Resistive internal kink modes

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For realistic parameters of magnetically confirmed plasmas the growth rates on internal kink modes, resulting from the effects of finite electrical resistivity, can be considerably larger than those evaluated from the idealized MHD theory. In addition, magnetic field lines can reconnect and produce configurations with magnetic islands.

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We consider, for simplicity, a magnetically confined, current carrying cylindrical plasma column of length $2\pi R$, in which $B_z(r)$ and $B_\theta(r)$ are the axial and azimuthal components of the magnetic field. The so-called internal kink mode is characterized by an azimuthal wave number $m = 1$ and by leaving the plasma boundary unperturbed. This mode can be excited whenever there is a region inside the plasma column where the value of the quantity $q(r) \equiv rB_z(r)/[RB_\theta(r)]$ falls below unity within the region, and increases monotonically outside it to a value above unity at the plasma boundary.[1]

The experimental work reported in Ref. 2 has confirmed that $|m| = 1$ oscillations do occur inside a toroidal plasma column when the value of q(r) falls below unity at the magnetic axis, and that the radial mode structure is in reasonable agreement with the theoretical predictions. However, the ideal MHD treatment of this mode including a nonlinear analysis of it[3] is not sufficient to explain other important features of the observed oscillations such as their frequency and the values of their amplitudes.

We recall that the ideal MHD theoretical growth rate of the internal kink mode, γ_{MHD}, is considerably smaller than that of the "free-boundary" kink mode, as $\gamma_{MHD} \sim (r_0/R)^2/\tau_H$, where $\tau_H \equiv r_0/v_{A\theta}$, $v_{A\theta}^2 \equiv [B_\theta^2/(4\pi\rho)] \times (q' r_0/q)^2$, $r = r_0$ is the surface at which $q(r) = 1$, ρ is the mass density, $q' = dq/dr$, and $v_{A\theta}$ is evaluated at r_0. In considering the effects of finite electrical resistivity η we define a magnetic diffusion coefficient $D_m \equiv \eta c^2/4\pi$, a resistive diffusion time $\tau_R \equiv r_0^2/D_m$, and a parameter $\varepsilon \equiv \tau_H/\tau_R$ that is taken to be small. Then on the basis of the general results obtained in Ref. 4, we expect the growth rate of this mode to be of order $\gamma \sim \varepsilon^{1/3}/\tau_H$ whenever $\gamma_{MHD}\tau_H \lesssim \varepsilon^{1/3}$, a condition that is frequently realized in current experiments.

In addition the effects of resistivity lead to a different topology of the perturbed magnetic field that is consistent with the formation of magnetic islands. The linear resistive treatment of this mode is still not sufficient to account for the frequencies and amplitudes of the observed oscillations but is the necessary first step in this direction and in order to establish the relevance of the m = 1 mode[3] to the onset of the so-called disruptive instability.[5]

We refer to the following linearized equations for small perturbations of the considered cylindrical plasma column:

$$\rho\frac{\partial}{\partial t}\mathbf{v}_1 = -\nabla p_1 + \frac{1}{c}\mathbf{J}_1\times\mathbf{B} + \frac{1}{c}\mathbf{J}\times\mathbf{B}_1,$$

$$\frac{\partial}{\partial t}\mathbf{B}_1 = \nabla\times(\mathbf{v}_1\times\mathbf{B}) - \eta c\nabla\times\mathbf{J}_1,$$

where the subscript 1 indicates perturbed quantities, $\mathbf{J}_1 = (\nabla\times\mathbf{B}_1)c/(4\pi)$, $\mathbf{J} = (\nabla\times\mathbf{B})c/4\pi$, and the resistivity η has been considered as constant for simplicity. In particular, we look for normal mode solutions of the form $\mathbf{v}_1(t, r, \theta, z) = \mathbf{v}_1(r)\exp[\gamma t + i(m\theta + kz)]$, where $k \equiv 1/R$; R is the major radius of the toroidal configuration that is simulated. Then, following a procedure outlined in Ref. 6, in order to reduce the stability problem to a couple of equations in the radial components v_{1r} and B_{1r}, we apply the operator $\mathbf{B}\cdot\nabla\times$ to the first of the equations given above and take the radial component of the second. We recall that $(kr)^2 = (r/R)^2 \ll m^2$ and obtain

$$4\pi\gamma\left[\frac{1}{r}\frac{d}{dr}\rho r^3\frac{dv_{1r}}{dr} - \rho(m^2-1)v_{1r}\right] = irF\left[\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}(rB_{1r}) - \frac{1}{Fr^2}\left\{m^2G + \frac{d}{dr}r\frac{d}{dr}(rF)\right\}(rB_{1r})\right], \quad (1)$$

$$\gamma B_{1r} = iFv_{1r} + \frac{\eta c^2}{4\pi}\left[\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{dB_{1r}}{dr}\right) - \frac{m^2-1}{r^2}B_{1r}\right]. \quad (2)$$

