frac{\delta^2 - \pi^2 R^2/L^2}{2w}\right)F.$$

This equation can be numerically solved, once ϵ and δ^2 are assigned, by requiring that F vanishes at $w = \frac{(R\pm a)^2}{2R^2}$ and that it has only one maximum inside the domain. However, to our purpose, there is an easily solvable case. This corresponds to $\delta^2 = \pi^2 R^2/L^2$, in this case we have

$$\frac{\partial^2 F}{\partial w^2} = -\alpha^2 e^{3\epsilon w} F,$$

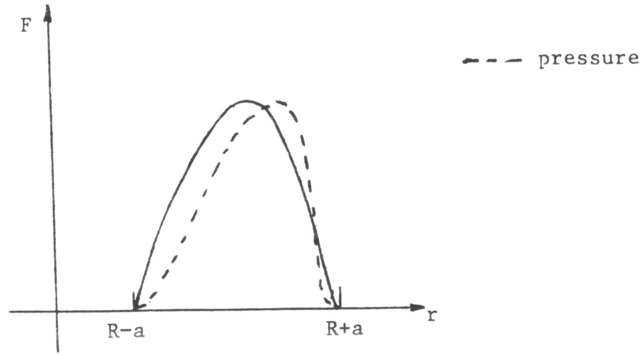


Figure 10:

whose general solution is:

$$F = AJ_0(\xi) + BY_0(\xi),$$

where $\xi = 2\alpha e^{3\epsilon w/2}$ and J_0, Y_0 are Bessel's functions of the 1^{st} and 2^{nd} kind respectively. Boundary conditions fix one of the constants A or B , for example, asking $F = 0$ at $r = R - a$ and defining $\xi_0 = 2\alpha \exp\left[\frac{3\epsilon(R-a)^2}{2R^2}\right]$ we have

$$F = A\left[J_0(\xi) - \frac{Y_0(\xi)J_0(\xi_0)}{Y_0(\xi_0)}\right].$$

Asking that $F = 0$ at $r = R + a$ gives an eigenvalue problem for α once ϵ is assigned.

One can therefore amuse oneself with tables and completely solve the problem. Qualitatively the behaviour of F is of the kind, figure (10).

and the magnetic surfaces are toroids of the kind, figure (11).

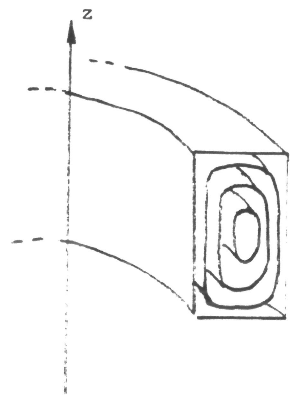


Figure 11:

Since $p = g(\psi)\exp\left[(m_e + m_i)\frac{\Omega^2 r^2}{2kT}\right]$, it follows that constant p surfaces will be drifted towards the outer boundary with respect to $\psi = const$ surfaces, the effect will increase with increasing ϵ.

Up to now we have considered stationary toroidal equilibria in general. From here on we will limit ourselves to static equilibria only, since their treatment is much simpler than the general case.

An axisymmetric static equilibrium can be described in terms of p and $\vec{B}$ only. $\vec{B}$ can be expressed in the same way as in the stationary case and we can repeat the formalism to obtain an equation for ψ. Such equation arises directly by taking the limit $\Omega \to 0$ in equation (39). We have

$$\lim_{\Omega\to 0} g(\psi) = \lim_{\Omega\to 0} p \exp\left[-\frac{(m_e + m_i)\Omega^2 r^2}{2kT}\right] = p(\psi)$$

Now $lim_{\Omega\to 0}$ [equation (39)] results

$$\nabla^{*}\psi = -\frac{8\pi^2 r}{c} j_\phi = -16\pi^3 r^2 \frac{dp}{d\psi} - \frac{8\pi^2}{c^2}\frac{dI^2}{d\psi}, \qquad (42)$$

which is the well known Grad-Shafranov equation (a very large number of work on its resolution have been published).

Looking at (42) we can distinguish 3 cases of interest

i) $\frac{dI^2}{d\psi} = 0$; This means that there are no poloidal currents in the plasma, and B_ϕ can be either zero or a vacuum field due for example to a wire in the center of the systema ($B_\phi \propto \frac{1}{r}$).

ii) $\frac{dp}{d\psi} = 0, \frac{dI^2}{d\psi} \neq 0$; in this case we have not a real confinement of the plasma ($p = const$) and it can be seen that $\vec{j} \parallel \vec{B}$. This kind of magnetic fields are named "force-free" and can be a good approximation for low β plasmas as in tokamaks or in some astrophysical situation.

iii) $| 16\pi^3 r^2 \frac{dp}{d\psi} | \sim | \frac{8\pi^2}{c}\frac{dI^2}{d\psi} | >> | \nabla^{*}\psi |$;
in this case currents in the plasma are essentially poloidal. A small j_ϕ must exists in order the equilibrium to be a closed one (if $j\phi = 0 \rightarrow \frac{dI^2}{d\psi} / \frac{dp}{d\psi} = -2\pi c^2 r^2 = f(\psi) \rightarrow \psi = \psi(r)$ only), but it is not the main cause of pressure confinement. Experimentally this situation is obtained in the high-β tokamaks where the plasma is diamagnetic with respect to the toroidal magnetic field.

In general, toroidal equilibria are assumed also symmetric to an equatorial plane ($z = 0$, for example), ψ should have only one extremum inside the plasma (but in some case it can have more than one, e.g. Doublet Machine), which corresponds to the magnetic axis. Since ψ can be defined apart from a constant, it is common to assume $\psi = 0$ at the plasma boundary (this does not modify $\vec{B}$). Also I can be assigned at the plasma boundary, where B_ϕ depends only on external conductors linked with the toroid (poloidal plasma currents do not modify B_ϕ outside the plasma). In order to solve (42), since p and I^2 must not be singular, one can think in some Taylor expansion for them.

$$\begin{aligned} p &= p_0 + p_1\psi + p_2\psi^2 + ... \\ I^2 &= I_0^2 + i_1\psi + i_2\psi^2 + ... \end{aligned}$$

if one truncates the series up to terms in ψ^2 it is possible to obtain a linear partial differential equation for ψ and analytical methods can be used in order to find solutions. If higher powers are considered numerical methods of integration must be used. There are numerical codes that, for given $p(\psi)$ and $I^2(\psi)$ and assigned plasma boundary, calculate ψ. (Moreover, it is also possible to calculate the external magnetic fields necessary to maintain the equilibrium by using the virtual casing principle which will be the subject of the next section.)

A relatively new kind of toroidal configurations are the compact tori. In this kind of equilibria B_ϕ vanishes at the plasma boundary. This means that no external conductors are linked with the toroid. In general, the plasma extends up to the symmetry axis and the plasma boundary may be a simply connected surface, on which $\vec{B}$ vanishes somewhere, of the kind, figure (12).

The $\psi = 0$ surface separates regions of closed ψ surfaces from open ψ surfaces and is named separatrix. In general, the separatrix is assumed to coincide with the plasma boundary but it may not be the case.

