3.3. *Derivatives*

$$\frac{d}{dx}\,\mathrm{ccos}\,(x)=C^{k}_{2n}\,\mathrm{scos}\,(x)-S^{k}_{2n}\,\mathrm{csin}\,(x) \qquad \frac{d}{dx}\,\mathrm{scos}\,(x)=C^{k}_{2n}\,\mathrm{ccos}\,(x)-S^{k}_{2n}\,\mathrm{ssin}\,(x)$$

$$\frac{d}{dx}\,\mathrm{csin}\,(x)=C^{k}_{2n}\,\mathrm{ssin}\,(x)+S^{k}_{2n}\,\mathrm{ccos}\,(x) \qquad \frac{d}{dx}\,\mathrm{ssin}\,(x)=C^{k}_{2n}\,\mathrm{csin}\,(x)+S^{k}_{2n}\,\mathrm{scos}\,(x)$$

$$\mathrm{ccos}\,(x)=\sum_{m=0}^{\infty}\frac{C^{mk}_{2n}}{m!}x^{m}$$

Further properties, analogous to those presented above for the circular functions are left as exercises for the interested reader.

4. Conclusion

For an extensive literature on the higher-order circular functions, the interested reader is referred to Kaufman [6]. In closing this note we suggest a further similar study of the parabolic functions. Eastham [7] and Deakin [8] have studied systems of dual numbers

$$z=x+\varepsilon y \qquad \varepsilon^2=0$$

(in which division by multiples of ε is impossible) to get parabolic functions which are related to circular functions

$$\mathrm{cosp}\,\theta=\cos\varepsilon\theta \qquad \varepsilon\,\mathrm{sinp}\,\theta=\sin\varepsilon\theta$$

by analogy with

$$\cosh\theta=\cos i\theta \qquad i\sinh\theta=\sin i\theta$$

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Some remarks on the two-dimensional Liouville's equation

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(*Received 2 July 1991*)

Using the method of finding solutions of the two-dimensional Liouville's equation in terms of solutions of Laplace's equation, new expressions that allow the use of real Laplace solutions are presented.

The two-dimensional Liouville's equation [1]

$$F_{xx}+F_{yy}=a\exp(bF) \tag{1}$$

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where F is a real function, a and b are real constants, applies to large numbers of problems in different branches of physics [2–6]. Its general solution, originally formulated by Liouville [1], has been the object of study by several mathematicians [7–11]. However, practically no modifications to the original expressions by Liouville were found. Here it is shown that the general solution to this equation can be expressed in different ways that, perhaps, may be more suitable in particular applications.

By means of the transformation

$$\chi=\frac{ab}{|ab|}(bF+\ln|ab|) \tag{2}$$

equation (1) transforms to

$$\chi_{xx}+\chi_{yy}=\exp\left(\frac{ab}{|ab|}\right)\chi=|ab|\exp(bF) \tag{3}$$

from which two general cases can be distinguished:

$$\chi_{xx}+\chi_{yy}=\exp(\chi) \tag{4}$$

and

$$\chi_{xx}+\chi_{yy}=\exp(-\chi) \tag{5}$$

Which case occurs in practice depends on the physical problem to be treated. Since χ must be real it follows that $\exp(\pm\chi)$ must be real and positive.

If ϕ is a solution of the two-dimensional Laplace equation, i.e. $\phi_{xx}+\phi_{yy}=0$, it follows also

$$[\ln(\nabla\phi)^2]_{xx}+[\ln(\nabla\phi)^2]_{yy}=0 \tag{6}$$

This remarkable property allows a solution for χ to be formulated as

$$\exp(\pm\chi)=G(\phi)(\nabla\phi)^2 \tag{7}$$

or

$$\chi=\pm[\ln G(\phi)+\ln(\nabla\phi)^2] \tag{8}$$

where G is a function of ϕ to be determined consistently.

Placing equation (8) with the + sign into equation (4) and taking into account equation (7) yields the following equation for G:

$$\left(\frac{G_\phi}{G}\right)_\phi=G \tag{9}$$

whose solution, generally considered in the literature, is

$$G=\frac{2}{\phi^2} \tag{10}$$

This allows the use of real ϕ directly in equations (7) and (8) in order to obtain real ϕ and positive real $\exp(\chi)$. However

$$G=\frac{2}{\sinh^2\phi} \tag{11}$$

is also a solution of equation (9) that allows the use of real ϕ in equations (7) and (8), and it yields another general solution to equation (4).

Placing equation (8) with the − sign into equation (5) and considering equation (7) yields

$$\left(\frac{G_\phi}{G}\right)_\phi=-G \tag{12}$$

whose solution, generally considered in the literature, is

$$G=-\frac{2}{\phi^2} \tag{13}$$

But now real ϕ cannot be used directly in equations (7) and (8) in order to get positive real $\exp(-\chi)$ and real ϕ. This can be accomplished allowing for complex ϕ of the form [2–3]

$$\phi=g+ih+(g-ih)^{-1} \tag{14}$$

where $g(x, y)$ and $h(x, y)$ are real conjugate functions resulting from $f(x+iy)=g+ih$, with f a sufficiently differentiable arbitrary function. The resulting $\exp(-\chi)$ is:

$$\exp(-\chi)=\frac{8(\nabla g)^2}{(g^2+h^2+1)^2} \tag{15}$$

However, another solution of equation (12) that allows the use of real ϕ in expressions (7) and (8) is:

$$G=\frac{2}{\cosh^2\phi} \tag{16}$$

which yields an alternative expression for the general solution of equation (5).

Solutions resulting from expressions (11) and (16) offer the possibility of constructing other new interesting expressions for the solutions of equations (4) and (5) simply by replacing ϕ with $\ln(\nabla\phi)^2$. In this case the following expressions can be obtained:

$$\exp(\pm\chi)=\frac{[8\nabla(\nabla\phi)^2]^2}{[(\nabla\phi)^4\mp 1]^2} \tag{17}$$

where ϕ is still a real solution of Laplace's equation and upper or lower signs must be taken at the same time.

New expressions for the general solution of the two-dimensional Liouville's equation have been presented. As in previous solutions, they are still expressed in terms of solutions of Laplace's equation, however they allow the use of real Laplace solutions directly. Whether the new expressions are more convenient in specific problems than the old ones, is a matter that should be solved by the imagination of applied physicists. In this respect it should be noted that existing azimuthally symmetric radial solutions [2–3] can be easily recovered using $\phi\propto\ln(x^2+y^2)$ when expressions (11) or (16) are employed, or $\phi\propto r^n\cos n\theta$ when expression (17) is used.

Acknowledgments

This work was partially supported by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico, Brazil).

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