The quantities F and G are defined as

$$F \equiv mB_\theta/r + kB_z, \quad (3)$$

$$G \equiv F\frac{m^2+k^2r^2-1}{m^2} + 2\frac{k^2r^2}{m^4}\left(kB_z - \frac{m}{r}B_\theta\right). \quad (3')$$

In the limit of zero resistivity we obtain simply[7]

$$\frac{d}{dr}\left[r^3(4\pi\rho\gamma^2+F^2)\frac{d\xi}{dr}\right] - g\xi = 0 \quad (4)$$

for $m = -1$ modes, where we have defined $v_{1r} = \gamma\xi$, $g \equiv FGr$,

$g/B_\theta \sim (kr)^2 F, \quad F = -B_\theta / r[1-q(r)]$.

The solution ξ of Eq. (4) is expanded in $(kr)^2$; $\xi = \xi_0 + \xi_1 + \ldots$. Then ξ_0 is taken to be a nonzero constant inside the region where $F(r) > 0$, $[q(r) < 1]$ and zero in the region where $F(r) < 0$, $[q(r) > 1]$. The discontinuity in ξ_0 is allowed because, neglecting the inertia term $\rho\gamma^2$, the point r_0, where $F(r_0) = 0$, is a singular point of Eq. (4). The first order solution ξ_1 is made to join smoothly the constant and zero solutions on each side of the surface $r = r_0$ through an "inner" layer where $4\pi\rho\gamma^2 \sim F^2$. The solution outside this layer is

$$\xi = \xi_\infty = \text{const}; \quad \frac{d\xi_1}{dr} = \frac{\xi_\infty}{r^3 F^2} \int_0^r g\, dr \quad \text{for} \quad r < r_0$$

and

$$\xi_0 = 0; \quad \frac{d\xi_1}{dr} = \frac{\xi_\infty}{r^3 F^2} \int_0^{r_0} g\, dr \quad \text{for} \quad r > r_0.$$

Since the parameter $\varepsilon \equiv \tau_H/\tau_R$ is small under realistic conditions, the resistive term in Eq. (2) tends to become important only within the considered "inner" layer. Then we may write the equations for this layer in the form

$$\frac{d^2\xi}{dx^2} = \frac{x}{\lambda^2} \frac{d^2\psi}{dx^2}, \tag{5}$$

$$\psi = -\xi x + \frac{\varepsilon}{\lambda} \frac{d^2\psi}{dx^2}, \tag{6}$$

where $x \equiv (r - r_0)/r_0$, $\psi \equiv iB_{1r}/(F' r_0^2)$, $F' = (dq/dr)B_\theta/r_0$, and $\lambda \equiv \gamma\tau_H$. We notive that the "resistive" ordering, for which all terms in Eqs. (5) and (6) are comparable, is

$$\psi/\xi \sim x \sim \lambda \sim \varepsilon^{1/3}. \tag{7}$$

We look for solutions of Eqs. (5) and (6) which can match smoothly the relevant MHD solutions outside the inner layer. In particular, if we reduce Eqs. (5) and (6) to a fourth order equation in ξ, there are four independent solutions with different asymptotic behavior as $|x| \to \infty$. One solution is ξ = constant, two others behave as exp $[\pm x^2/(4\varepsilon\lambda)^{1/2}]$, and the last solution is such that $d\xi/dx \approx (\text{const.})/x^2$ as $|x| \to \infty$. We are interested in the last solution that we write in the form $\xi = {}^1\!/_2\xi_\infty + \xi_{odd}(x)$, where the function ξ_{odd} is odd in x. Notice that the form of the solution outside the inner layer implies that

$$\frac{1}{\xi_\infty} \frac{d\xi}{dr}\bigg|_{r \to r_0} = -\frac{1}{\pi} \frac{\lambda_H}{x^2}, \tag{8}$$

where

$$\lambda_H = -\frac{\pi}{(B_\theta q r)^2_{r=r_0}} \int_0^{r_0} g(r)\, dr \leqslant \left(\frac{r_0}{R}\right)^2.$$

Thus we impose that the inner solution satisfy the condition

$$\frac{x^2}{2} \frac{d}{dx} (\ln \xi_{odd}) = -\frac{1}{\pi} \lambda_H \tag{9}$$

for $x/\lambda \to -\infty$ as $\xi_{odd} \to \xi_\infty/2$.