Compact Tori can be divided in two classes:

1) $I^2 \neq 0$:

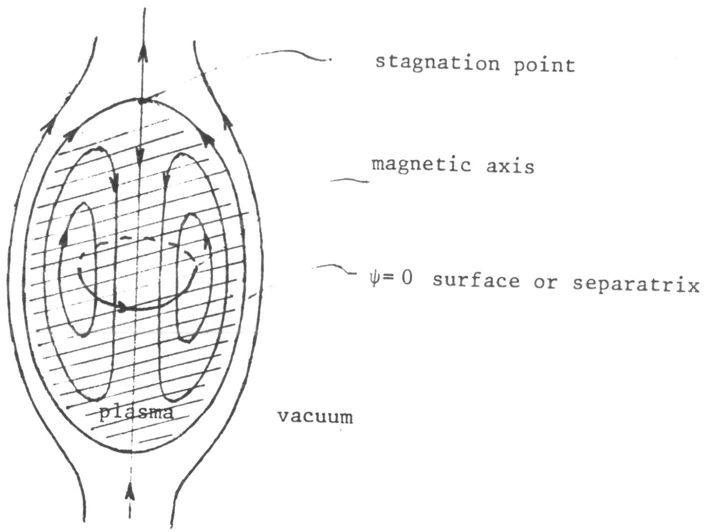


Figure 12:

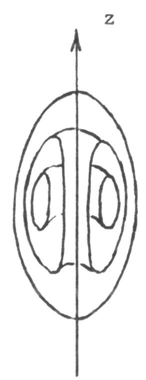


Figure 13:

$\vec{B}$ has both poloidal and toroidal components. In this case the equilibria are named spheromaks. Experimentally they have $\beta \sim 0.1$.

2) $I^2 = 0$:

$\vec{B}$ has only poloidal components. In this case the equilibria are known as field-reversed configuration(FRC). Experimentally they have $\beta \sim 1.0$.

A very simple equilibrium model for FRC can be obtained when p is linear with ψ and assuming the separatrix an ellipsoid of revolution of semi-axis a (along r) and b (along z). In this case we put $I^2 = 0$ and $p = \frac{3B_0}{16\pi^2}\left(\frac{1}{b^2} + \frac{4}{a^2}\right)\psi$, with B_0 a constant. It can be proved that a solution for ψ is given by

$$\psi = \frac{3\pi B_0}{2} r^2 \left(1 - \frac{r^2}{a^2} - \frac{z^3}{b^2}\right), \tag{43}$$

and magnetic surfaces look like, figure (13)

Solutions (43) is known in the literature as Hill's vortex model and has been

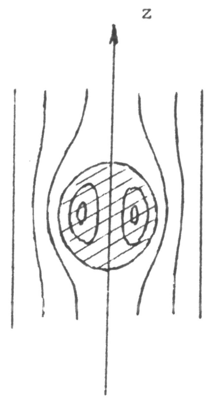


Figure 14:

used in a large number of studies. It corresponds to a situation in which $j_0 \propto r$, so the currents does not vanish at the separatrix, which is physically desirable. However, it has the advantage of being a very simple analytical model that can be used in stability and transport calcualtion. By using spheroidal coordinates it is also possible to find an analytical solution for the exterior problem that matches the Hill's model at the separatrix. This expressions are quite complicated and we present only the external solution for the cases in which the separatrix is a sphere of radius a. In this case a solution of $\nabla^2\psi = 0$ which matches with (43) is:

$$\psi_{ext} = -\pi B_0 r^2 \left[1 - \frac{a^3}{(r^2 + z^2)^{3/2}}\right]. \tag{44}$$

For $r^2 + z^2 >> a^2$ it can be seen that $\psi_{ext} \rightarrow -\pi B_0 r^2$ which is the flux resulting from a uniform longitudinal magnetic field $\vec{B} = -B_0\vec{e}_z$. Plasma contribution to the external $\vec{B}$ is equivalent to that of a magnetic dipole $\left[\pi B_0 r^2 a^3/(r^2 + z^2)^{3/2}\right]$. The complete configuration looks like. figure (14)

For prolate equilibria we have something like, figure (15) and for oblate equilibria, figure (16).

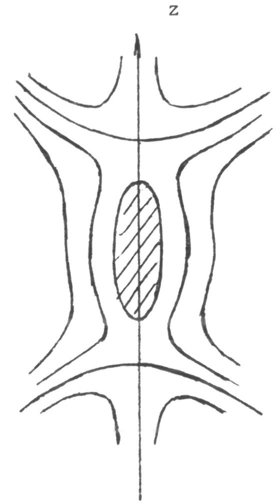


Figure 15:

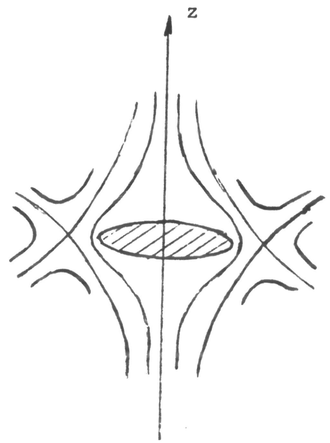


Figure 16:

2.5 VIRTUAL CASING PRINCIPLE

In a toroidal equilibrium $\vec{B}$ is the superposition of the magnetic field due to plasma currents, $\vec{B}_{self}$, and the magnetic field produced by external conductors, $\vec{B}_{vac}$. The solutions of the Grad-Shafranov equation give the total $\vec{B} = \vec{B}_{vac} + \vec{B}_{self}$ inside the plasma and allow to find $\vec{B}_{vac}$ inside the plasma through an analytic procedure known as virtual casing principle.

Let us assume that $p = 0$ defines the plasma boundary and that it coincides with a perfectly conducting and rigid wall, in such a way that $\vec{B}$ can be considered zero outside this wall. Then, the only $\vec{B}$ sources are plasma currents and surface currents induced in the wall that can be obtained by using Ampère's law:

$$\vec{i} = \frac{c}{4\pi}\vec{B}(p=0) \times \vec{n},$$

where $\vec{n}$ is the outward normal versor at the all. The magnetic field created by $\vec{i}$ must coincide with $\vec{B}_{vac}$ inside the plasma and it should be exactly opposite to $\vec{B}_{self}$ outside the plasma. $\vec{i}$ can be divided in a toroidal component $i_T\vec{e}_\phi$, which creates the $\vec{B}^{poloidal}_{vac}$, and a poloidal component $\vec{i}_p$ which create B_ϕ at the wall. Since $B_\phi(p=0) = \frac{2I(p=0)}{cr}$, it follows:

$$\vec{i}_p = \frac{I(p=0)}{2\pi r}\vec{e}_\phi \times \vec{n},$$

where r has to be computed at the wall. Therefore, the necessary $B_{\phi vac}$ can be produced by a conductor, linked with the toroid, carrying a total current $I(p=0)$.

For axisymmetric equilibria $\vec{B}^{poloidal} = -\frac{\vec{e}_\phi \times \nabla\psi}{2\pi r}$ and i_T results:

$$i_T = \frac{c}{8\pi^2 r}\vec{n} \cdot \nabla\psi, (at\ the\ wall)$$

and $\vec{B}^{poloidal}_{vac}$ in the plasma can be computed using Biot-Savart's law

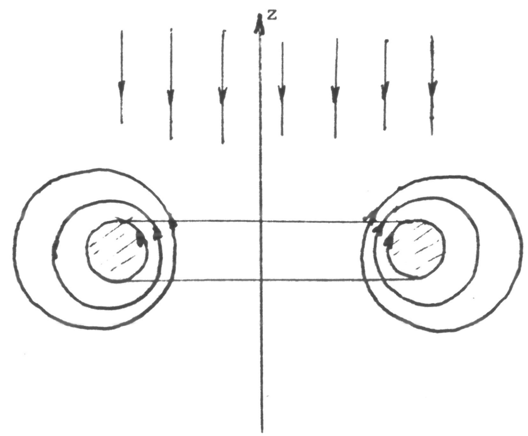


Figure 17:

$$\vec{B}^{poloidal}_{vac}(\vec{x}) = \oint_{wall-surface} d^2x' \frac{1}{8\pi r'}(\vec{n} \cdot \nabla\psi)\frac{\vec{e}'_\phi \times (\vec{x} - \vec{x}')}{\mid \vec{x} - \vec{x}' \mid^3}$$

or in terms of the vector potential $A_{vac}(\vec{x})\vec{e}_\phi$

$$A_{vac}(\vec{x})\vec{e}_\phi = \oint_{wall-surface} d^2x' \frac{(\vec{n} \cdot \nabla\psi)\vec{e}'_\phi}{8\pi^3 r' \mid \vec{x} - \vec{x}' \mid}$$

where r' has to be calculated at the point of integration. This equation formally solve the problem and in practice the integrals have to be done numerically.

In order to give an idea we show that to a first order approximation in order to maintain in equilibrium a term of circular corss-section a vertical magnetic field is necessary, figure (17).

Let us assume that the plasma is a torus with only surface currents, from $\nabla p = \frac{\nabla \times \vec{B} \times \vec{B}}{4\pi}$ it follows that $\left(p + \frac{B^2}{8\pi}\right)$ must be continuous at the plasma-vacuum interface. In absence of external fields $\mid \vec{B} \mid$ is greater inside of the torus than outside. The pressure can be assumed uniform on the cross-section so it would be impossible to satisfy the continuity condition of $\left(p + \frac{B^2}{8\pi}\right)$. The condition can be satisfied if we aggregate a vertical $\vec{B}$ oriented as in the figure. The plasma will

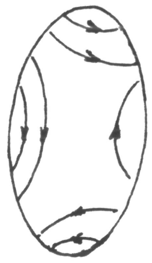


Figure 18:

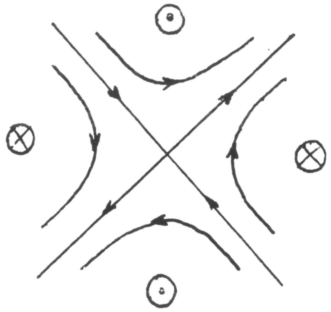


Figure 19:

exclude such field and the result will be that the magnetic pressure will increase outside and decrease inside the torus allowing for a constant B^2 at the interface.

For plasma cross-section different from circular, the situation is quite complicated, for example, in the case of elliptical cross-section $\vec{B}^{poloidal}_{vac}$ looks like, figure (18), which results from the superposition of a vertical $\vec{B}$ with the $\vec{B}$ arising from a quadrupole, figure (19).

The method is of interest in designing tokamaks of non-circular cross-section since it allows to find, for a given v, the $\vec{B}_{vac}$ inside the plasma. Then, it is possible to decide what number and configuration of external conductors should be suitable

in order to reproduce such a field within the constructing constraints.

3 STABILITY: TILTING MODE IN FRC

Studying the force-free model for spheromaks, Rosenbluth and Bussac [6] found that if the separatrix is prolate (elongation > 1), the plasma is unstable to an internal $m = 1$ (azimuthal number) instability called tilting mode, figure (20).

They also predicted that such mode should also be observed in elongated FRC. In spite of having been observed in almost all spheromak experiments, the tilting mode seems to be absent in FRC experiments. Theory confirmed the prevision of Rosenbluth and Bussac, firstly with MHD numerical simulation [7], [8] and shortly after with a simple analytic method , [9]. Here, we want to sketch the analytic demonstration which is based on the Energy Principle, [10], and the Hill's vortex as a model of equilibrium.

For an infinitesimal eigenmode $\vec{\xi} = \vec{\xi}(\vec{r})e^{i\omega t}$ resulting from solving the linearized MHD equations, which does not perturb the plasma boundary (internal mode) and with $\nabla \cdot \vec{\xi} = 0$, the Energy Principle gives the following expression for ω^2:

$$\omega^2 = \frac{\int d\vec{r} \left\{ \mid \nabla \times (\vec{\xi} \times \vec{B}) \mid^2 - \nabla \times \vec{B} \cdot \left[\nabla \times (\vec{\xi} \times \vec{B}) \right] \times \vec{\xi}^{*} \right\}}{4\pi \int d\vec{r} \; \rho \; \vec{\xi} \cdot \vec{\xi}^{*}}, \qquad (45)$$

where integrals are extended to the volume occupied by the plasma, ρ and $\vec{B}$ are equilibrium quantities. If $\omega^2 \geq 0$ the equilibrium will be stable, if $\omega^2 < 0$ it will be unstable to $\vec{\xi}$.

Applying (45) to a trial $\vec{\xi}$ which does not coincide with an eigenmode, the above conclusions on ω^2 will not be valid. However, if $\omega^2 < 0$ is found, instability will be demonstrated, since at least one of the eigenmodes, with which the trial $\vec{\xi}$ can be decomposed, will be unstable (due to the orthogonality properties of the eigenmodes).

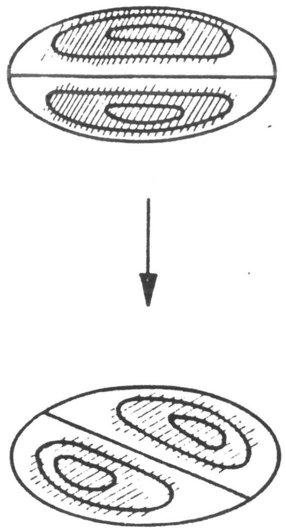


Figure 20:

A trial $\vec{\xi}$ which has been shown to be effective in demonstrating the tilting instability of FRC is:

$$\vec{\xi} \propto (\vec{e}_r \cos\phi - \vec{e}_\phi \sin\phi) \times \nabla \left(\frac{\psi}{r^2}\right), \tag{46}$$

where ψ is the poloidal magnetic flux representing the equilibrium. Formula (46) reproduces an internal rotation of the plasma around the axis $z = 0$, $\phi = 0$, π, without perturbing the separatrix.