The solutions of Eqs. (5) and (6) that are shown in Fig. 1 have been obtained numerically, after rewriting the same equation with the variables $\hat{x} \equiv x/\varepsilon^{1/3}$, $\hat{\psi} \equiv \psi/\varepsilon^{1/3}$, and $\hat{\lambda} = \lambda/\varepsilon^{1/3}$, fixing a value of $\hat{\lambda}$ and imposing the conditions $\xi_{odd}(0) = 0$, $\hat{\psi}(0) = 1$. Then $d\xi_{odd}/dx$ for $x = 0$ has been varied until the condition (9) has been met. We have followed closely the procedure described in Ref. 6 and confirmed the result predicted in general in Ref. 4. In particular, we notice that for $\lambda_H \gg \varepsilon^{1/3}$ we have $\lambda \simeq \lambda_H$ as λ_H is proportional to the purely MHD growth rate. In the opposite extreme corresponding to MHD marginal stability, $\lambda_H = 0$, we have $\hat{\lambda} = 1$, that is, $\lambda = \varepsilon^{1/3}$. We recall[2] that the integral $\int g(r)dr$ is negative and, therefore, $\lambda_H > 0$, for any current density distribution monotonically decreasing from the magnetic axis and such that $q(r = 0) < 1$.

In Fig. 1 we have also reported solutions corresponding to $\hat{\lambda} < 1$ and, therefore, to $d\ln\xi_{odd}/d\hat{x} > 0$ for $\hat{x} \to -\infty$. These can be meaningful to the extent that λ_H can be made negative, while $q(0) < 1$, by considering additional effects that add stabilizing terms to $\int g(r)dr$.

It is possible to reproduce analytically these results by introducing the even function

$$\chi(x) = x\frac{d\psi}{dx} - \psi = \lambda^2 \frac{d\xi}{dx} + \chi_\infty,$$

where χ_∞ = const. Then

$$\xi = \lambda^{-2} \int_x^\infty dx(\chi - \chi_\infty) \quad \text{and} \quad \psi = -\chi - x\int_x^\infty dx \frac{d\chi}{dx}\frac{1}{x}.$$

In addition

$$\xi_\infty \simeq 2\int_0^\infty \frac{d\chi}{dx}\frac{dx}{x} \quad \text{and} \quad \frac{d\xi}{dx} \to -\frac{\chi_\infty}{x^2}.$$

Thus condition (9) reduces to

$$\chi_\infty = \lambda_H \frac{2}{\pi} \int_0^\infty \frac{d\chi}{dx}\frac{dx}{x}, \tag{10}$$

and Eqs. (5) and (6) to

$$\varepsilon\lambda\left[\frac{d^2\chi}{dx^2} - \frac{2}{x}\frac{d\chi}{dx}\right] - (x^2+\lambda^2)\chi = -x^2\chi_\infty. \tag{11}$$

Thus in the MHD limit, $\lambda_H \gg \varepsilon^{1/3}$,

$$\chi \simeq \chi_\infty \frac{x^2}{x^2+\lambda^2} \quad \text{and} \quad \lambda \simeq \lambda_H. \tag{12}$$

In the MHD marginal stability case[5] $\lambda_H = 0$, $\chi_\infty = 0$,

$$\chi = -\xi_\infty \frac{\varepsilon^{1/3}}{2^{1/2}} \exp[-x^2/(2\varepsilon^{2/3})] \quad \text{and} \quad \lambda = \varepsilon^{1/3}. \tag{13}$$

In order to derive a general solution for Eq. (11) it is convenient to rewrite Eq. (11) as

$$4\left[\frac{d^2\chi}{d\zeta^2} + \frac{2}{\zeta}\frac{d\chi}{d\zeta}\right] - \left[\frac{\hat{\lambda}^{3/2}}{\zeta} + 1\right]\chi = -\chi_\infty \tag{14}$$

after adopting the independent variable $\zeta \equiv \hat{\chi}^2/\hat{\lambda}^{1/2}$. The solution can be obtained by expansion in Laguerre polynomials[6)] and is represented conveniently by

$$\chi = -\frac{\chi_\infty}{\hat{\lambda}^{3/2}-1}\left\{1-\frac{\hat{\lambda}^{3/2}}{2^{1/2}}\int_0^1 dy\cdot y^{(\hat{\lambda}^{3/2}-1)/4}\right.$$

$$\left.\times\frac{d}{dy}\left[(1+y)^{1/2}\exp\left[-\frac{\zeta(1-y)}{2(1+y)}\right]\right]\right\}. \quad (15)$$