If we use as ψ the Hill's vortex

$$\psi = \frac{3\pi B_0}{2} r^2 \left(1 - \frac{r^2}{a^2} - \frac{z^2}{b^2}\right), \tag{47}$$

and assume uniform the temperature of the plasma, $\rho = p(m_e + m_i)/k(T_e + T_i)$, it is possible to perform all the integrals appearing in formula (45) and the following expression for ω^2 results:

$$\omega^2 = \frac{63k(T_e + T_i)}{2b^2(m_e + m_i)\left(\frac{6a^2}{b^2} + \frac{a^4}{b^4} + 8\right)} \left(\frac{a^2}{b^2} - 1\right). \tag{48}$$

As it can be seen $\omega^2 < 0$ if $a^2/b^2 < 1$. Therefore, the Hill's vortex will be unstable if $b > a$, i.e., if the separatrix is prolate. Moreover, for large elongations

$$\omega^2 \simeq -\frac{3.9\, k\, (T_e + T_i)}{(m_e + m_i)\, b^2},$$

which mean that the growth rate should be inversely proportional to the separatrix length. For the FRX-B experiment at Los Alamos: $a = 5.4\ cm, b = 25\ cm$, $T_e + T_i = 310\ eV$ and deuterium. It follows $Im\{\omega\} \simeq 10^6$, a value in good agreement with the numerical results.

The method has also been applied to the Maschke-Hernegger solution for FRC [11], finding similar results, i.e. instability when $b/a > 0.97$.

In spite of the high predicted growth rates, the tilting mode is not observed in experiments. This represents an interesting paradox which still await for an explanation. Several attempts to explain it in terms of plasma rotation [12], kinetic effects[13] and Hall's terms in two fluid models [14], have been made, but no one has been conclusive. The paradox still persists.

4 CLASSICAL TRANSPORT

Particle diffusion across magnetic field and magnetic field dissipation are limiting factors in plasma confinement. An ideal MHD model cannot describe this kind of phenomena since in it σ is assumed infinite. A very simple picture of these transport properties can be obtained by using resistive MHD equations with scalar σ. To this respect it is worth to note that more sophisticated models (with σ as a tensor, or also taking into account thermal conductivity and radiation losses) exist, but in general, they are quite limited in application to bi-dimensional configurations as toroidal equilibria.

The purpose of transport theory is to obtain typical decay times of the equilibria as functions of the different experimental parameters, and also to reproduce, if possible, experimental measurements. Moreover, the results can be used as scaling laws, this means formulas that can predict typical lifetimes for new experimental parameters, as is the case when bigger machines are planned.

Heuristically, one can obtain estimates of the particle lifetime by comparing it with the typical $\vec{B}$ diffusion time which arises from

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) - \frac{c^2}{4\pi} \nabla \times \left(\frac{\nabla \times \vec{B}}{\sigma} \right),$$

assuming that $\vec{B}$ – *decay* does not produce plasma motion ($\vec{v} \simeq 0$) and taking L as a typical scale-length, we obtain:

$$\frac{\partial \vec{B}}{\partial t} \sim -\frac{c^2}{4\pi\sigma L^2}\vec{B} \;\rightarrow\; \tau_R = \frac{4\pi\sigma L^2}{c^2},$$

which is a typical resistive time. It is usual to assume a Spitzer conductivity, for example in the from given by Braginskii [15]:

$$\sigma = \frac{0.9 \times 10^{14} T^{3/2}(eV)}{\ln \Lambda Z_{eff}} \; (sec_{-1}), \qquad (49)$$

where $Z_{eff} = \sum_j n_j Z_j^2 / n_e$ (the sum is extended to all ion species present in the plasma), and for $T_e > 50\ eV$

$$\ln \Lambda = 25.3 + \ln \left[T_e(eV)/\sqrt{n_e} \right].$$

Using (49) we can evaluate $\tau_R = 4\pi\sigma L^2/c^2$ for typical data from experiments. In general such estimates are quite larger ($\sim$ 20 to 100 times) than the experimentally measured values for particle or energy lifetimes. Therefore, one can assume that the discrepancy should be due to some anomaly in the conductivity. Anyway, better agreement with experiments, using Spitzer's conductivity is possible (especially in high β plasmas) if one use a little more rigorous method for computing, for example, particle lifetime. It will be shown that the topological characteristics of closed equilibria are important in reducing the anomaly factor of the conductivity.

4.1 STATIONARY MODEL

This section is based on papers by Kruskal and Kulsrud [16] and Auerbach and Condit [17].

Let us assume that the plasma configuration can reach a stationary state $((\frac{\partial}{\partial t}) \equiv 0)$ in which a constant mass (or particle) flux across magnetic surfaces exists, and also the total mass of the system remains constant thanks to some source of particles inside the plasma. Therefore, we can identify the typical particle diffusion e-decay time as

$$\tau_p = \frac{\mathcal{M}}{\Gamma_m}, \tag{50}$$

where $\mathcal{M}$ is the total mass of the configuration and Γ_m the mass flux that leaves the system. The model assumes that once the particles reach the plasma boundary, they promptly leave the system through open $\vec{B}$-lines (we can also assume the system to be in contact with a perfectly absorbing wall). τ_p can be interpreted as an instantaneous e-decay time.

The problem is therefore reduced to compute $\mathcal{M}$ and Γ_m. Due to the stationarity assumption, $\frac{\partial}{\partial t} \equiv 0$ it follows $\nabla \times \vec{E} = 0 \;\rightarrow\; \vec{E} = \nabla\phi$ with ϕ single-valued (if $\oint d\vec{\ell}\cdot\vec{E} \;\neq\; 0 \;\rightarrow\; \nabla\times\vec{E} \;\neq\; 0$). If we assume that the diffusion velocity is sufficiently small in order to neglect inertial effects in the momentum balance equation we can decouple the equations governing equilibrium from those governing diffusion. In this case we have

$$\rho\vec{v}\cdot\nabla\vec{v} + \nabla p = \frac{\vec{j}\times\vec{B}}{c},$$

(for toroidal equation it will result the Grad-Shafravov equation.) and $\vec{v}$ will appear only in Ohm's law

$$\vec{j} = \sigma\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right).$$

Γ_m will depend only on the $\vec{v}$-component $\parallel$ to ∇p, and we can write for each $p = constant$-surface

$$\Gamma_m = -\oint_{p=const} dS\rho\frac{\vec{v}\cdot\nabla p}{\mid\nabla p\mid}.$$

$-\vec{v}\cdot\nabla p$ can be obtained from Ohm's law by taking the scalar product with $\vec{j}$

$$\begin{aligned}
\vec{j}\cdot\vec{j} = j^2 &= \sigma\left(\vec{j}\cdot\vec{E} + \vec{j}\cdot\frac{\vec{v}\times\vec{B}}{c}\right)\\
&= \sigma\left(\vec{j}\cdot\nabla\phi - \vec{v}\cdot\frac{\vec{j}\times\vec{B}}{c}\right)\\
&= \sigma\left(\vec{j}\cdot\nabla\phi - \vec{v}\cdot\nabla p\right)\\
\rightarrow &\quad -\vec{v}\cdot\nabla p = \frac{j^2}{\sigma} - \vec{j}\cdot\nabla\phi,
\end{aligned}$$

and therefore

$$\Gamma_m = -\oint_{p=const} dS\rho\left(\frac{j^2}{\sigma\mid\nabla p\mid} - \frac{\vec{j}\cdot\nabla\phi}{\mid\nabla p\mid}\right).$$

Let us assume that ρ is a surface quantity. Writing $\mid\nabla p\mid = \frac{\delta p}{\delta\ell}$ it follows

$$\begin{aligned}
\oint_{p=const} dS\frac{\vec{j}\cdot\nabla\phi}{\mid\nabla p} &= \oint_{p=const} dS\delta\ell\frac{\vec{j}\cdot\nabla\phi}{\delta p} && taking\ \delta p = const\\
&= \frac{1}{\delta p}\oint_{Vol\{p,p+\delta p\}} d^3x\vec{j}\cdot\nabla\phi && (d^3x = dS\delta\ell)\\
&= \frac{1}{\delta p}\oint_{Vol\{p,p+\delta p\}} d^3x\nabla\cdot\vec{j}\phi\\
&= \oint_{\Sigma} dS\vec{n}\cdot\vec{j}\phi = 0
\end{aligned}$$

(since ϕ is single-valued and we have defined Σ as the simply connected surface enclosing Vol $\{p, p+\delta p\}$.)

Therefore Γ_m is:

$$\Gamma_m = \frac{\rho}{\sigma}\oint_{p=const} dS\frac{j^2}{\mid\nabla p\mid}, \tag{51}$$

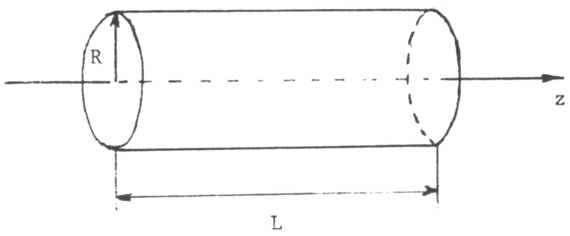


Figure 21:

where σ has also been assumed as a surface quantity (if p and ρ are surface quantities, T will also be). Γ_m will also be a surface quantity. If we assume that the source of particles is concentrated at the magnetic axis, owing to the stationarity, by hypothesis, Γ_m must be a constant (p^0) since we cannot admit rarefaction or compression of the plasma. In order to accomplish this, ρ and σ shall be proper functions of p. This in turn will allow us to obtain the functional dependence of ρ with p [or $\rho(\psi)$ if toroidal axisymmetric equilibria are considered] in order to compute $\mathcal{M}$ and obtain τ_p.

In order to understand better, let us give an example of application to FRC. Example[11].

Let us consider an axisymmetric FRC inside a cylindrical box of length L and radius R with $L >> R$. figure (21).

The first step is to find an equilibrium through solving the Grad-Shafranov equation inside the vessel. In order to simplify things we assume that ψ has only one extremum and also mirror symmetry around the center of the vessel at $z = 0$.

For an FRC $I^2 = 0$ and the Grad-Shafranov equation reduces to:

$$\nabla^* \psi = r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}\psi + \frac{\partial^2 \psi}{\partial z^2} = -16\pi^3 r^2 \frac{dp}{d\psi} = -\frac{8\pi^2 r}{c} j_\phi$$

j_ϕ is the only component of $\vec{j}$. If we assume $p = \frac{\alpha^2\psi^2}{32\pi^3}$ the equation transforms to $\nabla^*\psi = -\alpha^2 r^2 \psi$. Being a linear homogeneous partial differential equation for ψ we can propose a solution like:

$$\psi = F(r)\cos\left(\frac{\pi z}{L}\right),$$

which automatically satisfies the boundary condition $\psi = 0$ at $z = \pm L/2$. Using $x = r^2/R^2$ we obtain the following equation for F.

$$\frac{\partial^2 F}{\partial x^2} - \frac{\pi^2 R^2}{4L^2}\frac{F}{x} + \frac{\alpha^2 R^4}{4} F = 0, \tag{52}$$

which has to be solved requiring that $F(0) = F(1) = 0$ and that F has only one extremum in the internal $0 < x < 1$. It can be shown that (52) has an exact analytical solution:

$$F(x) = \mathcal{F}_0\left(\frac{\pi^2}{64L^2\alpha}, \frac{\alpha R^2 x}{2}\right)$$

where $\mathcal{F}_0$ is the regular Coulomb wave function of order zero, parameter $\eta = \frac{\pi^2}{64L^2\alpha}$ and argument $\frac{\alpha R^2 x}{2}$. For the present purpose we can restrict parameters to consider the case $R^2/L^2 << 1$. In this case it can be shown that $\eta << 1$ and that the Coulomb function can be very well approximated by $\sin(\pi x)$ [for $L/R \geq 6 \rightarrow \eta \leq 1/100$. $\mathcal{F}_0(1/100, 1.58) = 1.002$, corresponds to the first maximum and $\mathcal{F}_0(1/100, 3.17) = 0$ corresponds to the first zero]. This approximation allows us to compute Γ_m;

$$j^2 = j_\phi^2 = \mid 2\pi c r \frac{dp}{d\psi} \mid^2$$

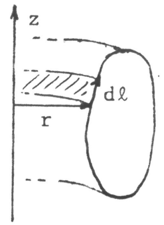
but $\frac{dp}{d\psi} = \frac{\alpha^2}{16\pi^3}\psi = \frac{4\pi^2}{16\pi^3 R^4}\psi$ and we have:

$$\frac{j^2}{|\nabla p|} = \frac{j_\phi^2}{|\frac{dp}{d\psi}||\nabla\psi|}$$
$$= \frac{4\pi^2c^2r^2\,|\frac{dp}{d\psi}|}{|\nabla\psi|} = \pi\frac{c^2r^2}{r^4}\frac{|\psi|}{|\nabla\psi|}$$

therefore Γ_m results (taking $\psi \geq 0$ for simplicity):

$$\Gamma_m = \frac{\rho}{\sigma}\frac{\pi c^2}{R^4}\psi\oint_{\psi=const} dS\frac{r^2}{|\nabla\psi|}$$

dS can be written as $2\pi r d\ell$, being $d\ell$ an element of a $\vec{B}$-line



From the magnetic field line equations

$$\frac{dr}{B_r} = \frac{dz}{B_z} \rightarrow d\ell = \sqrt{dr^2 + dz^2}$$
$$= dr\left(1 + \frac{B_z^2}{B_r^2}\right)^{1/2} = dr\frac{|\vec{B}|}{|B_r|}$$

but $|\vec{B}| = \frac{|\nabla\psi|}{2\pi r}$, so we have $\left(\vec{B} = \frac{\nabla\psi\times\vec{e}_\phi}{2\pi r}\right)$

$$\frac{dS}{|\nabla\psi|} = \frac{2\pi r d\ell}{|\nabla\psi|} = \frac{dr}{|B_r|} = \frac{2\pi r dr}{|\frac{\partial\psi}{\partial z}|}$$

Taking into account the equatorial symmetry we can write Γ_m as:

$$\Gamma_m = \frac{4\pi^2c^2}{\sigma R^4}\rho\psi\int_{r_1(\psi)}^{r_2(\psi)} dr\frac{r^3}{|\frac{\partial\psi}{\partial z}|},$$

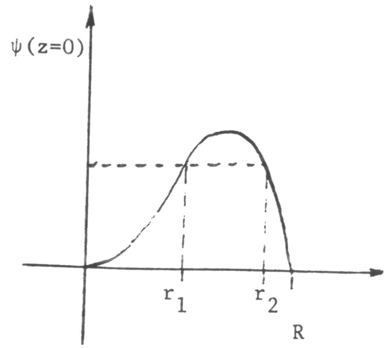


Figure 22:

where $\psi(r_1,0) = \psi(r_2,0)$, figura (22)