It is easy to verify by direct substitution that (15) satisfies Eq. (14) and that the asymptotic solutions (12) and (13) can be obtained as special cases for $\hat{\lambda} \gg 1$ and $\chi_\infty \to 0$, and $\hat{\lambda} \to 1$ respectively. The complete dispersion relation can be obtained by evaluating the integral

$$\int_0^\infty \frac{d\chi}{dx}\frac{dx}{x} = \varepsilon^{-1/3}\frac{\sqrt{\pi}\,\hat{\lambda}^{5/4}}{\hat{\lambda}^{3/2}-1}\int_0^1 dy\cdot y^{(\hat{\lambda}^{3/2}-1/4)}(1-y)^{-1/2} \quad (16)$$

in Eq. (10). Then

$$\lambda=\lambda_H\left\{\frac{\hat{\lambda}^{5/4}}{8}\frac{\Gamma\left(\frac{\hat{\lambda}^{3/2}-1}{4}\right)}{\Gamma\left(\frac{\hat{\lambda}^{3/2}+5}{4}\right)}\right\}. \quad (17)$$

In addition to recovering the previously mentioned limits $\lambda = \lambda_H$ for $\lambda_H \gg \varepsilon^{1/3}$, and $\hat{\lambda} = 1$ for $\lambda_H = 0$, we obtain

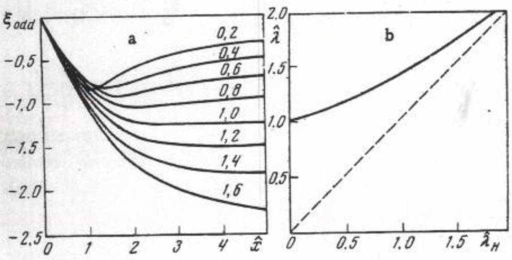

FIG. 1. a) Odd part, $\xi - \frac{1}{2}\xi_\infty$, of the resistive layer solution ξ as a function of $\hat{x} = (r - r_0)/(r_0\varepsilon^{1/3})$ for different values of the normalized growth rate $\hat{\lambda} = \lambda/\varepsilon^{1/3}$. The eigenfunctions that are represented here have been derived by direct numerical integration of Eqs. (5) and (6) as described in the paragraph preceding Eq. (9). These agree with the analytical representation (15). Notice the change in character of the eigensolutions as $\hat{\lambda}$ becomes smaller than unity, corresponding to $\lambda_H < 0$. b) The solid curve represents the growth rate as a function of the ideal MHD growth that is measured by $\hat{\lambda}_H \equiv \lambda_H/\varepsilon^{1/3}$. This curve has been obtained by the direct numerical integration of Eqs. (5) and (6) as indicated earlier and the analytical dispersion relation (17) agrees with it.

$$\lambda \simeq \varepsilon^{1/3}\{\Gamma(5/4)/[\lambda_H\Gamma(-1/4)]\}^{4/5}$$

for $\lambda_H < 0$ and $\hat{\lambda} < 1$. We recall that this scaling of λ, in terms of ε, is the same as that of the well-known tearing mode[8] and, as we can see from Fig. 1a, the function ξ_{odd} tends to become of the same type as that of this mode as $\hat{\lambda}$ goes below unity. We notice that in this limit the effects of the resistivity gradient are important and should be considered in order to have a realistic assessment of the mode stability. We note that Eq. (17) thus gives a unified description for MHD stable, neutral, and unstable m = 1 modes, providing a link between the internal kink and tearing mode theories. Present experiments (the rapid portion of the sawtooth[2] oscillations) tend to fall in the region $\lambda \gtrsim 1$, where the two effects are of comparable importance.

Finally, it is easy to verify that the considered resistive mode, in addition to having a larger growth rate than its purely MHD version for $\lambda_H \lesssim \varepsilon^{1/3}$, generates a completely different topology of the perturbed magnetic field within the resistive layer. In particular, while $B_{1r} = 0$ at $r = r_0$ when the effects of resistivity are neglected, we now have $B_{1r} \neq 0$, at $r = r_0$, and[4] of order $\varepsilon^{1/3} F' r_0^2 \xi_\infty$. This makes the considered instability a natural process to explain the formation of magnetic islands.

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