In the integrand we have to calculate $|\frac{\partial\psi}{\partial z}|$ for a constant ψ-surface. Using our solution $\psi = \psi_0\sin(\pi x)\cos(\frac{\pi z}{L})$, we have

$$\frac{\partial\psi}{\partial z} = -\frac{\pi}{L}\psi_0\sin(\pi x)\sin(\frac{\pi z}{L})$$

but $\cos(\frac{\pi z}{L}) = \frac{\psi}{\psi_0\sin(\pi x)}$ over a constant ψ-surface, so

$$\sin(\frac{\pi z}{L}) = \frac{\left[\psi_0^2\sin^2(\pi x) - \psi^2\right]^{1/2}}{\psi_0\sin(\pi x)}.$$

Finally we obtain

$$|\frac{\partial\psi}{\partial z}| = \frac{\pi}{L}\psi_0\left[\sin^2(\pi x) - \frac{\psi^2}{\psi_0^2}\right]^{1/2}$$

and Γ_m reduces to

$$\Gamma_m = \frac{4\pi^2c^2L\rho}{\sigma R^4}\frac{\psi}{\psi_0}\int_{r_1(\psi)}^{r_2(\psi)} dr r^3\left[\sin^2\left(\pi\frac{r^2}{R^2}\right) - \frac{\psi^2}{\psi_0^2}\right]^{-1/2},$$

using a new variable $w = \pi\left(\frac{r^2}{R^2} - \frac{1}{2}\right)$, Γ_m results:

$$\Gamma_m = \frac{2c^2}{\pi\sigma} L\rho \frac{\psi}{\psi_0} \int_{-w_1}^{w_1} dw (w + \frac{\pi}{2}) \left(\cos^2 w - \frac{\psi^2}{\psi_0^2}\right)^{-1/2}$$
$$= \frac{2c^2 L\rho}{\sigma} \frac{\psi}{\psi_0} \int_0^{w_1} dw \left[\left(1 - \frac{\psi^2}{\psi_0^2}\right)\left(1 - \frac{\sin^2 w}{1 - \frac{\psi^2}{\psi_0^2}}\right)\right]^{-1/2},$$

where the limit of integration is defined in such a way that $\sin^2 w_1 = 1 - \frac{\psi^2}{\psi_0^2}$.

We need further change of variables defined by $\sin^2\theta = \frac{\sin^2 w}{1 - \frac{\psi^2}{\psi_0^2}}$, in order to obtain:

$$\Gamma_m = \frac{2c^2 L\rho}{\sigma} \frac{\psi}{\psi_0} \int_0^{\pi/2} d\theta \left[1 - \left(1 - \frac{\psi^2}{\psi_0^2}\right)\sin^2\theta\right]^{-1/2}$$
$$= 2\frac{c^2 L\rho}{\sigma} \frac{\psi}{\psi_0} K\left(1 - \frac{\psi^2}{\psi_0^2}\right),$$

where K is the complete elliptic integral of the first kind and argument $1 - \frac{\psi^2}{\psi_0^2}$ (which varies between 0 and 1). For $\psi = \psi_0 \rightarrow K(0) = \pi/2$, for $\psi = 0$ $K(1) = \infty$.

In order to Γ_m be a constant, independent of ψ, we have to recall

$$\frac{\rho}{\sigma}\frac{\psi}{\psi_0} K\left(1 - \frac{\psi^2}{\psi_0^2}\right) \equiv \left(\frac{\psi}{\psi_0}\right)^0$$

If we assume a Spitzer's like conductivity, $\sigma \propto T_e^{3/2}$, and that $T_e \propto T_e + T_i = T$ is a surface quantity, since $p \propto \rho T \propto \psi^2$, it follows

$$\frac{\rho}{\sigma} \propto \frac{p}{T^{5/2}} \propto \frac{1}{\psi K\left(1 - \frac{\psi^2}{\psi_0^2}\right)} \rightarrow T \propto \left[\psi^3 K\left(1 - \frac{\psi^2}{\psi_0^2}\right)\right]^{2/5}$$

so we can make $\Gamma_m = \pi c^2 L\rho_0/\sigma_0$ constant by choosing:

$$\rho = \rho_0 \left(\frac{\psi}{\psi_0}\right)^{4/5} \left[\frac{2}{\pi} K\left(1 - \frac{\psi^2}{\psi_0^2}\right)\right]^{-2/5}, \text{ and}$$

$$\sigma = \sigma_0 \left[\left(\frac{\psi}{\psi_0}\right)^3 \frac{2}{\pi} K\left(1 - \frac{\psi^2}{\psi_0^2}\right)\right]^{3/5},$$

being $\rho_0 = \rho(\psi = \psi_0)$ and $\sigma_0 = \sigma(\psi = \psi_0)$ the maximum values of ρ and σ. It can be seen that, owing to the logarithmic divergence of K, ρ and σ vanish at the separatrix. Moreover, we can see that

$$\rho_0 \left(\frac{\psi}{\psi_0}\right)^{6/5} \leq \rho \leq \rho_0 \left(\frac{\psi}{\psi_0}\right)^{4/5}$$

and we can take as an approximation $\rho = \rho_0 \frac{\psi}{\psi_0}$, this allows us to compute $\mathcal{M}$ immediately:

$$\mathcal{M} = \rho_0 4\pi \int_0^R drr \int_0^{L/2} dz \sin\left(\pi \frac{r^2}{R^2}\right) \cos\left(\pi \frac{z}{L}\right)$$
$$= \rho_0 \frac{4R^2 L}{\pi},$$

and finally $\tau_p = \mathcal{M}/\Gamma_m$ results:

$$\tau_p = 0.4 \frac{\sigma_0 R^2}{c^2}$$

(Calculating $\mathcal{M}$ with the correct expression for ρ and using the Coulomb function would give, for $L/R \geq 6$, $\tau_p = 0.41\sigma_0 R^2/c^2$).

If we compare τ_p with heuristic τ_R taking as a characteristic length $R/2 \rightarrow \tau_R = \pi\sigma_0 R^2/c^2$, almost 8 times larger than τ_p.

Let us now make some comparison with experiments. τ_p has been measured for two distinct machines at Los Alamos, the FRX-B and FRX-C experiments, whose characteristics are summarized above.

Using formula (49) for σ_0 , and applying our formula to these parameters, we obtain:

.	FRX-B	FRX-C
L (cm)	50	150
R (cm)	5.4	10
$n_{e,max}(cm^{-3})$	3.5×10^{15}	5.0×10^{15}
T_e (eV)	110	100
T_i (eV)	200	250
τ_p (μsec)	39 ± 15	187 ± 25

$$\tau_p \simeq \frac{115}{Z_{eff}}\ \mu sec\ ,\ for\ FRX-B$$
$$\tau_p \simeq \frac{340}{Z_{eff}}\ \mu sec\ ,\ for\ FRX-C$$

Z_{eff} has not been measured in the experiments, but it is quite probable that $Z_{eff} \sim 1$ (deuterium was used as a filling gas). With $Z_{eff} \simeq 1$ the calculated times are approximately 2 times greater than measured ones. This allows to infer some anomaly in the conductivity. However the model has several faults; first, quite special T profiles are necessary to get a consistent result; second, the model does not allow to compute, for example, the magnetic flux decay (energy decay can be computed in a way similar to particle decay); third, some singularity in the diffusion velocity always occurs (for example, at the 0-point and at the separatrix). In the following section we will consider a somewhat better model to calculate non-stationary decay in FRC. The model applies only to FRC and it is an extension of Grad/Hogan theory [18] for slowly diffusing plasmas, originally formulated to treat plasma decay in tokamaks.

4.2 FRC GRAD-HOGAN DECAY

Let us now relax the stationarity assumption ($\frac{\partial}{\partial t} \neq 0$) but still neglect inertial effects in the momentum balance equation. Moreover, in the heat balance equation we neglect thermal conductivity, assuming a uniform but time dependent plasma temperature, and also neglect radiation losses. We keep only Joule's heating and

compressibility. We also restrict the analysis to FRC which implies configurations with only azimuthal current and assume that no plasma sources exist inside the plasma [19].

With these assumptions the plasma will be described by the following equations:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0, \qquad (53)$$
$$\nabla p = \frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi}, \qquad (54)$$
$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \qquad (55)$$
$$\nabla\times\vec{B} = \frac{c\sigma}{4\pi}\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right), \qquad (56)$$
$$p = \frac{\rho kT}{m_e + m'} \qquad (T = T_e + T_i), \qquad (57)$$
$$\frac{3}{2}\left(\frac{\partial p}{\partial t} + \vec{v}\cdot\nabla p\right) + \frac{5}{2}p\nabla\cdot\vec{v} = \frac{c^2}{16\pi^2\sigma}\,|\nabla\times\vec{B}|^2, \qquad (58)$$

where σ is assumed a scalar quantity. Assuming axisymmetry we can once more describe $\vec{B}(r,z,t)$ through a poloidal magnetic flux function $\psi(r,z,t)$ as:

$$\vec{B} = -\frac{\vec{e}_\phi\times\nabla\psi}{2\pi r}$$

and (54) will transform to the classical Grad-Shafranov equation

$$r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}\psi + \frac{\partial^2\psi}{\partial z^2} = -16\pi^3 r^2\frac{\partial p}{\partial\psi} = -\frac{8\pi^2 r}{c}j_\phi$$

where, instead of $\frac{dp}{d\psi}$ we have written $\frac{\partial p}{\partial\psi}$ since now $p(r,z,t) = p(\psi,t)$. Since $T(t)$ also $\rho(r,z,t) = \rho(\psi,t)$, and this allow to write terms like $\vec{v}\cdot\nabla(\rho\ or\ p)$ as $\frac{\partial}{\partial\psi}(\rho\ or\ p)\vec{v}\cdot\nabla\psi$. So, by using (53) and (58) it will result:

$$\frac{3}{2}\left(\frac{\partial p}{\partial t}+\frac{\partial p}{\partial \psi}\vec{v}\cdot\nabla\psi\right)-\frac{5}{2}p\left(\frac{1}{\rho}\frac{\partial \rho}{\partial t}+\frac{1}{\rho}\frac{\partial \rho}{\partial \psi}\vec{v}\cdot\nabla\psi\right)=\frac{c^2}{16\pi^2\sigma}\mid\nabla\times\vec{B}\mid^2, \tag{59}$$

but $\frac{1}{\rho}\frac{\partial\rho}{\partial\psi}=\frac{1}{p}\frac{\partial p}{\partial\psi}$ and $\frac{1}{\rho}\frac{\partial\rho}{\partial t}=\frac{1}{p}\frac{\partial p}{\partial t}-\frac{\dot{T}}{T}$, moreover from the Grad-Shafranov equation $\frac{c}{4\pi}(\nabla\times\vec{B})_\phi=2\pi rc\frac{\partial p}{\partial\psi}$, and (59) transforms to ($\dot{T}=\frac{dT}{dt}$)

$$-\left(\frac{\partial p}{\partial t}+\frac{\partial p}{\partial \psi}\vec{v}\cdot\nabla\psi\right)+\frac{5}{2}p\frac{\dot{T}}{T}=\frac{4\pi^2r^2c^2}{\sigma}\left(\frac{\partial p}{\partial\psi}\right)^2. \tag{60}$$

Equation (60) can further be reduced by using (55) and (56). Since

$$\vec{B}=\nabla\times\frac{\psi}{2\pi r}\vec{e}_\phi\ \rightarrow\ \vec{E}=-\frac{1}{2\pi rc}\frac{\partial\psi}{\partial t}\vec{e}_\phi$$

and Ohm's law can be written as

$$-\frac{4\pi^2r^2c^2}{\sigma}\frac{\partial p}{\partial\psi}=\frac{\partial\psi}{\partial t}+\vec{v}\cdot\nabla\psi,$$

where we used

$$\frac{\vec{v}\times\vec{B}}{c}=\vec{v}\times\frac{(\nabla\psi\times\vec{e}_\phi)}{2\pi rc}=-\vec{e}_\phi\frac{\vec{v}\cdot\nabla\psi}{2\pi rc},\ \textit{assuming}\ v_\phi=0.$$

Therefore (60) reduces to:

$$\frac{\partial p}{\partial t}-\frac{\partial p}{\partial\psi}\frac{\partial\psi}{\partial t}=\frac{5}{2}p\frac{\dot{T}}{T}. \tag{61}$$

This is a very workable relation. Le us see why. Since the most general expression for p will be:

$$p=\sum_{m=0}^{\infty}p_m(t)\psi^m.$$

it follows

$$\frac{\partial p}{\partial t}=\sum_{m=0}^{\infty}\left[\dot{p}_m(t)\psi^m+mp_m(t)\psi^{m-1}\frac{\partial\psi}{\partial t}\right],$$

and (61) transforms to:

$$\sum_{m=0}^{\infty}\dot{p}_m\psi^m=\frac{5}{2}\frac{\dot{T}}{T}\sum_{m=0}^{\infty}p_m\psi^m.$$

Equating the powers of ψ it results:

$$p(\psi,t)=\left(\frac{T}{T_0}\right)^{5/2}\sum_{m=0}^{\infty}k_m\psi^m, \tag{62}$$

where $T_0=T(0)$ and the k_m's are constants independent of time defined in such a way that

$$p[\psi(r,z,0),0]=\sum_{m=0}^{\infty}k_m\psi(r,z,0).$$

Equation (62) tells us, that for uniform T, plasma diffusion does not change the functional dependence of the pressure with ψ. This allows to use a family of equilibria (where a family corresponds to a certain choice for the k_m's) satisfying proper boundary conditions for describing the continuous sequence of equilibrium states that the system will reach during diffusion. In general, each equilibrium of a given family will be characterized by one proper parameter, and the problem will be solved by finding the temporal behaviour of such parameter. Therefore, we need a temporal equation for the proper parameter. Such equation arises from asking that the diffusion velocity has no singularities inside the plasma. From Ohm's law we get, $j_\phi=\sigma(E+\frac{v\times B}{c})\rightarrow -\frac{4\pi^2r^2c^2}{\sigma}\frac{\partial p}{\partial\psi}=\frac{\partial\psi}{\partial t}+v\cdot\nabla\psi$

$$\frac{\vec{v}\cdot\nabla\psi}{\mid\nabla\psi\mid}=-\frac{1}{\mid\nabla\psi\mid}\left(\frac{4\pi^2r^2c^2}{\sigma}\frac{\partial p}{\partial\psi}+\frac{\partial\psi}{\partial t}\right),$$

in order to avoid singularities at the O-point we must recall,

$$\left(\frac{4\pi^2r^2c^2}{\sigma}\frac{\partial p}{\partial\psi}+\frac{\partial\psi}{\partial t}\right)\Big|_{O-point}=0. \tag{63}$$

Since ψ at the O-point will be a function of the proper parameter, equation (63) will be a temporal equation for it. Of course, this is valid if the FRC presents only one O-point. It can be seen that $r = 0$ does not implies singularity for $\vec{v}$. Other singularities can arise if $\nabla\psi$ vanishes (or $\vec{B}$ vanishes) on the separatrix far from $r = 0$, but for our purpose we can disregard such possibility.

In order to get an idea of the method let us apply it to the

4.2.1 Spherical Hill's Vortex

In this case we can use the following family of equilibria:

$$\psi_{int} = \frac{3\pi B r^2}{2}\left(1 - \frac{r^2}{a^2} - \frac{z^2}{a^2}\right) \;;\; \psi_{ext} = \pi B r^2 \left[1 - \frac{a^3}{(r^2+z^2)^{3/2}}\right],$$

correspondingly $p = \frac{15B}{16\pi^2 a^2}\psi_{int}$ inside the separatrix and $p = 0$ outside. In this case in order to maintain $p = 0$ outside the separatrix we must introduce a sink at the separatrix. This may reflect the fact that, once particles reach the external region, they are lost along $\vec{B}$-lines on a typical thermal transit time, which can be assumed $<<$ typical decay time.

In principle B and a may be time-dependent, but if we assume that far from the separatrix $\vec{B} = -B\vec{e}_z$ remain constant during diffusion, only a will depend on t. Therefore a will be the proper parameter. Since $\frac{\partial p}{\partial \psi} \propto a^{-2}$, from equation (62) it follows $T \propto a^{-4/5}$.

In order to quantify equation (63) we need σ, if we assume T_e/T_i independent of time and a Spitzer-like conductivity, it follows essentially $\sigma = \sigma_0(T/T_0)^{3/2} = \sigma_0(a/a_0)^{-6/5}$, where the subscript 0 indicates initial values. Since the O-point is located at $r = a/\sqrt{2}$; $z = 0$, equation(63) reads:

$$\frac{2\pi^2 a^2}{\sigma_0}\left(\frac{a}{a_0}\right)^{6/5}\frac{15B}{16\pi a^2} + \frac{3\pi B a\dot{a}}{4} = 0,$$

$$\frac{\dot{a}}{a_0} = -\frac{5c^2}{2\pi\sigma_0 a_0^2}\left(\frac{a}{a_0}\right)^{1/5} \rightarrow a = a_0\left(1 - \frac{2}{5}\frac{t}{\tau_R}\right)^{5/4},$$

being $\tau_R = \pi\sigma_0 a_0^2/5c^2$ a typical resistive time of the initial comfiguration. a determines the behaviour of all quantities of interest, for example,

$$\psi\,|_{O-point} = \psi_0\left(1 - \frac{2}{5}\frac{t}{\tau_R}\right)^{5/2},$$

where $\psi_0 = 3\pi B a_0^2/8$. This implies a e-decay time $\tau_\phi = 0.825\tau_R$. Analogously the e-decay time of the number of confined particles and plasma thermal energy can be deduced (the calculation is left as an exercise):

$N = N_0\left(1 - 0{,}4\frac{t}{\tau_R}\right)^{19/4}$

$E = E_0\left(1 - 0{,}4\frac{t}{\tau_R}\right)^{15/4}$ $\tau_p = 0.475\tau_R$; $\tau_E = 0.585\tau_R$.

The stationary model would give $\tau_p = 0.7\tau_R$ [17] and σ_0 has to be interpreted as the conductivity at the O-point. This Grad-Hogan model gives much more information, since besides τ_ϕ, *and* τ_E the evolution of the separatrix is obtained, with less stringent assumptions than in the case of the stationary model. The model has also been applied to the prolate Hill's vortex, but the algebra is quite involved and also numerical integration is needed. This model is also amenable of application to numerically generated familes of equilibria.

$p_{max} \propto \frac{\psi_{max}}{a^2} \propto \frac{a^{+2}}{a^2}$ $\quad\int d^3x\, p \propto a^{+3} \propto a^3$

$N \propto \int d^3x \frac{p}{T} \propto \frac{a^3}{T} \propto \frac{a^3}{a^{-4/5}} \propto a^{19/5}$

Acknowledgement

This work was supported by Fundação de Amparo à Pesquisa do Estado de São Paulo, FAPESP, and Conselho Nacional de Desenvolvimento Científico e Tecnológico, CNPq (Brazil).

References

[1] Alexandroff, P. and Hopf, H., *Topologie*, Springer, Berlin, (1935), p. 552.

[2] Maschke, E. K. and Perrin, H., Plasma Phys. **22** (1980), 579.

[3] Ferraro, Monthly Notices of the Roy. Astr. Soc. **97** (1937) 458.

[4] Pantuso-Sudano, J., *Proc. of the 1st Latin-American Workshop on Plasma Physics and Controlled Fusion Research*, Soc. Bras. de Física, 1982, vol1, 264

[5] Clemente, R.A., and Farengo, R., Phys. Fluids **27** (1984) 776.

[6] Rosenbluth, M.N., and Bussac, M. Nuclear Fusion **19** (1979) 489.

[7] Shestakov, A.J. et al., Proc. of US-Japan Joint Symposium on Compact Toruses and Energetic Particles Injection, Princeton 1979,p. 126.

[8] Barnes, D. C. et al., Proc. of the 3rd Symposium on the Physics and Technology of Compact Toroids in the Magnetic Fusion Energy Program, Los Alamos 1980, p. 134.

[9] Clemente, R.A. and Milovich, J. L., Phys. Letters **85A** (1981) 148.

[10] Bernstein, I.B. et al., Proc. Royal Soc. **A244** (1958), 17.

[11] Clemente, R.A. and Grillo, C.E., Phys. Fluids **27** (1984) 658.

[12] Clemente, R. A. and Milovich, J.L., Phys. Fluids **26** (1983) 1874.

[13] Barnes,D.C. et al., Phys. Fluids **29** (1986) 2616.

[14] Ishida, A. et al., Phys. Fluids **31** (1988) 3024.

[15] Braginskii,S.I. in *Review of Plasma Physics* (Consultants Brurreau, New York, 1965) **1**, 205 (1965).

[16] Kruskal, M.D. and Kulsrud, R.M., Phys. Fluids **1** (1958) 265.

[17] Auerbach, S.P. and Condit, W.C., Nucl. Fusion **11** (1981) 927.

[18] Grad, H., and Hogan, J., Phys. Rev. Lett. **24** (1970) 1337.

[19] Clemente, R. A., Sakanaka, P.H., and Mania, A.J.. Nuclear Fusion **29** (1989) 1339.